A new approach to multivariate extreme value theory - f-implicit max-infinitely divisible distributions and f-implicit max-stable processes

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"Mathematics is the language with which God has written the universe"

\- Galileo Galilei (1564-1642) -

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Zusammenfassung

Es seien $X_1, ..., X_n$ unabhängig und identisch verteilte Zufallsvektoren mit Werten in $\mathbb{R}^d$ sowie $f : \mathbb{R}^d \to [0, \infty)$ eine geeignete, die Rolle einer *Verlustfunktion* spielende Abbildung. Zudem sei $k(n) = \mathrm{argmax}_{k=1,...,n} f(X_k)$. Entsprechend [SchSt14, Definition 4.1] ist ein $\mathbb{R}^d$-wertiger Zufallsvektor X f-implizit max-stabil, wenn zu jedem $n \geq 1$ Zahlen $a_n > 0$ existieren, sodass für unabhängige Kopien $X_1, ..., X_n$ von X die Zufallsvektoren $a_n^{-1} X_{k(n)}$ und X dieselbe Verteilung besitzen. Unser Ziel ist es nun an dieses Konzept anzuknüpfen und es auf zwei spezielle Arten zu verallgemeinern. Dazu entwickeln wir eine neue Theorie, die wir *f-implizite Extremwerttheorie* nennen. Diese ähnelt stark der klassischen Extremwerttheorie, ist jedoch hinsichtlich des konkreten Blickwinkels auf die Analyse extremer Ereignisse von grundlegendem Unterschied. Genauer gesagt handelt es sich um die bereits in [SchSt14] vorgeschlagene Alternative zur klassischen Extremwerttheorie, die sich weniger der Analyse von *Ausreißern*, d.h. maximalen Werten einer Stichprobe widmet, sondern vielmehr der Analyse von Ereignissen, die zu extremen *Verlustszenarien* führen.
Im ersten Teil der Arbeit stellen wir einige, für den weiteren Inhalt essentielle Grundlagen zusammen. Im Einzelnen handelt es sich um die *f-implizite max-Operation*, einer speziellen inneren zweistelligen Verknüpfung auf $\mathbb{R}^d$, die *f-implizite max-Faltung* sowie die *f-implizite max-Ordnung*. Außerdem studieren wir hier die Verteilung des *f-impliziten Maximums* $X_{k(n)}$ von $X_1, ..., X_n$.
Darauf aufbauend führen wir im zweiten Teil der Arbeit das Konzept *f-implizit max-unendlich teilbarer Verteilungen* ein und erweitern dadurch die Klasse der f-implizit max-stabilen Verteilungen. In diesem Kontext beweisen wir die Zugehörigkeit zweier spezieller Klassen von Verteilungen zur Klasse der f-implizit max-unendlich teilbaren Verteilungen. Ein wichtiges Hilfsmittel, neben den vorab genannten Grundlagen, stellt dabei das Konzept der *f-impliziten max-Faltungshalbgruppe* dar. Zunächst offen bleibt die Frage, ob tatsächlich sogar alle Verteilungen auf $\mathbb{R}^d$ f-implizit max-unendlich teilbar sind. In diesem Zusammenhang könnten die sogenannten *f-impliziten max-zusammengesetzten Poissonverteilungen* eine entscheidende Rolle spielen.
Im dritten Teil der Arbeit wenden wir uns den *f-implizit max-stabilen Prozessen* zu. Diese stellen das Analogon zu den max-stabilen Prozessen aus der klassischen Extremwerttheorie dar. Durch die Frage nach geeigneten Beispiele solcher Prozesse motiviert, führen wir die Konzepte der *f-impliziten sup-Maße* und *f-impliziten Extremwertintegrale* ein. Wir beweisen die Existenz solcher f-impliziten sup-Maße, um dann die f-impliziten Extremwertintegrale einführen zu können, d.h. Integrale von deterministischen Funktionen bezüglich eines f-impliziten sup-Maßes. Dies liefert uns letztlich, wie gewünscht, eine Vielzahl von Beispielen f-implizit max-stabiler Prozesse.
Zum Abschluss sprechen wir einige Inhalte an, die über jene dieser Arbeit hinausgehen und das Gebiet der f-impliziten Extremwerttheorie weiter vorantreiben könnten.

Abstract

Let $X_1, ..., X_n$ be independent and identically distributed random vectors in $\mathbb{R}^d$ and $f : \mathbb{R}^d \to [0, \infty)$ a suitable function being referred to as the *loss function*. Further, let $k(n) = \text{argmax}_{k=1,...,n} f(X_k)$. Referring to [SchSt14, Definition 4.1], we recall that a random vector X in $\mathbb{R}^d$ is f-implicit max-stable if for all $n \geq 1$ there exist $a_n > 0$ such that $a_n^{-1} X_{k(n)}$ and X are equal in distribution, with $X_1, ..., X_n$ being independent copies of X. Now, our aim is to expand on this notion and to advance it. To this end, we develop a new mathematical framework called *f-implicit extreme value theory*, which is closely related to multivariate extreme value theory but yet different as to the study of extremes. More precisely, adopting the approach suggested in [SchSt14], we pursue the idea of focusing on *extreme loss events* rather than *extreme values*. Actually, the motivation behind this stems from some kind of inverse problem where one wants to determine the extremal behavior of an $\mathbb{R}^d$-valued random vector X when only explicitly observing the extremal loss $f(X)$.

In the first part of the thesis, we provide some basics constituting the fundament of our further deliberations. In particular, we establish a specific (inner) binary operation on $\mathbb{R}^d$ called *f-implicit max-operation*. Based on this, we introduce an astute convolution concept as well as a distinctive partial order being referred to as *f-implicit max-convolution* and *f-implicit max-order*, respectively. Finally, we also provide various possibilities to estimate the distribution of the *f-implicit maximum* $X_{k(n)}$ of $X_1, ..., X_n$.

Equipped with these aspects, we develop the notion of *f-implicit max-infinite divisibility* in the second part of this thesis, thus extending the class of f-implicit max-stable distributions. Here, we are able to prove that all random vectors coming under one of two specific classes of random vectors are f-implicit max-infinitely divisible. To this end, we apply the notion of *f-implicit max-convolution semigroups* in addition to the preliminaries. The question whether all distributions on $\mathbb{R}^d$ are f-implicit max-infinitely divisible remains unsolved for the time being. For logical reasons, however, we propose working with *f-implicit max-compound Poisson distributions* to answer this question.

Abandoning the studies on f-implicit max-infinitely divisible distributions, we introduce the class of *f-implicit max-stable processes* in the third and last part of the thesis. These processes are the analogue of max-stable processes occurring in the context of multivariate extreme value theory. In order to provide non-trivial examples of such processes, we apply the ingenious concepts of *f-implicit sup-measures* and *f-implicit extremal integrals*. To be more precise, we firstly prove the existence of an f-implicit sup-measure and then secondly establish the notion of f-implicit extremal integrals, that is, integrals of non-random functions with respect to an f-implicit sup-measures. This ultimately gives plenty of applicable examples of f-implicit max-stable processes.

We conclude the thesis with several suggestions for additional research possibilities which might refine the novel field of f-implicit extreme value theory.

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Introduction and Motivation

As the title already presages, the present thesis approaches a basically new mathematical framework being closely related to multivariate extreme value theory and being referred to as *f-implicit extreme value theory*. More precisely, we will address two specific issues constituting different branches of f-implicit extreme value theory. On the one hand, we will introduce the notion of f-implicit max-infinite divisibility, and on the other hand we will concern ourselves with the notion of f-implicit max-stable processes. The underlying motivation and fundamental idea of approaching exactly these topics stems from [SchSt14].
The latter dealing with *implicit extremes and implicit max-stable laws* is a pioneering research work in the field of modern probability proposing a seminal, innovative and, in particular, entirely new approach to the study and evaluation of extremes. Occasioned by certain practical observations in, for example, actuarial science, the authors promote the idea of focusing on *extreme loss events* rather than extreme values and thus develop theory that helps to understand and model the joint behavior of those factors leading to extreme losses. To be more precise, the idea proposed in [SchSt14] is to study *implicit extremes* or also *implicit losses*, the actual loss being measured by some appropriate *loss function f*. The motivation behind this stems from some kind of inverse problem where one wants to determine the extremal behavior of an $\mathbb{R}^d$-valued random vector X when only explicitly observing the extremal loss $f(X)$.
In conclusion, the latter strategy caught our attention since it yields a new method of studying and estimating extremal events which is similar to the method pursued in multivariate extreme value theory but materially different as to the basic approach. Moreover, it provides plenty of further research possibilities as will definitely become apparent in the course of this thesis.
Similar ideas to the ones in [SchSt14] and thus to ours are seized in the recent work of Clemént Dombry and Mathieu Ribatet [DoRi12] on ℓ-Pareto processes. Whereas our concern is to work on implicit extremes, Dombry and Ribatet study *implicit exceedances*. Conceptually, implicit extremes correspond to the study of implicit maxima, while implicit exceedances go with the study of implicit peaks-over-treshold. Hence, our targeted viewpoint of a suitable f-implicit extreme value theory is the (implicit) counterpart of the framework of the annual maxima method. On the contrary the approach in [DoRi12] constitutes the (implicit) counterpart of the common peak-over-treshold method.

Extreme value theory is an elegant and mathematically fascinating theory pervading an enormous variety of applications. Consequently, it is used widely in many disciplines, such as structural engineering, finance, earth sciences, traffic prediction and geological engineering. Historically, first attempts in extreme value theory date back to the beginnings of the 20th century. Especially to be emphasized in this context are the seminal works of Fisher and Tippett [FiTi 28], Gnedenko [Gn43] and finally Gumbel [Gu58], all of which provide asymptotic results of the distribution of suitably normalized maxima of independent and identically distributed random variables. However, only the PhD thesis by Laurens de Haan entitled *On regular variation and its applications to the weak convergence of sample extremes* [deHa70] ultimately popularized these results and motivated to expand on them. Consequently, well-known mathematicians such as R. Davis, M.R.Leadbetter, T. Mikosch, S.I. Resnick, H. Rootzén, just to mention a few, refined the ideas considered in [FiTi 28], [Gn43], [Gu58] and [deHa70] and contributed to such an extensive, profound and popular theory as extreme value theory is today. Nevertheless, the possibilities still do not seem to be exhausted here. For example, the notion of max-stable processes constitutes a particular branch of extreme value theory supplying many issues that still need to be considered.

Max-stable processes are important models in spatial extreme value theory since they arise as the only possible nondegenerate limits of suitably normalized maxima of independent and identically distributed processes and can indeed be used to assess environmental risks. And even though the theory of max-stable processes has been developing rapidly over the last decades - thanks to a careful analysis of spectral representations of max-stable processes - there are still many questions to be solved, for example, as to ergodic and mixing properties. First studies on these issues can be found in [St08] and [St10]. An even more profound insight into the structure of max-stable processes, however, might be derived by using a method proposed in [StTa05], [Ka09] and [StWa10]. In these works there is talk of the notion of *association* of stable and max-stable processes revealing close connections between these two classes of processes. As it will turn out throughout this thesis, we will unsurprisingly also gain from the notion of stable processes when considering f-implicit max-stable processes.

At this point it would clearly be unfair to ignore all other scientists having contributed to the development of extreme value theory. Nevertheless, we do neither intend to give a thorough survey of the historical development of extreme value theory nor to provide an entire exposition of recent results in this field. Instead, we rather strive for concretizing the idea of a coequal f-implicit extreme value theory by developing the notion of f-implicit max-infinite divisibility and f-implicit max-stable processes. For the sake of completeness, we refer to the theoretical monographs [Re06], [Re07], [LeLiRo83], [deHaFe06], [EmKlMi12] and [FaHüRe11] as well as to the more statistically oriented works [BeGoSeTe04], [Co04] and [Gu58] constituting an account of the state-of-the-art of extreme value theory.

Coming to a first interim conclusion, we state that our purpose of establishing the framework of f-implicit extreme value theory is, except from the pioneering considerations in [SchSt14], new in literature. Furthermore, we underline that the resulting theory is closely related to both extreme value theory and the corresponding theory in which the maximum operation is exchanged by the summation operation (see for example [SaTa94] or [MeSch01]).

To motivate and concretize our idea of this different viewpoint, consider the following scenario carried over from [SchSt14, Section 1].
On January 21 in 1959, Ohio went through an extreme flood that claimed 16 lives and caused extensive damage. The root cause for that was not entirely due to extreme precipitation but rather due to a rare combination of different factors, only one of which is actually intensive rainfall. Not only in such but also in many other applications, extreme loss events are caused by an unusual combination of factors, all of which may or may not be extreme but their coordinated effect is extreme. This ultimately prompted us to develop and consider f-implicit extreme value theory, to incorporate the issues of [SchSt14] into our deliberations and to finally extend them.
In concrete terms, let f be a Borel measurable non-negative function on $\mathbb{R}^d$ modeling the loss $f(x_1, ..., x_d)$ accompanied by the values $x_1, ..., x_d \in \mathbb{R}$. As an example related to the previously mentioned application, let $d = 4$. Further, let x_1, x_2, x_3 and x_4 be the values of ground saturation, snow-melt, precipitation intensity and precipitation duration, respectively. Then $f(x_1, x_2, x_3, x_4)$ stands for the degree of flooding caused by the specific combination of the factors x_1, x_2, x_3 and x_4. Consequently, it is sensible to take the decision of whether a particular event is extreme only by means of $f(x_1, x_2, x_3, x_4)$. In other words, whether or not some of the individual factors are extreme, it is exclusively the coordinated effect of the factors that is crucial. In this particular example as well as in many other applications, however, the individual factors are anything but deterministic and, in general, stochastically dependent. Hence, it is necessary to establish a statistical model enabling us to determine the behavior of the factors of interest when only explicitly observing the corresponding loss. These considerations ultimately yield to the following construction which initiated the studies on implicit extremes and implicit max-stable laws in [SchSt14], and thus provided the basis for the issues and ideas considered in this thesis. Actually, it is even *the* underlying idea of f-implicit extreme value theory.
Let $X = (X^{(1)}, ..., X^{(d)})$ be a random vector in $\mathbb{R}^d$ modeling the joint behavior of the factors $x_1, ..., x_d$. Suppose further that $X_1, ..., X_n$ are independent copies of X. Unlike in multivariate extreme value theory in which the focus is on the behavior of the random vector $M_n = \max(X_1, ..., X_n)$, the maximum here being taken componentwise, we are rather interested in the structure and behavior of the random vector $X_{k(n)}$ with

$$k(n) := \operatorname{argmax}_{k=1,...,n} f(X_k) := \operatorname{argmax}(f(X_1), ..., f(X_n)).$$

In the case of ties $k(n)$ is taken as the smallest index yielding the maximum. To be more precise, $k(n) = i_1$ if $f(X_{i_1}) = ... = f(X_{i_\ell})$ for $1 \leq i_1 < ... < i_\ell \leq n$ and $f(X_j) < f(X_{i_1})$ for $j \in \{1, ..., n\} \setminus \{i_1, ..., i_\ell\}$, where $1 \leq \ell \leq n$. Note that $X_{k(n)}$ is actually the random vector leading to maximal loss, that is,

$$f(X_{k(n)}) = \max_{k=1,...,n} f(X_k) = \max(f(X_1), ..., f(X_n)).$$

Note further that $X_{k(n)}$ is indeed part of the given sample set in contrast to M_n.
As illustrated earlier, the events leading to extreme losses $f(X)$ are of vital importance in practice. Therefore, the focus in [SchSt14] is mainly on the asymptotic behavior of the *f-implicit maximum* $X_{k(n)}$ of $X_1, ..., X_n$ under appropriate normalization. The purpose

in this thesis, however, is to draw in particular on the notion of f-implicit max-stability, which occurs naturally in the context of studies on implicit extremes, and to extend the class of f-implicit max-stable distributions in two different ways. This eventually yields first concrete branches of f-implicit extreme value theory that are based on the pioneering considerations in [SchSt14]. Here, in accordance with Definition 4.1 in [SchSt14] a random vector X in $\mathbb{R}^d$ is referred to as *f-implicit max-stable* if for all $n \geq 1$ there exist $a_n > 0$ such that

$$a_n^{-1} X_{k(n)} \stackrel{d}{=} X.$$

The significance of such distributions, which caught our particular attention, becomes clear due to Theorem 4.2 in [SchSt14]. Namely, analogous to common results in extreme value theory, the class of f-implicit max-stable distributions coincides under some mild assumptions with the class of *f-implicit extreme value distributions*, that is, they arise as the only possible limits of $X_{k(n)}$ under appropriate normalization (see for example [SchSt14, Theorem 4.4]). In addition, a similar analogy does also exist in the context of stable distributions, where stable distributions can be viewed as the counterpart of max-stable distributions by just exchanging the maximum operation for the summation operation. These analogies now raise the question whether even more concepts occurring in, for example, extreme value theory can be transferred into the f-implicit context. Consequently, the idea of considering f-implicit max-infinitely divisible distributions as well as f-implicit max-stable processes is mainly based on these observations.

Now, let the (loss) function $f : \mathbb{R}^d \to [0, \infty)$ be fixed for the remainder of this thesis. Being guided by [SchSt14], we will henceforward make three standing assumptions on f. For convenience, we require f to be continuous and 1-homogeneous with $f(x) = 0$ if and only if $x = 0$. In view of conceivable applications these assumptions on f are not particularly restrictive but quite reasonable. Nevertheless, observe that certain results stated in [SchSt14] can even be derived under slightly relaxed assumptions on f. In this thesis, however, we will only concern that aspect marginally when proposing several extensions in Section 2.4, Section 3.3 and Chapter 4.

Specially geared to the preceding introductory deliberations and the resultant objectives, the present thesis is structured as follows.
As regards content, it is primarily composed of two main parts constituting Chapter 2 and 3. In particular, we refer to the notion of f-implicit max-infinitely divisible distributions on the one hand and f-implicit max-stable processes on the other, both of which will indeed extend the notion of f-implicit max-stable distributions in a certain way. Before attending to both these topics, we establish a profound basis in Chapter 1. We both prepare the content of Chapter 2 and 3 by providing tailor-made essentials and introduce some convenient notation. In addition, we also incorporate the contents of [SchSt14] here. Some new light is especially shed on the notion of f-implicit max-stable distributions.
The primary focus in Chapter 1 is on an operation $\vee_f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ being referred to as f-implicit max-operation. This specific operation depending on $f : \mathbb{R}^d \to [0, \infty)$ can be viewed as a suitably chosen counterpart of the summation and maximum operation

which in turn play a pivotal role in the context of stable and max-stable distributions, respectively. Actually, $\vee_f(x_1, x_2) := x_1 \vee_f x_2$ coincides with the f-implicit maximum of x_1 and x_2, that is, with $x_{k(2)}$ for $k(2) = \operatorname{argmax}(f(x_1), f(x_2))$. Having established this operation, we supply several properties. Inter alia, we show that $\mathbb{R}^d$ equipped with $\vee_f$ is in general a non-commutative, non-topological semigroup with identity element $e = 0$. Based on the f-implicit max-operation, we further establish a proper notion of convolution on the space of bounded measures as well as a distinctive partial order on $\mathbb{R}^d$. The conclusion of Chapter 1 is finally intended to provide a clearer insight into the structure of the f-implicit maximum $X_{k(n)}$. This is basically achieved by determining the distribution of $X_{k(n)}$. In fact, we derive several possibilities to compute $\mathbb{P}(X_{k(n)} \in A), A \in \mathcal{B}(\mathbb{R}^d)$, explicitly.

In Chapter 2 we apply the acquired basics of Chapter 1 to address the notion of f-implicit max-infinite divisibility. Considering the common classes of infinitely and max-infinitely divisible distributions, we observe that they arose from certain issues in probability theory and extended the already existing classes of stable and max-stable distributions, respectively. Analogously, this is the case with f-implicit max-stable and f-implicit max-infinitely divisible distribution. Here, the f-implicit max-convolution appears to be particularly relevant. Namely, recall that a probability measure μ on $\mathbb{R}^d$ is, for instance, referred to as infinitely divisible if for all $n \geq 1$ there exist probability measures μ_n on $\mathbb{R}^d$ such that

$$\mu = (\mu_n)^{*n}.$$

Consequently, transferring the previous definition into the f-implicit context, we recognize that defining the notion of f-implicit max-infinite divisibility by means of the f-implicit max-convolution is indeed an appropriate approach. Next, we consider the main question in Chapter 2. More precisely, we investigate whether the class of f-implicit max-infinitely divisible distributions can be characterized as is the case with infinitely and max-infinitely divisible distributions (see [MeSch01, Theorem 3.1.11] and [Re07, Proposition 5.8]). As to that, we present two results. First, we prove that all random vectors X in $\mathbb{R}^d$ such that $x \mapsto \mathbb{P}(f(X) \leq x)$ is continuous on (ℓ, ∞), $\ell \geq 0$ here being the left endpoint of $f(X)$, are f-implicit max-infinitely divisible. Second, we show that all random vectors X in $\mathbb{R}^d$ such that the mass of $\mathbb{P}_{f(X)}$ is concentrated on a countable subset of $[0, \infty)$ are also f-implicit max-infinitely divisible. Apart from certain substitution rules for Stieltjes integrals, a beneficial tool in this context is the notion of f-implicit max-convolution semigroups. At the end of Chapter 2, we finally make an assertion yielding an open question that remains unsolved for the time being. We suggest that probably all distributions on $\mathbb{R}^d$ are f-implicit max-infinitely divisible.

In Chapter 3 we eventually attend to f-implicit max-stable processes. As before, we profit immensely from the preparations of Chapter 1. Whereas the f-implicit max-order is not relevant in Chapter 2, it is actually important here. Gaining intuition from the notion of α-stable and max-stable processes, we first provide a proper notion of f-implicit max-stable processes. Motivated by the fact that each f-implicit max-stable process supplies automatically a max-stable process with α-Fréchet marginals, we formulate our overarching goal which consists of constructing non-trivial examples of f-implicit max-stable processes. In order to meet this challenge, we develop an ingenious technique. We firstly establish the notion of f-implicit sup-measures and then secondly that

of f-implicit extremal integrals, that is, integrals of non-random functions with respect to an f-implicit sup-measure. Actually, this idea originates from [SaTa94] and [StTa05] in which appropriate counterparts of f-implicit sup-measures and f-implicit extremal integrals have been introduced in order to study α-stable and max-stable processes, respectively. In particular, this refers to α-stable random measures and α-stable stochastic integrals as well as to random α-Fréchet sup-measures and extremal stochastic integrals. In conclusion, both the notion of f-implicit sup-measures and f-implicit extremal integrals turn out to be an efficient tool to supply plenty of examples of f-implicit max-stable processes. Moreover, both these concepts are ultimately also a sophisticated construct of independent interest in f-implicit extreme value theory.
In the last and fourth chapter we summarize our findings particularly from the viewpoint of an applicable f-implicit extreme value theory. We also give a final outlook in which we propose considering primarily two promising issues in addition to the ones constituting Section 2.4 and Section 3.3. In doing so, we highlight the still existing possibilities in f-implicit extreme value theory and thus point out that this fairly new field is not exhausted at all.
In addition to an extensive list of symbols, one can also find a complete bibliography consisting of all used literature at the end of this thesis.

1 Preliminaries

In the preceding and detailed introduction we proposed considering two attractive topics in this thesis which refine the idea of an f-implicit extreme value theory and constitute two fundamental branches in this field. In particular, there is talk of f-implicit max-infinitely divisible distributions and f-implicit max-stable processes. Here, the first will be addressed in Chapter 2 to extend the notion of f-implicit max-stable distributions which has first been studied in [SchSt14, Section 4] as part of a new approach to multivariate extreme value theory. The latter, on the contrary, will be studied in Chapter 3. In preparation for this ambitious scheme, Chapter 1 is intended to provide the theoretical basics. In addition, some specific and repeatedly used notation is introduced. Accordingly, Chapter 1 laying the groundwork for the content of Chapter 2 and 3 is structured as follows.

In Section 1.1 we concern ourselves with the study of a distinctive (inner) binary operation on $\mathbb{R}^d$ constituting the basis of all subsequent examinations. As it will turn out, this operation depends considerably on our fixed loss function $f : \mathbb{R}^d \to [0, \infty)$.

Equipped with the notion introduced in Section 1.1, we proceed with an appropriate convolution concept depending on f as well. This constitutes the content of Section 1.2.

In Section 1.3 we attend to another theoretical aspect. Here, the focus is on the study of a specific binary relation between $\mathbb{R}^d$ and $\mathbb{R}^d$ gaining center stage in Chapter 3. This relation turns out to be a partial order on $\mathbb{R}^d$, thus providing $\mathbb{R}^d$ with a supplemental structure in addition to the operation introduced in Section 1.1.

In Section 1.4 we finally address ourselves to a detailed study of the distribution of the particular $\mathbb{R}^d$-valued random vector $X_{k(n)}$. This will be hugely useful in Chapter 2.

Before proceeding with Section 1.1, we recall that $f : \mathbb{R}^d \to [0, \infty)$ is assumed to be continuous and 1-homogeneous with $f(x) = 0$ if and only if $x = 0$. Here, a function $g : \mathbb{R}^d \to \mathbb{C}$ is referred to as 1-*homogenous* if $g(\lambda x) = \lambda g(x)$ for every $\lambda > 0$ and $x \neq 0$.

It should once be mentioned that some of the results in the subsequent deliberations are even valid under weaker assumptions on f. This, however, will not be considered in detail but will merely be mentioned on occasion.

For a deeper discussion on the role of the loss function f occurring in the context of an f-implicit extreme value theory, we refer to [SchSt14].

1.1 The f-implicit max-operation

As we are permanently guided by two securely established theories of statistics and probability theory, it seems reasonable to consider them in more detail. On the hand hand, there is, inter alia, the notion of stable distributions on $\mathbb{R}^d$ originating in examinations concerning limit theorems for sums of independent and identically distributed random vectors. One the other hand, there is the elaborately investigated extreme val-

ue theory and, in particular, the notion of max-stable distributions on $\mathbb{R}^d$. The notion of max-stable distributions on $\mathbb{R}^d$ is also engendered by considerations of limit theorems. This time, however, there is talk of limit theorems for the (componentwise) maxima of independent and identically distributed random vectors. For a recent account of these issues we refer to either [MeSch01] and [SaTa94] or to [deHaFe06] and [Re07].
Having a closer look at these concepts, we recognize that they share striking parallels and pursue similar ideas. In addition, several further connections between these two branches have been established during the last decades (see for example [StTa05] and [Ka09]). Nevertheless, the fundamental difference between the notion of stable and max-stable distributions on $\mathbb{R}^d$ is actually the underlying operation. Stable distributions are defined by means of the summation operation

$$+ : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, \quad (x_1, x_2) \mapsto x_1 + x_2,$$

whereas max-stable distribution are defined by means of the componentwise maximum operation

$$\max : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, \quad (x_1, x_2) \mapsto \max(x_1, x_2) := x_1 \vee x_2,$$

both being continuous and commutative. Considering the concepts from this particular point of view, we are encouraged to introduce the notion of an appropriate operation substituting the two latter ones in the setting of f-implicit extreme value theory. Complying with this task, we obtain what will be referred to as the f-implicit max-operation.

Definition 1.1.1
The (inner) binary operation $\vee_f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ on $\mathbb{R}^d$, defined by

$$\vee_f(x_1, x_2) := x_1 \vee_f x_2 := \begin{cases} x_1, & \text{if } f(x_2) \le f(x_1) \\ x_2, & \text{if } f(x_2) > f(x_1), \end{cases}$$

is called *f-implicit max-operation*. Inductively, for all $n \ge 2$, we define

$$\bigvee_{i=1}^{n}{}_f\, x_i := (x_1 \vee_f \ldots \vee_f x_{n-1}) \vee_f x_n$$

and further use the sensible convention

$$\bigvee_{i=1}^{1}{}_f\, x_i = x_1.$$

Remark 1.1.2. Note, the notion of f-implicit max-stable distributions (see [SchSt14, Section 4]), which has already been recalled in the introduction and will be taken up again in Chapter 2 and 3, can now be rewritten by using the f-implicit max-operation. Indeed, a random vector X in $\mathbb{R}^d$ is f-implicit max-stable in accordance with [SchSt14, Definition 4.1] if for all $n \ge 1$ there exist $a_n > 0$ such that

$$a_n^{-1} \bigvee_{i=1}^{n}{}_f\, X_i \overset{d}{=} X,$$

the random vectors $X_1, ..., X_n$ being independent copies of X. This is actually because of the fact that

$$\bigvee_{i=1}^{n}{}_f \, X_i = X_{k(n)}$$

for all $n \geq 1$ which in turn follows immediately from Definition 1.1.1 and from the definition of the random variable $k(n)$. Note that the latter equation is still true even if we dispense with the particular assumptions on the random vectors $X_1, ..., X_n$.

Remark 1.1.3. The algebraic structure $(\mathbb{R}^d, \vee_f)$ consisting of the set $\mathbb{R}^d$ together with the binary f-implicit max-operation $\vee_f$ is a semigroup with identity element $e = 0$, that is, we have

(1) $$x_1 \vee_f (x_2 \vee_f x_3) = (x_1 \vee_f x_2) \vee_f x_3 \tag{1.1.1}$$

(2) $$x \vee_f 0 = 0 \vee_f x = x \tag{1.1.2}$$

for all $x, x_1, x_2, x_3 \in \mathbb{R}^d$. The first property is clear by Definition 1.1.1. The second one follows from $\{f = 0\} = \{0\}$. Thus, for all $n \geq 2$, we may henceforth unambiguously write $(x_1 \vee_f ... \vee_f x_{n-1}) \vee_f x_n = x_1 \vee_f ... \vee_f x_{n-1} \vee_f x_n$.

Whereas the summation and maximum operation are continuous and commutative, the f-implicit max-operation is in general both discontinuous and non-commutative. Thus, $(\mathbb{R}^d, \vee_f)$ is neither a topological semigroup nor a commutative one. This involves the most aspects since continuity and, of course, commutativity are desirable properties. The f-implicit max-operation being non-commutative follows directly from Definition 1.1.1 as f does not need to be injective. The fact that the mapping $\vee_f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is in general discontinuous is illustrated by the subsequent example.

Example 1.1.4

Suppose that $f : \mathbb{R}^d \to [0, \infty)$ is explicitly given by $f(x) = \|x\|_2$, the norm $\|\cdot\|_2$ here being the common Euclidean norm. This is clearly a suitable choice for a loss function. Further, let $(x^{(m)})_{m \geq 1}$ denote the particular sequence in $\mathbb{R}^d \times \mathbb{R}^d$ given by $x^{(m)} = (x_1^{(m)}, x_2^{(m)})$ with

$$x_1^{(m)} = \left(-\frac{1}{\sqrt{d}} + \frac{1}{m}, ..., -\frac{1}{\sqrt{d}} + \frac{1}{m}\right) \qquad \text{and} \qquad x_2^{(m)} = \left(\frac{1}{\sqrt{d}} - \frac{1}{m^2}, ..., \frac{1}{\sqrt{d}} - \frac{1}{m^2}\right)$$

for all $m \geq 1$. Then

$$x^{(m)} \xrightarrow[(m \to \infty)]{} x = (x_1, x_2),$$

where

$$x_1 = \left(-\frac{1}{\sqrt{d}}, ..., -\frac{1}{\sqrt{d}}\right) \qquad \text{and} \qquad x_2 = \left(\frac{1}{\sqrt{d}}, ..., \frac{1}{\sqrt{d}}\right).$$

Consequently, we obtain

$$x_1 \vee_f x_2 = x_1$$

as $f(x_1) = 1 = f(x_2)$. Conversely, however, we have

$$x_1^{(m)} \vee_f x_2^{(m)} = x_2^{(m)}$$

for all m sufficiently large (depending on the dimension $d \geq 1$), thus implying

$$x_1^{(m)} \vee_f x_2^{(m)} \xrightarrow[(m\to\infty)]{} x_2 \neq x_1.$$

Note, however, that the mapping $\vee_f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is clearly continuous at all points $(x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d$ with $f(x_1) \neq f(x_2)$.
Having investigated the structure of the f-implicit max-operation by Remark 1.1.3 and Example 1.1.4 and having connected the f-implicit max-operation with the concept of f-implicit max-stable distributions by Remark 1.1.2, we proceed to elaborate on these observations in order to get an even better understanding of the f-implicit max-operation. To start with, we provide several properties of the f-implicit max-operation that will gain in interest in Chapter 2 and Chapter 3.

Lemma 1.1.5
Fix $x, x_1, x_2 \in \mathbb{R}^d$.

(a) For all $a, b, c \geq 0$, we have

(i) $(a \vee b)\, x = ax \vee_f bx$ (1.1.3)

(ii) $c\,(x_1 \vee_f x_2) = cx_1 \vee_f cx_2$ (1.1.4)

(b) If $f(x_1) \neq f(x_2)$, then $x_1 \vee_f x_2 = x_2 \vee_f x_1$.

Proof. (a) Fix $a, b, c \geq 0$. We begin by proving (1.1.3). Clearly, (1.1.3) is true if $x = 0$. Hence, assume $x \neq 0$. If $a = b = 0$, $a = 0 < b$ or $b = 0 < a$, then (1.1.3) is an easy consequence of Definition 1.1.1 and the assumption on the null set of f. Otherwise, if $a, b > 0$, then the 1-homogeneity of f yields the desired conclusion. Indeed, by assuming $0 < a < b$, we get $f(ax) = af(x) < bf(x) = f(bx)$ as well as $a \vee b = b$ and thus (1.1.3). The case $0 < b \leq a$ can be treated similarly. To prove (1.1.4), we may assume both $c > 0$ and $x_1, x_2 \neq 0$, for if not, the asserted claim is again already clear by the fact that $f(x) = 0$ if and only if $x = 0$. Let $f(x_1) \geq f(x_2)$ and hence $x_1 \vee_f x_2 = x_1$. Then we deduce, by 1-homogeneity of f, that $cx_1 \vee_f cx_2 = cx_1$ yielding (1.1.4). The case $f(x_1) < f(x_2)$ can be handled similarly, and the proof of part (a) is complete.
(b) This is clear by Definition 1.1.1. □

Note, the second part of Lemma 1.1.5 shows that $f(x_1) \neq f(x_2)$ is a sufficient condition for x_1, x_2 to commute under the in general non-commutative f-implicit max-operation. This will be crucial in particular with regard to the deliberations in Chapter 3.
The next lemma may first seem dispensable. However, it will prove necessary for Chapter 2, especially for the notion of f-implicit max-convolution.

Lemma 1.1.6
For all $n \geq 1$, the mapping $T_f^{(n)} := T^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}^d$, defined by

$$T^{(n)}(x_1, ..., x_n) := \bigvee_{f\, i=1}^{n} x_i,$$

is $\mathcal{B}(\mathbb{R}^d)^{\otimes n}$-$\mathcal{B}(\mathbb{R}^d)$-measurable.

Proof. The proof is by induction on $n \geq 2$. To show the asserted claim for $n = 2$, let $\mathrm{pr}_i : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, i=1,2, denote the canonical projections on the ith coordinate, that is,

$$\mathrm{pr}_1(x_1, x_2) = x_1 \qquad \text{and} \qquad \mathrm{pr}_2(x_1, x_2) = x_2.$$

Further, let $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ denote the mapping defined by

$$g(x_1, x_2) := f(x_1) - f(x_2) = f(\mathrm{pr}_1(x_1, x_2)) - f(\mathrm{pr}_2(x_1, x_2)).$$

Then g is continuous and hence particularly measurable as it is a composition of continuous functions. Referring to Definition 1.1.1, we finally deduce that

$$T^{(2)}(x_1, x_2) = x_1 \vee_f x_2 = \mathrm{pr}_1(x_1, x_2) \cdot \mathbb{1}_{g^{-1}([0,\infty))}(x_1, x_2) + \mathrm{pr}_2(x_1, x_2) \cdot \mathbb{1}_{g^{-1}((-\infty,0))}(x_1, x_2),$$

thus yielding the desired measurability of $T^{(2)}$. Now, let $T^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}^d$ be measurable for some $n \geq 2$. In order to show measurability of

$$T^{(n+1)}(x_1, ..., x_{n+1}) = (x_1 \vee_f ... \vee_f x_n) \vee_f x_{n+1} = T^{(n)}(x_1, ..., x_n) \vee_f x_{n+1},$$

we define the mappings

$$\mathrm{pr}_{1,...,n} : (\mathbb{R}^d)^{n+1} \to (\mathbb{R}^d)^n, \quad (x_1, ..., x_{n+1}) \mapsto (x_1, ..., x_n)$$

and

$$\mathrm{pr}_{n+1} : (\mathbb{R}^d)^{n+1} \to \mathbb{R}^d, \quad (x_1, ..., x_{n+1}) \mapsto x_{n+1}.$$

These mappings are continuous and hence measurable. By induction hypothesis, we therefore conclude that the mapping

$$h : (\mathbb{R}^d)^{n+1} \to \mathbb{R}^d \times \mathbb{R}^d, \quad (x_1, ..., x_{n+1}) \mapsto \left(T^{(n)} \circ \mathrm{pr}_{1,...,n}(x_1, ..., x_{n+1}), \mathrm{pr}_{n+1}(x_1, ..., x_{n+1})\right)$$

is measurable as well. Consequently, applying the induction basis and the relation

$$T^{(n+1)} = T^{(2)} \circ h,$$

we get the measurability of $T^{(n+1)}$. In addition, $T^{(1)}$ coinciding with the identity function is clearly measurable, which completes the proof. □

With view to Proposition 3.2.4 providing some properties of the f-implicit extremal stochastic integral we additionally require the following statement which is actually the result of the measurability of $T^{(n)}$.

Corollary 1.1.7

Fix $n \geq 2$. Let $X_1, ..., X_n$ be independent random vectors in $\mathbb{R}^d$ defined on some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. Suppose that K_1 and K_2 are two disjoint and non-empty subsets of $\{1, ..., n\}$, that is, $K_1 = \{i_1, ..., i_{k_1}\}$ and $K_2 = \{j_1, ..., j_{k_2}\}$ for pairwise distinct indices $i_1, ..., i_{k_1}, j_1, ..., j_{k_2} \in \{1, ..., n\}$ with $1 \leq k_1, k_2 \leq n-1$. Then the random vectors Z_1 and Z_2, defined by

$$Z_1 = \bigvee_{m=1}^{k_1}{}_f \, X_{i_m} \qquad \text{and} \qquad Z_2 = \bigvee_{m=1}^{k_2}{}_f \, X_{j_m}$$

are independent.

Proof. Lemma 1.1.6 shows that the mappings

$$T^{(k_p)} : (\mathbb{R}^d)^{k_p} \to \mathbb{R}^d, \quad (x_1, ..., x_{k_p}) \mapsto \bigvee_{m=1}^{k_p}{}_f \, x_m,$$

p=1,2, are measurable. Therefore, we can apply [Ba91, Theorem 9.6] to conclude that the random vectors

$$Z_1 = T^{(k_1)}(X_{i_1}, ..., X_{i_{k_1}}) \qquad \text{and} \qquad Z_2 = T^{(k_2)}(X_{j_1}, ..., X_{j_{k_2}})$$

are independent. □

Remark 1.1.8. Note that the assertion of Corollary 1.1.7 does not depend on the specific choice of order of the indices $i_1, ..., i_{k_1}, j_1, ..., j_{k_2}$ of the sets K_1 and K_2, but the random vectors Z_1 and Z_2 actually do. By this we mean that

$$\tilde{Z}_1 = \bigvee_{m=1}^{k_1}{}_f \, X_{i_{\pi_1(m)}} \qquad \text{and} \qquad \tilde{Z}_2 = \bigvee_{m=1}^{k_2}{}_f \, X_{j_{\pi_2(m)}}$$

are also independent, π_1 and π_2 here being permutations of $\{1, ..., k_1\}$ and $\{1, ..., k_2\}$, respectively. The random vectors $\tilde{Z}_1$ and $\tilde{Z}_2$, however, will in general not coincide with Z_1 and Z_2, respectively, as the f-implicit max-operation is usually non-commutative. If the random vectors $X_1, ..., X_n$ commute almost surely under the f-implicit max-operation, as it will be the case in Proposition 3.2.4, then we even have $Z_1 = \tilde{Z}_1$ and $Z_2 = \tilde{Z}_2$ almost surely.

In addition, we finally need the subsequent, inconspicuous lemma that will particularly gain in interest in Chapter 3.

Lemma 1.1.9
If $(x_n)_{n\geq1}$ and $(y_n)_{n\geq1}$ are two $\mathbb{R}^d$-valued sequences such that

$$x_n \xrightarrow[(n\to\infty)]{} x \qquad \text{and} \qquad y_n \xrightarrow[(n\to\infty)]{} 0$$

for some $x \in \mathbb{R}^d$, then

$$x_n \vee_f y_n \xrightarrow[(n\to\infty)]{} x.$$

Proof. The proof is straightforward but will, however, strongly depend on the assumption that one of the involved sequences tends to zero.
We start by assuming $x = 0$. Fix $\varepsilon > 0$. By assumption, there exist integers $N_1 \geq 1$ and $N_2 \geq 1$ such that $|x_n| < \varepsilon$ for all $n \geq N_1$ and $|y_n| < \varepsilon$ for all $n \geq N_2$. Since each element $x_n \vee_f y_n$ of the sequence $(x_n \vee_f y_n)_{n\geq1}$ is either x_n or y_n, we conclude that $|x_n \vee_f y_n| < \varepsilon$ for all $n \geq \max(N_1, N_2)$. This is precisely the assertion of the lemma, provided $x = 0$.
We proceed with the remaining case, the real number x being nonzero. Let $\delta := \|x\| > 0$, where $\|\cdot\|$ denotes some norm on $\mathbb{R}^d$. Applying the assumption on the sequence $(x_n)_{n\geq1}$, we may find an integer $N_3 \geq 1$, depending on $\delta > 0$, such that

$$x_n \in K_{\frac{\delta}{2}}(x)$$

for all $n \geq N_3$, where $K_{\delta/2}(x)$ denotes the open ball in $\mathbb{R}^d$ with radius $\frac{\delta}{2}$ and center x. Accordingly, we deduce that $\|x_n\| > \frac{\delta}{2}$ for all $n \geq N_3$ and hence $x_n \neq 0$ for all $n \geq N_3$. Now, let c denote the infimum of f on the sphere $S^1 := \{x \in \mathbb{R}^d : \|x\| = 1\}$. Since S^1 is compact and f continuous, this infimum is actually a minimum. Applying the assumption on the null set of f, we see that $c > 0$. Then, f being 1-homogeneous, we eventually deduce that

$$f(x_n) = \|x_n\| \cdot f\left(\frac{x_n}{\|x_n\|}\right) \geq \frac{\delta}{2}c$$

for all $n \geq N_3$. Since f is continuous with $f(0) = 0$ and $(y_n)_{n\geq 1}$ is a null sequence, there exists another integer $N_4 \geq 1$, depending on $\frac{\delta}{2}c > 0$, such that $f(y_n) < \frac{\delta}{2}c$ for all $n \geq N_4$. Consequently, we conclude that

$$f(y_n) < f(x_n)$$

for all $n \geq \max(N_3, N_4)$. Thus, the sequence $(x_n \vee_f y_n)_{n\geq 1}$ actually coincides with the sequence $(x_n)_{n\geq 1}$ for almost all $n \geq 1$, which completes the proof. □

Remark 1.1.10. Obviously, the assertion of Lemma 1.1.9 remains true if $(x_n)_{n\geq 1}$ is the null sequence and $(y_n)_{n\geq 1}$ converges to some $x \in \mathbb{R}^d$.

Now, we proceed with a more detailed study of an appropriate convolution concept. In this context, the significance of the f-implicit max-operation becomes apparent and thus the importance of this precursory section.

1.2 The f-implicit max-convolution

In preparation for the notion of f-implicit max-infinitely divisible distributions, which will extend the already tremendously investigated class of f-implicit max-stable distributions [SchSt14, Section 4] in an appropriate way, we need to establish a suitable convolution concept. This approach seems to be sensible as it originates from the notion of infinitely and max-infinitely divisible distributions. We will expand on these concepts and its close connections to the notion of f-implicit max-infinitely divisible distributions in Chapter 2. In doing so, we will also propose considering the infinite divisibility concept from a more abstract point of view (see [BeChRe84]).
To start with, let $M^b(\mathbb{R}^d)$ denote the set of all bounded measures on $\mathbb{R}^d$ equipped with the Borel σ-Algebra $\mathcal{B}(\mathbb{R}^d)$ and $M^1(\mathbb{R}^d)$ denote the set of all probability measures on $\mathbb{R}^d$.

Definition 1.2.1
For $\mu_1, \mu_2 \in M^b(\mathbb{R}^d)$, we refer to the convolution induced by $T^{(2)}$ as *f-implicit max-convolution* being denoted by $\mu_1 *_f \mu_2$ for the remainder of this thesis. More precisely,

$$\mu_1 *_f \mu_2 = T^{(2)}(\mu_1 \otimes \mu_2). \tag{1.2.1}$$

For $\mu_1, ..., \mu_n \in M^b(\mathbb{R}^d)$, we further define the multiple f-implicit max-convolution inductively by

$$\mu_1 *_f ... *_f \mu_n := (\mu_1 *_f ... *_f \mu_{n-1}) *_f \mu_n = T^{(n)}(\mu_1 \otimes ... \otimes \mu_n) \tag{1.2.2}$$

for all $n \geq 2$.

Remark 1.2.2. (i) Note that the f-implicit max-convolution is well-defined since $T^{(2)}$ is measurable by Lemma 1.1.6.

(ii) The operator $\mathcal{F} : M^b(\mathbb{R}^d) \times M^b(\mathbb{R}^d) \to M^b(\mathbb{R}^d)$, defined by

$$\mathcal{F}(\mu_1, \mu_2) = \mu_1 *_f \mu_2,$$

is called *f-implicit max-convolution operator*.

(iii) Clearly, the f-implicit max-convolution inherits the properties of being associative but generally non-commutative from the f-implicit max-operation. However, Lemma 1.1.5 (b) provides a sufficient condition for two bounded measures μ_1 and μ_2 to commute under the f-implicit max-convolution.

(iv) For convenience, let $\mu^{*_f n}$ denote the n-fold f-implicit max-convolution of $\mu \in M^b(\mathbb{R}^d)$ for all $n \geq 1$ with the sensible convention $\mu^{*_f 1} = \mu$.

Remark 1.2.3. Note, Definition 1.2.1 can be used to establish a third possible definition of an f-implicit max-stable random vector X in $\mathbb{R}^d$. In fact, X is f-implicit max-stable in accordance with [SchSt14, Definition 4.1] or equivalently in accordance with Remark 1.1.2 if for all $n \geq 1$ there exist $a_n > 0$ such that

$$(a_n \mu) = \mu^{*_f n}, \tag{1.2.3}$$

where μ denotes the distribution of X. The easy calculations yielding (1.2.3) will be skipped.

Having established the f-implicit max-convolution of bounded measures on $\mathbb{R}^d$, we now attend to a more detailed study of this particular convolution. To be more precise, we will consider the question whether the f-implicit max-convolution $\mu_1 *_f \mu_2$ of $\mu_1, \mu_2 \in M^b(\mathbb{R}^d)$ has an explicit representation. Similar to classical results concerning the common convolution, the next lemma provides such a representation.

Lemma 1.2.4 (a) Suppose that $\mu_1, \mu_2 \in M^b(\mathbb{R}^d)$ and fix $B \in \mathcal{B}(\mathbb{R}^d)$. Then we have

$$\mu_1 *_f \mu_2(B) = \int_B f(\mu_2)\big([0, f(x)]\big)\, \mu_1(dx) + \int_B f(\mu_1)\big([0, f(x))\big)\, \mu_2(dx). \tag{1.2.4}$$

(b) If X, Y are independent random vectors in $\mathbb{R}^d$, then

$$\mathbb{P}_X *_f \mathbb{P}_Y = \mathbb{P}_{X \vee_f Y}. \tag{1.2.5}$$

Proof. (a) Fix $B \in \mathcal{B}(\mathbb{R}^d)$. Applying Definition 1.2.1, subsequently the classical change of variables formula and finally Definition 1.1.1, we conclude that

$$\mu_1 *_f \mu_2(B) = \int_{\mathbb{R}^d} \mathbb{1}_B(z)\, T^{(2)}(\mu_1 \otimes \mu_2)(dz)$$

$$= \int\limits_{\mathbb{R}^d \times \mathbb{R}^d} \mathbb{1}_B \left(x \vee_f y\right) (\mu_1 \otimes \mu_2)(dx, dy)$$

$$= \int\limits_{\{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d:\, f(x)\geq f(y)\}} \mathbb{1}_B (x) (\mu_1 \otimes \mu_2)(dx, dy)$$

$$+ \int\limits_{\{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d:\, f(x)< f(y)\}} \mathbb{1}_B (y) (\mu_1 \otimes \mu_2)(dx, dy)$$

$$= \int\limits_{\mathbb{R}^d} \mathbb{1}_B (x) \int\limits_{\mathbb{R}^d} \mathbb{1}_{f^{-1}([0,f(x)])} (y)\, \mu_2(dy)\, \mu_1(dx)$$

$$+ \int\limits_{\mathbb{R}^d} \mathbb{1}_B (y) \int\limits_{\mathbb{R}^d} \mathbb{1}_{f^{-1}([0,f(y)))} (x)\, \mu_1(dx)\, \mu_2(dy)$$

$$= \int\limits_B f(\mu_2)\Big([0, f(x)]\Big)\, \mu_1(dx) + \int\limits_B f(\mu_1)\Big([0, f(x))\Big)\, \mu_2(dx).$$

for any two bounded measures μ_1, μ_2, which establishes (1.2.4).
(b) Equation (1.2.5) follows immediately from Definition 1.2.1 together with the assumed independence of X, Y. Indeed,

$$\mathbb{P}_X *_f \mathbb{P}_Y = T^{(2)} (\mathbb{P}_X \otimes \mathbb{P}_Y) = T^{(2)} \left(\mathbb{P}_{(X,Y)}\right) = \mathbb{P}_{T^{(2)}(X,Y)} = \mathbb{P}_{X \vee_f Y}.$$

□

The subsequent remarks concerning Lemma 1.2.4 will eventually complete Section 1.2. Having established them, we will then proceed with Section 1.3 and thus with the f-implicit max-order.

Remark 1.2.5. Equation (1.2.4) especially yields

$$\mu *_f \varepsilon_0 = \varepsilon_0 *_f \mu = \mu \tag{1.2.6}$$

for all $\mu \in M^b(\mathbb{R}^d)$ being in accord with equation (1.1.2).

Remark 1.2.6. Clearly, (1.2.5) can be iterated, that is,

$$\mathbb{P}_{X_1} *_f \ldots *_f \mathbb{P}_{X_n} = \mathbb{P}_{X_1 \vee_f \ldots \vee_f X_n} \tag{1.2.7}$$

for all $n \geq 1$, where $X_1, \ldots, X_n$ are independent. This follows from Definition 1.1.1 and Corollary 1.1.7 by using induction on n, or can directly be deduced from (1.2.2).

1.3 The f-implicit max-order

This section is intended to establish an interesting binary relation between $\mathbb{R}^d$ and $\mathbb{R}^d$, which heavily depends on our fixed loss function $f : \mathbb{R}^d \to [0, \infty)$ and will therefore referred to as the f-implicit max-order. The latter term indicates that this relation is not merely any binary relation between $\mathbb{R}^d$ and $\mathbb{R}^d$ but actually a relation that possesses additional properties and will thus turn out to be a partial order. We elaborate on this in Proposition 1.3.2. Before doing so, we first introduce the f-implicit max-order in Definition 1.3.1. To this end, we adopt the common terms of the theory of relations, especially those of binary relations and order relations. However, we dispense with going into detail here altogether since we pursue another goal.

Definition 1.3.1
The binary relation $\leq_f$ between $\mathbb{R}^d$ (*the set of departure*) and $\mathbb{R}^d$ (*the set of destination* or *codomain*) specified by its graph $G_{\leq_f} \subset \mathbb{R}^d \times \mathbb{R}^d$ is called *f-implicit max-order* if by definition x is $\leq_f$-related to y if and only if $f(x) < f(y)$ or $x = y$. This will formally be expressed by

$$(x, y) \in G_{\leq_f} \quad :\Leftrightarrow \quad x \leq_f y \quad :\Leftrightarrow \quad \begin{cases} f(x) < f(y) \\ \text{or } x = y, \end{cases}$$

thus following the notation of the classical theory of relations.

In a departure from convention, we will not distinguish between the two different definitions of a binary relation - the first one that defines $\leq_f$ by its graph $G_{\leq_f}$, the second one that defines $\leq_f$ to be the triple $(\mathbb{R}^d, \mathbb{R}^d, G_{\leq_f})$ - since it will not be important for our purpose. We rather concern ourselves with a more detailed analysis of the f-implicit max-order and begin proving that $\leq_f$ is a partial order on $\mathbb{R}^d$. Note, we simply say $\leq_f$ *is a binary relation on* $\mathbb{R}^d$ instead of $\leq_f$ *is a binary relation between* $\mathbb{R}^d$ *and* $\mathbb{R}^d$ since the set of departure and the codomain are the same.

Proposition 1.3.2
The binary relation $\leq_f$ on $\mathbb{R}^d$ is reflexive, antisymmetric and transitive. To be more precise, we have

(1) $x \leq_f x$ *(reflexivity)*

(2) if $x \leq_f y$ and $y \leq_f x$, then $x = y$ *(antisymmetry)*

(3) if $x \leq_f y$ and $y \leq_f z$, then $x \leq_f z$ *(transitivity)*

for all $x, y, z \in \mathbb{R}^d$. Hence, the relation $\leq_f$ is a partial order on $\mathbb{R}^d$. However, $\leq_f$ does not have the additional property of being *total*, that is,

$$x \leq_f y \quad \text{or} \quad y \leq_f x$$

for all $x, y \in \mathbb{R}^d$.

Proof. The reflexivity of $\leq_f$ is evident by Definition 1.3.1. In order to show the antisymmetry of $\leq_f$, fix $x, y \in \mathbb{R}^d$ such that $x \leq_f y$ and $y \leq_f x$. From $x \leq_f y$ it follows by definition that either $f(x) < f(y)$ or $x = y$. In the second case, we are done. Therefore, it remains to show that the first case can not occur. This will be done by contradiction. If it were true that $f(x) < f(y)$, we would have $x \neq y$. The additional condition $y \leq_f x$, however, yields $f(y) < f(x)$ or $y = x$, the latter being impossible because of our assumption. Hence, we would have $f(x) < f(y)$ and $f(y) < f(x)$ that is a contradiction. Finally, to prove the transitivity of $\leq_f$, fix $x, y, z \in \mathbb{R}^d$ such that $x \leq_f y$ and $y \leq_f z$. From $x \leq_f y$ we have either $f(x) < f(y)$ or $x = y$. Additionally, $y \leq_f z$ yields $f(y) < f(z)$ or $y = z$. Thus, the proof of transitivity is completed by showing that either $f(x) < f(z)$ or $x = z$ results from each of the disjoint cases

(i) $x = y$ and $y = z$,

(ii) $f(x) < f(y)$ and $f(y) < f(z)$,

(iii) $x = y$ and $f(y) < f(z)$,

(iv) $f(x) < f(y)$ and $y = z$.

To this end, we clearly need only consider the latter two cases in more detail. If $x = y$ and $f(y) < f(z)$, we see that $f(x) = f(y) < f(z)$, thus obtaining the desired conclusion. Similarly, $f(x) < f(y)$ and $y = z$ imply $f(x) < f(y) = f(z)$, and the asserted transitivity is proved. As a consequence, $\leq_f$ turns out to be a partial order.
What is left is to show that the property of being total does not apply to $\leq_f$. Fix $x, y \in \mathbb{R}^d$. We start by assuming that $x \nleq_f y$ and have to show that this does not necessarily imply $y \leq_f x$. Indeed, the fact that f does not need to be injective yields an applicable argument. To be more precise, $x \nleq_f y$ gives both $f(x) \geq f(y)$ and $x \neq y$. Since f may not be injective, it is possible that both $f(x) = f(y)$ and $x \neq y$. Consequently, $y \nleq_f x$, which is our claim. □

Apart from the preceding proposition, we do well to establish the following lemma that will gain in importance in Chapter 3 as it provides some useful properties.

Lemma 1.3.3
For all $x, y \in \mathbb{R}^d$, we have

(a) $0 \leq_f x,$ (1.3.1)

(b) $x \leq_f y \quad \Rightarrow \quad f(x) \leq f(y),$ (1.3.2)

(c) $x \leq_f x \vee_f y,$ (1.3.3)

(d) $x \leq_f y \quad \Rightarrow \quad y = x \vee_f y.$ (1.3.4)

Proof. Fix $x, y \in \mathbb{R}^d$. Assertion (a) is an easy consequence of the assumption $f(x) = 0$ if and only if $x = 0$. If $x = 0$, the claim follows due to the already proven reflexivity. Otherwise, we have $f(x) > 0 = f(0)$ and hence $0 \leq_f x$. The property asserted under (b) is valid due to Definition 1.3.1 as $x \leq_f y$ implies either $f(x) < f(y)$ or $x = y$ and therefore actually $f(x) \leq f(y)$. To verify (c), namely $x \leq_f x \vee_f y$, we refer to Definition 1.1.1. On account of this, we have to distinguish between the two cases $f(x) \geq f(y)$ and $f(x) < f(y)$. In the first case, we have $x \vee_f y = x$. Applying the reflexivity of $\leq_f$, we conclude that $x \leq_f x = x \vee_f y$. In the second case, we deduce that $x \vee_f y = y$. Because of

Definition 1.3.1, we obtain $x \leq_f y = x \vee_f y$, which establishes (c). Finally, it remains to prove (d). By assumption, we have either $f(x) < f(y)$ or $x = y$. If $f(x) < f(y)$, then $y = x \vee_f y$. Otherwise, we have $y = x = x \vee_f y$. Consequently, we obtain the desired conclusion in both cases, and the proof is complete. $\square$

Particularly with regard to Section 3.1 we further require the next definition that is intended to introduce some helpful terminology.

Definition 1.3.4
An $\mathbb{R}^d$-valued sequence $(x_n)_{n\geq 1}$ is said to be

(a) $\leq_f$*-increasing* if $x_n \leq_f x_{n+1}$ for all $n \geq 1$.

(b) $\leq_f$*-decreasing* if $x_{n+1} \leq_f x_n$ for all $n \geq 1$.

For convenience, we will write $x_n \uparrow_f$ and $x_n \downarrow_f$, respectively.

Lemma 1.3.5
Suppose that $(x_n)_{n\geq 1}$ is an $\leq_f$-increasing ($\leq_f$-decreasing) sequence. Then $(f(x_n))_{n\geq 1}$ is an increasing (decreasing) sequence of non-negative real numbers.

Proof. The assertion results directly from Definition 1.3.1 and Definition 1.3.4. $\square$

With the preceding lemma we want to complete our considerations concerning the f-implicit max-order. The next and last section of the preliminaries pursues another aim than providing the basic structures that we are consistently benefiting from. Actually, we will rather be concerned with the study of the distribution of the random vector

$$X_{k(n)} = \bigvee_{i=1}^{n}{}_f\, X_i$$

occurring in the introduction as well as in Remark 1.1.2.

1.4 Distribution of $X_{k(n)}$

As previously said, this section is devoted to a detailed investigation of the distribution of the random vector $X_{k(n)}$. This is especially necessary with regard to Chapter 2 in which we will introduce the notion of f-implicit max-infinitely divisible distributions, thus extending the concept of f-implicit max-stable distributions in an appropriate way.
To start with, fix $n \geq 1$. Furthermore, let $X_1, ..., X_n$ be independent and identically distributed random vector in $\mathbb{R}^d$. Recall from the introduction that the random variable $k(n)$ depending on the loss function f is given by

$$k(n) = \mathrm{argmax}(f(X_1), ..., f(X_n)),$$

where in the case of ties $k(n)$ is taken as the smallest index yielding the maximum. Then the random vector $X_{k(n)}$ referred to as the f-implicit maximum of $X_1, ..., X_n$ is the

subject of interest. In Remark 1.1.2 we mentioned that $X_{k(n)}$ can be expressed in term of the f-implicit max-operation. However, following the procedure of [SchSt14], we conveniently use the notation $X_{k(n)}$ for the remainder of this section.
In order to establish the central lemma of this section gaining in importance in Chapter 2, we first introduce some convenient notation. Given an arbitrary cumulative distribution function $H : \mathbb{R} \to [0,1]$, we define the sets $\mathcal{C}_f(H), \mathcal{D}_f(H) \subset \mathbb{R}^d$ by

$$\mathcal{C}_f(H) := \mathcal{C}(\mathrm{H}) := \{x \in \mathbb{R}^d : H(f(x)-) = H(f(x))\}$$

and

$$\mathcal{D}_f(H) := \mathcal{D}(\mathrm{H}) := \{x \in \mathbb{R}^d : H(f(x)-) < H(f(x))\},$$

respectively. Here, $H(f(x)-)$ denotes the one-sided limit from below of H at $f(x) \geq 0$, that is,

$$H(f(x)-) := \lim_{t \uparrow f(x)} H(t).$$

Clearly, $\mathcal{C}(\mathrm{H})$ and $\mathcal{D}(\mathrm{H})$ constitute a disjoint partition of $\mathbb{R}^d$. Equipped with this notation, we can proceed to establish the already announced lemma providing different representations of the distribution of the random vector $X_{k(n)}$.

Lemma 1.4.1
Let $G : \mathbb{R} \to \mathbb{R}_+$ denote the cumulative distribution function of $f(X)$, where $X \overset{d}{=} X_1$. Then the subsequent assertions hold.

(a) For all $x \in \mathbb{R}$, we have

$$\mathbb{P}(f(X_{k(n)}) \leq x) = G^n(x). \tag{1.4.1}$$

(b) For all $A \in \mathcal{B}(\mathbb{R}^d)$, we have

$$\mathbb{P}(X_{k(n)} \in A) = \sum_{\ell=1}^{n} \binom{n}{\ell} \int_A \mathbb{P}(f(X) = f(x))^{\ell-1} \mathbb{P}(f(X) < f(x))^{n-\ell}\, \mathbb{P}_X(dx) \tag{1.4.2}$$

and

$$\mathbb{P}(X_{k(n)} \in A) = \sum_{\ell=1}^{n} \int_A \mathbb{P}(f(X) < f(x))^{\ell-1} \mathbb{P}(f(X) \leq f(x))^{n-\ell}\, \mathbb{P}_X(dx) \tag{1.4.3}$$

and finally in addition

$$\begin{aligned} \mathbb{P}(X_{k(n)} \in A) = n & \int_{A \cap \mathcal{C}(\mathrm{G})} \mathbb{P}(f(X) \leq f(x))^{n-1}\, \mathbb{P}_X(dx) \\ & + \int_{A \cap \mathcal{D}(\mathrm{G})} \left(\frac{\mathbb{P}(f(X) \leq f(x))^n - \mathbb{P}(f(X) < f(x))^n}{\mathbb{P}(f(X) = f(x))} \right) \mathbb{P}_X(dx), \end{aligned} \tag{1.4.4}$$

(c) For all $A \in \mathcal{B}(\mathbb{R}^d)$, we have

$$n \int_A \mathbb{P}(f(X) < f(x))^{n-1}\, \mathbb{P}(X \in dx) \leq \mathbb{P}(X_{k(n)} \in A) \leq n \int_A \mathbb{P}(f(X) \leq f(x))^{n-1}\, \mathbb{P}_X(dx). \tag{1.4.5}$$

(d) If the cumulative distribution function G of $f(X)$ is continuous, we have

$$\mathbb{P}(X_{k(n)} \in A) = n \int_A \mathbb{P}(f(X) \leq f(x))^{n-1}\, \mathbb{P}_X(dx). \tag{1.4.6}$$

Remark 1.4.2. (i) Clearly, Lemma 1.4.1 is still true if it is just assumed that the loss function f is measurable as none of the required assumptions on f were necessary for the proof.

(ii) Both assertion (c) and (d) have already been proved in [SchSt14, Lemma 2.1], (1.4.6) being an easy consequence of (1.4.5). However, Lemma 1.4.1 expands (c) and (d) by the assertions (a) and (b). This is especially necessary with regard to Chapter 2. In the subsequent proof we will see that (1.4.5) can also be deduced directly from (1.4.2).

Proof of Lemma 1.4.1. (a) For all $x \in \mathbb{R}^d$, equation (1.4.1) is obvious by definition of $k(n)$ since

$$f(X_{k(n)}) = \bigvee_{i=1}^{n} f(X_i).$$

(b) Fix $A \in \mathcal{B}(\mathbb{R}^d)$. By assumptions on $X_1, \ldots, X_n$, we get

$$\begin{aligned}
&\mathbb{P}(X_{k(n)} \in A) \\
&= \sum_{\ell=1}^{n} \mathbb{P}\Bigg(X_{k(n)} \in A, \exists\, 1 \leq i_1 < \ldots < i_\ell \leq n\, \forall\, 1 \leq j \leq \ell : f\left(X_{i_j}\right) = \bigvee_{m=1}^{n} f(X_m) \text{ and} \\
&\qquad\qquad f(X_i) < \bigvee_{m=1}^{n} f(X_m)\, \forall\, i \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_\ell\}\Bigg) \\
&= \sum_{\ell=1}^{n} \binom{n}{\ell} \mathbb{P}(X_1 \in A, f(X_1) = \ldots = f(X_\ell), f(X_i) < f(X_1)\, \forall\, i = \ell+1, \ldots, n) \\
&= \sum_{\ell=1}^{n} \binom{n}{\ell} \int_A \mathbb{P}(f(X_1) = \ldots = f(X_\ell), f(X_i) < f(X_1)\, \forall\, i = \ell+1, \ldots, n \mid X_1 = x)\, \mathbb{P}_{X_1}(dx) \\
&= \sum_{\ell=1}^{n} \binom{n}{\ell} \int_A \mathbb{P}(f(x) = f(X_2) = \ldots = f(X_\ell), f(X_i) < f(x)\, \forall\, i = \ell+1, \ldots, n)\, \mathbb{P}_{X_1}(dx) \\
&= \sum_{\ell=1}^{n} \binom{n}{\ell} \int_A \left(\prod_{i=2}^{\ell} \mathbb{P}(f(X_i) = f(x)) \prod_{i=\ell+1}^{n} \mathbb{P}(f(X_i) < f(x)) \right) \mathbb{P}_{X_1}(dx)
\end{aligned}$$

$$= \sum_{\ell=1}^{n} \binom{n}{\ell} \int_A \mathbb{P}(f(X) = f(x))^{\ell-1} \mathbb{P}(f(X) < f(x))^{n-\ell} \, \mathbb{P}_X(dx),$$

and (1.4.2) is proved. In addition to $X_1, ..., X_n$ being independent, the second step of the previous calculation follows from the fact that there are exactly $\binom{n}{\ell}$ ways to choose ℓ indices, disregarding their order, from $\{1, ..., n\}$. Now, we proceed to prove (1.4.3). To this end, we consider the subsequent computation yielding the desired claim. Here, we gain from the definition of $k(n)$ in addition to the assumptions on $X_1, ..., X_n$.

$$\begin{aligned}
\mathbb{P}(X_{k(n)} \in A) &= \sum_{\ell=1}^{n} \mathbb{P}(X_{k(n)} \in A, k(n) = \ell) \\
&= \sum_{\ell=1}^{n} \mathbb{P}(X_\ell \in A, f(X_i) < f(X_\ell) \,\forall\, i = 1, ..., \ell - 1, f(X_i) \le f(X_\ell) \,\forall\, i = \ell + 1, ..., n) \\
&= \sum_{\ell=1}^{n} \int_A \mathbb{P}\Big(f(X_i) < f(X_\ell) \,\forall\, i = 1, ..., \ell - 1, \\
&\qquad\qquad f(X_i) \le f(X_\ell) \,\forall\, i = \ell + 1, ..., n \mid X_\ell = x\Big) \, \mathbb{P}_{X_\ell}(dx) \\
&= \sum_{\ell=1}^{n} \int_A \mathbb{P}(f(X) < f(x))^{\ell-1} \mathbb{P}(f(X) \le f(x))^{n-\ell} \, \mathbb{P}_X(dx).
\end{aligned}$$

Finally, (1.4.4) can be deduced directly from (1.4.2). Indeed, we have

$$\begin{aligned}
&\mathbb{P}(X_{k(n)} \in A) \\
&= \sum_{\ell=1}^{n} \binom{n}{\ell} \int_A \mathbb{P}(f(X) = f(x))^{\ell-1} \mathbb{P}(f(X) < f(x))^{n-\ell} \, \mathbb{P}_X(dx) \\
&= \sum_{\ell=1}^{n} \binom{n}{\ell} \int_{A \cap \mathcal{C}(G)} \mathbb{P}(f(X) = f(x))^{\ell-1} \mathbb{P}(f(X) < f(x))^{n-\ell} \, \mathbb{P}_X(dx) \\
&\quad + \sum_{\ell=1}^{n} \binom{n}{\ell} \int_{A \cap \mathcal{D}(G)} \mathbb{P}(f(X) = f(x))^{\ell-1} \mathbb{P}(f(X) < f(x))^{n-\ell} \, \mathbb{P}_X(dx) \\
&= \binom{n}{1} \int_{A \cap \mathcal{C}(G)} \mathbb{P}(f(X) \le f(x))^{n-1} \, \mathbb{P}_X(dx) \\
&\quad + \int_{A \cap \mathcal{D}(G)} \Bigg(\sum_{\ell=0}^{n} \binom{n}{\ell} \mathbb{P}(f(X) = f(x))^{\ell-1} \mathbb{P}(f(X) < f(x))^{n-\ell} \\
&\qquad\qquad - \frac{\mathbb{P}(f(X) < f(x))^n}{\mathbb{P}(f(X) = f(x))} \Bigg) \, \mathbb{P}_X(dx) \\
&= n \int_{A \cap \mathcal{C}(G)} \mathbb{P}(f(X) \le f(x))^{n-1} \, \mathbb{P}_X(dx)
\end{aligned}$$

$$+ \int\limits_{A\cap\mathcal{D}(G)} \left(\frac{\mathbb{P}(f(X) \le f(x))^n}{\mathbb{P}(f(X) = f(x))} - \frac{\mathbb{P}(f(X) < f(x))^n}{\mathbb{P}(f(X) = f(x))}\right) \mathbb{P}_X(dx)$$

$$= n \int\limits_{A\cap\mathcal{C}(G)} \mathbb{P}(f(X) \le f(x))^{n-1}\, \mathbb{P}_X(dx)$$

$$+ \int\limits_{A\cap\mathcal{D}(G)} \left(\frac{\mathbb{P}(f(X) \le f(x))^n - \mathbb{P}(f(X) < f(x))^n}{\mathbb{P}(f(X) = f(x))}\right) \mathbb{P}_X(dx),$$

the fourth step being due to the classical binomial theorem.
(c) Both inequalities in (1.4.5) are easy consequences of (1.4.2) and will be proved separately. First, we establish the lower bound in (1.4.5).

$$\begin{aligned}
\mathbb{P}(X_{k(n)} \in A) &= \sum_{\ell=1}^{n} \binom{n}{\ell} \int\limits_A \mathbb{P}(f(X) = f(x))^{\ell-1} \mathbb{P}(f(X) < f(x))^{n-\ell}\, \mathbb{P}_X(dx) \\
&= n \int\limits_A \mathbb{P}(f(X) < f(x))^{n-1}\, \mathbb{P}_X(dx) \\
&\quad + \underbrace{\sum_{\ell=2}^{n} \binom{n}{\ell} \int\limits_A \mathbb{P}(f(X) = f(x))^{\ell-1} \mathbb{P}(f(X) < f(x))^{n-\ell}\, \mathbb{P}_X(dx)}_{\ge 0} \\
&\ge n \int\limits_A \mathbb{P}(f(X) < f(x))^{n-1}\, \mathbb{P}_X(dx).
\end{aligned}$$

Applying the elementary inequality

$$\binom{n}{l+1} \le n \binom{n-1}{l},$$

which is true for all $0 \le \ell \le n-1$, we further deduce that

$$\begin{aligned}
\mathbb{P}(X_{k(n)} \in A) &= \sum_{\ell=1}^{n} \binom{n}{\ell} \int\limits_A \mathbb{P}(f(X) = f(x))^{\ell-1} \mathbb{P}(f(X) < f(x))^{n-\ell}\, \mathbb{P}_X(dx) \\
&= \int\limits_A \sum_{\ell=0}^{n-1} \binom{n}{\ell+1} \mathbb{P}(f(X) = f(x))^{\ell} \mathbb{P}(f(X) < f(x))^{n-(\ell+1)}\, \mathbb{P}_X(dx) \\
&\le n \int\limits_A \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \mathbb{P}(f(X) = f(x))^{\ell} \mathbb{P}(f(X) < f(x))^{(n-1)-\ell}\, \mathbb{P}_X(dx) \\
&= n \int\limits_A \mathbb{P}(f(X) \le f(x))^{n-1}\, \mathbb{P}_X(dx),
\end{aligned}$$

which completes the proof of part (c). The last step of the preceding estimation is once again due to the classical binomial theorem.
(d) Clearly, (1.4.6) follows from each of the already proved claims (1.4.2), (1.4.3), (1.4.4) or (1.4.5). □

In addition to the previous lemma, the subsequent one provides another possibility to represent the distribution of $X_{k(n)}$, thus yielding a still better insight into the structure of $\mathbb{P}_{X_{k(n)}}$. In contrast to Lemma 1.4.1, however, we will not apply this statement explicitly. Nevertheless, it clearly constitutes an additional result of independent interest in f-implicit extreme value theory.

Lemma 1.4.3
Let $\mu \in M^1(\mathbb{R})$ denote the distribution of $f(X)$, $X \overset{d}{=} X_1$. Furthermore, set

$$\rho_u(A) = \mathbb{P}(X \in A \,|\, f(X) = u) \tag{1.4.7}$$

for arbitrary $A \in \mathcal{B}(\mathbb{R}^d)$ and $u \geq 0$. Then, for all $A \in \mathcal{B}(\mathbb{R}^d)$, we have

$$\mathbb{P}(X_{k(n)} \in A) = \sum_{\ell=1}^{n} \int_{[0,\infty)} \dots \int_{[0,\infty)} \rho_{u_\ell}\Big(A \cap f^{-1}((\max(u_1, ..., u_{\ell-1}), \infty)) \cap f^{-1}([\max(u_{\ell+1}, ..., u_n), \infty))\Big)\, \mu(du_1) \dots \mu(du_n). \tag{1.4.8}$$

Especially, if $n = 2$, then

$$\mathbb{P}(X_1 \vee_f X_2 \in A) = \int_{[0,\infty)} \left(\rho_{u_2}(A)\, \mu(\{u_2\}) + 2 \int_{(u_2,\infty)} \rho_{u_1}(A)\, \mu(du_1) \right) \mu(du_2). \tag{1.4.9}$$

Remark 1.4.4. (i) Instead of delving into the common field of regular conditional probability here, we refer to [Al07] for a deeper discussion of such expressions as given in (1.4.7). We shall further point out that we have also already tacitly applied the notion of regular conditional probability in the proof of Lemma 1.4.1 and will further do so in the proof of Proposition 3.1.20.

(ii) Observe that the expression $f^{-1}((\max(u_1, ..., u_{\ell-1}), \infty))$ does actually not exist and is therefore not part of the intersection occurring in (1.4.8) if $\ell = 1$. Similarly, $f^{-1}([\max(u_{\ell+1}, ..., u_n), \infty))$ does actually not exist and is thus not part of the intersection occurring in (1.4.8) if $\ell = n$.

Proof of Lemma 1.4.3. Fix $A \in \mathcal{B}(\mathbb{R}^d)$. Then

$$\begin{aligned}\mathbb{P}(X_{k(n)} \in A) &= \sum_{\ell=1}^{n} \mathbb{P}(X_{k(n)} \in A, k(n) = \ell) \\ &= \sum_{\ell=1}^{n} \mathbb{P}(X_\ell \in A, f(X_i) < f(X_\ell)\, \forall\, i = 1, ..., \ell-1, f(X_i) \leq f(X_\ell)\, \forall\, i = \ell+1, ..., n),\end{aligned}$$

the last step being an easy consequence of the definition of $k(n)$. Note further that

$$\begin{aligned}
f(X_i) < f(X_\ell)\,\forall\, i = 1,...,\ell-1 \quad &\Leftrightarrow \quad X_\ell \in f^{-1}((f(X_i),\infty))\,\forall\, i = 1,...,\ell-1\\
&\Leftrightarrow \quad X_\ell \in \bigcap_{i=1}^{\ell-1} f^{-1}((f(X_i),\infty))\\
&\Leftrightarrow \quad X_\ell \in f^{-1}\left(\bigcap_{i=1}^{\ell-1}(f(X_i),\infty)\right)\\
&\Leftrightarrow \quad X_\ell \in f^{-1}((\max(f(X_1),...,f(X_{\ell-1})),\infty)),
\end{aligned}$$

and similarly that

$$f(X_i) \le f(X_\ell)\,\forall\, i = \ell+1,...,n \quad \Leftrightarrow \quad X_\ell \in f^{-1}([\max(f(X_{\ell+1}),...,f(X_n)),\infty)).$$

Accordingly, we conclude, by referring to the assumptions on the random vectors $X_1,...,X_n$, that

$$\begin{aligned}
&\mathbb{P}(X_{k(n)} \in A)\\
&= \sum_{\ell=1}^{n} \mathbb{P}(X_\ell \in A, X_\ell \in f^{-1}((\max(f(X_1),...,f(X_{\ell-1})),\infty)),\\
&\qquad\qquad X_\ell \in f^{-1}([\max(f(X_{\ell+1}),...,f(X_n)),\infty)))\\
&= \sum_{\ell=1}^{n} \mathbb{P}(X_\ell \in A \cap f^{-1}((\max(f(X_1),...,f(X_{\ell-1})),\infty))\\
&\qquad\qquad \cap f^{-1}([\max(f(X_{\ell+1}),...,f(X_n)),\infty)))\\
&= \sum_{\ell=1}^{n} \int\limits_{[0,\infty)^{n-1}} \mathbb{P}\Big(X_\ell \in A \cap f^{-1}((\max(u_1,...,u_{\ell-1}),\infty))\\
&\qquad\qquad \cap f^{-1}([\max(u_{\ell+1},...,u_n),\infty))\Big)\,\mu^{\otimes(n-1)}(du_1,...,du_{j-1},du_{j+1},...,du_n)\\
&= \sum_{\ell=1}^{n} \int\limits_{[0,\infty)^{n-1}} \int\limits_{[0,\infty)} \rho_{u_\ell}\Big(A \cap f^{-1}((\max(u_1,...,u_{\ell-1}),\infty))\\
&\qquad\qquad \cap f^{-1}([\max(u_{\ell+1},...,u_n),\infty))\Big)\,\mu(du_\ell)\,\mu^{\otimes(n-1)}(du_1,...,du_{j-1},du_{j+1},...,du_n)\\
&= \sum_{\ell=1}^{n} \int\limits_{[0,\infty)} \cdots \int\limits_{[0,\infty)} \rho_{u_\ell}\Big(A \cap f^{-1}((\max(u_1,...,u_{\ell-1}),\infty))\\
&\qquad\qquad \cap f^{-1}([\max(u_{\ell+1},...,u_n),\infty))\Big)\,\mu(du_1)\ldots\mu(du_n),
\end{aligned}$$

and (1.4.8) is proved. To deduce (1.4.9) from (1.4.8), we first refer to Remark 1.4.4 (ii). In doing so, we get

$$\mathbb{P}(X_1 \vee_f X_2 \in A) = \int\limits_{[0,\infty)} \int\limits_{[0,\infty)} \rho_{u_1}\Big(A \cap f^{-1}([u_2,\infty))\Big)\,\mu(du_1)\,\mu(du_2)$$

$$+ \int_{[0,\infty)} \int_{[0,\infty)} \rho_{u_2}\Big(A \cap f^{-1}((u_1,\infty))\Big)\, \mu(du_1)\, \mu(du_2)$$

$$= \int_{[0,\infty)} \int_{[0,\infty)} \rho_{u_1}\Big(A \cap f^{-1}([u_2,\infty))\Big)\, \mu(du_1)\, \mu(du_2)$$

$$+ \int_{[0,\infty)} \int_{[0,\infty)} \rho_{u_1}\Big(A \cap f^{-1}((u_2,\infty))\Big)\, \mu(du_1)\, \mu(du_2),$$

where the last equality is merely due to a change of designation. Now, observe that

$$\rho_{u_1}\Big(A \cap f^{-1}([u_2,\infty))\Big) = 0$$

for all $u_1 < u_2$ and also

$$\rho_{u_1}\Big(A \cap f^{-1}((u_2,\infty))\Big) = 0$$

for all $u_1 \le u_2$. This can be seen easily by just applying (1.4.7) as well as the assumptions on f. In addition, we have

$$\rho_{u_1}\Big(A \cap f^{-1}([u_2,\infty))\Big) = \rho_{u_1}(A)$$

for all $u_1 \ge u_2$ and

$$\rho_{u_1}\Big(A \cap f^{-1}((u_2,\infty))\Big) = \rho_{u_1}(A)$$

for all $u_1 > u_2$, both being again an easy consequence of the definition of ρ_{u_1} given in (1.4.7). Therefore, we conclude immediately that

$$\mathbb{P}(X_1 \vee_f X_2 \in A) = \int_{[0,\infty)} \int_{[u_2,\infty)} \rho_{u_1}\Big(A \cap f^{-1}([u_2,\infty))\Big)\, \mu(du_1)\, \mu(du_2)$$

$$+ \int_{[0,\infty)} \int_{(u_2,\infty)} \rho_{u_1}\Big(A \cap f^{-1}((u_2,\infty))\Big)\, \mu(du_1)\, \mu(du_2)$$

$$= \int_{[0,\infty)} \int_{[u_2,\infty)} \rho_{u_1}(A)\, \mu(du_1)\, \mu(du_2) + \int_{[0,\infty)} \int_{(u_2,\infty)} \rho_{u_1}(A)\, \mu(du_1)\, \mu(du_2)$$

$$= \int_{[0,\infty)} \left(\rho_{u_2}(A)\, \mu(\{u_2\}) + 2 \int_{(u_2,\infty)} \rho_{u_1}(A)\, \mu(du_1) \right) \mu(du_2),$$

which establishes (1.4.9). □

Having investigated the distribution of the random vector $X_{k(n)}$ extensively, we complete this chapter as we are eventually prepared to proceed with the detailed study of f-implicit max-infinitely divisible distributions and subsequently with the study of f-implicit max-stable processes. Starting with an appropriate approach to f-implicit max-infinitely divisible distributions, we will be guided by the two theories of infinitely divisible and max-infinitely divisible distributions. The probably best general references on these concepts are [MeSch01, Chapter 3] or [Sat99, Chapter 2] for the first and [Re07, Chapter 5] for the latter. As it will turn out, there exist striking parallels between these concepts and the one that we are about to establish within a short time. In addition to the latest observations concerning the distribution of the random vector $X_{k(n)}$, the examinations of Section 1.1 as well as of Section 1.2. will gain in importance here as well. Indeed, the next chapter will reveal the particular relevance of the f-implicit max-operation for our purpose. Considering the theories of infinitely divisible and max-infinitely divisible distributions from a more abstract point of view, we recognize that the summation and maximum operation constitute the underlying basis. In our context these operations have to be substituted by another appropriate operation. Unsurprisingly, this will be the f-implicit max-operation. Therefore, the detailed study of the f-implicit max-operation has been a necessary assignment in order to be able to establish a more profound insight into the two branches of f-implicit extreme value theory that will be approached in Chapter 2 and Chapter 3. In the deliberations of Chapter 3 we will have recourse to the results of Section 1.3.

2 f-implicit max-infinitely divisible distributions

While the first chapter was intended to provide the necessary fundamentals including the essential f-implicit max-operation, the f-implicit max-convolution, the f-implicit max-order and finally also several different representations of the distribution of the random vector $X_{k(n)}$, the present Chapter is devoted to the notion of f-implicit max-infinitely divisible distributions. Here, we are considerably guided by the two theories of infinitely divisible and max-infinitely divisible distributions.

Chapter 2 is structured as follows. In Section 2.1, we begin by introducing the central notion of f-implicit max-infinitely divisible distributions. In particular, we establish two equivalent definitions, the first one using the notion of measures and the second one the notion of random vectors. Subsequently, we illustrate this new concept with basic examples, one of them incorporating the notion of f-implicit max-stable distributions.

Section 2.2 is devoted to a more detailed study of the class of f-implicit max-infinitely divisible distributions. As infinitely divisible distributions can be characterized in terms of its Lévy-Khintchine triplet and max-infinitely divisible distributions in terms of its spectral measure, it seems reasonable to ask whether similar characterizations exist for f-implicit max-infinitely divisible distributions. Unfortunately, we are not able to provide a complete solution as to this question. Nevertheless, we show that all random vectors in $\mathbb{R}^d$ coming under one of two specific classes of random vectors are f-implicit max-infinitely divisible distributions.

In Section 2.3, we introduce the notion of f-implicit max-compound Poisson distributions and subsequently the notion of f-implicit max-compound Poisson processes. Here, the idea originates from the class of generalized Poisson distributions which have been studied in [MeSch01, Chapter 3].

Finally, Section 2.4 consists of an extensive outlook providing several ideas and unsolved problems for further research projects. In particular, we formulate the hypothesis that all distributions on $\mathbb{R}^d$ are f-implicit max-infinitely divisible and demonstrate its reasonableness.

2.1 The f-implicit max-infinite divisibility

As recently proposed, we do well to take the theories of infinitely and max-infinitely divisible distributions into account while establishing an appropriate notion of f-implicit max-infinite divisibility. The idea here is actually to provide an analogous concept sharing striking parallels to the initially mentioned ones.

Considering the historical development during the 20th century, we recognize that first

stable and max-stable distributions and only then infinitely divisible and max-infinitely divisible distributions occurred in literature. The notion of stable distributions first arose from investigations of limit theorems for sums of independent and identically distributed random vectors, whereas the notion of max-stable distributions grew out of studies concerning limit theorems for the componentwise maxima of such random vectors. Besides, this is exactly the same with f-implicit stable distributions as can be seen in [SchSt14]. Motivated by further problems, such as convergence criteria for triangular arrays (see for example [MeSch01, Section 3.2]) or Lévy processes (see for example [Sat99]), it was then necessary to develop a more general class of distributions. Accordingly, the notion of infinitely divisible distributions was developed extending the class of stable distributions. Similar circumstances gave occasion to the introduction of the class of max-infinitely divisible distributions enlarging the class of max-stable distributions. From the historical point of view it is therefore reasonable to take the class of f-implicit max-stable distribution as a starting point for our studies regarding f-implicit max-infinitely divisible distributions.
In addition to the latter observations, the mathematical definition of infinitely and max-infinitely divisible distributions is useful as well. Indeed, it turns out to be the right approach to adopt a slight modification of the respective definitions. We just have to apply the f-implicit max-operation instead of the summation or maximum operation, and we have to substitute the classical convolution concepts induced by the summation or maximum operation with the f-implicit max-convolution. This is again an evidence of the necessity of the extensive groundwork addressed in Chapter 1.
Summarizing the previous aspects, we will thus establish a notion of f-implicit max-infinite divisibility which extends the class of f-implicit max-stable distributions and emerges from just a slight modification of the classical concepts. This will definitely be achieved by Definition 2.1.1.
Furthermore, it is worth to emphasize that the general concept of infinite divisibility can be considered from an even more technical point of view. This exciting but very theoretical aspect occurs in considerations concerning harmonic analysis on semigroups (see [BeChRe84]). However, we do not go into detail here. Instead, we proceed to establish the pivotal concept supplying the title of Chapter 2.

Definition 2.1.1 (a) A probability measure $\mu \in M^1(\mathbb{R}^d)$ is called *f-implicit max-infinitely divisible* if for all $n \geq 1$ there exist probability measures $\mu_n \in M^1(\mathbb{R}^d)$ such that

$$\mu = (\mu_n)^{*_f n}. \tag{2.1.1}$$

Henceforth, μ_n will be referred to as *nth root of μ*.

(b) A random vector X in $\mathbb{R}^d$ is meant to be f-implicit max-infinitely divisible if its distribution $\mathbb{P}_X \in M^1(\mathbb{R}^d)$ is f-implicit max-infinitely divisible in accordance with (a).

With regard to the upcoming deliberations, the following reformulation of Definition 2.1.1 (b) is most convenient as Lemma 1.4.1 provides several explicit representations of the right hand-side of (2.1.2).

Lemma 2.1.2
A random vector X in $\mathbb{R}^d$ is f-implicit max-infinitely divisible if and only if for all $n \geq 1$ there exist independent and identically distributed random vectors $X_1^{(n)}, ..., X_n^{(n)}$ in $\mathbb{R}^d$ such that

$$X \stackrel{d}{=} \bigvee_{i=1}^{n}{}_f \, X_i^{(n)} = X_{k(n)}^{(n)}, \tag{2.1.2}$$

the random variable $k(n)$ here being defined as

$$k(n) = \operatorname{argmax}\left(f\left(X_1^{(n)}\right), ..., f\left(X_n^{(n)}\right)\right) \tag{2.1.3}$$

with the common convention that in the case of ties $k(n)$ is taken as the smallest index yielding the maximum.

Proof of Lemma 2.1.2. Both implication are easy consequences of (1.2.7). □

Using the same notation for two slightly different objects - on the one hand at the beginning of Section 1.4 and on the other hand in (2.1.3) - shall cause no misunderstandings since unless $k(n)$ is explicitly defined the concrete definition of $k(n)$ will always be clear from context.

Remark 2.1.3. (i) With regard to Definition 2.1.1 (a), it is reasonable to refer to the random vector $X_1^{(n)}$ as *nth root of X*, thus being consistent in our terminology.

(ii) Having introduced Definition 2.1.1 as well as the equivalent formulation of the notion of f-implicit max-infinite divisibility in terms of random vectors, we must confess that one crucial aspect remained unconsidered. So far, we have not concerned ourselves with the question whether the roots μ_n or $X_1^{(n)}$ are unique. Unfortunately, this remains as an open question for the time being. Typically, considerations concerning the notion of infinite divisibility on non-commutative (semi-) groups deal with the same open problem as non-commutativity involves things a lot. Although uniqueness is clearly a desirable property, this open question will not constrain the further deliberations.

In order to illustrate Definition 2.1.1 and Lemma 2.1.2, we do well to proceed with an easy example. Throughout this section we will, however, derive a much broader class of distributions coming under the class of f-implicit max-infinitely divisible distributions. Similar to the fact that all distributions on $\mathbb{R}$ are max-infinitely divisible (see for example [Re07, Chapter 5]), we will even state the reasonable conjecture that probably all distributions on $\mathbb{R}^d$ are f-implicit max-infinitely divisible distributions.

Example 2.1.4
Fix $\ell_0 \geq 0$ and let L_{ℓ_0} denote the set $\{x \in \mathbb{R}^d : f(x) = \ell_0\}$. Then every random vector X such that $\operatorname{supp} \mathbb{P}_X \subset L_{\ell_0}$ is f-implicit max-infinitely divisible. Broadly speaking, every random vector with support being a subset of some level set of f is f-implicit max-infinitely divisible. Indeed, let $X_1^{(n)}, ..., X_n^{(n)}$ be independent copies of X. Then

$$k(n) = \operatorname{argmax}\left(f\left(X_1^{(n)}\right), ..., f\left(X_n^{(n)}\right)\right) = 1$$

almost surely showing that $X \overset{d}{=} X_1^{(n)} \overset{d}{=} X_{k(n)}^{(n)}$. Clearly, the latter shows that X is actually even f-implicit max-stable (see also Lemma 2.1.5).

Having established Definition 2.1.1, subsequently Lemma 2.1.2 and finally Example 2.1.4, it is reasonable to ask whether there exist more examples of distributions on $\mathbb{R}^d$ being f-implicit max-infinitely divisible. The subsequent lemma shows that there are in fact quite a lot of such distributions. In addition, it makes the technical definition of f-implicit max-infinitely divisible distributions more tangible as it establishes a connection between the already extensively studied class of f-implicit max-stable distributions and the class of f-implicit max-infinitely divisible distributions.

Lemma 2.1.5
All f-implicit max-stable distributions are f-implicit max-infinitely divisible.

Proof. Fix $n \geq 1$ and let X be some f-implicit max-stable random vector in $\mathbb{R}^d$, that is,

$$X \overset{d}{=} a_n^{-1} X_{k(n)}$$

for some constant $a_n > 0$ and

$$k(n) = \operatorname{argmax}\left(f(X_1), ..., f(X_n)\right),$$

the random vectors $X_1, ..., X_n$ here being independent copies of X. Now, let $X_1^{(n)}, ..., X_n^{(n)}$ be defined as

$$X_i^{(n)} = a_n^{-1} X_i \tag{2.1.4}$$

for each $i = 1, ..., n$ which are clearly independent and identically distributed. Moreover, we have

$$k(n) = \operatorname{argmax}\left(f\left(X_1^{(n)}\right), ..., f\left(X_n^{(n)}\right)\right),$$

being an easy consequence of the 1-homogeneity of f and the positivity of a_n. Thus, we conclude that

$$X_{k(n)}^{(n)} = a_n^{-1} X_{k(n)} \overset{d}{=} X.$$

□

In what follows, we expand on the notion of f-implicit max-infinitely divisible distribution. Considering infinitely and max-infinitely divisible distributions, we recall that both these types of distributions can fully be characterized. Indeed, infinitely divisible distributions can be characterized by means of their unique Lévy-Khintchine triplet occurring in the context of the well-known Lévy-Khintchine representation (see for example [MeSch01, Theorem 3.1.11]). Max-infinitely divisible distributions, on the contrary, are characterized by means of their unique exponent measure. In particular, in the univariate case all distributions are max-infinitely divisible (see for instance [Re07, Section 5.1]). Accordingly, the legitimate question arises whether the previously introduced class of f-implicit max-infinitely divisible distributions allows a similar characterization. At present, this question is far from being solved. However, we prove that two particular classes of distributions on $\mathbb{R}^d$ pertain to the class of f-implicit max-infinitely divisible distributions, thus providing first attractive theorems concerning the notion of f-implicit max-infinitely divisible distributions.

2.2 Main results

As already announced, this section deals with two specific classes of distributions on $\mathbb{R}^d$ which will turn out to be part of the class of f-implicit max-infinitely divisible distributions. In order to be able to introduce these two particular classes in more detail, we first establish the notion of the so called left end point of the distribution of a random variable Z in $\mathbb{R}$.

Definition 2.2.1
Suppose that Z is an arbitrary random variable in $\mathbb{R}$. We refer to $\ell \in \mathbb{R}$ as the *left endpoint of the distribution of* Z if

$$\mathbb{P}(Z \leq x) \quad \begin{cases} = 0, & \text{if } x < \ell \\ > 0, & \text{if } x > \ell, \end{cases}$$

whether or not Z has an atom in $\ell \in \mathbb{R}$.

Now, we proceed to introduce some notation in addition to Definition 2.2.1. In account with the notation of Section 1.4, let $G : \mathbb{R} \to \mathbb{R}$ denote the cumulative distribution function of $f(X)$ and $\ell \geq 0$ denote the left end point of the distribution of $f(X)$, where X is a random vector in $\mathbb{R}^d$. Furthermore, let $L_{\ell_0}, \Gamma_{\ell_0} \subset \mathbb{R}^d$ be defined by

$$L_{\ell_0} = \{x \in \mathbb{R}^d : f(x) = \ell_0\} \tag{2.2.1}$$

and

$$\Gamma_{\ell_0} = \{x \in \mathbb{R}^d : f(x) > \ell_0\}, \tag{2.2.2}$$

the non-negative real number ℓ_0 here being arbitrary. Note that the set L_{ℓ_0} is actually nothing but the level set of our homogeneous loss function f to the level ℓ_0. From a more geometrical or topological point of view, the set L_{ℓ_0} is the boundary of a star-shaped set in $\mathbb{R}^d$ including the origin. The set Γ_{ℓ_0}, however, is the complement of the closure of this star-shaped set. For convenience, we will moreover simply write L and Γ if $\ell_0 = \ell$, that is, we refrain from the index if we consider specifically the left end point. Clearly,

$$\overline{\Gamma}_{\ell_0} = \Gamma_{\ell_0} \cup \partial\Gamma_{\ell_0} = \Gamma_{\ell_0} \cup L_{\ell_0} = \{x \in \mathbb{R}^d : f(x) \geq \ell_0\},$$

as f is continuous. Moreover,

$$\Gamma_{\ell_1} \supset \Gamma_{\ell_2}$$

for all $0 \leq \ell_1 \leq \ell_2$ and

$$L_{\ell_1} \cap L_{\ell_2} = \emptyset$$

for all $\ell_1 \neq \ell_2$. Finally, note that

$$\operatorname{supp} \mathbb{P}_X \subset \overline{\Gamma} \tag{2.2.3}$$

whether or not $f(X)$ has an atom in its left end point ℓ. This is an easy consequence of the definitions of ℓ, L and Γ and of the assumptions on f.
The next figure is intended to illustrate the geometrical structure of the recently introduced sets for the special case $d = 2$. Note that the considered example actually reflects the fact that $\mathbb{R}^2 \setminus \Gamma_{\ell_0}$ can be even more general than just convex.

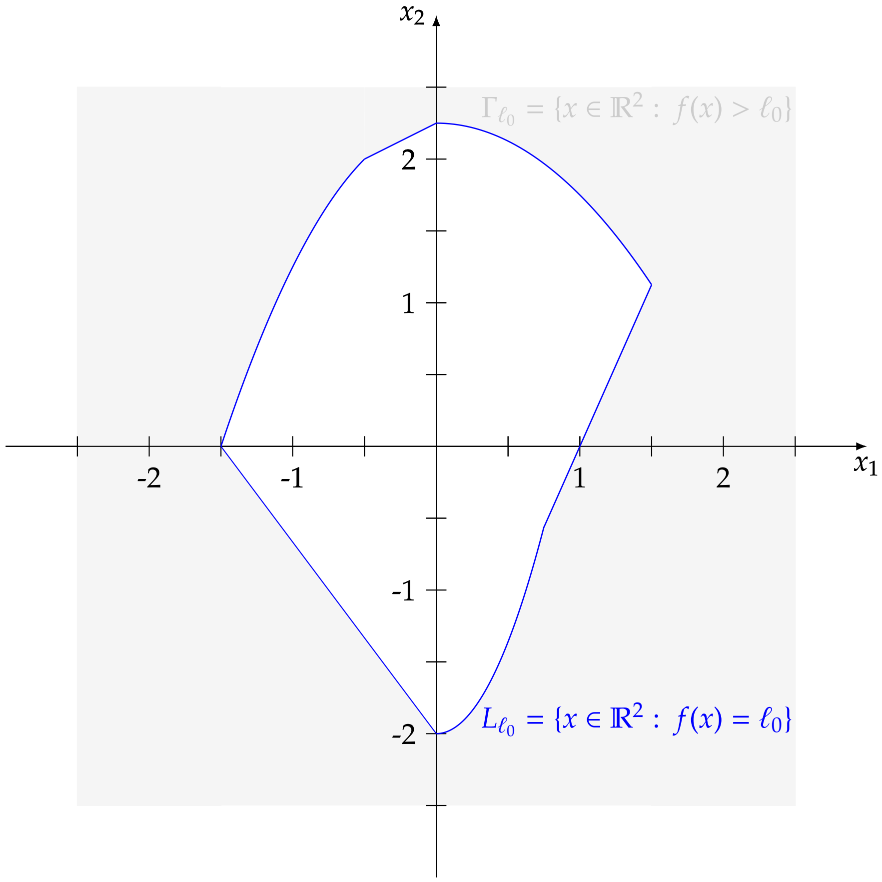

Figure 2.1: An example of possible sets L_{ℓ_0} and Γ_{ℓ_0} in $\mathbb{R}^2$

Having completed these preliminary studies, we are now able to specify the two already mentioned classes of random vectors in $\mathbb{R}^d$ that will be investigated in this section. In particular, we will focus on the two subsequent ones.

(1.) X is a random vector in $\mathbb{R}^d$ such that the cumulative distribution function G of $f(X)$ is continuous on (ℓ, ∞) whether or not $f(X)$ has an atom in ℓ.

(2.) X is a random vector in $\mathbb{R}^d$ such that the mass of $\mathbb{P}_{f(X)}$ is concentrated on a countable subset of $[0, \infty)$.

Remark 2.2.2. Concerning the second class of random vectors we underline that the expression *countable set* is understood as either a finite or a countably infinite set.

For the rest of this section we devote ourselves to proving that all random vectors X in $\mathbb{R}^d$ coming under the first or the second class are f-implicit max-infinitely divisible. To this end, we first need to provide some auxiliary tools. More precisely, we begin by establishing specific substitution rules for the Riemann-Stieltjes integral which will prove extremely beneficial. In doing so, we will be guided by [DuNo11, Chapter 2] providing an extensive amount of results concerning this particular issue. Without going into great detail here, we will derive some formulas being specially geared to our

further course of action.
Starting point of the next deliberations is the following, fairly general result making an assertion about the representation of a *full Stieltjes integral* with respect to the composition $V \circ v$ of functions $V : \mathbb{R}^d \to \mathbb{R}$ and $v : [a,b] \to \mathbb{R}^d, -\infty < a \leq b < \infty$, where v has bounded variation and V is a C^1-function.

Theorem 2.2.3 ([DuNo11, Theorem 2.87])
Let $v : [a,b] \to \mathbb{R}^d$ be of bounded variation, let V be a real-valued C^1 function on $\mathbb{R}^d$ and let $h \in \mathcal{R}[a,b]$. Then the composition $V \circ v$ is of bounded variation and

$$(S)\int_a^b h\, d(V \circ v) = (S)\int_a^b \langle h\,(\nabla V \circ v), dv\rangle + \sum_{(a,b]} h\Big(\Delta^-(V \circ v) - \langle \nabla V \circ v, \Delta^- v\rangle\Big)$$
$$+ \sum_{[a,b)} h\Big(\Delta^+(V \circ v) - \langle \nabla V \circ v, \Delta^+ v\rangle\Big), \tag{2.2.4}$$

where the two sums converge absolutely if $a < b$ and equal 0 if $a = b$.

Remark 2.2.4. At this point we clearly need to clarify some notation. Following [DuNo11], we mean by C^1 the class of all real-valued, continuously differentiable functions on $\mathbb{R}^d$. The space $\mathcal{R}[a,b]$ is defined as the set of all functions $h : [a,b] \to \mathbb{R}$ being *regulated* in accordance with [DuNo11, Part 1 of Section 2.1]. Furthermore,

$$(\Delta^- F)(t) := F(t) - F(t-) \qquad \text{with} \qquad F(t-) := \lim_{s \uparrow t} F(s)$$

for all $t \in (a,b]$ and

$$(\Delta^+ F)(t) := F(t+) - F(t) \qquad \text{with} \qquad F(t+) := \lim_{s \downarrow t} F(s),$$

for all $t \in [a,b)$, where F is a regulated function on $[a,b]$ with values in some Banach space B. Finally, the general notion of full Stieltjes integrals of two regulated functions h and g on $[a,b]$ with values in some Banach spaces B_1 and B_2, respectively, denoted by

$$(S)\int_a^b h \cdot dg,$$

can be found in [DuNo11, Definition 2.41]. Note that (2.2.4) differs slightly from Equation (2.91) in [DuNo11] since we used an easier formulation which is based on the fact that the involved functions are invariably $\mathbb{R}$- or $\mathbb{R}^d$-valued. However, we skip the technical details here.

Since we will apply the latter and quite technical result for a particular setting only, (2.2.4) simplifies significantly. To be more precise, if $d = 1$ and if $v : [a,b] \to \mathbb{R}$ is continuous as well as of bounded variation and if finally $h \equiv 1$, then (2.2.4) reduces to

$$\int_a^b d(V \circ v) = \int_a^b (V' \circ v)\, dv, \tag{2.2.5}$$

where both integrals are understood as classical Riemann-Stieltjes integrals. Note that the full Stieltjes integral coinciding with the common Riemann-Stieltjes integral follows from a combination of various results in [DuNo11, Chapter 2] and the particular assumptions on v, h and V. Indeed, applying Theorem 2.17, subsequently Theorem 2.18 as well as Theorem 2.42, and finally Definition 2.41, we can deduce that both integrals in (2.2.5) exist and can either be understood as full Stieltjes or Riemann-Stieltjes integrals. Here, we shall also refer to Corollary 2.43 in [DuNo11] which combines the latter arguments once again.
In what follows, we concern specific improper integrals and thus need a slight extension of the common Riemann-Stieltjes integral. Adopting the classical proceeding, we obtain the subsequent definition.

Definition 2.2.5 (a) Let $-\infty < a < b \leq \infty$ and let $g, h : [a, b) \to \mathbb{R}$ be two functions such that the Riemann-Stieltjes integral

$$\int_a^{b^*} h\, dg$$

exists for all $a \leq b^* < b$. Then we define the improper Riemann-Stieltjes integral on $[a, b)$ by

$$\int_a^b h\, dg = \int_a^{b-} h\, dg = \lim_{b^* \uparrow b} \int_a^{b^*} h\, dg,$$

provided the limit exists.

(b) Let $-\infty \leq a < b < \infty$ and let $g, h : (a, b] \to \mathbb{R}$ be two functions such that the Riemann-Stieltjes integral

$$\int_{a^*}^{b} h\, dg$$

exists for all $a < a^* \leq b$. Then we define the improper Riemann-Stieltjes integral on $(a, b]$ by

$$\int_a^b h\, dg = \int_{a+}^{b} h\, dg = \lim_{a^* \downarrow a} \int_{a^*}^{b} h\, dg,$$

provided the limit exists.

(c) Let $-\infty \leq a < b \leq \infty$ and let $g, h : (a, b) \to \mathbb{R}$ be two functions such that the improper Riemann-Stieltjes integrals

$$\int_a^c h\, dg \qquad \text{and} \qquad \int_c^b h\, dg$$

exist for some $a < c < b$. Then we define the improper Riemann-Stieltjes integral on (a, b) by

$$\int_a^b h\,dg = \int_{a+}^{b-} h\,dg = \lim_{a^*\downarrow a} \int_{a^*}^{c} h\,dg + \lim_{b^*\uparrow b} \int_c^{b^*} h\,dg.$$

The latter is well-defined as it does not depend on c.

Remark 2.2.6. More details concerning the notion of improper Riemann-Stieltjes integrals can be found in [BuBu70, Section 6.3] or in [Ho72, Paragraph 29].

Remark 2.2.7. Note that (2.2.5) can actually be extended to improper Riemann-Stieltjes integrals. For a particular setting, this will be explained in Corollary 2.2.8 completing the preparatory considerations concerning substitution rules for Riemann-Stieltjes integrals.

Corollary 2.2.8
Let X be a random vector in $\mathbb{R}^d$. Referring to the notation introduced at the beginning of Section 2.2, we obtain the subsequent claims.

(a) If G is continuous on (ℓ, ∞), we have

$$\int_s^\infty \frac{1}{G}\,dG = \int_s^\infty d(\ln \circ G) \tag{2.2.6}$$

for all $s > \ell$.

(b) If ρ is a measure on Γ being finite on regions bounded away from L such that the mapping $V_\rho : (\ell, \infty) \to \mathbb{R}_+$, defined by

$$V_\rho(u) = f(\rho)\big((u, \infty)\big) := f(\rho)(u, \infty),$$

is continuous, then we have

$$-\kappa \int_\ell^{f(x)} e^{-\kappa V_\rho}\,dV_\rho = \int_\ell^{f(x)} d\left(e^{-\kappa V_\rho}\right) \tag{2.2.7}$$

for all $\kappa \geq 0$ and $x \in \Gamma$. In addition, we even have

$$-\kappa \int_\ell^{\infty} e^{-\kappa V_\rho}\,dV_\rho = \int_\ell^{\infty} d\left(e^{-\kappa V_\rho}\right) \tag{2.2.8}$$

for all $\kappa \geq 0$.

Proof. (a) Fix $s > \ell$. Since $G > 0$ on $[s, \infty)$, the continuity of G and (2.2.5) ensure that both

$$\int_s^{b^*} \frac{1}{G}\,dG \qquad \text{and} \qquad \int_s^{b^*} d(\ln \circ G)$$

exist for all $\ell < s \leq b^* < \infty$ and that the integrals are equal. Note that G being a cumulative distribution function implies that G is non-decreasing and hence of bounded variation. Furthermore,

$$\int_s^{b^*} d(\ln \circ G) = \ln(G(b^*)) - \ln(G(s)) \xrightarrow[(b^* \to \infty)]{} -\ln(G(s)) < \infty.$$

Thus, by Definition 2.2.5 (a), we have

$$\int_s^{\infty} d(\ln \circ G) = \lim_{b^* \to \infty} \int_s^{b^*} d(\ln \circ G)$$

showing that the improper Riemann-Stieltjes integral on the right-hand side in (2.2.6) exists. Moreover, since

$$\int_s^{b^*} \frac{1}{G}\, dG = \int_s^{b^*} d(\ln \circ G)$$

for all $\ell < s \leq b^* < \infty$, the limit

$$\lim_{b^* \to \infty} \int_s^{b^*} \frac{1}{G}\, dG$$

exists as well and hence the improper Riemann-Stieltjes integral on the left-hand side in (2.2.6). Combining the recent findings and taking into account Definition 2.2.5 (a) once again, we conclude that

$$\int_s^{\infty} \frac{1}{G}\, dG = \lim_{b^* \to \infty} \int_s^{b^*} \frac{1}{G}\, dG = \lim_{b^* \to \infty} \int_s^{b^*} d(\ln \circ G) = \int_s^{\infty} d(\ln \circ G),$$

and (2.2.6) is proved.

(b) We start by proving (2.2.7). To this end, fix $\kappa \geq 0$ and $x \in \Gamma$. Without loss of generality, we can certainly assume that $\kappa > 0$, for if not, (2.2.7) is evident. Similar to the proof of (a), the continuity of V_ρ on (ℓ, ∞) and (2.2.5) imply that both

$$-\kappa \int_{a^*}^{f(x)} e^{-\kappa V_\rho}\, dV_\rho \qquad \text{and} \qquad \int_{a^*}^{f(x)} d\left(e^{-\kappa V_\rho}\right)$$

exist for all $\ell < a^* \leq f(x) < \infty$ and that the integrals are equal. Furthermore,

$$\int_{a^*}^{f(x)} d\left(e^{-\kappa V_\rho}\right) = e^{-\kappa V_\rho(f(x))} - e^{-\kappa V_\rho(a^*)} \xrightarrow[(a^* \downarrow \ell)]{} e^{-\kappa V_\rho(f(x))} - e^{-\kappa \rho(\Gamma)} < \infty.$$

Applying Definition 2.2.5 (b), we hence obtain

$$\int_{\ell}^{f(x)} d\left(e^{-\kappa V_\rho}\right) = \lim_{a^*\downarrow \ell} \int_{a^*}^{f(x)} d\left(e^{-\kappa V_\rho}\right),$$

so the improper Riemann-Stieltjes integral on the right-hand side in (2.2.7) exists. Since

$$-\kappa \int_{a^*}^{f(x)} e^{-\kappa V_\rho}\, dV_\rho = \int_{a^*}^{f(x)} d\left(e^{-\kappa V_\rho}\right)$$

for all $\ell < a^* \leq f(x) < \infty$, the limit

$$\lim_{a^*\downarrow \ell} \left(-\kappa \int_{a^*}^{f(x)} e^{-\kappa V_\rho}\, dV_\rho \right)$$

exists as well and therefore the improper Riemann-Stieltjes integral on the left-hand side in (2.2.7). As before, we deduce that

$$-\kappa \int_{\ell}^{f(x)} e^{-\kappa V_\rho}\, dV_\rho = \lim_{a^*\downarrow \ell} \left(-\kappa \int_{a^*}^{f(x)} e^{-\kappa V_\rho}\, dV_\rho \right) = \lim_{a^*\downarrow \ell} \int_{a^*}^{f(x)} d\left(e^{-\kappa V_\rho}\right) = \int_{\ell}^{f(x)} d\left(e^{-\kappa V_\rho}\right),$$

which establishes (2.2.7).
In order to prove (2.2.8), let $c \in (\ell, \infty)$ be fixed. Again, we see that both

$$-\kappa \int_{c}^{b^*} e^{-\kappa V_\rho}\, dV_\rho \qquad \text{and} \qquad \int_{c}^{b^*} d\left(e^{-\kappa V_\rho}\right)$$

exist for all $c \leq b^* < \infty$ and that the integrals are equal. We further have

$$\int_{c}^{b^*} d\left(e^{-\kappa V_\rho}\right) = e^{-\kappa V_\rho(b^*)} - e^{-\kappa V_\rho(c)} \xrightarrow[(b^*\to\infty)]{} 1 - e^{-\kappa V_\rho(c)} < \infty$$

showing that

$$\int_{c}^{\infty} d\left(e^{-\kappa V_\rho}\right) = \lim_{b^*\to\infty} \int_{c}^{b^*} d\left(e^{-\kappa V_\rho}\right).$$

Analogous to the proof of (a), we therefore get

$$-\kappa \int_{c}^{\infty} e^{-\kappa V_\rho}\, dV_\rho = \lim_{b^*\to\infty} \left(-\kappa \int_{c}^{b^*} e^{-\kappa V_\rho}\, dV_\rho \right) = \lim_{b^*\to\infty} \int_{c}^{b^*} d\left(e^{-\kappa V_\rho}\right) = \int_{c}^{\infty} d\left(e^{-\kappa V_\rho}\right),$$

and consequently

$$
\begin{aligned}
-\kappa \int_{\ell}^{\infty} e^{-\kappa V_\rho}\, dV_\rho &= -\kappa \int_{\ell}^{c} e^{-\kappa V_\rho}\, dV_\rho + \left(-\kappa \int_{c}^{\infty} e^{-\kappa V_\rho}\, dV_\rho \right) \\
&= \int_{\ell}^{c} d\left(e^{-\kappa V_\rho}\right) + \int_{c}^{\infty} d\left(e^{-\kappa V_\rho}\right) \\
&= \int_{\ell}^{\infty} d\left(e^{-\kappa V_\rho}\right),
\end{aligned}
$$

the latter following from Definition 2.2.5 (c) and from equation (2.2.7). □

In addition to the recently established substitution formulas, the subsequent lemma is also of great importance with regard to the upcoming main results.

Lemma 2.2.9
Let X be a random vector in $\mathbb{R}^d$ such that G is continuous on (ℓ, ∞). Further, let ν denote the measure on Γ defined as

$$
\nu(dx) = \mathbb{P}(f(X) \le f(x))^{-1} \mathbb{P}_X(dx) = G(f(x))^{-1} \mathbb{P}_X(dx).
$$

Then the subsequent properties apply.

(a) For all $s > \ell$,
$$
V_\nu(s) = f(\nu)(s, \infty) = -\ln G(s).
$$

(b) $\nu \in M^b(\Gamma)$ if and only if $\mathbb{P}(f(X) = \ell) = G(\ell) > 0$.

(c) The measure ν is finite on regions bounded away from L, that is, $\nu(A) < \infty$ for all $A \in \mathcal{B}(\Gamma)$ with $\operatorname{dist}(L, A) := \inf\{d(x, y) : x \in L, y \in A\} > 0$.

(d) The measure ν is σ-finite on Γ.

Remark 2.2.10. Note that $\mathbb{P}(f(X) \le f(x)) = G(f(x)) > 0$ for all $x \in \Gamma$ following immediately from the definition of the set Γ. Therefore, the measure ν is actually well-defined.

Proof of Lemma 2.2.1. (a) To start with, fix $s > \ell$. Accordingly, $f^{-1}((s, \infty)) \subset \Gamma$. Applying the common change of variables formula and subsequently (2.2.6), we hence obtain

$$
\begin{aligned}
V_\nu(s) &= f(\nu)(s, \infty) \\
&= \int_{f^{-1}((s,\infty))} G(f(x))^{-1}\, \mathbb{P}_X(dx) \\
&= \int_{s}^{\infty} \frac{1}{G(u)}\, dG(u)
\end{aligned}
$$

$$= \int_{s}^{\infty} d(\ln \circ G)(u)$$
$$= -\ln G(s).$$

(b) The asserted equivalence follows immediately from the previously proven claim. Indeed, we have

$$\nu(\Gamma) = f(\nu)(\ell, \infty) = \lim_{s \downarrow \ell} f(\nu)(s, \infty) = -\lim_{s \downarrow \ell} \ln G(s) = -\ln G(\ell) = -\ln \mathbb{P}(f(X) = \ell),$$

where the second step is due to the fact that $f(\nu)$ is continuous from below, the forth step due to the right-continuity of G and the last step finally due to the definition of $\ell \geq 0$. Clearly, the latter equation yields the desired conclusion.
(c) In order to prove this part of the lemma, we need to consider the cases $\ell > 0$ and $\ell = 0$ separately. Fix $\varepsilon > 0$. To begin with, let $\ell > 0$. We have to show that $\nu(A_\varepsilon) < \infty$, the set $A_\varepsilon \subset \Gamma$ being defined by

$$A_\varepsilon = \{\lambda x \in \mathbb{R}^d : x \in L, \lambda > 1 + \varepsilon\}.$$

To this end, let $y \in A_\varepsilon$ be arbitrary. Accordingly, $y = \lambda_0 x_0$ for some $\lambda_0 > 1 + \varepsilon$ and some $x_0 \in L$. Then we have

$$\mathbb{P}(f(X) \leq f(y)) = G(\lambda_0 f(x_0)) = G(\lambda_0 \ell) \geq G((1 + \varepsilon)\ell) > 0$$

following from the 1-homogeneity of f. Therefore, we deduce that

$$\inf_{y \in A_\varepsilon} \mathbb{P}(f(X) \leq f(y)) = \inf_{y \in A_\varepsilon} G(f(y)) > 0,$$

and consequently

$$\sup_{y \in A_\varepsilon} \mathbb{P}(f(X) \leq f(y))^{-1} = \sup_{y \in A_\varepsilon} G(f(y))^{-1} < \infty.$$

Applying the latter, we finally conclude that

$$\nu(A_\varepsilon) = \int_{A_\varepsilon} G(f(x))^{-1}\, \mathbb{P}_X(dx) \leq \sup_{y \in A_\varepsilon} G(f(y))^{-1} \mathbb{P}_X(A_\varepsilon) < \infty,$$

which is the desired claim, provided $\ell > 0$. Now, let $\ell = 0$. As $L = \{0\}$, we only need to show that $\nu(\tilde{A}_\varepsilon) < \infty$, the set $\tilde{A}_\varepsilon \subset \Gamma$ here being defined by

$$\tilde{A}_\varepsilon = \{\lambda x \in \mathbb{R}^d : \|x\| = 1, \lambda > \varepsilon\}$$

with some norm $\|\cdot\|$ on $\mathbb{R}^d$. Using similar ideas as in the case $\ell > 0$ as well as the assumption on the null set of f, we get

$$\sup_{y \in \tilde{A}_\varepsilon} \mathbb{P}(f(X) \leq f(y))^{-1} = \sup_{y \in \tilde{A}_\varepsilon} G(f(y))^{-1} < \infty$$

and hence

$$\nu(\tilde{A}_\varepsilon) < \infty.$$

This finishes the proof of (c), the details in the case $\ell = 0$ being skipped.
(d) Showing σ-finiteness of ν on Γ is straightforward. Indeed, let $(A_n)_{n\geq 1}$ be the countable sequence of sets $A_n \in \mathcal{B}(\Gamma)$ defined by

$$A_n = f^{-1}\left(\left(\ell + \frac{1}{n}, \infty\right)\right).$$

Then we have $A_n \uparrow f^{-1}((\ell, \infty)) = \Gamma$ and, by part (c), $\nu(A_n) < \infty$ for all $n \geq 1$. This is precisely the assertion. □

Remark 2.2.11. Clearly, assertion (b) is equivalent to saying $\nu(\Gamma) = \infty$ if and only if $\mathbb{P}(f(X) = \ell) = G(\ell) = 0$.

Finally, we are prepared to elaborate on the two above-mentioned classes of random vectors and to prove that all such random vectors are f-implicit max-infinitely divisible. We begin by focusing on the first class of random vectors. Our investigations concerning the second class of random vectors are postponed for the time being.
The first step is to show that the distribution of all such random vectors X satisfies a particular representation. Following this, we establish the notion of f-implicit max-convolution semigroups on $\mathbb{R}^d$ that eventually enables us to prove that X is in fact f-implicit max-infinitely divisible.

Theorem 2.2.12
Let X be a random vector in $\mathbb{R}^d$ coming under the first class of random vectors, that is, the cumulative distribution function $G : \mathbb{R} \to \mathbb{R}$ of $f(X)$ is continuous on (ℓ, ∞). Further, let ν be the corresponding measure occurring in Lemma 2.2.9.

(a) If $\nu(\Gamma) = \infty$, the mass of $\mathbb{P}_X$ is concentrated on Γ and we have

$$\mathbb{P}_X(dx) = e^{-f(\nu)(f(x),\infty)}\nu(dx), \tag{2.2.9}$$

that is,

$$\mathbb{P}(X \in A) = \int\limits_{A\cap\Gamma} e^{-f(\nu)(f(x),\infty)}\, \nu(dx) \tag{2.2.10}$$

for all $A \in \mathcal{B}(\mathbb{R}^d)$.

(b) If $\nu(\Gamma) < \infty$, there exists a measure $\rho_L \in M^1(\mathbb{R}^d)$ with supp $\rho_L \subset L$ such that

$$\mathbb{P}_X(dx) = e^{-\nu(\Gamma)}\rho_L(dx) + e^{-f(\nu)(f(x),\infty)}\nu(dx), \tag{2.2.11}$$

that is,

$$\mathbb{P}(X \in A) = e^{-\nu(\Gamma)}\rho_L(A) + \int\limits_{A\cap\Gamma} e^{-f(\nu)(f(x),\infty)}\, \nu(dx) \tag{2.2.12}$$

for all $A \in \mathcal{B}(\mathbb{R}^d)$.

Proof. (a) Fix $A \in \mathcal{B}(\mathbb{R}^d)$ and let $\nu(\Gamma) = \infty$. First, we deduce that $\mathbb{P}(f(X) = \ell) = 0$ following immediately from Remark 2.2.11. By (2.2.3), this yields $\mathbb{P}(X \in \Gamma) = 1$, thus showing that the mass of $\mathbb{P}_X$ is indeed concentrated on Γ. Moreover, we have

$$G(f(x)) = e^{-V_\nu(f(x))} = e^{-f(\nu)(f(x),\infty)}$$

for all $x \in \Gamma$ which is an easy consequence of Lemma 2.2.9 (a). Hence, we can conclude that

$$\mathbb{P}(X \in A) = \int\limits_{A\cap\Gamma} \mathbb{P}_X(dx) = \int\limits_{A\cap\Gamma} G(f(x))\,\nu(dx) = \int\limits_{A\cap\Gamma} e^{-f(\nu)(f(x),\infty)}\,\nu(dx),$$

and (2.2.10) is proved.
(b) Now, fix $A \in \mathcal{B}(\mathbb{R}^d)$ and let $\nu(\Gamma) < \infty$. Referring to (2.2.3) once again, we see that the mass of $\mathbb{P}_X$ is concentrated on $\overline{\Gamma}$. Therefore, we have

$$\begin{aligned}
\mathbb{P}(X \in A) &= \mathbb{P}\left(X \in A \cap \overline{\Gamma}\right) \\
&= \mathbb{P}(X \in A \cap L) + \int\limits_{A\cap\Gamma} \mathbb{P}_X(dx) \\
&= \mathbb{P}(X \in A \cap L) + \int\limits_{A\cap\Gamma} e^{-f(\nu)(f(x),\infty)}\,\nu(dx). \qquad (2.2.13)
\end{aligned}$$

The proof is completed by showing that

$$\mathbb{P}(X \in A \cap L) = e^{-\nu(\Gamma)}\rho_L(A) \qquad (2.2.14)$$

for some probability measure ρ_L with supp $\rho_L \subset L$. This is straightforward. Indeed, applying (2.2.13), subsequently common properties of the theory of Lebesgue- and Riemann-Stieltjes integration and finally (2.2.8) for $\kappa = 1$ and $\rho = \nu$, we get

$$\begin{aligned}
\mathbb{P}(X \in \mathbb{R}^d) &= \mathbb{P}(X \in L) + \int\limits_{\Gamma} e^{-f(\nu)(f(x),\infty)}\,\nu(dx) \\
&= \mathbb{P}(X \in L) + \int\limits_{\ell}^{\infty} e^{-f(\nu)(u,\infty)}\,f(\nu)(du) \\
&= \mathbb{P}(X \in L) - \int\limits_{\ell}^{\infty} e^{-V_\nu(u)}\,dV_\nu(u) \\
&= \mathbb{P}(X \in L) + \int\limits_{\ell}^{\infty} d\left(e^{-V_\nu(u)}\right) \\
&= \mathbb{P}(X \in L) + 1 - e^{-\nu(\Gamma)}.
\end{aligned}$$

Equivalently, we have
$$\mathbb{P}(X \in L) = e^{-\nu(\Gamma)}.$$
Therefore, the measure ρ_L on $\mathbb{R}^d$, defined by
$$\rho_L(A) = e^{\nu(\Gamma)}\,\mathbb{P}(X \in A \cap L),$$
is actually a probability measure with supp $\rho_L \subset L$. This proves (2.2.14), and the desired conclusion follows. $\square$

Remark 2.2.13. Having proved Theorem 2.2.12, we do well to elaborate on one argument used in its proof. In particular, we need to clarify the applicability of (2.2.8) in the preceding computations. To this end, just note that $V_\nu : (\ell, \infty) \to \mathbb{R}_+$ is continuous, so that the assumptions of Corollary 2.2.8 (b) are fulfilled for the particular case $\rho = \nu$. The continuity of V_ν is clear by Lemma 2.2.9 (a) and the fact that G is continuous. In addition, Lemma 2.2.9 (c) shows that ν is finite on regions bounded away from L.

Remark 2.2.14. As a test, we can actually convince ourselves, by using similar calculations as in the preceding proof, that both the right-hand side in (2.2.9) and the right-hand side in (2.2.11) constitute appropriate probability distributions.

The next proposition is intended to extend the assertions of Theorem 2.2.12 in a particular way. Actually, we can prove that there cannot exist another measure ν' on Γ with the property of being finite on regions bounded away from L and with $V_{\nu'}$ being continuous such that (2.2.9) holds, provided $\nu'(\Gamma) = \infty$. In addition, we can show that there cannot exist another measure $\nu' \in M^b(\Gamma)$ with $V_{\nu'}$ being continuous such that (2.2.11) holds. In this case even the probability measure ρ_L on $\mathbb{R}^d$ occurring in (2.2.11) turns out to be unique.

Proposition 2.2.15
Differing the two relevant cases, we obtain the subsequent statements.

(a) Suppose that X is a random vector with $\mathbb{P}(X \in \Gamma) = 1$ and
$$e^{-f(\nu_1)(f(x),\infty)}\nu_1(dx) = \mathbb{P}_X(dx) = e^{-f(\nu_2)(f(x),\infty)}\nu_2(dx)$$
for two unbounded measures ν_1 and ν_2 on Γ, which are finite on regions bounded away from L such that V_{ν_1} and V_{ν_2} are continuous. Then we obtain $\nu_1 = \nu_2$.

(b) Suppose that X is a random vector with
$$e^{-\nu_1(\Gamma)}\rho_{L,1}(dx) + e^{-f(\nu_1)(f(x),\infty)}\nu_1(dx) = \mathbb{P}_X(dx) = e^{-\nu_2(\Gamma)}\rho_{L,2}(dx) + e^{-f(\nu_2)(f(x),\infty)}\nu_2(dx)$$
for two measures $\nu_1, \nu_2 \in M^b(\Gamma)$ such that V_{ν_1} and V_{ν_2} are continuous and for two measures $\rho_{L,1}, \rho_{L,2} \in M^1(\mathbb{R}^d)$ such that supp $\rho_{L,1} \subset L$ and supp $\rho_{L,2} \subset L$, respectively. Then we obtain $\rho_{L,1} = \rho_{L,2}$ and $\nu_1 = \nu_2$.

Proof. To start with, we remark that both equation (2.2.7) and equation (2.2.8) can clearly be generalized. Indeed, by repeating the argumentation in the corresponding proofs, we obtain

$$-\kappa \int_{y}^{f(x)} e^{-\kappa V_\rho}\, dV_\rho = \int_{y}^{f(x)} d\left(e^{-\kappa V_\rho}\right) \tag{2.2.15}$$

for all $\kappa \geq 0$, all $x \in \Gamma$ and all $\ell \leq y \leq f(x)$ as well as

$$-\kappa \int_{y}^{\infty} e^{-\kappa V_\rho}\, dV_\rho = \int_{y}^{\infty} d\left(e^{-\kappa V_\rho}\right) \tag{2.2.16}$$

for all $\kappa \geq 0$ and all $\ell \leq y < \infty$. Remember that ρ was assumed to be a measure on Γ such that V_ρ is continuous and such that ρ is finite on regions bounded away from L. Therefore, both (2.2.15) and (2.2.16) can be applied to both ν_1 and ν_2. Especially, we will benefit from (2.2.16).

(a) For the rest of the proof of part (a), we shall consider $\mathbb{P}_X$ as a probability measure on Γ leading to no loss of generality. Fix $\ell \leq y < \infty$ and let $A_y \subset \Gamma$ be the Borel set defined by

$$A_y = f^{-1}((y,\infty)).$$

Applying the assumed representation of $\mathbb{P}_X$ and subsequently equation (2.2.16) for $\kappa = 1$ and $\rho = \nu_1$, we get

$$\begin{aligned}
\mathbb{P}(X \in A_y) &= \int_{f^{-1}((y,\infty))} e^{-f(\nu_1)(f(x),\infty)}\, \nu_1(dx) \\
&= \int_{y}^{\infty} e^{-f(\nu_1)(u,\infty)}\, f(\nu_1)(du) \\
&= -\int_{y}^{\infty} e^{-V_{\nu_1}(u)}\, dV_{\nu_1}(u) \\
&= \int_{y}^{\infty} d\left(e^{-V_{\nu_1}(u)}\right) \\
&= 1 - e^{-\nu_1\left(f^{-1}((y,\infty))\right)}.
\end{aligned} \tag{2.2.17}$$

Similarly, we have

$$\mathbb{P}(X \in A_y) = 1 - e^{-\nu_2\left(f^{-1}((y,\infty))\right)}. \tag{2.2.18}$$

Combining (2.2.17) and (2.2.18), we deduce immediately that

$$\nu_1\left(f^{-1}((y,\infty))\right) = \nu_2\left(f^{-1}((y,\infty))\right). \tag{2.2.19}$$

In other words, we have proved that ν_1 and ν_2 already coincide on the set of Borel sets

$$\left\{A_y = f^{-1}((y,\infty)) :\ \ell \leq y < \infty\right\}.$$

Thus, we can conclude that

$$\begin{aligned}\int_A e^{-f(\nu_1)(f(x),\infty)}\,\nu_1(dx) = \mathbb{P}(X \in A) &= \int_A e^{-f(\nu_2)(f(x),\infty)}\,\nu_2(dx)\\ &= \int_A e^{-\nu_2\left(f^{-1}((f(x),\infty))\right)}\,\nu_2(dx)\\ &= \int_A e^{-\nu_1\left(f^{-1}((f(x),\infty))\right)}\,\nu_2(dx)\\ &= \int_A e^{-f(\nu_1)(f(x),\infty)}\,\nu_2(dx)\end{aligned}$$

for all $A \in \mathcal{B}(\Gamma)$. Consequently, we obtain

$$\mathbb{P}_X(dx) = h(x)\nu_1(dx) \qquad \text{and} \qquad \mathbb{P}_X(dx) = h(x)\nu_2(dx), \tag{2.2.20}$$

where the function $h : \Gamma \to (0,\infty)$ is given by

$$h(x) = e^{-f(\nu_1)(f(x),\infty)}.$$

Since h is positive, (2.2.20) yields $\nu_1 \ll \mathbb{P}_X$ and $\nu_2 \ll \mathbb{P}_X$ which in turn ensures the existence of a Radon-Nikodým derivate $g : \Gamma \to (0,\infty)$ of ν_1 with respect to $\mathbb{P}_X$ and of ν_2 with respect to $\mathbb{P}_X$. Applying common properties concerning the classical Radon-Nikodým theorem, we actually have $g = h^{-1}$. As a consequence, we get

$$\nu_1(A) = \int_A h^{-1}(x)\,\mathbb{P}_X(dx) = \int_A h^{-1}(x)h(x)\,\nu_2(dx) = \nu_2(A) \tag{2.2.21}$$

for all $A \in \mathcal{B}(\Gamma)$.
(b) To start with, note that (2.2.17), (2.2.18) and (2.2.19) clearly also apply in the setting of part (b) as $A_y \subset \Gamma$ for all $\ell \le y < \infty$. Since (2.2.19) implies in particular that

$$\nu_1(\Gamma) = \nu_2(\Gamma),$$

we deduce that

$$e^{-\nu_1(\Gamma)}\rho_{L,1}(B) = \mathbb{P}(X \in B) = e^{-\nu_2(\Gamma)}\rho_{L,2}(B) = e^{-\nu_1(\Gamma)}\rho_{L,2}(B)$$

and thus

$$\rho_{L,1}(B) = \rho_{L,2}(B)$$

for all Borel sets $B \subset L$. The latter, however, shows that $\rho_{L,1} = \rho_{L,2}$ as supp $\rho_{L,1} \subset L$ and supp $\rho_{L,2} \subset L$. Actually, this completes the proof of part (b). Indeed, $\nu_1 = \nu_2$ can be established similar to the way of proceeding in part (a). Applying (2.2.19), we obtain

$$\mathbb{P}_X\big|_\Gamma(dx) = h(x)\nu_1(dx) \qquad \text{and} \qquad \mathbb{P}_X\big|_\Gamma(dx) = h(x)\nu_2(dx)$$

Therefore, (2.2.21) follows instantly by using an analogous argumentation as above. □

Remark 2.2.16. Combining Theorem 2.2.12 and the latter proposition, we can finally deduce that the assertion stated before Proposition 2.2.15 holds. In other words, for all X in $\mathbb{R}^d$ coming under the first class of random vectors, the measure ν on Γ introduced in Lemma 2.2.9 is the only unbounded measure on Γ with the property of being finite on regions bounded away from L and with V_ν being continuous such that (2.2.9) holds, provided $f(X)$ has no atom in $\ell \geq 0$. If $f(X)$ has an atom in $\ell \geq 0$, however, then the measure ν on Γ introduced in Lemma 2.2.9 is the only bounded measure on Γ with V_ν being continuous such that (2.2.11) holds. In this case, the probability measure ρ_L occurring in (2.2.11) is also unique and moreover explicitly given by

$$\rho_L(A) = e^{\nu(\Gamma)}\, \mathbb{P}(X \in A \cap L)$$

as can be deduced from the proof of Theorem 2.2.12.

The latter remark decides us to establish the following, convenient notation. Let X be a random vector in $\mathbb{R}^d$ coming under the first class of random vectors, that is, the cumulative distribution function G of $f(X)$ is continuous on (ℓ, ∞). Then we write

$$X \sim \begin{cases} [\nu]_f, & \text{if } G(\ell) = 0, \\ [\rho_L, \nu]_f, & \text{if } G(\ell) > 0. \end{cases} \tag{2.2.22}$$

Having investigated the structure of all random vectors X in $\mathbb{R}^d$, which come under the first class of random vectors, we proceed to introduce the notion of f-implicit max-convolution semigroups. In doing so, we will be guided by the general and commonly known notion of classical convolution semigroups (see for example [Kl08, Definition 14.46]). However, we need to formulate the definition in a slightly different form since the f-implicit max-convolution does not have the same structure as the common convolution.

Definition 2.2.17
Let I be either $(0, \infty)$ or $[0, \infty)$ and let $(\mu_t)_{t \in I}$ be a family of probability measures on $\mathbb{R}^d$. If

$$\mu_{s+t} = \mu_s *_f \mu_t \tag{2.2.23}$$

for all $s, t \in I$, we refer to $(\mu_t)_{t \in I}$ as *f-implicit max-convolution semigroup of probability measures on* $\mathbb{R}^d$. Moreover, an f-implicit max-convolution semigroup of probability measures on $\mathbb{R}^d$ is said to be *continuous* if the mapping $\iota : I \to M^1(\mathbb{R}^d)$, defined by $\iota(t) = \mu_t$, is (weakly) continuous. That is,

$$\int_{\mathbb{R}^d} h(x)\, \mu_s(dx) \xrightarrow[(s \to t)]{} \int_{\mathbb{R}^d} h(x)\, \mu_t(dx)$$

for all bounded and continuous functions h on $\mathbb{R}^d$.

Remark 2.2.18. (i) Unlike in the theory of classical convolution semigroups (see for example [Ba91, Definition 29.5]) we do not necessarily have $\mu_0 = \varepsilon_0$, provided $I = [0, \infty)$. A suitable counterexample will be given in Theorem 2.2.19 (b).

(ii) Clearly, each probability measure μ_t of an f-implicit max-convolution semigroup $(\mu_t)_{t\in I}$ is f-implicit max-infinitely divisible being an easy consequence of (2.2.23). In Remark 3.1.5 we will have recourse to this aspect when considering a particular f-implicit max-convolution semigroup.

(iii) Note that property (2.2.23) actually yields $\mu_s *_f \mu_t = \mu_t *_f \mu_s$ for all $s, t \in I$.

Theorem 2.2.19
Fix $\ell_0 \geq 0$.

(a) Let ν_0 be an unbounded measure on Γ_{ℓ_0} such that ν_0 is finite on regions bounded away from L_{ℓ_0} and such that the mapping $V_{\nu_0} : (\ell_0, \infty) \to \mathbb{R}_+$, defined by

$$V_{\nu_0}(s) = f(\nu_0)\Big((s, \infty)\Big) := f(\nu_0)(s, \infty),$$

is continuous. Then the family $(\mu_t)_{t>0}$ of probability measures μ_t on $\mathbb{R}^d$, defined by

$$\mu_t(dx) = te^{-tf(\nu_0)(f(x),\infty)}\nu_0(dx) = te^{-tV_{\nu_0}(f(x))}\nu_0(dx) \tag{2.2.24}$$

for all $t > 0$, is an f-implicit max-convolution semigroup.

(b) Let $\nu_0 \in M^b(\Gamma_{\ell_0})$ such that V_{ν_0} is continuous. Further, let $\rho_0 := \rho_{L_{\ell_0}} \in M^1(\mathbb{R}^d)$ such that $\operatorname{supp} \rho_0 \subset L_{\ell_0}$. Then the family $(\mu_t)_{t\geq 0}$ of probability measures μ_t on $\mathbb{R}^d$, defined by

$$\begin{aligned}\mu_t(dx) &= e^{-t\nu_0(\Gamma_{\ell_0})}\rho_0(dx) + te^{-tf(\nu_0)(f(x),\infty)}\nu_0(dx)\\ &= e^{-t\nu_0(\Gamma_{\ell_0})}\rho_0(dx) + te^{-tV_{\nu_0}(f(x))}\nu_0(dx)\end{aligned} \tag{2.2.25}$$

for all $t \geq 0$, is a continuous f-implicit max-convolution semigroup.

Remark 2.2.20. (i) As before, the formal notation used in (2.2.24) and in (2.2.25) is a convenient abbreviation for

$$\mu_t(A) = \int\limits_{A\cap\Gamma_{\ell_0}} te^{-tV_{\nu_0}(f(x))}\,\nu_0(dx)$$

and

$$\mu_t(A) = e^{-t\nu_0(\Gamma_{\ell_0})}\rho_0(A) + \int\limits_{A\cap\Gamma_{\ell_0}} te^{-tV_{\nu_0}(f(x))}\,\nu_0(dx),$$

respectively. In particular, this involves $\mu_t(\Gamma_{\ell_0}) = 1$ and $\mu_t(\overline{\Gamma_{\ell_0}}) = 1$, respectively.

(ii) Note that part (b) of Theorem 2.2.19 yields $\mu_0 = \rho_0$, the measure ρ_0 here being an arbitrary probability measure on $\mathbb{R}^d$ with $\operatorname{supp} \rho_0 \subset L_{\ell_0}$. If $\ell_0 > 0$ and thus $L_{\ell_0} \neq \{0\}$, this provides a suitable counterexample for the issue stated in Remark 2.2.18.

Proof of Theorem 2.2.19. As already mentioned in Remark 2.2.14, simple calculations yield immediately that both the family $(\mu_t)_{t>0}$ given in part (a) and the family $(\mu_t)_{t\geq 0}$ given in part (b) are families of probability measures on $\mathbb{R}^d$. An essential tool helping to verify these facts is in particular given by Corollary 2.2.8. However, it must be underlined that this is just ensured due to the required assumptions on ν_0 and ρ_0.
(a) Fix $s, t > 0$ and $A \in \mathcal{B}(\mathbb{R}^d)$. Observe, by applying similar arguments as in Theorem 2.2.12 or in Proposition 2.2.15 and by using a slight modification of Corollary 2.2.8 with ℓ, L, Γ replaced by $\ell_0, L_{\ell_0}, \Gamma_{\ell_0}$, that

$$\begin{aligned}
f(\mu_t)\big([0, f(x)]\big) &= \underbrace{\mu_t\left(f^{-1}([0,\ell_0])\right)}_{=0} + \mu_t\left(f^{-1}((\ell_0, f(x)])\right) \\
&= t \int\limits_{f^{-1}((\ell_0, f(x)])} e^{-tf(\nu_0)(f(y),\infty)}\,\nu_0(dy) \\
&= t \int\limits_{\ell_0}^{f(x)} e^{-tf(\nu_0)(u,\infty)}\,f(\nu_0)(du) \\
&= -t \int\limits_{\ell_0}^{f(x)} e^{-tV_{\nu_0}(u)}\,dV_{\nu_0}(u) \\
&= \int\limits_{\ell_0}^{f(x)} d\left(e^{-tV_{\nu_0}(u)}\right) \\
&= e^{-tV_{\nu_0}(f(x))} - e^{-t\nu_0(\Gamma_{\ell_0})} \\
&= e^{-tf(\nu_0)(f(x),\infty)}
\end{aligned}$$

for all $x \in \Gamma_{\ell_0}$. Similarly, we have

$$f(\mu_s)\big([0, f(x))\big) = e^{-sf(\nu_0)(f(x),\infty)}$$

for all $x \in \Gamma_{\ell_0}$ since V_{ν_0} is left-continuous in $f(x)$. Using (1.2.4) and (2.2.24), we can hence conclude that

$$\begin{aligned}
\mu_s *_f \mu_t(A) &= \int\limits_A f(\mu_t)\big([0, f(x)]\big)\,\mu_s(dx) + \int\limits_A f(\mu_s)\big([0, f(x))\big)\,\mu_t(dx) \\
&= s \int\limits_{A\cap\Gamma_{\ell_0}} e^{-tf(\nu_0)(f(x),\infty)} e^{-sf(\nu_0)(f(x),\infty)}\,\nu_0(dx) \\
&\quad + t \int\limits_{A\cap\Gamma_{\ell_0}} e^{-tf(\nu_0)(f(x),\infty)} e^{-sf(\nu_0)(f(x),\infty)}\,\nu_0(dx)
\end{aligned}$$

$$= (s+t) \int\limits_{A\cap\Gamma_{\ell_0}} e^{-(s+t)f(\nu_0)(f(x),\infty)}\, \nu_0(dx)$$
$$= \mu_{s+t}(A).$$

(b) Now, fix $s, t \geq 0$ and $A \in \mathcal{B}(\mathbb{R}^d)$. Similar as before, we first observe that

$$\begin{aligned} f(\mu_t)\big([0, f(x)]\big) &= \mu_t\left(f^{-1}([0,\ell_0])\right) + \mu_t\left(f^{-1}((\ell_0, f(x)])\right) \\ &= e^{-t\nu_0(\Gamma_{\ell_0})} \underbrace{\rho_0(\mathbb{R}^d \setminus \Gamma_{\ell_0})}_{=1} + \left(e^{-tV_{\nu_0}(f(x))} - e^{-t\nu_0(\Gamma_{\ell_0})}\right) \\ &= e^{-tf(\nu_0)(f(x),\infty)} \end{aligned}$$

for all $x \in \Gamma_{\ell_0}$. Note, however, that ν_0 being a finite measure is actually important here. For all $x \in \Gamma_{\ell_0}$, we also have

$$f(\mu_s)\big([0, f(x))\big) = e^{-sf(\nu_0)(f(x),\infty)}$$

once again following in exactly the same way as above. Therefore, we can deduce, by applying (1.2.4) and (2.2.25), that

$$\begin{aligned} \mu_s *_f \mu_t(A) &= \int\limits_A f(\mu_t)\big([0,f(x)]\big)\,\mu_s(dx) + \int\limits_A f(\mu_s)\big([0,f(x))\big)\,\mu_t(dx) \\ &= \int\limits_{A\cap L_{\ell_0}} f(\mu_t)\big([0,f(x)]\big)\,\mu_s(dx) + \int\limits_{A\cap\Gamma_{\ell_0}} f(\mu_t)\big([0,f(x)]\big)\,\mu_s(dx) \\ &\quad + \int\limits_{A\cap L_{\ell_0}} f(\mu_s)\big([0,f(x))\big)\,\mu_t(dx) + \int\limits_{A\cap\Gamma_{\ell_0}} f(\mu_s)\big([0,f(x))\big)\,\mu_t(dx) \\ &= e^{-s\nu_0(\Gamma_{\ell_0})}\rho_0(A)\, \underbrace{f(\mu_t)\big([0,\ell_0]\big)}_{=e^{-t\nu_0(\Gamma_{\ell_0})}} + s \int\limits_{A\cap\Gamma_{\ell_0}} f(\mu_t)\big([0,f(x)]\big)\, e^{-sf(\nu_0)(f(x),\infty)}\,\nu_0(dx) \\ &\quad + e^{-t\nu_0(\Gamma_{\ell_0})}\rho_0(A)\, \underbrace{f(\mu_s)\big([0,\ell_0)\big)}_{=0} + t \int\limits_{A\cap\Gamma_{\ell_0}} f(\mu_s)\big([0,f(x))\big)\, e^{-tf(\nu_0)(f(x),\infty)}\,\nu_0(dx) \\ &= e^{-(s+t)\nu_0(\Gamma_{\ell_0})}\rho_0(A) + s \int\limits_{A\cap\Gamma_{\ell_0}} e^{-tf(\nu_0)(f(x),\infty)}\, e^{-sf(\nu_0)(f(x),\infty)}\,\nu_0(dx) \\ &\quad + t \int\limits_{A\cap\Gamma_{\ell_0}} e^{-sf(\nu_0)(f(x),\infty)}\, e^{-tf(\nu_0)(f(x),\infty)}\,\nu_0(dx) \end{aligned}$$

$$= e^{-(s+t)\nu_0(\Gamma_{\ell_0})}\rho_0(A) + (s+t) \int\limits_{A\cap\Gamma_{\ell_0}} e^{-(s+t)f(\nu_0)(f(x),\infty)}\,\nu_0(dx)$$

$$= \mu_{s+t}(A).$$

The continuity of $(\mu_t)_{t\geq 0}$ is an easy consequence of Lebesgue's dominated convergence theorem since

$$\int\limits_{\mathbb{R}^d} h(x)\,\mu_s(dx) = e^{-s\nu_0(\Gamma_{\ell_0})} \int\limits_{L_{\ell_0}} h(x)\,\rho_0(dx) + s \int\limits_{\Gamma_{\ell_0}} h(x) e^{-sf(\nu_0)(f(x),\infty)}\,\nu_0(dx)$$

$$\xrightarrow[(s\to t)]{} e^{-t\nu_0(\Gamma_{\ell_0})} \int\limits_{L_{\ell_0}} h(x)\,\rho_0(dx) + t \int\limits_{\Gamma_{\ell_0}} h(x) e^{-tf(\nu_0)(f(x),\infty)}\,\nu_0(dx)$$

$$= \int\limits_{\mathbb{R}^d} h(x)\,\mu_t(dx)$$

for all bounded and continuous functions h on $\mathbb{R}^d$. This completes the proof of part (b). □

Finally, Theorem 2.2.12 in combination with Theorem 2.2.19 yields immediately that all X in $\mathbb{R}^d$ coming under the first class of random vectors are f-implicit max-infinitely divisible. Note that we adopt the notation introduced in Theorem 2.2.19.

Corollary 2.2.21
Suppose that X is a random vector in $\mathbb{R}^d$ coming under the first class of random vectors.

(a) If $X \sim [\nu]_f$, then X is f-implicit max-infinitely divisible. Moreover, for all $n \geq 1$,

$$\mathbb{P}_{X_1^{(n)}}(dx) := \mu_{\frac{1}{n}}(dx) = \frac{1}{n} e^{-\frac{1}{n} f(\nu)(f(x),\infty)} \nu(dx) = \frac{1}{n} e^{-\frac{1}{n} V_\nu(f(x))} \nu(dx) \tag{2.2.26}$$

is a suitable nth root of X.

(b) If $X \sim [\rho_L, \nu]_f$, then X is f-implicit max-infinitely divisible. Furthermore, for all $n \geq 1$,

$$\mathbb{P}_{X_1^{(n)}}(dx) := \mu_{\frac{1}{n}}(dx) = e^{-\frac{1}{n}\nu(\Gamma)} \rho_L(dx) + \frac{1}{n} e^{-\frac{1}{n} f(\nu)(f(x),\infty)} \nu(dx)$$
$$= e^{-\frac{1}{n}\nu(\Gamma)} \rho_L(dx) + \frac{1}{n} e^{-\frac{1}{n} V_\nu(f(x))} \nu(dx) \tag{2.2.27}$$

is a a suitable nth root of X.

Proof. The proof is straightforward after having already established the previous results. (a) Applying Theorem 2.2.19 for $\ell_0 = \ell$ and $\nu_0 = \nu$ as well as using the representation of the distribution of X, we deduce immediately that

$$\left(\mathbb{P}_{X_1^{(n)}}\right)^{*_f n} = \mu_{\frac{1}{n}} *_f \ldots *_f \mu_{\frac{1}{n}} = \mu_1 = \mathbb{P}_X$$

for all $n \geq 1$, which is precisely the asserted claim.
(b) This is once again an easy consequence of Theorem 2.2.19 in combination with the representation of the distribution of X. □

Remark 2.2.22. Note that both Example 2.1.4 and Lemma 2.1.5 are special cases of Corollary 2.2.21.
Clearly, any random vector X in $\mathbb{R}^d$ with a distribution as described in Example 2.1.4 comes under the first class of random vectors. Actually, we have $X \sim [\rho_L, 0]_f$ so that (2.2.27) reduces to

$$\mathbb{P}_{X_1^{(n)}}(dx) = \rho_L(dx)$$

which coincides with the findings in Example 2.1.4 since $X \sim \rho_L$.
Moreover, even all f-implicit max-stable distributions come under the first class of distributions. Indeed, X being f-implicit max-stable implies immediately that $f(X)$ is α-Fréchet, provided the support of $\mathbb{P}_X$ is not confined to L_{ℓ_0} for some $\ell_0 \geq 0$ (see for example [SchSt14, Theorem 4.2]). Accordingly, the cumulative distribution function G of $f(X)$ is continuous on $(0, \infty)$ proving that X comes under the first class of distributions and is therefore f-implicit max-infinitely divisible. For all $n \geq 1$, both (2.1.4) and (2.2.26) can be chosen as nth root of X. In this case, they even actually coincide.
Although Example 2.1.4 and Lemma 2.1.5 now appear to be redundant, we did well to include them at the beginning of Chapter 2 in order to illustrate Definition 2.1.1 as well as the assertion of Lemma 2.1.2. In addition, this underlines the verisimilitude of our recent findings.
More aspects concerning the notion of α-Fréchet random variables and its connection to f-implicit max-stable distributions will be given in Chapter 3 or can also be found in [SchSt14].

Remark 2.2.23. The assertion that all random vectors X in $\mathbb{R}^d$ coming under the first class of random vectors are f-implicit max-infinitely divisible could have also been proved in a different way. In fact, for all $n \geq 1$,

$$\mathbb{P}_{X_1^{(n)}}(dx) = \begin{cases} (1 - \mathbb{P}(X \in \Gamma))^{\frac{1}{n}} \rho_L(dx) + \frac{1}{n}\mathbb{P}(f(X) \leq f(x))^{\frac{1}{n}-1}\mathbb{P}_X\big|_\Gamma(dx), & \text{if } G(\ell) > 0, \\ \frac{1}{n}\mathbb{P}(f(X) \leq f(x))^{\frac{1}{n}-1}\mathbb{P}_X(dx), & \text{if } G(\ell) = 0 \end{cases}$$

serves as an nth root of X which can be seen by applying Lemma 1.4.1 as well as the well-established substitution rules for Riemann-Stieltjes integrals. Here, ρ_L is an appropriate probability measure on $\mathbb{R}^d$ with supp $\rho_L \subset L$. We will pick up on this aspect in Section 2.4. Note, even in the case $G(\ell) = 0$ the distribution of the nth root is indeed well-defined since $\mathbb{P}(X \in \Gamma) = 1$ and hence $\mathbb{P}_X$ can be considered as a measure on Γ. The function $x \mapsto \mathbb{P}(f(X) \leq f(x))^{1/n-1}$ is de facto just relevant if $x \in \Gamma$. Clearly, the formal notation used above shall involve this observation. A similar consideration will occur in Remark 2.4.2.

Having proved the latter corollary and thus having completed our considerations concerning the first class of distributions, we can now proceed to focus on the second one. In addition to the previous findings concerning the first class of random vectors, it is our aim to show that all X in $\mathbb{R}^d$ coming under the second class of random vectors

are f-implicit max-infinitely divisible. Here, however, we will pursue a completely different approach as before. To be more precise, Lemma 1.4.1 and especially (1.4.4) will gain in importance as can be seen in the subsequent Theorem.
For convenience and in a break from tradition, let $\ell_1 \geq 0$ denote the left end point of $f(X)$ in the upcoming theorem.

Theorem 2.2.24
Let X be a random vector in $\mathbb{R}^d$ coming under the second class of random vectors, that is, the mass of $\mathbb{P}_{f(X)}$ is concentrated on a countable subset of $[0, \infty)$ and hence

$$\mathbb{P}_{f(X)}(dx) = \sum_{i=1}^{\infty} p_i \, \varepsilon_{\ell_i}(dx) \tag{2.2.28}$$

for a suitable sequence $(p_i)_{i\geq 1} \subset [0, 1]$ with

$$\sum_{i=1}^{\infty} p_i = 1$$

and for an appropriate, strictly increasing sequence $(\ell_i)_{i\geq 1}$ of non-negative real numbers. Then X is f-implicit max-infinitely divisible. Moreover, for all $n \geq 1$,

$$\mu_n(dx) = \sum_{i=1}^{\infty} \left[\left(\sum_{j=1}^{i} p_j \right)^{\frac{1}{n}} - \left(\sum_{j=1}^{i-1} p_j \right)^{\frac{1}{n}} \right] \rho_{L_{\ell_i}}(dx) \tag{2.2.29}$$

is a suitable choice for the corresponding nth root of X, where $\rho_{L_{\ell_i}} \in M^1(\mathbb{R}^d)$ such that $\operatorname{supp} \rho_{L_{\ell_i}} \subset L_{\ell_i}$ for all $i \geq 1$.

Proof. To start with, we observe that (2.2.28) actually yields

$$\mathbb{P}_X(dx) = \sum_{i=1}^{\infty} p_i \, \rho_{L_{\ell_i}}(dx) \tag{2.2.30}$$

for suitable probability measures $\rho_{L_{\ell_i}}$ on $\mathbb{R}^d$ with $\operatorname{supp} \rho_{L_{\ell_i}} \subset L_{\ell_i}$. For convenience, assume that $p_i > 0$ for all $i \geq 1$. Now, fix $n \geq 1$ and let $X_1, ..., X_n$ be independent copies of X. Then we deduce, by applying (1.4.4), (2.2.30) and finally (2.2.28), that

$$\begin{aligned}
\mathbb{P}(X_{k(n)} \in A) &= n \int_{A \cap \mathcal{C}(G)} \mathbb{P}(f(X) \leq f(x))^{n-1} \, \mathbb{P}_X(dx) \\
&\quad + \int_{A \cap \mathcal{D}(G)} \left(\frac{\mathbb{P}(f(X) \leq f(x))^n - \mathbb{P}(f(X) < f(x))^n}{\mathbb{P}(f(X) = f(x))} \right) \mathbb{P}_X(dx) \\
&= \int_{A \cap \left(\bigcup_{i=1}^{\infty} L_{\ell_i} \right)} \left(\frac{\mathbb{P}(f(X) \leq f(x))^n - \mathbb{P}(f(X) < f(x))^n}{\mathbb{P}(f(X) = f(x))} \right) \mathbb{P}_X(dx)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} \int_{A\cap L_{\ell_i}} \left(\frac{\mathbb{P}(f(X) \le f(x))^n - \mathbb{P}(f(X) < f(x))^n}{\mathbb{P}(f(X) = f(x))} \right) \mathbb{P}_X(dx) \\
&= \sum_{i=1}^{\infty} p_i \rho_{L_{\ell_i}}(A) \left(\frac{\mathbb{P}(f(X) \le \ell_i)^n - \mathbb{P}(f(X) < \ell_i)^n}{\mathbb{P}(f(X) = \ell_i)} \right) \\
&= \sum_{i=1}^{\infty} \left[\left(\sum_{j=1}^{i} p_j \right)^n - \left(\sum_{j=1}^{i-1} p_j \right)^n \right] \rho_{L_{\ell_i}}(A)
\end{aligned}
$$

for all $A \in \mathcal{B}(\mathbb{R}^d)$. Using the latter calculation, we can eventually complete the proof. Indeed, let $X_1^{(n)}, \ldots, X_n^{(n)}$ be independent and identically distributed random vectors in $\mathbb{R}^d$ with

$$
\mathbb{P}_{X_1^{(n)}}(dx) = \mu_n(dx).
$$

Then

$$
\begin{aligned}
&\mathbb{P}\left(X_{k(n)}^{(n)} \in A\right) \\
&= \sum_{i=1}^{\infty} \left\{ \left(\sum_{j=1}^{i} \left[\left(\sum_{k=1}^{j} p_k \right)^{\frac{1}{n}} - \left(\sum_{k=1}^{j-1} p_k \right)^{\frac{1}{n}} \right] \right)^n - \left(\sum_{j=1}^{i-1} \left[\left(\sum_{k=1}^{j} p_k \right)^{\frac{1}{n}} - \left(\sum_{k=1}^{j-1} p_k \right)^{\frac{1}{n}} \right] \right)^n \right\} \rho_{L_{\ell_i}}(A) \\
&= \sum_{i=1}^{\infty} \left\{ \left[\left(\sum_{k=1}^{i} p_k \right)^{\frac{1}{n}} \right]^n - \left[\left(\sum_{k=1}^{i-1} p_k \right)^{\frac{1}{n}} \right]^n \right\} \rho_{L_{\ell_i}}(A) \\
&= \sum_{i=1}^{\infty} p_i \, \rho_{L_{\ell_i}}(A) \\
&= \mathbb{P}(X \in A)
\end{aligned}
$$

for all $A \in \mathcal{B}(\mathbb{R}^d)$. Note that the second equality is due to the method of differences which typically occurs in the context of telescoping series. According to Lemma 2.1.2, the latter is precisely the assertion of the theorem. □

Remark 2.2.25. Similar to Remark 2.2.22, Example 2.1.4 also rates as a special case of Theorem 2.2.24. In fact, the mass of $\mathbb{P}_{f(X)}$ is concentrated on the single point $\ell \ge 0$, thus showing that X comes under the second class of distributions and is therefore f-implicit max-infinitely divisible. Adjusting the different notations used in Example 2.1.4 and in Theorem 2.2.24, we moreover conclude that (2.2.29) actually coincides with the representation of the nth root given in Example 2.1.4.

Eventually, Corollary 2.2.21 and Theorem 2.2.24 supplied two tremendous classes of f-implicit max-infinitely divisible distributions. Nevertheless, there clearly exist much more distributions on $\mathbb{R}^d$ that do not come under one of these two classes. Therefore, those distributions have to be considered separately. Our expectation is that all those distributions are also f-implicit max-infinitely divisible which will be considered in more detail in Section 2.4. However, as we have already confessed, this hypothesis remains unproven for the time being and is therefore an exciting object of study for future research projects. For all that, we can provide satisfactory indications substantiating the fact that all distributions on $\mathbb{R}^d$ might be f-implicit max-infinitely divisible. In concrete terms, this will be achieved with the aid of the subsequent proposition as it supplies several examples of distributions which do not come under one of the previously studied classes of distributions but yet are f-implicit max-infinitely divisible. In addition to this, Remark 2.2.23 and Section 2.4 are also intended to comply with the latter purpose.

Proposition 2.2.26
Let $0 \leq \ell < r$ and $p, p_1, p_2 > 0$. Then any random vector X in $\mathbb{R}^d$ with either

$$\mathbb{P}_{f(X)}(dx) = \mathbb{1}_{[\ell,r)}(x)\, h(x)\lambda^1(dx) + p\varepsilon_r(dx)$$

or

$$\mathbb{P}_{f(X)}(dx) = p_1\varepsilon_\ell(dx) + \mathbb{1}_{(\ell,r)}(x)\, h(x)\lambda^1(dx) + p_2\varepsilon_r(dx)$$

or even

$$\mathbb{P}_{f(X)}(dx) = \mathbb{1}_{[\ell,r)}(x)\, h(x)\lambda^1(dx) + p\varepsilon_r(dx) + \mathbb{1}_{(r,\infty)}(x)\, h(x)\lambda^1(dx)$$

is f-implicit max-infinitely divisible. Here, h denotes a suitable, non-negative and measurable function on $\mathbb{R}$. Moreover, in any of the above cases, we assume that $\ell \geq 0$ is really the left endpoint of $f(X)$ in conformity with our convention established at the beginning of this section. That is, $\mathbb{P}_{f(X)}([\ell, \ell + \varepsilon)) > 0$ for all $\varepsilon > 0$.

Remark 2.2.27. Clearly, each of the random vectors X considered in Proposition 2.2.26 does neither come under the first nor under the second class of distributions, provided the function h is appropriately chosen. Of course, there are many more of such distributions. However, those can be handled in much the same manner so that we can restrict ourselves to the consideration of the three above ones.

Proof of Proposition 2.2.26. In what follows, we apply slight extensions of Corollary 2.2.8 which are straightforward and do not need to be considered in more detail. Moreover, we repeatedly avail ourselves of the notation introduced during the preceding deliberations.
In order to be brief, we give the proof only for the first case and do not carry out every detail here. As to the other cases, one may proceed similarly.
We begin by estimating the distribution of the random vector X. To this end, fix $A \in \mathcal{B}(\mathbb{R}^d)$. Then we obtain

$$\begin{aligned}
\mathbb{P}(X \in A) &= \mathbb{P}(X \in A \cap \{x \in \mathbb{R}^d : \ell \leq f(x) \leq r\}) \\
&= \mathbb{P}(X \in A \cap \{x \in \mathbb{R}^d : \ell < f(x) < r\}) + \mathbb{P}(X \in A \cap \{x \in \mathbb{R}^d : f(x) = r\}) \\
&= \int\limits_{A \cap \{x \in \mathbb{R}^d :\, \ell < f(x) < r\}} \mathbb{P}(f(X) \leq f(x))\, \mathbb{P}(f(X) \leq f(x))^{-1}\, \mathbb{P}_X(dx) + p\, \rho_{L_r}(A)
\end{aligned}$$

$$:= \int_{A\cap\{x\in\mathbb{R}^d:\,\ell<f(x)<r\}} \mathbb{P}(f(X)\le f(x))\,\eta(dx) + p\,\rho_{L_r}(A)$$

for an appropriate probability measure ρ_{L_r} on $\mathbb{R}^d$ with supp $\rho_{L_r} \subset L_r$. Note that η is a measure being similar to the specific one introduced in Lemma 2.2.9. In the present context, however, η is understood as a measure on $\{x \in \mathbb{R}^d : \ell < f(x) < r\} = \Gamma \setminus (\Gamma_r \cup L_r)$. Note that

$$f(\eta)(s,r) = \ln G(r-) - \ln G(s) = \ln(1-p) - \ln G(s) \tag{2.2.31}$$

for all $\ell < s < r$ and, as a consequence, $\eta(\Gamma \setminus (\Gamma_r \cup L_r)) = \infty$. This can be seen similar to Lemma 2.2.9 (a) by using a slight modification of Corollary 2.2.8 (a). Accordingly, we have actually proved

$$\mathbb{P}(X \in A) = (1-p) \int_{A\cap\{x\in\mathbb{R}^d:\,\ell<f(x)<r\}} e^{-f(\eta)(f(x),r)}\,\eta(dx) + p\,\rho_{L_r}(A). \tag{2.2.32}$$

Now, fix $n \ge 1$ and let $X_1, ..., X_n$ be independent copies of X. Applying (1.4.4), (2.2.31) and (2.2.32), we deduce immediately that

$$\begin{aligned}
\mathbb{P}(X_{k(n)} \in A) &= n \int_{A\cap\mathcal{C}(G)} \mathbb{P}(f(X)\le f(x))^{n-1}\,\mathbb{P}_X(dx) \\
&\quad + \int_{A\cap\mathcal{D}(G)} \left(\frac{\mathbb{P}(f(X)\le f(x))^n - \mathbb{P}(f(X)< f(x))^n}{\mathbb{P}(f(X)=f(x))}\right)\mathbb{P}_X(dx) \\
&= n \int_{A\cap(\Gamma\setminus(\Gamma_r\cup L_r))} \mathbb{P}(f(X)\le f(x))^{n-1}\,\mathbb{P}_X(dx) \\
&\quad + \int_{A\cap L_r} \left(\frac{\mathbb{P}(f(X)\le f(x))^n - \mathbb{P}(f(X)< f(x))^n}{\mathbb{P}(f(X)=f(x))}\right)\mathbb{P}_X(dx) \\
&= n \int_{A\cap(\Gamma\setminus(\Gamma_r\cup L_r))} (1-p)^{n-1}\left(e^{-f(\eta)(f(x),r)}\right)^{n-1}(1-p)\,e^{-f(\eta)(f(x),r)}\,\eta(dx) \\
&\quad + p\,\rho_{L_r}(A)\,\frac{\mathbb{P}(f(X)\le r)^n - \mathbb{P}(f(X)< r)^n}{\mathbb{P}(f(X)=r)} \\
&= n\,(1-p)^n \int_{A\cap(\Gamma\setminus(\Gamma_r\cup L_r))} e^{-nf(\eta)(f(x),r)}\,\eta(dx) + (1-(1-p)^n)\;\rho_{L_r}(A).
\end{aligned}$$

This finding eventually enables us to make an educated guess of how to choose a suitable nth root of X. Indeed, let $X_1^{(n)}, ..., X_n^{(n)}$ be independent and identically distributed random vectors in $\mathbb{R}^d$ with

$$\mathbb{P}_{X_1^{(n)}}(dx) = \frac{1}{n}\,(1-p)^{\frac{1}{n}} e^{-\frac{1}{n}f(\eta)(f(x),r)}\eta(dx) + \left(1-(1-p)^{\frac{1}{n}}\right)\,\rho_{L_r}(dx). \tag{2.2.33}$$

Applying the latter computation to $X_1^{(n)}$ instead of X we conclude, by simple calculations, that

$$\mathbb{P}\left(X_{k(n)}^{(n)} \in A\right) = (1-p) \int\limits_{A \cap \{x \in \mathbb{R}^d :\, \ell < f(x) < r\}} e^{-f(\eta)(f(x),r)}\, \eta(dx) + p\, \rho_{L_r}(A) = \mathbb{P}(X \in A).$$

Here, the details can be skipped as the corresponding computations closely resemble the previous ones. Taking into account Lemma 2.1.2, we have thus proved that X is f-implicit max-infinitely divisible. □

Remark 2.2.28. Similar to (2.2.33), we can estimate possible nth roots for the two remaining distributions of Proposition 2.2.26. In particular,

$$\mathbb{P}_{X_1^{(n)}}(dx) = p_1^{\frac{1}{n}} \rho_L(dx) + \frac{1}{n}\,(1-p_2)^{\frac{1}{n}} e^{-\frac{1}{n} f(\eta)(f(x),r)} \eta(dx) + \left(1 - (1-p_2)^{\frac{1}{n}}\right)\, \rho_{L_r}(dx)$$

serves as an nth root for the second distribution, ρ_{L_r} being an appropriate probability measure on $\mathbb{R}^d$ with supp $\rho_{L_r} \subset L_r$, $\eta \in M^b(\Gamma \setminus (\Gamma_r \cup L_r))$ being defined as in the preceding proof and ρ_L being a suitable probability measure on $\mathbb{R}^d$ with supp $\rho_L \subset L$. In contrast,

$$\mathbb{P}_{X_1^{(n)}}(dx) = \frac{1}{n}\, p_0^{\frac{1}{n}} e^{-\frac{1}{n} f(\eta_1)(f(x),r)} \eta_1(dx) + \left((p_0+p)^{\frac{1}{n}} - p_0^{\frac{1}{n}}\right)\, \rho_{L_r}(dx) + \frac{1}{n} e^{-\frac{1}{n} f(\eta_2)(f(x),\infty)} \eta_2(dx)$$

serves as an nth root for the third distribution. Here, the positive constant p_0 is given by

$$p_0 = G(r-) = \mathbb{P}(f(X) < r),$$

the unbounded measure η_1 on $\Gamma \setminus (\Gamma_r \cup L_r)$ by

$$\eta_1(dx) = \mathbb{P}(f(X) \leq f(x))^{-1} \mathbb{P}_X(dx)$$

and finally the measure $\eta_2 \in M^b(\Gamma_r)$ by

$$\eta_2(dx) = \mathbb{P}(f(X) \leq f(x))^{-1} \mathbb{P}_X(dx)$$

as well.

These observations shall finally complete Section 2.2. We will now proceed with a new issue, the notion of f-implicit max-compound Poisson distributions and the notion of f-implicit max-compound Poisson processes.

2.3 The f-implicit max-compound Poisson process

During our previous deliberations we frequently broached the notion of f-implicit max-compound Poisson distributions and the notion of f-implicit max-compound Poisson process. Now, we eventually concretize these concepts. In particular, we are concerned with an accurate introduction of f-implicit max-compound Poisson distributions and f-implicit max-compound Poisson processes. Building upon this, we derive certain

properties providing a more profound insight into the structure of f-implicit max-compound Poisson processes.
The idea here is to approach a field that might be beneficial within the context of a possible proof of the still unsolved conjecture that all distributions on $\mathbb{R}^d$ are f-implicit max-infinitely divisible. This strategy originates from the notion of generalized Poisson distributions specified in [MeSch01, Chapter 3] as these distributions are the analogue of f-implicit max-compound Poisson distributions and as they essentially contribute to a successful proof of the common Lévy-Khintchine formula.
To start with, we give the following Proposition which subsequently enables us to introduce the central concepts of this section in terms of Definition 2.3.2 and Definition 2.3.4.

Proposition 2.3.1
Let $c \geq 0$, $\rho_1 \in M^1(\mathbb{R}^d)$ and $\rho_2 \in M^b(\mathbb{R}^d)$. Then the measure $\Pi_f(c, \rho_1, \rho_2)$, defined by

$$\Pi_f(c, \rho_1, \rho_2)(dx) = e^{-c\rho_2(\mathbb{R}^d)}\rho_1(dx) + e^{-c\rho_2(\mathbb{R}^d)} \sum_{n=1}^{\infty} \frac{c^n}{n!}(\rho_2)^{*_f n}(dx), \tag{2.3.1}$$

is a probability measure on $\mathbb{R}^d$.

Proof. From (1.2.2) we have

$$(\rho_2)^{*_f n}(\mathbb{R}^d) = (\rho_2(\mathbb{R}^d))^n$$

proving that

$$\begin{aligned}\Pi_f(c, \rho_1, \rho_2)(\mathbb{R}^d) &= e^{-c\rho_2(\mathbb{R}^d)}\rho_1(\mathbb{R}^d) + e^{-c\rho_2(\mathbb{R}^d)} \sum_{n=1}^{\infty} \frac{\left(c\,\rho_2(\mathbb{R}^d)\right)^n}{n!} \\ &= e^{-c\rho_2(\mathbb{R}^d)} + e^{-c\rho_2(\mathbb{R}^d)}\left(e^{c\rho_2(\mathbb{R}^d)} - 1\right) \\ &= 1.\end{aligned}$$

□

The latter proposition gives occasion to the following definition.

Definition 2.3.2
The class $\{\Pi_f(c, \rho_1, \rho_2) : c \geq 0, \rho_1 \in M^1(\mathbb{R}^d), \rho_2 \in M^b(\mathbb{R}^d)\}$ of probability measures on $\mathbb{R}^d$ defined by (2.3.1) is called the *class of f-implicit max-compound Poisson distributions*. Any random vector having a distribution of the form (2.3.1) is referred to as *f-implicit max-compound Poisson distributed with parameters c, ρ_1, ρ_2*.

Remark 2.3.3. At this point it should be underlined that the recently introduced class of f-implicit max-compound Poisson distributions shares striking parallels to the particular class of generalized Poisson distributions introduced in [MeSch01, Definition 3.1.7]. This is no coincidence but accounted for by the repeatedly mentioned, close connections between the underlying theories of infinite divisibility on the one hand and f-implicit max-infinite divisibility on the other. Similar to the theory of infinitely divisible distributions (see for example [MeSch01, Chapter 3]), the notion of f-implicit max-compound Poisson distributions as well as the notion of f-implicit max-compound Poisson processes might therefore contribute to further achievements in the field of f-implicit max-infinite divisibility. We will expand on these observations in the context of Conjecture 2.4.1.

Having supplied the notion of f-implicit max-compound Poisson distributions, we proceed to establish the notion of f-implicit max-compound Poisson processes. This is actually the central object of study here as the corresponding headline of Section 2.3 already anticipates.

Definition 2.3.4
Let $(N_t)_{t\geq 0}$ be a homogeneous Poisson process with rate $\lambda > 0$ and suppose that X_0 is random vector in $\mathbb{R}^d$ being independent of $(N_t)_{t\geq 0}$. Furthermore, let $X, X_1, X_2, \ldots$ be independent and identically distributed random vectors in $\mathbb{R}^d$ being independent of $(N_t)_{t\geq 0}$ as well. Then the process $\mathbb{Y} = (Y_t)_{t\geq 0}$, defined by

$$Y_t = \begin{cases} X_0, & \text{if } N_t = 0, \\ \bigvee_{i=1}^{N_t} {}_f \, X_i, & \text{if } N_t \geq 1, \end{cases}$$

is referred to as *f-implicit max-compound Poisson process*.

Remark 2.3.5. Note that the random variable N_t has a Poisson distribution with parameter λt for all $t \geq 0$ since $(N_t)_{t\geq 0}$ is assumed to be a homogeneous Poisson process with rate $\lambda > 0$.

In the remainder of this section we exclusively concern ourselves with a detailed investigation of f-implicit max-compound Poisson processes. We establish, inter alia, a connection between those processes and the notion of f-implicit max-infinite divisibility. To start with, we provide a more profound insight into the structure of f-implicit max-compound Poisson processes by calculating the distribution of the marginals Y_t for all $t \geq 0$.

Lemma 2.3.6
Let $\mathbb{Y} = (Y_t)_{t\geq 0}$ be an f-implicit max-compound Poisson process with $X_0 \sim \mu_0$ and $X \sim \mu$. Then, for all $t \geq 0$, we obtain

$$\mathbb{P}_{Y_t}(dx) = e^{-\lambda t}\mu_0(dx) + e^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}\mu^{*_f n}(dx) \tag{2.3.2}$$

Proof. Fix $A \in \mathcal{B}(\mathbb{R}^d)$ and $t \geq 0$. Referring to the fundamental law of total probability and observing the required assumptions on μ_0 and μ, we obtain

$$\mathbb{P}(Y_t \in A) = \sum_{n=0}^{\infty} \mathbb{P}(Y_t \in A \,|\, N_t = n)\mathbb{P}(N_t = n)$$

$$= e^{-\lambda t}\mathbb{P}(X_0 \in A) + e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \mathbb{P}\left(\bigvee_{i=1}^{N_t}{}_f \ X_i \in A \,\middle|\, N_t = n \right)$$

$$= e^{-\lambda t}\mu_0(A) + e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \mathbb{P}\left(\bigvee_{i=1}^{n}{}_f \ X_i \in A \right)$$

$$= e^{-\lambda t}\mu_0(A) + e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \mu^{*_f n}(A).$$

The latter equality follows instantly from (1.2.7). In the third step we further benefited from Lemma 1.1.6. □

Corollary 2.3.7
Let $\mathbb{Y} = (Y_t)_{t\geq 0}$ be an f-implicit max-compound Poisson process with $X_0 \sim \mu_0$ and $X \sim \mu$. Then, for all $t \geq 0$, Y_t is f-implicit max-compound Poisson distributed with parameters λt, μ_0 and μ. That is,

$$Y_t \sim \Pi_f(\lambda t, \mu_0, \mu)$$

for all $t \geq 0$. □

Having proved the latter representation of the distribution of the marginals Y_t of an f-implicit max-compound Poisson process $\mathbb{Y} = (Y_t)_{t\geq 0}$ with $X_0 \sim \mu_0$ and $X \sim \mu$, we may now proceed to refine this result considerably. To this end, recall from Section 1.4 that the sets $\mathcal{C}(G)$ and $\mathcal{D}(G)$ are given by

$$\mathcal{C}(G) = \{x \in \mathbb{R}^d : \ G(f(x)-) = G(f(x))\} = \{x \in \mathbb{R}^d : \ f(\mu)(\{f(x)\}) = 0\}$$

and

$$\mathcal{D}(G) = \{x \in \mathbb{R}^d : \ G(f(x)-) < G(f(x))\} = \{x \in \mathbb{R}^d : \ f(\mu)(\{f(x)\}) > 0\},$$

the function G here being the cumulative distribution function of $f(X)$. For convenience, let further

$$f(\mu)\big([0, f(x)]\big) := f(\mu)[0, f(x)]$$

as well as

$$f(\mu)\big([0, f(x))\big) := f(\mu)[0, f(x))$$

and finally also

$$f(\mu)\big([f(x), \infty)\big) := f(\mu)[f(x), \infty)$$

as well as

$$f(\mu)\big(\{f(x)\}\big) := f(\mu)\{f(x)\}$$

in compliance with the notational convention introduced in Corollary 2.2.8 (b).

Theorem 2.3.8
Let A denote some Borel set in $\mathbb{R}^d$. Further, suppose that $\mathbb{Y} = (Y_t)_{t\geq 0}$ an f-implicit max-compound Poisson process with $X_0 \sim \mu_0$ and $X \sim \mu$. Then we have

$$\mathbb{P}(Y_t \in A) = e^{-\lambda t}\mu_0(A) + \lambda t \int_{A\cap\mathcal{C}(G)} e^{-\lambda t f(\mu)(f(x),\infty)}\, \mu(dx)$$

$$+ \int\limits_{A\cap\mathcal{D}(G)} \left(\frac{e^{-\lambda t f(\mu)(f(x),\infty)} - e^{-\lambda t f(\mu)[f(x),\infty)}}{f(\mu)\{f(x)\}} \right) \mu(dx) \tag{2.3.3}$$

for all $t \geq 0$.

Proof. The proof is straightforward. To start with, let $A \in \mathcal{B}(\mathbb{R}^d)$ and $t \geq 0$ be fixed. Then a combination of (1.2.7), Lemma 1.4.1 and (2.3.2) yields immediately

$$\begin{aligned}
\mathbb{P}(Y_t \in A) &= e^{-\lambda t}\mu_0(A) + e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \mu^{*_f n}(A) \\
&= e^{-\lambda t}\mu_0(A) + e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!}\, n \int\limits_{A\cap\mathcal{C}(G)} \Big(f(\mu)[0,f(x)]\Big)^{n-1} \mu(dx) \\
&\quad + e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \int\limits_{A\cap\mathcal{D}(G)} \left(\frac{\Big(f(\mu)[0,f(x)]\Big)^n - \Big(f(\mu)[0,f(x))\Big)^n}{f(\mu)\{f(x)\}} \right) \mu(dx) \\
&= e^{-\lambda t}\mu_0(A) + \lambda t\, e^{-\lambda t} \int\limits_{A\cap\mathcal{C}(G)} \sum_{n=1}^{\infty} \frac{\Big(\lambda t\, f(\mu)[0,f(x)]\Big)^{n-1}}{(n-1)!} \mu(dx) \\
&\quad + e^{-\lambda t} \int\limits_{A\cap\mathcal{D}(G)} \frac{1}{f(\mu)\{f(x)\}} \left(\sum_{n=1}^{\infty} \frac{\Big(\lambda t\, f(\mu)[0,f(x)]\Big)^n}{n!} - \sum_{n=1}^{\infty} \frac{\Big(\lambda t\, f(\mu)[0,f(x))\Big)^n}{n!} \right) \mu(dx) \\
&= e^{-\lambda t}\mu_0(A) + \lambda t\, e^{-\lambda t} \int\limits_{A\cap\mathcal{C}(G)} e^{\lambda t\, f(\mu)[0,f(x)]} \mu(dx) \\
&\quad + e^{-\lambda t} \int\limits_{A\cap\mathcal{D}(G)} \left(\frac{e^{\lambda t\, f(\mu)[0,f(x)]} - e^{\lambda t\, f(\mu)[0,f(x))}}{f(\mu)\{f(x)\}} \right) \mu(dx) \\
&= e^{-\lambda t}\mu_0(A) + \lambda t \int\limits_{A\cap\mathcal{C}(G)} e^{-\lambda t(1-f(\mu)[0,f(x)])} \mu(dx) \\
&\quad + \int\limits_{A\cap\mathcal{D}(G)} \left(\frac{e^{-\lambda t(1-f(\mu)[0,f(x)])} - e^{-\lambda t(1-f(\mu)[0,f(x)))}}{f(\mu)\{f(x)\}} \right) \mu(dx) \\
&= e^{-\lambda t}\mu_0(A) + \lambda t \int\limits_{A\cap\mathcal{C}(G)} e^{-\lambda t f(\mu)(f(x),\infty)} \mu(dx) \\
&\quad + \int\limits_{A\cap\mathcal{D}(G)} \left(\frac{e^{-\lambda t f(\mu)(f(x),\infty)} - e^{-\lambda t f(\mu)[f(x),\infty)}}{f(\mu)\{f(x)\}} \right) \mu(dx).
\end{aligned}$$

This is precisely the desired claim. Note that the latter equality in the previous calculation is due to the fact that $f(\mu)$ is a probability measure on $\mathbb{R}$ having support confined to the non-negative real numbers. □

Under some additional assumptions on the two probability measures μ_0 and μ the assertion of the latter theorem, especially formula (2.3.3), simplifies significantly. Indeed, complying with the notation introduced in Section 2.2, we obtain the next result.

Corollary 2.3.9
Fix $\ell \geq 0$ and suppose that supp $\mu_0 \subset L$ and $\mu(\Gamma) = 1$. Furthermore, let V_μ be continuous. Then, for all $t \geq 0$, we have

$$\mathbb{P}_{Y_t}(dx) = e^{-\lambda t}\mu_0(dx) + \lambda t e^{-\lambda t f(\mu)(f(x),\infty)}\mu(dx). \tag{2.3.4}$$

Especially, $Y_t \sim [\mu_0, \nu_t]_f$ for all $t \geq 0$, the finite measure ν_t on Γ here being defined by

$$\nu_t(dx) = \lambda t \mu(dx). \tag{2.3.5}$$

Moreover, every random vector $Y_t, t \geq 0$, is f-implicit max-infinitely divisible.

Proof. The proof follows immediately from Theorem 2.3.8. To start with, fix $t \geq 0$. Then the required assumptions on μ give $C(G) = \mathbb{R}^d$ which yields (2.3.4). Furthermore, (2.3.4) in combination with the assumptions on μ_0 and μ actually shows that the random vector Y_t comes under the first class of f-implicit max-infinitely divisible random vectors. We skip the details here as the corresponding calculations are similar to the ones that repeatedly occurred in Section 2.2. On account of Proposition 2.2.15, this observation proves that $Y_t \sim [\mu_0, \nu_t]_f$ with ν_t being defined by (2.3.5). Applying Corollary 2.2.21 (b), we eventually see that Y_t is f-implicit max-infinitely divisible. □

Remark 2.3.10. Referring to Remark 2.2.16, we also obtain another possibility to represent μ_0 and ν_t. We can convince ourselves of the fact that the different representations for both μ_0 and ν_t coincide, thus revealing the validity of our deliberations once again.

Instead of amplifying the latter aspects, we will now conclude our brief considerations concerning the notion of f-implicit max-compound Poisson distributions and f-implicit max-compound Poisson processes. As we could see, however, this particular branch of f-implicit extreme value theory seems to hold a lot of promise as to further research possibilities. Especially, we might benefit from the notion of f-implicit max-compound Poisson distributions and from the notion of f-implicit max-compound Poisson processes in the context of further studies on f-implicit max-infinitely divisible distributions. In addition to several suggestions for possible and attractive extension of the content of Chapter 2, elaborating on this hypothesis will be part of the next Section.

2.4 Outlook

Up to this point, we concentrated on pioneering the idea of an f-implicit extreme value theory motivated by the seminal studies on implicit extremes and f-implicit max-stable laws in [SchSt14]. In particular, in Chapter 1 we established a profound, mathematical basis consisting of the f-implicit max-operation, the f-implicit max-convolution and the

f-implicit max-order. Following this, we proceeded to introduce the notion of f-implicit max-infinitely divisible distributions extending the class of f-implicit max-stable distributions. In this context, we have already been able to prove attractive results. With regard to Conjecture 2.4.1, we then also introduced f-implicit max-compound Poisson distributions as well as f-implicit max-compound Poisson processes. Finally, the next chapter will be devoted to the notion of f-implicit max-stable processes being another attractive branch of f-implicit extreme value theory and constituting the second part of the present thesis. Prior to this, we both return to some open problems and, subsequently, give suggestions for additional research possibilities extending the content of Chapter 2. In doing so, we mainly concentrate on alleging sensible reasons for the conjecture that probably all random vectors in $\mathbb{R}^d$ are f-implicit max-infinitely divisible. To start with, we underline that the class of f-implicit max-infinitely divisible distributions still needs to be studied in more detail. In fact, in the context of investigations of f-implicit max-infinitely divisible distributions the legitimate question arises whether there exist results going beyond our preliminary findings. In particular, there is talk of a potential result that improves the assertions of Corollary 2.2.21, Theorem 2.2.24 and Proposition 2.2.26 and that characterizes the class of f-implicit max-infinitely divisible distributions in more detail. Our educated guess regarding this issue is actually that all distributions on $\mathbb{R}^d$ are f-implicit max-infinitely divisible. More precisely, we consider the subsequent hypothesis possible.

Conjecture 2.4.1
Every random vector X in $\mathbb{R}^d$ is f-implicit max-infinitely divisible. Moreover, for all $n \geq 1$, a suitable nth root is given by

$$\mathbb{P}_{X_1^{(n)}}(dx) = g_n(x)\mathbb{P}_X(dx) \tag{2.4.1}$$

with

$$g_n(x) = \begin{cases} \frac{\mathbb{P}(f(X) \leq f(x))^{\frac{1}{n}} - \mathbb{P}(f(X) < f(x))^{\frac{1}{n}}}{\mathbb{P}(f(X) = f(x))}, & \text{if } \mathbb{P}(f(X) = f(x)) > 0, \\ \frac{1}{n}\mathbb{P}(f(X) \leq f(x))^{\frac{1}{n}-1}, & \text{if } \mathbb{P}(f(X) = f(x)) = 0. \end{cases}$$

Remark 2.4.2. According to (2.2.3), we have $\operatorname{supp} \mathbb{P}_X \subset \overline{\Gamma}$. If $G(\ell) = 0$, we even have $\mathbb{P}(X \in \Gamma) = 1$. Therefore, the function g_n is in effect just relevant if $x \in \overline{\Gamma}$ or if $x \in \Gamma$ depending on whether $G(\ell) > 0$ or $G(\ell) = 0$. Note that this observations is important to assure that the nth root is well-defined. Actually, the function g_n is just relevant for those $x \in \mathbb{R}^d$ pertaining to the set on which the mass of $\mathbb{P}_X$ is concentrated. Hence, we shall expressly underline that the definition of the distribution of the nth root naturally involves these observations. This is similar to the considerations stated in Remark 2.2.23.

As already mentioned, the assertion of Conjecture 2.4.1 is yet unproved and therefore a possible starting point for further promising research projects. Up to this point, however, we can at least notice that both the claim that all distributions are f-implicit max-infinitely divisible and (2.4.1) seem quite sensible. Indeed, Conjecture 2.4.1 is grounded on several cogent reasons.

First, the assumption that all random vectors X in $\mathbb{R}^d$ are f-implicit max-infinitely divisible stems from the fact that we have not been able to find any counterexample so far. Instead, we rather provided a wealth of examples of f-implicit max-infinitely divisible distributions as can be seen from Corollary 2.2.21, Theorem 2.2.24 and Proposition 2.2.26 in combination with Remark 2.2.27. Furthermore, for all random vectors X in $\mathbb{R}^d$, the non-negative random variable $f(X)$ is max-infinitely divisible since all distributions on $\mathbb{R}$ are max-infinitely divisible (see for instance [Re07, Section 5.1]). So, the frequently mentioned close connection between the theories of max-infinitely divisible distributions and f-implicit max-infinitely divisible distributions makes the suggestion that all random vectors X in $\mathbb{R}^d$ might be f-implicit max-infinitely divisible appear even more reasonable.
Second, Remark 2.2.23, (2.2.29), (2.2.33) and finally Remark 2.2.28 suggest the verisimilitude of (2.4.1). Indeed, all nth roots occurring therein coincide with the one in (2.4.1), provided the respective assumptions on X are fulfilled. For example, if X is a random vector such that (2.2.28) holds, we actually have

$$\mathbb{P}\left(X_1^{(n)} \in A\right) = \int_A g_n(x)\,\mathbb{P}_X(dx) = \sum_{i=1}^{\infty}\left[\left(\sum_{j=1}^{i} p_j\right)^n - \left(\sum_{j=1}^{i-1} p_j\right)^n\right]\rho_{L_{\ell_i}}(A)$$

for all $A \in \mathcal{B}(\mathbb{R}^d)$. As a side note, the outer sum on the right-hand side is actually only taken over those $i \geq 1$ with $p_i > 0$ as can be seen by following the proof of Theorem 2.2.24 once again.
Since all the latter reasonable arguments do clearly not serve as proof for Conjecture 2.4.1, its assertion remains as an open problem for the time being. Finding a proper proof for Conjecture 2.4.1 is therefore an ambitious project for future studies in the novel field of f-implicit extreme value theory. In this context, the notion of f-implicit max-compound Poisson distributions and f-implicit max-compound Poisson processes introduced in Section 2.3 might become important as we have already suggested. This suggestion is especially based on Corollary 3.1.8, Remark 3.1.9 and Theorem 3.1.11 in [MeSch01]. Therefore, the notion of f-implicit max-compound Poisson distributions and f-implicit max-compound Poisson processes constitutes another exciting possibility for future research projects as a more profound insight of the latter might be indispensable. In compliance with [MeSch01, Corollary 3.1.8], an important question to be answered is whether all distributions on $\mathbb{R}^d$ can be realized as a weak limit of an appropriate sequence of f-implicit max-compound Poisson distributions. In fact, this is for example true for all random vectors X coming under the first class of distributions with $X \sim [\rho_L, \nu]$ which can easily be seen by applying the constant sequence

$$\left(\Pi_f\left(\nu(\Gamma), \rho_L, \frac{\nu}{\nu(\Gamma)}\right)\right)_{n\geq 1}$$

of f-implicit max-compound Poisson distributions. We shall note that ν should here be viewed as a finite measure in $\mathbb{R}^d$ with $\nu(\mathbb{R}^d \setminus \Gamma) = 0$ rather than a finite measure on Γ as in Section 2.2.
Another approach to proving Conjecture 2.4.1 could be to investigate whether the class

of f-implicit max-infinitely divisible distributions is closed with respect to weak convergence and, subsequently, whether the second class of f-implicit max-infinitely divisible distributions can be applied to approximate any distribution on $\mathbb{R}^d$.
In addition to the latter open problem, the yet unsolved question of uniqueness of the nth roots needs to be answered as well. Here, it seems easily conceivable that the nth roots are at least unique if $f(X)$ has a continuous cumulative distribution function. Furthermore, it appears reasonable to postulate that the nth roots are unique modulo transformations which leave $f(X)$ unchanged, that is, modulo transformations preserving the level sets of the loss function f.
Having considered the previous open problems concerning the notion of f-implicit max-infinitely divisibility, we further would like to broach some issues for additional research possibilities extending the content of Chapter 2. To start with, potential convergence criteria for triangular arrays in an analogous manner as in [MeSch01, Section 3.2] might be a promising aspect. To be more precise, transferring the results in [MeSch01, Section 3.2] into results coming under the field of f-implicit extreme value theory are a challenging assignment. However, we should keep in mind that this is just an idea for a possible research subject and it is not clear whether this might work. Nevertheless, the notion of f-implicit max-compound Poisson distributions as well as the notion of f-implicit max-compound Poisson processes could gain in interest here again.
Moreover, extensions concerning the assumptions on the loss function f are also conceivable. It is an attractive question whether the preliminary findings can be extended under more relaxed assumptions on f. For example, the continuity of f could be replaced by some weaker condition. Moreover, the null set of f could be chosen non-trivially. That is, we could admit the case $D := \{f = 0\} \neq \{0\}$, where D would then be a closed cone in $\mathbb{R}^d$ which does not coincide with the trivial cone $\{0\}$. Finally, the 1-homogeneity of f could be substituted by a more general condition such as the concept of E-homogeneity, E here being some suitable matrix (for a deeper discussion of this suggestion we refer to Section 4.2). Since our deliberations in Chapter 1 depend on the loss function f as well, we would therefore also need to check the correctness of the theoretical framework established there. Potentially, a complete new basis extending the already existing one would have to be formed.
On a final note, it might hold promise to investigate to what extent the underlying space $\mathbb{R}^d$ matters in the results obtained so far. In particular, can $\mathbb{R}^d$ be replaced by a metric space (E, d) equipped with a suitable scalar multiplication operation? What are the involved key challenges here? Ultimately, this could be a formidable extension of the present results being worthy of another doctoral dissertation.
Instead of amplifying these consideration, we proceed with the next chapter dealing with the notion of f-implicit max-stable processes and thus with the second main part of the present thesis. More on feasible extensions will be given in Chapter 4. At this point, however, we can already conclude that the possibilities for further studies in the field of f-implicit extreme value theory are not exhausted at all. This fact becomes even more obvious in Chapter 3.

3 f-implicit max-stable processes

While the first Chapter was intended to provide a theoretical base frame and the second one to establish the notion of f-implicit infinitely divisible distributions, we are now concerned with the second main part of the this thesis being the notion of f-implicit max-stable processes.

As already mentioned in the introduction, it is of interest to know whether there exists an extension of f-implicit max-stable distributions to f-implicit max-stable processes as is the case with stable and max-stable distributions, respectively. Our purpose in this chapter is to develop theory as to those processes by drawing on the current studies on f-implicit max-stable distributions (see [SchSt14]) and by using the extensively studied theories of stable and max-stable processes as an aid. Instead of giving a brief exposition of these theories, which can be reviewed in the extensive monograph of Samorodnitsky and Taqqu [SaTa94] as well as in numerous papers (see for instance [DaMi08], [deHaFe06], [EmKlMi12], [Ka09], [StTa05] or [StWa10]) resulting from the seminal work on max-stable processes by Laurens de Haan [deHa84], we proceed with the central definition. Before doing so, it is worth to mention that not all different definitions of stable or max-stable processes can be adopted.

Recall that a stochastic process $\mathbb{X} := (X_t)_{t\in T}$ is said to be *stable* if all finite dimensional distributions $(X_{t_1}, ..., X_{t_n}), t_1, ..., t_n \in T, n \geq 1$, are stable in accordance with Definition 2.1.1 in [SaTa94]. Here, T is an arbitrary and non-empty index set. Under the restriction $\alpha \geq 1$ on the stability index $\alpha \in (0, 2]$ this is equivalent to the condition that all linear combinations

$$\sum_{i=1}^{n} \alpha_i X_{t_i},\ t_1, ..., t_n \in T, \alpha_1, ..., \alpha_n \in \mathbb{R}, n \geq 1,$$

are α-stable (see for instance [SaTa94, Theorem 3.1.2]). Similarly, a stochastic process $\mathbb{X} := (X_t)_{t\in T}$ is said to be *max-stable* if all finite dimensional distributions $(X_{t_1}, ..., X_{t_n})$, $t_1, ..., t_n \in T, n \geq 1$, are max-stable in accordance with identity (1.1) in [StTa05]. If, for some $\alpha > 0$, all marginals are α-Fréchet (with in general different scale parameters), this is equivalent to the condition that $\mathbb{X}$ is an *α-Fréchet process* meaning that all linear combinations

$$\bigvee_{i=1}^{n} \alpha_i X_{t_i},\ t_1, ..., t_n \in T, \alpha_1, ..., \alpha_n \geq 0, n \geq 1,$$

are α-Fréchet (see [StTa05, Definition 1.2 and Proposition 6.1]). Remember that a random variable Z in $\mathbb{R}$ is said to be *α-Fréchet with scale parameter $\kappa > 0$* if

$$\mathbb{P}(Z \leq x) = \begin{cases} \exp(-\kappa^{\alpha} x^{-\alpha}), & x > 0 \\ 0, & x \leq 0. \end{cases}$$

Adopting the common notation, we write $Z \sim \Phi_\alpha(\kappa)$. If $\kappa = 1$, then Z is commonly called *standard α-Fréchet* and we simply write $Z \sim \Phi_\alpha$ instead of $Z \sim \Phi_\alpha(1)$. For our purpose it is convenient to extend the notion of α-Fréchet random variables by allowing $\kappa = 0$. More precisely, we set $\mathbb{P}_Z = \varepsilon_0$ if $\kappa = 0$. For a more detailed treatment of the latter concepts we refer to [SaTa94] and [deHaFe06].
Taking into account the above-mentioned definitions of stable and max-stable processes, we get an idea of how to formulate an appropriate definition for the f-implicit setting. Namely, the definitions using the notion of linear combinations seem suitable to be adopted.

Definition 3.0.1
Let T be an arbitrary, non-empty index set and $\mathbb{X} := (X_t)_{t\in T}$ a stochastic process with values in $\mathbb{R}^d$. We call $\mathbb{X}$ *f-implicit max-stable* if for all $n \geq 1$, $\alpha_1, ..., \alpha_n \geq 0$ and $t_1, ..., t_n \in T$ the random vector

$$\xi := \bigvee_{i=1}^{n}{}_f \, \alpha_i X_{t_i}$$

is f-implicit max-stable.

Note that in general *every* choice of indices $t_1, ..., t_n \in T$ has to be considered as the f-implicit max-operation is non-commutative. To be more precise, the equation

$$\alpha_1 X_{t_1} \vee_f \alpha_2 X_{t_2} \vee_f \alpha_3 X_{t_1} = \alpha_1 X_{t_1} \vee_f \alpha_3 X_{t_1} \vee_f \alpha_2 X_{t_2} \overset{(1.1.3)}{=} (\alpha_1 \vee \alpha_3) X_{t_1} \vee_f \alpha_2 X_{t_2}$$

does not need to hold almost surely. With regard to Lemma 1.1.5, this would only be the case if $f(\alpha_2 X_{t_2}) \neq f(\alpha_3 X_{t_1})$ almost surely. Accordingly, we cannot assume the indices $t_1, ..., t_n \in T$ to be pairwise distinct as could be done in the case of linear combinations with respect to commutative operations. However, equation (1.1.2) assures that we can restrict our choice of real numbers $\alpha_1, ..., \alpha_n \geq 0$ to real positive ones.
Having established the notion of f-implicit max-stable processes, the crucial question arises whether there exist non-trivial examples which can be applied to other scientific areas. In addition, it is of great interest to make an effort in finding examples of f-implicit max-stable processes as motivated by the following remark.

Remark 3.0.2. Suppose that $\mathbb{X} := (X_t)_{t\in T}$ is an f-implicit max-stable process such that, for all $t \in T$, the support of $\mathbb{P}_{X_t}$ is not confined to L_{ℓ_0} for some $\ell_0 \geq 0$. Define $\mathbb{Y} := (Y_t)_{t\in T}$ by

$$Y_t := f(X_t), \; t \in T.$$

Then it is easy to check that $\mathbb{Y}$ is an α-Fréchet process. Indeed, following ideas used in [SchSt14, Theorem 4.2] and applying the subsequent equation

$$\bigvee_{i=1}^{n} \alpha_i Y_{t_i} = \bigvee_{i=1}^{n} \alpha_i f(X_{t_i}) = f\left(\bigvee_{i=1}^{n}{}_f \, \alpha_i X_{t_i}\right),$$

we obtain the desired conclusion.

In other words, Remark 3.0.2 proves that each f-implicit max-stable process yields automatically a max-stable process with α-Fréchet marginals.
Consequently, the remainder of this chapter is devoted to the construction of non-trivial examples of f-implicit max-stable processes. For that reason, following the idea of stable processes (see for instance [SaTa94, Chapter 3]) or the idea of max-stable processes (see for example [StTa05]), we introduce the concept of f-implicit, α-Fréchet, random sup-measures which turn out to be an efficient approach. For convenience, we just use the term f-implicit sup-measure in the remainder of this chapter.
Following this, we introduce the notion of f-implicit extremal stochastic integrals, that is to say, integrals of non-random functions with respect to an f-implicit sup-measure. Here, we pursue a way sharing striking parallels to the common constructions of α-stable stochastic integrals (see for instance [SaTa94, Chapter 3]) and extremal stochastic integrals (see for instance [StTa05]), respectively.
As the overarching goal is to construct non-trivial examples of f-implicit max-stable processes, we concern ourselves both in Section 3.1 and in Section 3.2 with an attractive one resulting from the previously mentioned concepts, thus re-emphasizing the strength and benefit of f-implicit sup-measures and f-implicit extremal stochastic integrals, respectively.
Finally, we finish this chapter with an extensive outlook suggesting some ideas for additional research work. This points out the wealth of possibilities existing in the field of f-implicit extreme value theory.
Following this way of proceeding, we start with Section 3.1 being devoted to the study of f-implicit sup-measure.

3.1 The concept of f-implicit, α-Fréchet, random sup-measures

As previously said, this section provides a detailed exposition of the concept of f-implicit sup-measures which brings us closer to the accomplishment of our initial purpose, that is, to the construction of non-trivial examples of f-implicit max-stable processes.
Before introducing the notion of f-implicit sup-measures, we remark that the studies on f-implicit max-stable distributions in [SchSt14] and the theories of α-stable random measures as well as random α-Fréchet sup-measures (see [SaTa94] and [StTa05]) are central to our next steps. Hence, we recall the relevant aspects about f-implicit max-stable distributions, thus making our exposition self-contained. This also includes some basic facts about generalized polar coordinates in $\mathbb{R}^d \setminus \{0\}$. However, we do not review the concepts of α-stable random measures and random α-Fréchet sup-measures.
We start by introducing the notion of generalized polar coordinates in $\mathbb{R}^d \setminus \{0\}$ following [SchSt14, Definition 3.4] in a slightly different way. Namely, we neither use the notion of compactification nor that of closed cones $D \subset [-\infty, \infty]^d$ distinguishing from $D = \{0\}$. Suppose that $\tau : \mathbb{R}^d \to [0, \infty)$ is continuous and 1-homogeneous with $\tau(x) = 0 \Leftrightarrow x = 0$. For $x \in \mathbb{R}^d \setminus \{0\}$, its *generalized polar coordinates* in terms of τ can be defined as

$$(\tau, \theta) := (\tau(x), \theta(x)) := \left(\tau(x), \frac{x}{\tau(x)}\right). \tag{3.1.1}$$

Here, τ is referred to as the *radial part* and θ as the *angular part* of $x = \tau\theta$. Observe that θ heavily depends on τ. Defining the *unit sphere*

$$S := S_\tau := \{x \in \mathbb{R}^d \setminus \{0\} : \tau(x) = 1\},$$

which is bounded and closed and hence compact, we readily obtain a homeomorphism

$$T := T_\tau : \mathbb{R}^d \setminus \{0\} \to (0,\infty) \times S, \quad x \mapsto (\tau(x), \theta(x)) = (\tau, \theta).$$

Indeed, T is one-to-one and onto, and both T and its inverse

$$T^{-1} : (0,\infty) \times S \to \mathbb{R}^d \setminus \{0\}, \quad (\tau, \theta) \mapsto \tau\theta$$

are continuous which is merely clear by definition. This observation can be used to derive an efficient disintegration formula for all non-trivial measures ν on $\mathbb{R}^d \setminus \{0\}$ that are finite on regions bounded away from zero and that have the scaling property

$$\nu(\lambda \cdot) = \lambda^{-\alpha}\nu(\cdot) \quad \forall\, \lambda > 0 \tag{3.1.2}$$

for some $\alpha > 0$. We skip the details here and refer to [SchSt14, Fact 3.5]. Anyhow, it is important to mention that the disintegration formula yields a non-trivial measure $\sigma := \sigma_S \in M^b(S)$ depending on ν. This turns out to be crucial in the subsequent deliberations concerning f-implicit max-stable distributions and f-implicit sup-measures. In literature σ is commonly referred to as *spectral measure of ν with respect to some fixed polar coordinates*. For a somewhat different treatment of generalized polar coordinates and its applications to operator-scaling limit theorems we refer to [MeSch01, Chapter 6].
Now, we set our focus on f-implicit max-stable distributions, our notion differing from the one used in [SchSt14]. Namely, we take the space $\mathbb{R}^d \setminus \{0\}$ equipped with the topology introduced in [MeSch01, Definition 1.2.17] as a basis rather than $[-\infty,\infty]^d \setminus D$ together with the topology of vague convergence. Nevertheless, these approaches are equivalent both in the classical case $D = \{0\}$, which can be found in [Sch02, Chapter 2], and in the more general case in which we observe $\mathbb{R}^d \setminus D$ equipped with the topology introduced in [LiReRo14, Chapter 2].
Although we have already recalled the definition of f-implicit max-stable distributions, we address this concept once again in order to initiate the central Definition 3.1.2. Remember that a random vector X with values in $\mathbb{R}^d$ is f-implicit max-stable if for all $n \geq 1$ there exist $a_n > 0$ such that

$$a_n^{-1} X_{k(n)} = a_n^{-1} \bigvee_{i=1}^{n}{}_f\, X_i = \bigvee_{i=1}^{n}{}_f\, a_n^{-1} X_i \overset{d}{=} X,$$

the random vectors $X_1, ..., X_n$ being independent copies of X. Referring to Example 2.1.4, we can clearly exclude the tedious case of random vectors X with $\mathbb{P}(X \in L_{\ell_0}) = 1$ for some $\ell_0 \geq 0$. Using classical results from extreme value theory we obtain $a_n = n^{1/\alpha}$ for some $\alpha > 0$. Referring to Theorem 4.2 in [SchSt14], we further point out that these distributions can be characterized completely. Indeed, under the assumptions on f a distribution is f-implicit max-stable if and only if it is an (f,ν)-implicit extreme value

distribution. As usual in the context of (f, ν)-implicit extreme value distributions, we point out that ν is a non-trivial measure on $\mathbb{R}^d \setminus \{0\}$ being finite on regions bounded away from zero and satisfying the scaling property (3.1.2) for some $\alpha > 0$ (see [SchSt14, Definition 3.2 and Definition 3.17]). More general and at the same time more detailed considerations concerning the notion of (f, ν)-implicit extreme value distributions and its significance in the context of limit theorems for implicit extremes can be found in [SchSt14].
Now, fix some generalized polar coordinates as in (3.1.1). In order to motivate the basic definition of f-implicit-Fréchet distributions, which is intended to prepare the notion of f-implicit sup-measure, we use Proposition 3.18 of [SchSt14]. To be more precise, the latter provides a probabilistic characterization of (f, ν)-implicit extreme value distributions. Referring especially to the *only if-part* as well as to [SchSt14, Equation (3.21)], we get the next proposition.

Proposition 3.1.1
Any random vector Y in $\mathbb{R}^d \setminus \{0\}$ having an (f, ν)-implicit extreme value distribution can be represented as

$$Y \stackrel{d}{=} Z \frac{\Theta}{g(\Theta)},$$

where σ denotes the spectral measure of ν with respect to the given polar coordinates, $g : S \to [0, \infty)$ is measurable with $\int_S g^\alpha(\theta)\sigma(d\theta) = 1$, Z is standard α-Fréchet and Θ is a random vector, being independent of Z, taking values in S and having the distribution $\mathbb{P}_\Theta(d\theta) := g^\alpha(\theta)\sigma(d\theta)$. In addition, g is actually given by

$$g(\theta) = \nu(\{f > 1\})^{-\frac{1}{\alpha}} f(\theta)$$

σ-almost surely.

Combining parts of the preceding observations and referring to some additional aspects considered in [SchSt14], especially to the notion of regular varying measures on $\mathbb{R}^d \setminus \{0\}$, we conclude that every f-implicit max-stable distribution Y in $\mathbb{R}^d$ can be expressed by

$$Y \stackrel{d}{=} C^{\frac{1}{\alpha}} Z \frac{\Theta}{f(\Theta)} \tag{3.1.3}$$

with some constant $0 < C < \infty$ and Z, Θ as specified above. Conversely, every such random vector Y is f-implicit max-stable which can be seen easily by just following the proof of the *if-part* of Proposition 3.18 in [SchSt14]. Here, we refer specifically to equation (3.32).
In view of our purpose it is sufficient and convenient at the same time to consider a special class of distributions given by (3.1.3). To be more precise, we will henceforth consider in particular the generalized polar coordinates (τ, θ) in terms of f. Clearly, $(f(x), \frac{x}{f(x)})$ can serve as generalized polar coordinates because of our assumptions on f. Thus, we fix the following notation for the remainder of this chapter:

$$(\tau, \theta) = \left(f(x), \frac{x}{f(x)}\right), \qquad S = \{f = 1\} = \{x \in \mathbb{R}^d \setminus \{0\} : f(x) = 1\}.$$

Referring to (3.7) in [SchSt14], we deduce that C now actually coincides with $\sigma(S)$. Accordingly, (3.1.3) results in

$$Y \stackrel{d}{=} \sigma(S)^{\frac{1}{\alpha}} Z\Theta,$$

where Z is standard α-Fréchet and $\mathbb{P}_\Theta(d\theta) := \sigma(S)^{-1}\sigma(d\theta)$. If we finally even assume σ to be a probability measure (see [SchSt14, Remark 3.21]), we get exactly those distributions which are of interest for our purpose. As this class of distributions will definitely be crucial for our considerations, we dedicate the following definition to them.

Definition 3.1.2
Let $\alpha > 0$ and $\sigma \in M^1(S)$ be fixed.

(a) A random vector X in $\mathbb{R}^d$ is said to have a *standard f-implicit α-Fréchet distribution with angular part $\sigma \in M^1(S)$* if

$$X \stackrel{d}{=} Z\Theta, \tag{3.1.4}$$

where Z is standard α-Fréchet and Θ a random vector with values in S, being independent of Z and having distribution $\mathbb{P}_\Theta := \sigma$. For abbreviation, X is referred to as *f-implicit α-Fréchet with angular part $\sigma \in M^1(S)$* if it has a standard f-implicit α-Fréchet distribution with angular part $\sigma \in M^1(S)$. In this case, we write $X \sim \Phi^f_{\alpha,\sigma}$.

(b) More generally, a random vector X in $\mathbb{R}^d$ is said to have an *f-implicit α-Fréchet distribution with scale $\kappa > 0$ and angular part $\sigma \in M^1(S)$* if

$$X \stackrel{d}{=} \kappa Z\Theta, \tag{3.1.5}$$

with Z, Θ as above. If X has an f-implicit α-Fréchet distribution with scale $\kappa > 0$ and angular part $\sigma \in M^1(S)$, we simply call X *f-implicit α-Fréchet with scale $\kappa > 0$ and angular part $\sigma \in M^1(S)$* and write $X \sim \Phi^f_{\alpha,\sigma}(\kappa)$. Hence, we have $\Phi^f_{\alpha,\sigma}(1) = \Phi^f_{\alpha,\sigma}$.

By convention, let X have distribution $\mathbb{P}_X = \varepsilon_0$ if $X \sim \Phi^f_{\alpha,\sigma}(0)$. This extends the notion of f-implicit α-Fréchet distributions with scale $\kappa > 0$ and angular part $\sigma \in M^1(S)$. In fact, the scale $\kappa = 0$ is now permitted which is definitely necessary for our upcoming considerations.

Remark 3.1.3. (i) For all $\kappa > 0$, equation (3.1.5) means nothing but

$$X \stackrel{d}{=} Z_\kappa\Theta$$

with $Z_\kappa \sim \Phi_\alpha(\kappa)$. By convention, this is even true if $\kappa = 0$. Consequently, we deduce that $f(X) \sim \Phi_\alpha(\kappa)$ for all $\kappa \geq 0$, provided $X \sim \Phi^f_{\alpha,\sigma}(\kappa)$. This conclusion follows directly from the assumptions on f. Thus, our terminology regarding the distributions introduced in Definition 3.1.2 becomes more reasonable.

(ii) Observe that $\mathbb{P}(X = 0) = 0$ for all $\kappa > 0$, provided $X \sim \Phi^f_{\alpha,\sigma}(\kappa)$. In fact, we have $\mathbb{P}(X = 0) = \mathbb{P}(f(X) = 0)$. Taking into account both the assumed 1-homogeneity of f and (3.1.5), we conclude that $\mathbb{P}(X = 0) = \mathbb{P}(\kappa Z = 0) = 0$. Actually, the distribution of all such random vectors is therefore completely determined on $\mathbb{R}^d \setminus \{0\}$.

(iii) Note that the distributions introduced in Definition 3.1.2 (a) are closely related to the *standard max-stable distributions* derived at the beginning of Section 6 in [SchSt14].

Equipped with this important definition, we are now interested in first properties of these distributions serving as a preparation for our further course of action. Lemma 3.1.4 is intended to meet this purpose. It provides some formulas proving extremely useful in the following considerations. Especially, the proof of existence of an f-implicit sup-measure benefits from these results. In addition, it assures the existence of two families of sets which are π-systems generating the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d \setminus \{0\})$. To be more precise, fix $r > 0$ as well as $F \in \mathcal{B}(S)$ and consider the cylinder sets defined by

$$D(r, F) := \{x = \tau\theta \in \mathbb{R}^d \setminus \{0\} : \tau \le r, \theta \in F\} \subset \mathbb{R}^d \setminus \{0\}$$
$$D^*(r, F) := \{x = \tau\theta \in \mathbb{R}^d \setminus \{0\} : \tau > r, \theta \in F\} \subset \mathbb{R}^d \setminus \{0\}.$$

Furthermore, set

$$\mathcal{E}_1 := \{D(r, F) : r > 0, F \in \mathcal{B}(S)\}$$
$$\mathcal{E}_2 := \{D^*(r, F) : r > 0, F \in \mathcal{B}(S)\}.$$

Lemma 3.1.4
Fix $\alpha > 0$ and $\sigma \in M^1(S)$.

(a) Both $\mathcal{E}_1$ and $\mathcal{E}_2$ are π-systems generating the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d \setminus \{0\})$, that is, $\mathfrak{A}(\mathcal{E}_1) = \mathfrak{A}(\mathcal{E}_2) = \mathcal{B}(\mathbb{R}^d \setminus \{0\})$. Hence, two probability measures μ_1, μ_2 on $(\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\}))$ coincide if $\mu_1(A) = \mu_2(A)$ for all $A \in \mathcal{E}_1$ or all $A \in \mathcal{E}_2$.

(b) If $X \sim \Phi^f_{\alpha,\sigma}\left(\kappa^{1/\alpha}\right)$ with $\kappa > 0$, then

$$\mathbb{P}_X(A) = \int\limits_S \int\limits_0^\infty \mathbb{1}_A(\tau\theta) \frac{\kappa\alpha}{\tau^{\alpha+1}} e^{-\kappa\tau^{-\alpha}} \, d\tau \, \sigma(d\theta). \tag{3.1.6}$$

for all $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$. For convenience, we shortly write

$$\mathbb{P}_X(d\tau, d\theta) = \frac{\kappa\alpha}{\tau^{\alpha+1}} e^{-\kappa\tau^{-\alpha}} \, d\tau \, \sigma(d\theta). \tag{3.1.7}$$

(c) Let $(\mu_t)_{t\ge0}$ be the family of probability measures on $\mathbb{R}^d$ defined by

$$\mu_t = \Phi^f_{\alpha,\sigma}\left(t^{\frac{1}{\alpha}}\right), \ t \ge 0.$$

Then $(\mu_t)_{t\ge0}$ is an f-implicit max-convolution semigroup, that is,

$$\mu_s *_f \mu_t = \mu_{s+t}, \ \forall\, s, t \ge 0 \tag{3.1.8}$$

Proof. (a) Let us start with the following observation. Choose $A \in \mathcal{E}_2$. The definition of $\mathcal{E}_2$ yields $A = D^*(r, F)$ for some $r > 0$ and $F \in \mathcal{B}(S)$. Therefore,

$$\begin{aligned} A^c &= D^*(r,F)^c \\ &= (\mathbb{R}^d \setminus \{0\}) \setminus D^*(r,F) \\ &= D(r,F) \cup \{x = \tau\theta \in \mathbb{R}^d \setminus \{0\} : \tau < \infty,\ \theta \in F^c\} \\ &= D(r,F) \cup \bigcup_{n \geq 1} D(n, F^c). \end{aligned}$$

Since $D(r,F) \in \mathcal{E}_1$ for all $r > 0$ and all $F \in \mathcal{B}(S)$, we conclude that $A^c \in \mathfrak{A}(\mathcal{E}_1)$. Hence, $A \in \mathfrak{A}(\mathcal{E}_1)$ showing $\mathcal{E}_2 \subset \mathfrak{A}(\mathcal{E}_1)$ and finally $\mathfrak{A}(\mathcal{E}_2) \subset \mathfrak{A}(\mathcal{E}_1)$. Applying analogous arguments, we also verify $\mathfrak{A}(\mathcal{E}_1) \subset \mathfrak{A}(\mathcal{E}_2)$. Thus, it suffices to proof $\mathfrak{A}(\mathcal{E}_1) = \mathcal{B}(\mathbb{R}^d \setminus \{0\})$. In order to do so, let us introduce the family of sets

$$\mathcal{E}' := \{(0, r] \times F : r > 0, F \in \mathcal{B}(S)\}.$$

It is commonly known that $\{(0, r] : r > 0\}$ generates the Borel σ-algebra $\mathcal{B}((0,\infty))$ and therefore

$$\mathfrak{A}(\mathcal{E}') = \mathcal{B}((0,\infty)) \otimes \mathcal{B}(S) = \mathcal{B}((0,\infty) \times S).$$

The first equality is due to [Kl08, Theorem 14.12(i)] and the second due to [Kl08, Theorem 14.8]. Moreover, we have $\mathcal{E}_1 = T^{-1}(\mathcal{E}') := \{E_1 \in \mathbb{R}^d \setminus \{0\} : E_1 = T^{-1}(E'), E' \in \mathcal{E}'\}$. With

$$\begin{aligned} \mathcal{O}_1 &:= \{O \in (0,\infty) \times S : O \text{ open}\} \\ \mathcal{O}_2 &:= \{O \in \mathbb{R}^d \setminus \{0\} : O \text{ open}\} \end{aligned}$$

and by applying [Sc11, Theorem 2.3.2] twice, we obtain

$$\begin{aligned} \mathfrak{A}(\mathcal{E}_1) &= \mathfrak{A}\left(T^{-1}(\mathcal{E}')\right) \\ &= T^{-1}(\mathfrak{A}(\mathcal{E}')) \\ &= T^{-1}(\mathcal{B}((0,\infty) \times S)) \\ &= T^{-1}(\mathfrak{A}(\mathcal{O}_1)) \\ &= \mathfrak{A}\left(T^{-1}(\mathcal{O}_1)\right) \\ &= \mathfrak{A}(\mathcal{O}_2) \\ &= \mathcal{B}(\mathbb{R}^d \setminus \{0\}). \end{aligned}$$

As usual, $(0,\infty) \times S$ is equipped with the product topology. Observe that we essentially gain from the fact that $T : \mathbb{R}^d \setminus \{0\} \to (0,\infty) \times S$, $x \mapsto (\tau, \theta)$ is a homeomorphism. Taking into account [MeSch01, Theorem 1.1.3], we get immediately the additional statement.
(b) Applying (a), we conclude that it is sufficient to verify (3.1.6) for all $A \in \mathcal{E}_1$. In doing so, fix an arbitrary $A \in \mathcal{E}_1$, that is, $A = D(r,F)$ for some $r > 0$ and $F \in \mathcal{B}(S)$. Then the estimation of the right-hand side yields

$$\int_S \int_0^\infty \mathbb{1}_{D(r,F)}(\tau\theta) \frac{\kappa\alpha}{\tau^{\alpha+1}} e^{-\kappa\tau^{-\alpha}}\, d\tau\, \sigma(d\theta) = \int_F \int_0^r \frac{\kappa\alpha}{\tau^{\alpha+1}} e^{-\kappa\tau^{-\alpha}}\, d\tau\, \sigma(d\theta)$$

$$
\begin{aligned}
&= \sigma(F) \int_0^r \frac{\kappa\alpha}{\tau^{\alpha+1}} e^{-\kappa\tau^{-\alpha}}\, d\tau \\
&= \sigma(F)\, e^{-\kappa r^{-\alpha}} \\
&= \mathbb{P}(\Theta \in F)\, \mathbb{P}(\kappa^{\frac{1}{\alpha}} Z \leq r) \\
&= \mathbb{P}(\kappa^{\frac{1}{\alpha}} Z \leq r, \Theta \in F) \\
&= \mathbb{P}(\kappa^{\frac{1}{\alpha}} Z\Theta \in D(r,F)) \\
&= \mathbb{P}(X \in D(r,F)),
\end{aligned}
$$

which proves the assertion.
(c) In order to prove the asserted equality (3.1.8), fix $s, t \geq 0$. On account of (1.2.6), we recognize that the proof is already completed by showing the desired equality (3.1.8) for $s, t > 0$. To this end, fix $s, t > 0$ and write

$$
\mu_s = \mathbb{P}_{s^{\frac{1}{\alpha}} Z_1 \Theta_1} \quad \text{and} \quad \mu_t = \mathbb{P}_{t^{\frac{1}{\alpha}} Z_2 \Theta_2}
$$

with $Z_1, Z_2 \sim \Phi_\alpha$ and $\Theta_1, \Theta_2 \sim \sigma \in M^1(S)$ such that Z_1 and Θ_1 as well as Z_2 and Θ_2 are independent. Applying the independence of Z_1 and Θ_1 as well as of Z_2 and Θ_2, we may assume that

$$
\mu_s = \mathbb{P}_{s^{\frac{1}{\alpha}} Z \Theta} \quad \text{and} \quad \mu_t = \mathbb{P}_{t^{\frac{1}{\alpha}} Z \Theta}
$$

for Z, Θ as stated in Definition 3.1.2 (b). Since $\mu_t(\{0\}) = 0$ for all $t > 0$, we can use the result of (a) once again to see that it remains to show that

$$
\mu_s *_f \mu_t(D(r,F)) = \mu_{s+t}(D(r,F))
$$

for an arbitrary $r > 0$ and any $F \in \mathcal{B}(S)$. For that purpose, we apply (1.2.4) and (3.1.6) to conclude that

$$
\begin{aligned}
\mu_s *_f \mu_t(D(r,F)) &= \int_{D(r,F)} f(\mu_t)([0, f(x)])\, \mu_s(dx) + \int_{D(r,F)} f(\mu_s)([0, f(x)))\, \mu_t(dx) \\
&= \int_{D(r,F)} \mathbb{P}\left(f\left(t^{\frac{1}{\alpha}} Z\Theta\right) \in [0, f(x)]\right) \mathbb{P}_{s^{\frac{1}{\alpha}} Z\Theta}(dx) \\
&\qquad + \int_{D(r,F)} \mathbb{P}\left(f\left(s^{\frac{1}{\alpha}} Z\Theta\right) \in [0, f(x))\right) \mathbb{P}_{t^{\frac{1}{\alpha}} Z\Theta}(dx) \\
&= \int_{D(r,F)} \mathbb{P}\left(t^{\frac{1}{\alpha}} Z \in [0, f(x)]\right) \mathbb{P}_{s^{\frac{1}{\alpha}} Z\Theta}(dx) \\
&\qquad + \int_{D(r,F)} \mathbb{P}\left(s^{\frac{1}{\alpha}} Z \in [0, f(x))\right) \mathbb{P}_{t^{\frac{1}{\alpha}} Z\Theta}(dx)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_F \int_0^r \underbrace{\mathbb{P}\left(t^{\frac{1}{\alpha}} Z \leq \tau\right)}_{=e^{-t\tau^{-\alpha}}} \frac{s\alpha}{\tau^{\alpha+1}} e^{-s\tau^{-\alpha}} \, d\tau \, \sigma(d\theta) \\
&\qquad\qquad + \int_F \int_0^r \underbrace{\mathbb{P}\left(s^{\frac{1}{\alpha}} Z < \tau\right)}_{=e^{-s\tau^{-\alpha}}} \frac{t\alpha}{\tau^{\alpha+1}} e^{-t\tau^{-\alpha}} \, d\tau \, \sigma(d\theta) \\
&= \sigma(F) \int_0^r \frac{s\alpha}{\tau^{\alpha+1}} e^{-(s+t)\tau^{-\alpha}} \, d\tau + \sigma(F) \int_0^r \frac{t\alpha}{\tau^{\alpha+1}} e^{-(s+t)\tau^{-\alpha}} \, d\tau \\
&= \sigma(F) \left(\frac{s}{s+t} e^{-(s+t)\tau^{-\alpha}} \Big|_0^r + \frac{t}{s+t} e^{-(s+t)\tau^{-\alpha}} \Big|_0^r \right) \\
&= \sigma(F) e^{-(s+t)r^{-\alpha}}.
\end{aligned}
$$

Considering the proof of (b), we finally get

$$
\begin{aligned}
&= \mathbb{P}\left((s+t)^{\frac{1}{\alpha}} Z\Theta \in D(r,F)\right) \\
&= \mu_{s+t}(D(r,F)),
\end{aligned}
$$

which is the desired conclusion. □

Remark 3.1.5. (i) Note that the claim of Lemma 3.1.4 (a) also holds if we replace the generalized polar coordinates in terms of f by other ones.

(ii) A result being similar to the one stated in (b) has already been formulated in [SchSt14, Fact 3.5]. As there was only given a rough sketch of the proof, we included a detailed one here, thus making our exposition self-contained. Note further that there clearly exist other suitable ways to prove (b) and even (c).

(iii) Referring to Remark 2.2.18 (iii), we observe that (3.1.8) yields the commutativity of μ_s and μ_t under the f-implicit max-convolution.

(iv) Fix $t \geq 0$. Iterating (3.1.8), we obtain

$$
\mu_{\frac{t}{n}} *_f \dots *_f \mu_{\frac{t}{n}} = \left(\mu_{\frac{t}{n}}\right)^{*_f n} = \mu_t
$$

for all $n \geq 1$ showing that μ_t is f-implicit max-infinitely divisible with nth root $\mu_{t/n}$. The property of being f-implicit max-infinitely divisible is already clear by Lemma 2.1.5 as μ_t is f-implicit max-stable. Moreover, this has also already been asserted in Remark 2.2.18 (ii). In addition, we even have an explicit formula for the corresponding nth root.

(v) On a final note, we shall mention that the class of probability measures on $\mathbb{R}^d$ introduced in Lemma 3.1.4 (c) is even a continuous f-implicit max-convolution semigroup. Clearly, without loss of generality, we may assume once again that

$$
\mu_s = \mathbb{P}_{s^{\frac{1}{\alpha}} Z\Theta} \quad \text{and} \quad \mu_t = \mathbb{P}_{t^{\frac{1}{\alpha}} Z\Theta}
$$

for Z, Θ as stated in Definition 3.1.2 (b) and $s, t \geq 0$. Since $s^{1/\alpha} Z\Theta$ converges pointwise to $t^{1/\alpha} Z\Theta$ as s tends to t, the desired claim follows.

Adopting the previously introduced notation $\mu_t = \Phi^f_{\alpha,\sigma}\left(t^{1/\alpha}\right)$ for all $t \geq 0$, we obtain the ensuing statements that are indispensable for the upcoming studies on f-implicit sup-measures. In particular, statement (a) turns out to be extremely beneficial. On the contrary, statement (b) is less essential but an observation that is of independent interest.

Corollary 3.1.6
Fix $\alpha > 0$ and $\sigma \in M^1(S)$.

(a) If X, Y are random vectors in $\mathbb{R}^d$ with $X \sim \mu_s$ and $Y \sim \mu_t$ for $s, t > 0$, then

$$\mathbb{P}_X \otimes \mathbb{P}_Y \left(\left\{ (x, y) \in \mathbb{R}^d \times \mathbb{R}^d : \ f(x) = f(y) \right\} \right) = 0.$$

In particular, if X and Y are in addition independent, then

$$X \vee_f Y = Y \vee_f X \tag{3.1.9}$$

almost surely. That is, X and Y commute almost surely under the f-implicit max-operation.

(b) Let $h : \mathbb{R}^d \to [0, \infty)$ be measurable and $\rho_1, \rho_2 \in M^b(\mathbb{R}^d)$ such that $\rho_1 *_f \rho_2 = \rho_2 *_f \rho_1$. Then we have

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(x \vee_f y)\, \rho_1(dx)\, \rho_2(dy) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(y \vee_f x)\, \rho_1(dx)\, \rho_2(dy), \tag{3.1.10}$$

In particular, if $\rho_1 = \mu_s$ and $\rho_2 = \mu_t$ for some $s, t \geq 0$, then

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(x \vee_f y)\, \mu_s(dx)\, \mu_t(dy) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(y \vee_f x)\, \mu_s(dx)\, \mu_t(dy). \tag{3.1.11}$$

Proof. (a) Let $s, t > 0$ be fixed. By Remark 3.1.3 (i), $f(X) \sim \Phi_\alpha\left(s^{1/\alpha}\right)$ and $f(Y) \sim \Phi_\alpha\left(t^{1/\alpha}\right)$. Hence,

$$\begin{aligned}\mathbb{P}_{f(X)}(dx) &= g_s(x) \lambda^1(dx) \\ &= \alpha s x^{-\alpha-1} e^{-s x^{-\alpha}} \mathbb{1}_{(0,\infty)}(x)\, \lambda^1(dx)\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}_{f(Y)}(dx) &= g_t(x) \lambda^1(dx) \\ &= \alpha t x^{-\alpha-1} e^{-t x^{-\alpha}} \mathbb{1}_{(0,\infty)}(x)\, \lambda^1(dx).\end{aligned}$$

Therefore, we get

$$\mathbb{P}_{f(X)} \otimes \mathbb{P}_{f(Y)}(dx, dy) = g_s(x) g_t(y) \lambda^2(dx, dy)$$

and conclude that

$$
\begin{aligned}
&\mathbb{P}_X \otimes \mathbb{P}_Y \left(\left\{ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \; f(x) = f(y) \right\} \right) \\
&= \int\limits_{\mathbb{R}^d \times \mathbb{R}^d} \mathbb{1}_{\{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d :\, f(x)=f(y)\}} (x,y) \, (\mathbb{P}_X \otimes \mathbb{P}_Y) \, (dx, dy) \\
&= \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \mathbb{1}_{\{(u,v)\in\mathbb{R}^2 :\, u=v\}} \left(f(x), f(y)\right) \mathbb{P}_Y(dy) \, \mathbb{P}_X(dx) \\
&= \int\limits_{\mathbb{R}^d} \int\limits_{(0,\infty)} \mathbb{1}_{\{(u,v)\in\mathbb{R}^2 :\, u=v\}} \left(f(x), z_2\right) \mathbb{P}_{f(Y)}(dz_2) \, \mathbb{P}_X(dx) \\
&= \int\limits_{(0,\infty)} \int\limits_{\mathbb{R}^d} \mathbb{1}_{\{(u,v)\in\mathbb{R}^2 :\, u=v\}} \left(f(x), z_2\right) \mathbb{P}_X(dx) \, \mathbb{P}_{f(Y)}(dz_2) \\
&= \int\limits_{(0,\infty)} \int\limits_{(0,\infty)} \mathbb{1}_{\{(u,v)\in\mathbb{R}^2 :\, u=v\}} \left(z_1, z_2\right) \mathbb{P}_{f(X)}(dz_1) \, \mathbb{P}_{f(Y)}(dz_2) \\
&= \int\limits_{\mathbb{R}^2} \mathbb{1}_{\{(u,v)\in\mathbb{R}^2 :\, u=v\}} \left(z_1, z_2\right) g_s(z_1) g_t(z_2) \, \lambda^2(dz_1, dz_2) \\
&= 0.
\end{aligned}
$$

The last equality is due to the fact that $\{(u,v) \in \mathbb{R}^2 : \; u = v\}$ is a null set in $\mathbb{R}^2$ with respect to Lebesgue measure on $\mathbb{R}^2$.
Now, we deduce (3.1.9) from the preceding result. By Lemma 1.1.5 (b) it is sufficient to show that $f(X) \neq f(Y)$ almost surely which in turn follows from

$$
\begin{aligned}
0 &= \mathbb{P}_X \otimes \mathbb{P}_Y \left(\left\{ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \; f(x) = f(y) \right\} \right) \\
&= \mathbb{P}_{(X,Y)} \left(\left\{ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \; f(x) = f(y) \right\} \right) \\
&= \mathbb{P}(f(X) = f(Y)).
\end{aligned}
$$

(b) We begin by proving (3.1.10). Applying the classical change of variables formula, Definition 1.2.1 and the assumed requirements, we conclude that

$$
\begin{aligned}
\int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} h(x \vee_f y) \, \rho_1(dx) \, \rho_2(dy) &= \int\limits_{\mathbb{R}^d \times \mathbb{R}^d} h\left(T^{(2)}(x,y)\right) (\rho_1 \otimes \rho_2)(dx, dy) \\
&= \int\limits_{\mathbb{R}^d} h(z) \, T^{(2)}(\rho_1 \otimes \rho_2)(dz) \\
&= \int\limits_{\mathbb{R}^d} h(z) \, (\rho_1 *_f \rho_2)(dz)
\end{aligned}
$$

$$= \int_{\mathbb{R}^d} h(z)\,(\rho_2 *_f \rho_1)(dz)$$
$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(y \vee_f x)\,\rho_1(dx)\,\rho_2(dy).$$

Finally, (3.1.11) is a consequence of (3.1.10) and Remark 3.1.5 (iii). Indeed, the latter provides the commutativity of the f-implicit max-convolution applied to μ_s and μ_t. □

Remark 3.1.7. (i) Once more, we shall point out that conclusion (a) of Corollary 3.1.6 will essentially contribute to the proof of existence of an f-implicit sup-measure.

(ii) The first assertion in part (a) is obviously also true if we let $s \geq 0$ and $t > 0$ or vice versa. Thus, $s = t = 0$ is the only case for which the assertion does not hold. However, (3.1.9) is valid for all $s, t \geq 0$.

(iii) Clearly, (3.1.9) implies
$$X \vee_f Y \overset{d}{=} Y \vee_f X.$$
However, the latter equation is even true under less restrictive assumptions. We only need to require that both X, Y are independent and that the f-implicit max-convolution applied to $\mathbb{P}_X$ and $\mathbb{P}_Y$ is commutative. Indeed, using Lemma 1.2.4 (b), we get
$$\mathbb{P}_{X\vee_f Y} = \mathbb{P}_X *_f \mathbb{P}_Y = \mathbb{P}_Y *_f \mathbb{P}_X = \mathbb{P}_{Y\vee_f X}.$$
Conversely, if X, Y are independent with $X \vee_f Y \overset{d}{=} Y \vee_f X$, then the f-implicit max-convolution applied to $\mathbb{P}_X$ and $\mathbb{P}_Y$ is commutative.

After all these elaborate preparations, we can finally proceed to establish the notion of f-implicit sup-measures. For convenience, we shall fix $\alpha > 0$ and $\sigma \in M^1(S)$ for the remainder of this chapter. To simplify notation, we further write

$$\mu_\kappa := \Phi^f\left(\kappa^{\frac{1}{\alpha}}\right) := \Phi^f_{\alpha,\sigma}\left(\kappa^{\frac{1}{\alpha}}\right)$$

for all $\kappa \geq 0$. Moreover, we call $X \sim \mu_\kappa$ simply f-implicit α-Fréchet with scale $\kappa^{1/\alpha} \geq 0$, that is, we neglect the angular part in our way of speaking. In this way, we follow the preassigned notation of Lemma 3.1.4 and Corollary 3.1.6.
Let $(E, \mathcal{E}, m)$ be an arbitrary measure space, m being a measure on the σ-algebra $\mathcal{E}$. Following the ideas of [SaTa94, Chapter 3.3], let $\mathcal{E}_0$ denote the set of all measurable sets $A \subset E$ for which $m(A)$ is finite, that is,

$$\mathcal{E}_0 := \{A \in \mathcal{E} :\ m(A) < \infty\}.$$

Furthermore, define

$$L_0^d := L_0^d(\Omega, \mathfrak{A}) := \{X : \Omega \to \mathbb{R}^d \,|\, X \text{ is a random vector}\}$$

for an appropriate probability space $(\Omega, \mathfrak{A}, \mathbb{P})$.

Definition 3.1.8 (f-implicit, α-Fréchet, random sup-measure)
An *f-implicit, α-Fréchet, random sup-measure* or just *f-implicit sup-measure* is a mapping

$$M^f_{\alpha,\sigma} := M : \mathcal{E}_0 \to L^d_0$$

with the following properties:

(i) (*independently scattered*)
For any collection of disjoint sets $A_1, ..., A_n \in \mathcal{E}_0, n \geq 1$, the random vectors $M(A_1), ..., M(A_n)$ are independent.

(ii) (*f-implicit α-Fréchet*)
For every $A \in \mathcal{E}_0$, we have

$$M(A) \sim \mu_{m(A)} := \mu_A. \tag{3.1.12}$$

(iii) (*f-max σ-sup-additive*)
For any collection of disjoint sets $A_n, n \geq 1$, such that $A_n \in \mathcal{E}_0$ for all $n \geq 1$ and $\bigcup_{n\geq 1} A_n \in \mathcal{E}_0$, we have

$$M\left(\bigcup_{n\geq 1} A_n\right) = \bigvee_{n=1}^{\infty}{}_f\, M(A_n) = M\left(A_{n_0}\right) \tag{3.1.13}$$

almost surely, where n_0 is a random index.

The measure m is referred to as the *control measure*.

Definition 3.1.8 instantly raises the crucial question regarding the existence of an f-implicit sup-measure. Finding an answer to this issue occupies center stage in Chapter 3. Before elaborating on this problem, we shall first note the following remarks concerning Definition 3.1.8.

Remark 3.1.9. (i) Equation (3.1.12) means nothing else but

$$M(A) \stackrel{d}{=} m(A)^{\frac{1}{\alpha}} Z\Theta$$

for all $A \in \mathcal{E}_0$, where Z and Θ are defined as in Definition 3.1.2. In other words, $M(A)$ is f-implicit α-Fréchet with scale $m(A)^{1/\alpha} \geq 0$ for every $A \in \mathcal{E}_0$.

(ii) Applying equation (3.1.8), we have

$$\mu_A *_f \mu_B = \mu_{A\cup B} \tag{3.1.14}$$

for any choice of disjoint sets $A, B \in \mathcal{E}_0$ as $m(A) + m(B) = m(A \cup B)$.

(iii) The assumption $\bigcup_{n\geq 1} A_n \in \mathcal{E}_0$ in property (iii) of Definition 3.1.8 is certainly anything but superfluous and cannot be dropped since $\mathcal{E}_0$ is in general no σ-algebra.

(iv) For reasons of clarity and comprehensibility, we point out once again that independent f-implicit α-Fréchet random vectors commute under the f-implicit max-operation.

Remark 3.1.10. Even more important and necessary to be clarified is the expression

$$\bigvee_{n=1}^{\infty}{}_f\, M(A_n)$$

of an infinite f-implicit maximum occuring in (3.1.13). Briefly speaking, the legitimate question arises why the infinite f-implicit maximum in (3.1.13) exists almost surely and why it equals $M(A_{n_0})$ for some random index n_0. On that point, we firstly focus on deterministic infinite sequences $(x_n)_{n\geq 1} \subset \mathbb{R}^d$. Clearly, the infinite f-implicit maximum

$$\bigvee_{n=1}^{\infty}{}_f\, x_n$$

does not make sense unless $\sup_{n\geq 1} f(x_n)$ is attained, that is, unless there exists an $n_0 \geq 1$ such that

$$f(x_{n_0}) = \sup_{n\geq 1} f(x_n).$$

Thus, it does not make sense defining the infinite f-implicit maximum for general sequences. Even the class of bounded sequences is unsuitable to apply the infinite f-implicit maximum. An appropriate counterexample showing the latter is given by the sequences $(x_n)_{n\geq 1}$ with

$$x_n := \left(1 - \frac{1}{n}\right)\theta, n \geq 1,$$

for some $\theta \in S$. However, the infinite f-implicit maximum is obviously well defined for the class of sequences

$$\chi_0^f := \left\{(x_n)_{n\geq 1} \subset \mathbb{R}^d : \lim_{n\to\infty} f(x_n) = 0\right\}.$$

In fact, it is defined for an even larger class of sequences, but it will be sufficient for our purpose to consider χ_0^f. Besides, in view of Lemma 3.1.14 χ_0^f coincides with the set of all null sequences in $\mathbb{R}^d$. However, this does not need to be the case for more general loss functions f.

Coming back to equation (3.1.13), we turn our attention in particular to the sequence $(M(A_n))_{n\geq 1}$ of random vectors $M(A_n)$ in $\mathbb{R}^d$. On account of the previous deliberations, we only need to ensure that

$$(M(A_n))_{n\geq 1} \in \chi_0^f \tag{3.1.15}$$

almost surely, thus obtaining

$$\bigvee_{n=1}^{\infty}{}_f\, M(A_n) = M\left(A_{n_0}\right)$$

almost surely for some random index $n_0 \geq 1$.

The next lemma is intended to verify (3.1.15).

Lemma 3.1.11
For any collection of disjoint sets $A_1, A_2, \ldots$ belonging to $\mathcal{E}_0$ such that $\bigcup_{n\geq 1} A_n \in \mathcal{E}_0$, the sequence of corresponding random vectors $(M(A_n))_{n\geq 1}$ with $M(A_n) \sim \mu_{A_n}, n \geq 1$, belongs to χ_0^f almost surely. That is,

$$f(M(A_n)) \xrightarrow[(n\to\infty)]{} 0$$

almost surely.

Proof. First, note that

$$\mathbb{P}(f(M(A_n)) > \varepsilon) = \mathbb{P}\left(m(A_n)^{\frac{1}{\alpha}} Z > \varepsilon\right) = 1 - e^{-m(A_n)\varepsilon^{-\alpha}} \leq m(A_n)\varepsilon^{-\alpha}$$

for all $\varepsilon > 0$. Since the sets $A_1, A_2, \ldots$ are disjoint with $\bigcup_{n\geq 1} A_n \in \mathcal{E}_0$, we further have

$$\sum_{n=1}^{\infty} \mathbb{P}(f(M(A_n)) > \varepsilon) \leq \varepsilon^{-\alpha} \sum_{n=1}^{\infty} m(A_n) = \varepsilon^{-\alpha} m\left(\bigcup_{n\geq 1} A_n\right) < \infty$$

for all $\varepsilon > 0$. Thus, applying the Borel-Cantelli lemma, we conclude that

$$\mathbb{P}\left(\limsup_{n\to\infty} \{f(M(A_n)) > \varepsilon\}\right) = 0$$

for all $\varepsilon > 0$. Referring to [Bi12, Theorem 5.2 (i)], we obtain the desired conclusion. □

Remark 3.1.10 and Lemma 3.1.11 in combination justify the reasonableness of Definition 3.1.8. Accordingly, we can proceed to investigate the fundamental question whether an f-implicit sup-measure exists. To this end, we establish the next theorem ensuring the existence of an f-implicit sup-measure.

Theorem 3.1.12
Let $(E, \mathcal{E}, m)$ be an arbitrary measure space. Then there exists an f-implicit sup-measure M with control measure m defined over an appropriate probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. In other words, there exists a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ and a mapping $M : \mathcal{E}_0 \to L_0^d$ such that it fulfills the properties (i)-(iii) listed in Definition 3.1.8.

In order to prove Theorem 3.1.12, we shall first establish the following lemmas since their assertions will essentially contribute to our argumentation.

Lemma 3.1.13
If $X, Y : \Omega \to \mathbb{R}^d$ are two random vectors on an appropriate probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ such that

$$\mathbb{P}(X \in D) = \mathbb{P}(Y \in D) = \mathbb{P}(X \in D, Y \in D)$$

for all $D \in \mathcal{B}(\mathbb{R}^d)$, then $X = Y$ almost surely.

Proof. The proof is straightforward and falls into two parts. We first show the assertion for $d = 1$ and then infer the general case by applying the case $d = 1$. Therefore, let X, Y be real-valued random variables on an appropriate probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ such that

$$\mathbb{P}(X \in D) = \mathbb{P}(Y \in D) = \mathbb{P}(X \in D, Y \in D)$$

for all $D \in \mathcal{B}(\mathbb{R})$. Using this assumption particularly for the Borel sets $D = (-\infty, x]$, $x \in \mathbb{R}$, we get

$$\mathbb{P}(X \leq x, Y \leq x) = \mathbb{P}(X \leq x) = \mathbb{P}(X \leq x, Y \leq x) + \mathbb{P}(X \leq x, Y > x)$$

and hence conclude that

$$\mathbb{P}(X \leq x, Y > x) = 0$$

for all $x \in \mathbb{R}$. Proceeding similarly, we further have

$$\mathbb{P}(X > x, Y \leq x) = 0$$

for all $x \in \mathbb{R}$, thus obtaining

$$\operatorname{supp} \mathbb{P}_{(X,Y)} \subset \{(x, x) : \ x \in \mathbb{R}\}.$$

In other words, the support of $\mathbb{P}_{(X,Y)} \in M^1(\mathbb{R}^2)$ is a subset of the diagonal in $\mathbb{R}^2$ proving that $X = Y$ almost surely.

The more general case is an easy consequence of the recently proved assertion. To see this, fix $d > 1$. Further, write $X = (X_1, ..., X_d)$ and $Y = (Y_1, ..., Y_d)$. Applying the assumption specially for the Borel sets $D_1 \times \mathbb{R} \times \cdots \times \mathbb{R}$, $D_1 \in \mathcal{B}(\mathbb{R})$, we deduce that

$$\mathbb{P}(X_1 \in D_1) = \mathbb{P}(Y_1 \in D_1) = \mathbb{P}(X_1 \in D_1, Y_1 \in D_1)$$

for all $D_1 \in \mathcal{B}(\mathbb{R})$ and hence $X_1 = Y_1$ almost surely. Likewise, we get $X_i = Y_i$ almost surely for all $i = 2, ..., d$, which completes the proof. $\square$

Lemma 3.1.14
Let $(x_n)_{n\geq 1}$ be a sequence in $\mathbb{R}^d$. If $(f(x_n))_{n\geq 1}$ is a null sequence, then so is $(x_n)_{n\geq 1}$, that is,

$$f(x_n) \xrightarrow[(n\to\infty)]{} 0 \qquad \Rightarrow \qquad x_n \xrightarrow[(n\to\infty)]{} 0.$$

Proof. Note, $(x_n)_{n\geq 1}$ being a null sequence implies $(f(x_n))_{n\geq 1}$ is a null sequence. This follows from the continuity of f. However, even the converse is true. The simple proof is done by contradiction. Suppose, contrary to our claim, that $(x_n)_{n\geq 1}$ is not a null sequence. Consequently, there exist $\varepsilon_0 > 0$ and a subsequence $(x_{n_k})_{k\geq 1}$ of $(x_n)_{n\geq 1}$ such that

$$\|x_{n_k}\| \geq \varepsilon_0 > 0$$

for all $k \geq 1$, where $\|\cdot\|$ denotes some norm on $\mathbb{R}^d$. Therefore, f being 1-homogeneous, we conclude that

$$f\left(x_{n_k}\right) = f\left(\|x_{n_k}\| \frac{x_{n_k}}{\|x_{n_k}\|}\right) = \|x_{n_k}\| f\left(\frac{x_{n_k}}{\|x_{n_k}\|}\right)$$

for all $k \geq 1$. Similarly to the proof of Lemma 1.1.9, the assumptions on f and the compactness of the sphere $S^1 = \{x \in \mathbb{R}^d : \|x\| = 1\}$ yield

$$f\left(\frac{x_{n_k}}{\|x_{n_k}\|}\right) \geq c > 0$$

for all $k \geq 1$, the positive real number c denoting the minimum of f on S^1. Accordingly, we obtain

$$f\left(x_{n_k}\right) \geq \varepsilon_0 \cdot c > 0$$

for all $k \geq 1$ contradicting the assumption that $(f(x_n))_{n\geq 1}$ is a null sequence. □

Proof of Theorem 3.1.12. Since proving the assertion is quite involved, we do well to outline the proof first. To this end, we refer to [StTa05, Proposition 2.1] or to [SaTa94, Chapter 3] as the principal idea originates from the considerations given there. Indeed, it is most convenient to view an f-implicit sup-measure as a stochastic process indexed by the sets $A \in \mathcal{E}_0$, that is, as a family $(M(A))_{A\in\mathcal{E}_0}$ of $\mathbb{R}^d$-valued random vectors $M(A), A \in \mathcal{E}_0$, defined on an appropriate probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. More precisely, the basic idea is to define a consistent family of probability measures

$$\left\{\rho_{A_1,\ldots,A_m} \in M^1\left((\mathbb{R}^d)^m\right) : A_1, \ldots, A_m \in \mathcal{E}_0, m \geq 1\right\} \tag{3.1.16}$$

and then to apply Kolmogorov's extension theorem that guarantees the existence of a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ together with a stochastic process $(M(A))_{A\in\mathcal{E}_0}$ on $(\Omega, \mathfrak{A}, \mathbb{P})$ such that its finite dimensional distributions are given by (3.1.16). Finally, it remains to show that the latter process fulfills the properties (i), (ii) and (iii) of Definition 3.1.8. $(M(A))_{A\in\mathcal{E}_0}$ can actually be viewed as a mapping

$$M : \mathcal{E}_0 \to L_0^d,$$

thus yielding the desired conclusion.
Having formulated this brief sketch of the proof, which is intended to provide a clearer comprehension of the subsequent explanations, we now expand on the latter deliberations. First, we remark that the proof falls naturally into 5 parts.

Part 1: (Construction of a consistent family of finite dimensional distributions)
We begin by choosing any finite collection $A_1, \ldots, A_m \in \mathcal{E}_0$ of not necessarily disjoint sets $A_1, \ldots, A_m$ for some $m \geq 1$. It is always possible to find disjoint sets $B_1, \ldots, B_n \in \mathcal{E}_0$ such that

$$A_j = \bigcup_{i\in a(j)} B_i$$

for all $j = 1, \ldots, m$ with suitable index sets $a(j) \subset \{1, \ldots, n\}, j = 1, \ldots, m$. For all $j = 1, \ldots, m$, $a(j)$ is consequently the set of indices $\{i_1, \ldots, i_l\} \subset \{1, \ldots, n\}$ depending on $j \in \{1, \ldots, m\}$ for which $B_{i_k}, k = 1, \ldots, l, 1 \leq l \leq n$ is part of the disjoint partition of A_j. Using this construction as well as the notation introduced in (3.1.12), we define a probability measure $\rho_{A_1,\ldots,A_m}$ on $(\mathbb{R}^d)^m \cong \mathbb{R}^{md}$ by

$$\rho_{A_1,\ldots,A_m}(D) := \int\limits_{\mathbb{R}^d\times\cdots\times\mathbb{R}^d} \mathbb{1}_D\left(\bigvee_{f\,i\in a(1)} x_i, \ldots, \bigvee_{f\,i\in a(m)} x_i\right)(\mu_{B_1} \otimes \cdots \otimes \mu_{B_n})\,(dx_1, \ldots, dx_n)$$

$$= \int\limits_{\mathbb{R}^d} \dots \int\limits_{\mathbb{R}^d} \mathbb{1}_D \left(\bigvee\nolimits_{f}{}_{i \in a(1)} x_i, \dots, \bigvee\nolimits_{f}{}_{i \in a(m)} x_i \right) \mu_{B_n}(dx_n) \dots \mu_{B_1}(dx_1) \tag{3.1.17}$$

with $D \in \mathcal{B}\Big((\mathbb{R}^d)^m\Big) = \mathcal{B}(\mathbb{R}^d)^{\otimes m}$. Consequently, any collection $A_1, ..., A_m \in \mathcal{E}_0$ yields a probability measure $\rho_{A_1,...,A_m}$ on $\Big((\mathbb{R}^d)^m, \mathcal{B}(\mathbb{R}^d)^{\otimes m}\Big)$. Indeed, both the property

$$\rho_{A_1,...,A_m}\Big((\mathbb{R}^d)^m\Big) = 1$$

and the countable additivity of $\rho_{A_1,...,A_m}$ are evident and do not need further explanations. It is worth to mention that in some parts of the proof it will be most convenient to rewrite the definition of $\rho_{A_1,...,A_m}$ as

$$\begin{aligned}
\rho_{A_1,...,A_m}(D) &= \int\limits_{\mathbb{R}^d} \dots \int\limits_{\mathbb{R}^d} \varepsilon_{\left(\bigvee_{f}{}_{i \in a(1)} x_i, \dots, \bigvee_{f}{}_{i \in a(m)} x_i \right)}(D)\, \mu_{B_n}(dx_n) \dots \mu_{B_1}(dx_1) \\
&= \int\limits_{\mathbb{R}^d} \dots \int\limits_{\mathbb{R}^d} \left(\varepsilon_{\bigvee_{f}{}_{i \in a(1)} x_i} \otimes \dots \otimes \varepsilon_{\bigvee_{f}{}_{i \in a(m)} x_i} \right)(D)\, \mu_{B_n}(dx_n) \dots \mu_{B_1}(dx_1).
\end{aligned}$$

We refer to this particularly in Part 5. Further note that a combination of (1.1.2) and Corollary 3.1.6 (a) ensures that the variables $x_1, ..., x_n$, occurring in (3.1.17), commute under the in general non-commutative f-implicit max-operation. This becomes more evident in the subsequent deliberations. In addition, the latter observation justifies the above-mentioned expressions

$$\bigvee\nolimits_{f}{}_{i \in a(j)} x_i$$

for all $j = 1, ..., m$.

The next important step is to show that our definition of $\rho_{A_1,...,A_m}$ in (3.1.17) does not depend on the choice of a representation of the sets $A_1, ..., A_m$ in terms of disjoint sets $B_1, ..., B_n$, thus ensuring that the definition of $\rho_{A_1,...,A_m}$ in (3.1.17) is unambiguous. Suppose that

(A) $B_1, ..., B_{n_1} \in \mathcal{E}_0$ such that

$$A_j = \bigcup_{i \in a(j)} B_i$$

for all $j = 1, ..., m$ with suitable index sets $a(j) \subset \{1, ..., n_1\}$, $j = 1, ..., m$

and

(B) $\tilde{B}_1, ..., \tilde{B}_{n_2} \in \mathcal{E}_0$ such that

$$A_j = \bigcup_{\ell \in b(j)} \tilde{B}_\ell$$

for all $j = 1, ..., m$ with suitable index sets $b(j) \subset \{1, ..., n_2\}$, $j = 1, ..., m$

are two arbitrary representations of the sets $A_1, ..., A_m \in \mathcal{E}_0$ in terms of disjoint sets $B_1, ..., B_{n_1} \in \mathcal{E}_0$ and $\tilde{B}_1, ..., \tilde{B}_{n_2} \in \mathcal{E}_0$, respectively. In order to show that the definition of $\rho_{A_1,...,A_m}$ in (3.1.17) is well-defined, we have to prove that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \mathbb{1}_D \left(\bigvee_{f\, i\in a(1)} x_i, ..., \bigvee_{f\, i\in a(m)} x_i \right) \mu_{B_{n_1}}(dx_{n_1}) \dots \mu_{B_1}(dx_1) \\
&\quad = \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \mathbb{1}_D \left(\bigvee_{f\, \ell\in b(1)} x_\ell, ..., \bigvee_{f\, \ell\in b(m)} x_\ell \right) \mu_{\tilde{B}_{n_2}}(dx_{n_2}) \dots \mu_{\tilde{B}_1}(dx_1)
\end{aligned}$$

for all $D \in \mathcal{B}(\mathbb{R}^d)^{\otimes m}$. Since $\{D_1 \times \cdots \times D_m : D_i \in \mathcal{B}(\mathbb{R}^d), 1 \le i \le m\}$ is a π-system generating the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)^{\otimes m}$, the only point remaining is to check

$$\begin{aligned}
&\int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \mathbb{1}_{D_1\times\cdots\times D_m} \left(\bigvee_{f\, i\in a(1)} x_i, ..., \bigvee_{f\, i\in a(m)} x_i \right) \mu_{B_{n_1}}(dx_{n_1}) \dots \mu_{B_1}(dx_1) \\
&\quad = \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \mathbb{1}_{D_1\times\cdots\times D_m} \left(\bigvee_{f\, \ell\in b(1)} x_\ell, ..., \bigvee_{f\, \ell\in b(m)} x_\ell \right) \mu_{\tilde{B}_{n_2}}(dx_{n_2}) \dots \mu_{\tilde{B}_1}(dx_1),
\end{aligned}$$

that is,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \prod_{j=1}^m \mathbb{1}_{D_j} \left(\bigvee_{f\, i\in a(j)} x_i \right) \mu_{B_{n_1}}(dx_{n_1}) \dots \mu_{B_1}(dx_1) \\
&\quad = \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \prod_{j=1}^m \mathbb{1}_{D_j} \left(\bigvee_{f\, \ell\in b(j)} x_\ell \right) \mu_{\tilde{B}_{n_2}}(dx_{n_2}) \dots \mu_{\tilde{B}_1}(dx_1) \qquad (3.1.18)
\end{aligned}$$

for any choice of sets $D_1, ..., D_m \in \mathcal{B}(\mathbb{R}^d)$. To this end, we define the disjoint sets

$$C_{i,\ell} := B_i \cap \tilde{B}_\ell$$

for all $1 \le i \le n_1$ and $1 \le \ell \le n_2$. Observe that

$$B_i = \bigcup_{\ell=1}^{n_2} C_{i,\ell} \qquad \text{and} \qquad \tilde{B}_\ell = \bigcup_{i=1}^{n_1} C_{i,\ell}$$

for all $1 \le i \le n_1$ and $1 \le \ell \le n_2$, respectively, and therefore

$$A_j = \bigcup_{i\in a(j)} B_i = \bigcup_{i\in a(j)} \bigcup_{\ell=1}^{n_2} C_{i,\ell} = \bigcup_{\substack{i\in a(j) \\ \ell\in\{1,...,n_2\}}} C_{i,\ell} \qquad (3.1.19)$$

$$A_j = \bigcup_{\ell\in b(j)} \tilde{B}_\ell = \bigcup_{\ell\in b(j)} \bigcup_{i=1}^{n_1} C_{i,\ell} = \bigcup_{\substack{\ell\in b(j) \\ i\in\{1,...,n_1\}}} C_{i,\ell} \qquad (3.1.20)$$

for all j=1,...,m. Accordingly, we obtain another representation of the sets $A_1,...,A_m \in \mathcal{E}_0$ in terms of disjoint sets $C_{i,\ell} \in \mathcal{E}_0, 1 \leq i \leq n_1, 1 \leq \ell \leq n_2$. Here, (3.1.19) and (3.1.20) are just two different ways to write down the disjoint unions in terms of the sets $C_{i,\ell} \in \mathcal{E}_0, 1 \leq i \leq n_1, 1 \leq \ell \leq n_2$ which represent A_j for all $j = 1,...,m$. This is exactly the principal aspect enabling us to bridge the gap between the two prespecified choices of representations of the sets $A_1,...,A_m$ given in (A) and (B).
Fix arbitrary sets $D_1,...,D_m \in \mathcal{B}(\mathbb{R}^d)$. Applying the definition of $\rho_{A_1,...,A_m}$ given in (3.1.17) specifically to (3.1.19) and to (3.1.20), we get with $D = D_1 \times \cdots \times D_m$

$$\underbrace{\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d}}_{(n_1 \cdot n_2)\text{-times}} \prod_{j=1}^{m} \mathbb{1}_{D_j}\left(\bigvee_{i\in a(j)}{}_{f}\left(\bigvee_{\ell\in\{1,...,n_2\}}{}_{f} \; x_{i,\ell}\right)\right) \mu_{C_{n_1,n_2}}(dx_{n_1,n_2}) \ldots \mu_{C_{1,1}}(dx_{1,1}) \tag{I}$$

and

$$\underbrace{\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d}}_{(n_1 \cdot n_2)\text{-times}} \prod_{j=1}^{m} \mathbb{1}_{D_j}\left(\bigvee_{\ell\in b(j)}{}_{f}\left(\bigvee_{i\in\{1,...,n_1\}}{}_{f} \; x_{i,\ell}\right)\right) \mu_{C_{n_1,n_2}}(dx_{n_1,n_2}) \ldots \mu_{C_{1,1}}(dx_{1,1}), \tag{II}$$

respectively. Note, generally most of the sets $C_{i,\ell}$ are empty. For those sets, integrating with respect to $\mu_{C_{i,\ell}}$ means integrating with respect to ε_0 as $\mu_{C_{i,\ell}} = \mu_{m(C_{i,\ell})} = \mu_0 = \varepsilon_0$. Referring to (1.1.2), we are therefore actually dealing with considerably less than $n_1 \cdot n_2$ integrals in (I) and (II). This observation together with Corollary 3.1.6 (a) implies (I)=(II). Now, we show that

$$(\mathrm{I}) = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{j=1}^{m} \mathbb{1}_{D_j}\left(\bigvee_{i\in a(j)}{}_{f} \; x_i\right) \mu_{B_{n_1}}(dx_{n_1}) \ldots \mu_{B_1}(dx_1)$$

and

$$(\mathrm{II}) = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{j=1}^{m} \mathbb{1}_{D_j}\left(\bigvee_{\ell\in b(j)}{}_{f} \; x_\ell\right) \mu_{\tilde{B}_{n_2}}(dx_{n_2}) \ldots \mu_{\tilde{B}_1}(dx_1),$$

thus obtaining (3.1.18) and thereby the desired conclusion. To this end, we need only consider the first equality since the second can be handled in much the same way. From the mathematical point of view, we are dealing with just a few arguments in order to obtain the first equation. Apart from the common change of variables formula and equation (3.1.14), we benefit from only two more aspects. First, we gain from the already mentioned commutativity of the f-implicit max-operation which is valid in our concrete context and guarantees that the specific order in the expressions

$$\bigvee_{i\in a(j)}{}_{f}\left(\bigvee_{\ell\in\{1,...,n_2\}}{}_{f} \; x_{i,\ell}\right) \qquad j = 1,...,m$$

is negligible. Besides, this justifies other ways of notation such as

$$\bigvee_{\substack{i\in a(j)\\ \ell\in\{1,...,n_2\}}}{}_{f} \; x_{i,\ell}$$

for all $j = 1, ..., m$ disregarding the special order of the variables $x_{i,\ell}$. Second, we profit from both (A) and the construction of the sets $C_{i,\ell}, 1 \leq i \leq n_1, 1 \leq \ell \leq n_2$.
To be more accurate, we first collect all $a(j), j = 1, ..., m$, such that $1 \in a(j)$. In other words, we start to collect all sets A_j of our initial choice of not necessary disjoint sets $A_1, ..., A_m, m \geq 1$, which include the set B_1 in their disjoint partition. Taking into account the construction of the sets $C_{i,\ell}$, we obtain the crucial property that either all sets $C_{1,\ell} \neq \emptyset$ are part of the disjoint partition of A_j in terms of $C_{i,\ell}, 1 \leq i \leq n_1, 1 \leq \ell \leq n_2$, or none of them. Thus, the invariably applicable associativity of the f-implicit max-operation and the already justified aspect that we are de facto dealing with considerably less than $n_1 \cdot n_2$ integrals in (I) yield

$$
\begin{aligned}
(\mathrm{I}) &= \int\limits_{\mathbb{R}^d} \dots \int\limits_{\mathbb{R}^d} \prod_{j=1}^{m} \mathbb{1}_{D_j}\left(\vee_{f \atop i \in a(j)} \left(\vee_{f \atop \ell \in \{1,\dots,n_2\}} x_{i,\ell} \right)\right) \mu_{C_{n_1,n_2}}(dx_{n_1,n_2}) \dots \mu_{C_{1,1}}(dx_{1,1}) \\
&= \int\limits_{\mathbb{R}^d} \dots \int\limits_{\mathbb{R}^d} \prod_{j=1}^{m} \mathbb{1}_{D_j}\left(\vee_{f \atop {i \in a(j) \atop \ell \in \{1,\dots,n_2\}}} x_{i,\ell} \right) \mu_{C_{n_1,n_2}}(dx_{n_1,n_2}) \dots \mu_{C_{1,1}}(dx_{1,1}) \\
&= \int\limits_{\mathbb{R}^d} \dots \int\limits_{\mathbb{R}^d} \Bigg\{ \int\limits_{\mathbb{R}^d} \dots \int\limits_{\mathbb{R}^d} \prod_{j=1,\dots,m: \atop 1 \in a(j)} \mathbb{1}_{D_j}\left(\left[\vee_{f \atop \ell \in \{1,\dots,n_2\}} x_{1,\ell} \right] \vee_f \left[\vee_{f \atop {i \in a(j)\setminus\{1\} \atop \ell \in \{1,\dots,n_2\}}} x_{i,\ell} \right] \right) \\
&\qquad \cdot \prod_{j=1,\dots,m: \atop 1 \notin a(j)} \mathbb{1}_{D_j}\left(\vee_{f \atop {i \in a(j) \atop \ell \in \{1,\dots,n_2\}}} x_{i,\ell} \right) \mu_{C_{1,n_2}}(dx_{1,n_2}) \dots \mu_{C_{1,1}}(dx_{1,1}) \Bigg\} \prod_{i \geq 2 \atop \ell \in \{1,\dots,n_2\}} \mu_{C_{i,\ell}}(dx_{i,\ell}).
\end{aligned}
$$

Observe that we used the convenient notation

$$
\begin{aligned}
&\prod_{i \geq 2 \atop \ell \in \{1,\dots,n_2\}} \mu_{C_{i,\ell}}(dx_{i,\ell}) \\
&= \mu_{C_{n_1,n_2}}(dx_{n_1,n_2}) \dots \mu_{C_{n_1,1}}(dx_{n_1,1}) \mu_{C_{n_1-1,n_2}}(dx_{n_1-1,n_2}) \dots \mu_{C_{2,n_2}}(dx_{2,n_2}) \dots \mu_{C_{2,1}}(dx_{2,1}).
\end{aligned}
$$

Note further that we include all integrals in our notation, even those which can be neglected. Nevertheless, they have no effect, thus allowing us to apply Corollary 3.1.6 over and over again. To benefit from the change of variables formula, we recollect the mapping $T^{(n_2)} : (\mathbb{R}^d)^{n_2} \to \mathbb{R}^d$ defined by

$$
T^{(n_2)}(x_1, ..., x_{n_2}) := \bigvee_{\ell=1}^{n_2}{}_f \, x_\ell
$$

and proceed as follows

$$
(\mathrm{I}) = \int\limits_{\mathbb{R}^d} \dots \int\limits_{\mathbb{R}^d} \Bigg\{ \int\limits_{\mathbb{R}^d \times \dots \times \mathbb{R}^d} \prod_{j=1,\dots,m: \atop 1 \in a(j)} \mathbb{1}_{D_j}\left(T^{n_2}(x_{1,1}, ..., x_{1,n_2}) \vee_f \left[\vee_{f \atop {i \in a(j)\setminus\{1\} \atop \ell \in \{1,\dots,n_2\}}} x_{i,\ell} \right] \right)
$$

$$\cdot \prod_{\substack{j=1,\dots,m:\\ 1\notin a(j)}} \mathbb{1}_{D_j}\left(\bigvee_{\substack{i\in a(j)\\ \ell\in\{1,\dots,n_2\}}}{}_f \; x_{i,\ell}\right) \bigotimes_{\ell=1}^{n_2} \mu_{C_{1,\ell}}(dx_{1,1},\dots,dx_{1,n_2})\Bigg\} \prod_{\substack{i\geq 2\\ \ell\in\{1,\dots,n_2\}}} \mu_{C_{i,\ell}}(dx_{i,\ell})$$

$$= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \Bigg\{ \int_{\mathbb{R}^d} \prod_{\substack{j=1,\dots,m:\\ 1\in a(j)}} \mathbb{1}_{D_j}\left(z_1 \vee_f \left[\bigvee_{\substack{i\in a(j)\setminus\{1\}\\ \ell\in\{1,\dots,n_2\}}}{}_f \; x_{i,\ell}\right]\right)$$

$$\cdot \prod_{\substack{j=1,\dots,m:\\ 1\notin a(j)}} \mathbb{1}_{D_j}\left(\bigvee_{\substack{i\in a(j)\\ \ell\in\{1,\dots,n_2\}}}{}_f \; x_{i,\ell}\right) T^{n_2}\left(\bigotimes_{\ell=1}^{n_2} \mu_{C_{1,\ell}}\right)(dz_1)\Bigg\} \prod_{\substack{i\geq 2\\ \ell\in\{1,\dots,n_2\}}} \mu_{C_{i,\ell}}(dx_{i,\ell}).$$

Referring to (3.1.14), we may rewrite the latter expression as

$$= \int_{\mathbb{R}^d} \Bigg\{ \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{\substack{j=1,\dots,m:\\ 1\in a(j)}} \mathbb{1}_{D_j}\left(z_1 \vee_f \left[\bigvee_{\substack{i\in a(j)\setminus\{1\}\\ \ell\in\{1,\dots,n_2\}}}{}_f \; x_{i,\ell}\right]\right)$$

$$\cdot \prod_{\substack{j=1,\dots,m:\\ 1\notin a(j)}} \mathbb{1}_{D_j}\left(\bigvee_{\substack{i\in a(j)\\ \ell\in\{1,\dots,n_2\}}}{}_f \; x_{i,\ell}\right) \prod_{\substack{i\geq 2\\ \ell\in\{1,\dots,n_2\}}} \mu_{C_{i,\ell}}(dx_{i,\ell})\Bigg\} \mu_{B_1}(dz_1)$$

since

$$T^{n_2}\left(\bigotimes_{l=1}^{n_2} \mu_{C_{1,l}}\right) = \mu_{C_{1,1}} *_f \cdots *_f \mu_{C_{1,n_2}} = \mu_{\bigcup_{\ell=1}^{n_2} C_{1,\ell}} = \mu_{B_1}.$$

After this computation, we proceed in the same manner as before. That is, we continue to collect all $a(j)$, $j = 1,\dots,m$, such that $2 \in a(j)$. Using the same notation and similar arguments as in the preceding calculations, we get

$$(\mathrm{I}) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Bigg\{ \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{\substack{j=1,\dots,m:\\ 1,2\in a(j)}} \mathbb{1}_{D_j}\left(z_1 \vee_f z_2 \vee_f \left[\bigvee_{\substack{i\in a(j)\setminus\{1,2\}\\ \ell\in\{1,\dots,n_2\}}}{}_f \; x_{i,\ell}\right]\right)$$

$$\cdot \prod_{\substack{j=1,\dots,m:\\ 1\in a(j),2\notin a(j)}} \mathbb{1}_{D_j}\left(z_1 \vee_f \left[\bigvee_{\substack{i\in a(j)\setminus\{1\}\\ \ell\in\{1,\dots,n_2\}}}{}_f \; x_{i,\ell}\right]\right) \cdot \prod_{\substack{j=1,\dots,m:\\ 1\notin a(j),2\in a(j)}} \mathbb{1}_{D_j}\left(z_2 \vee_f \left[\bigvee_{\substack{i\in a(j)\setminus\{2\}\\ \ell\in\{1,\dots,n_2\}}}{}_f \; x_{i,\ell}\right]\right)$$

$$\cdot \prod_{\substack{j=1,\dots,m:\\ 1,2\notin a(j)}} \mathbb{1}_{D_j}\left(\bigvee_{\substack{i\in a(j)\\ \ell\in\{1,\dots,n_2\}}}{}_f \; x_{i,\ell}\right) \prod_{\substack{i\geq 3\\ \ell\in\{1,\dots,n_2\}}} \mu_{C_{i,\ell}}(dx_{i,\ell})\Bigg\} \mu_{B_2}(dz_2)\mu_{B_1}(dz_1).$$

At this point we do well to point out that some of the products occurring in the last expression might be empty. Then we adopt the common convention

$$\prod_{\emptyset} \equiv 1.$$

Iterating the previous procedure concerning the sets B_1, B_2 and omitting the details once again, we finally conclude that

$$(\mathrm{I}) = \underbrace{\int\limits_{\mathbb{R}^d} \dots \int\limits_{\mathbb{R}^d}}_{n_1\text{-times}} (*) \quad \prod_{i=1}^{n_1} \mu_{B_i}(dz_i),$$

where the integrand is given by

$$\begin{aligned}
(*) = &\left(\prod_{\substack{j=1,\dots,m: \\ 1,\dots,n_1 \in a(j)}} \mathbb{1}_{D_j} \left(\bigvee\nolimits_{f\, i=1,\dots n_1} z_i \right) \right) \cdot \left(\prod_{p=1}^{n_1} \prod_{\substack{j=1,\dots,m: \\ p \notin a(j) \\ \{1,\dots,n_1\} \setminus \{p\} \subset a(j)}} \mathbb{1}_{D_j} \left(\bigvee\nolimits_{f\, \substack{i=1,\dots,n_1: \\ i \neq p}} z_i \right) \right) \\
&\cdot \left(\prod_{\substack{p_1, p_2 \in \{1,\dots,n_1\} \\ p_1 \neq p_2}} \prod_{\substack{j=1,\dots,m: \\ p_1, p_2 \notin a(j) \\ \{1,\dots,n_1\} \setminus \{p_1, p_2\} \subset a(j)}} \mathbb{1}_{D_j} \left(\bigvee\nolimits_{f\, \substack{i=1,\dots,n_1: \\ i \notin \{p_1, p_2\}}} z_i \right) \right) \\
&\cdot \dots \\
&\cdot \left(\prod_{\substack{p_1,\dots,p_q \in \{1,\dots,n_1\} \\ p_r \neq p_k \, \forall \, r \neq k}} \prod_{\substack{j=1,\dots,m: \\ p_1,\dots,p_q \notin a(j) \\ \{1,\dots,n_1\} \setminus \{p_1,\dots,p_q\} \subset a(j)}} \mathbb{1}_{D_j} \left(\bigvee\nolimits_{f\, \substack{i=1,\dots,n_1: \\ i \notin \{p_1,\dots,p_q\}}} z_i \right) \right) \\
&\cdot \dots \\
&\cdot \left(\prod_{\substack{p_1,\dots,p_{n_1-1} \in \{1,\dots,n_1\} \\ p_r \neq p_k \, \forall \, r \neq k}} \prod_{\substack{j=1,\dots,m: \\ p_1,\dots,p_{n_1-1} \notin a(j) \\ \{1,\dots,n_1\} \setminus \{p_1,\dots,p_{n_1-1}\} \subset a(j)}} \mathbb{1}_{D_j} \left(\bigvee\nolimits_{f\, \substack{i=1,\dots,n_1: \\ i \notin \{p_1,\dots,p_{n_1-1}\}}} z_i \right) \right).
\end{aligned}$$

Fortunately, the last representation of (*) is anything but complicated. Most of the products are empty and hence equal to one. The only non-trivial product remaining is actually

$$\prod_{j=1}^{m} \mathbb{1}_{D_j}\left(\bigvee_{i\in a(j)}{}_f \, x_i\right)$$

being an easy consequence of the interaction of the definitions and constructions concerning the sets $A_1, ..., A_m$, the sets $B_1, ..., B_n$ and the set $C_{i,\ell}, 1 \leq i \leq n_1, 1 \leq \ell \leq n_2$. To apprehend this conclusion, it is worthwhile to study an easier situation first. Indeed, it is extremely recommendable to assure oneself of the latter conclusion by studying an elucidatory example providing an idea of how the general case works. Summarizing all aspects, we have

$$(\mathrm{I}) = \int\limits_{\mathbb{R}^d} \dots \int\limits_{\mathbb{R}^d} \prod_{j=1}^{m} \mathbb{1}_{D_j}\left(\bigvee_{i\in a(j)}{}_f \, x_i\right) \mu_{B_{n_1}}(dx_{n_1}) \dots \mu_{B_1}(dx_1).$$

Concluding the observations of Part 1, we obtain the family

$$\left\{\rho_{A_1,...,A_m} \in M^1\left((\mathbb{R}^d)^m\right) : \, A_1, ..., A_m \in \mathcal{E}_0, m \geq 1\right\}$$

of well-defined probability measures.

Part 2: (Consistency)
In this part of the proof we show that the previously constructed family

$$\left\{\rho_{A_1,...,A_m} \in M^1\left((\mathbb{R}^d)^m\right) : \, A_1, ..., A_m \in \mathcal{E}_0, m \geq 1\right\}$$

of well-defined probability measures is consistent. Referring to [Bi95, Theorem 36.2], we need to prove that this family satisfies the two consistency conditions:

(i) For all $m \geq 1$, any collection of sets $A_1, ..., A_m \in \mathcal{E}_0$, any choice of sets $D_1, ..., D_m \in \mathcal{B}(\mathbb{R}^d)$ and any permutation π of $\{1, ..., m\}$, we have

$$\rho_{A_1,...,A_m}(D_1 \times \dots \times D_m) = \rho_{A_{\pi(1)},...,A_{\pi(m)}}(D_{\pi(1)} \times \dots \times D_{\pi(m)}). \tag{3.1.21.a}$$

(ii) For all $m \geq 1$, any collection of sets $A_1, ..., A_m \in \mathcal{E}_0$ and any choice of sets $D_1, ..., D_m \in \mathcal{B}(\mathbb{R}^d)$, we have

$$\rho_{A_1,...,A_m}(D_1 \times \dots \times D_{m-1} \times \mathbb{R}^d) = \rho_{A_1,...,A_{m-1}}(D_1 \times \dots \times D_{m-1}). \tag{3.1.21.b}$$

To this end, we first remark that condition (3.1.21.a) is already clear because of Part 1. It follows immediately from the definition of $\rho_{A_1,...A_m}$ and the fact that this definition is independent of the specific representation of the sets $A_1, ..., A_m$ in terms of appropriate disjoint sets. Similar considerations as to (3.1.21.a) can be found in the proof of Proposition 2.1 in [StTa05] and can therefore be skipped.
In order to prove (3.1.21.b), we start by fixing $m \geq 1$, a collection of sets $A_1, ..., A_m \in \mathcal{E}_0$ and a collection of sets $D_1, ..., D_m \in \mathcal{B}(\mathbb{R}^d)$. Moreover, let $B_1, ..., B_n \in \mathcal{E}_0$ be disjoint sets constituting an appropriate representation of the sets $A_1, ..., A_m$, that is to say,

$$A_j = \bigcup_{i\in a(j)} B_i$$

for all $j = 1, ..., m$ and for suitable index sets $a(j)$. Following the definition of $\rho_{A_1,...,A_m}$ in (3.1.17), we obtain

$$\begin{aligned}
&\rho_{A_1,...,A_m}(D_1 \times \cdots \times D_{m-1} \times \mathbb{R}^d) \\
&= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \underbrace{\mathbb{1}_{\mathbb{R}^d}\left(\bigvee_{i \in a(m)}{}_f \, x_i\right)}_{=1} \cdot \prod_{j=1}^{m-1} \mathbb{1}_{D_j}\left(\bigvee_{i \in a(j)}{}_f \, x_i\right) \mu_{B_n}(dx_n) \ldots \mu_{B_1}(dx_1) \\
&= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{j=1}^{m-1} \mathbb{1}_{D_j}\left(\bigvee_{i \in a(j)}{}_f \, x_i\right) \mu_{B_n}(dx_n) \ldots \mu_{B_1}(dx_1) \\
&= \rho_{A_1,...,A_{m-1}}(D_1 \times \cdots \times D_{m-1}),
\end{aligned}$$

which already completes the proof of the second consistency condition (3.1.21.b). The last equality in the previous calculation follows immediately from the already proven fact that $\rho_{A_1,...,A_{m-1}}$ is independent of the specific representation of the sets $A_1, ..., A_{m-1}$ in terms of appropriate disjoint sets. Indeed, since the sets $B_1, ..., B_n$ constitute a representation of the sets $A_1, ..., A_m$, they also constitute an appropriate representation of the sets $A_1, ..., A_{m-1}$.
Kolmogorov's extension theorem now guarantees the existence of a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ together with a stochastic process $(M(A))_{A \in \mathcal{E}_0}$ on $(\Omega, \mathfrak{A}, \mathbb{P})$ such that

$$\mathbb{P}_{(M(A_1),...,M(A_m))} = \rho_{A_1,...,A_m}$$

for any $m \geq 1$ and any choice of sets $A_1, ..., A_m \in \mathcal{E}_0$. This process can actually be viewed as a mapping

$$M : \mathcal{E}_0 \to L_0^d.$$

The remainder of the proof is intended to verify that $(M(A))_{A \in \mathcal{E}_0}$ fulfills the properties (i), (ii) and (iii) in Definition 3.1.8, thus completing the proof of Theorem 3.1.12.

Part 3: (independently scattered)
For any $n \geq 1$ and any collection of disjoint sets $A_1, ..., A_n \in \mathcal{E}_0$ we have to show that the corresponding random vectors $M(A_1), ..., M(A_n)$ are independent. To this end, fix $n \geq 1$ and a collection of disjoint sets $A_1, ..., A_n$. Then we obtain the desired independency of the random vectors $M(A_1), ..., M(A_n)$ by proving

$$\mathbb{P}_{(M(A_1),...,M(A_n))} = \bigotimes_{i=1}^{n} \mathbb{P}_{M(A_i)}. \tag{3.1.22}$$

Here, we substantially benefit from Part 1 and from the definition of $\rho_{A_1,...A_n}$. To be more precise, we point out that the sets $A_1, ..., A_n$ are for their part already disjoint and hence a suitable choice for a representation in terms of disjoint sets such as described in Part 1. That is,

$$B_j = A_j \qquad \text{and} \qquad a(j) = \{j\}$$

for all $j = 1, ..., n$. Thus, applying the definition in (3.1.17) to this simple situation, we conclude that

$$\begin{aligned}
\mathbb{P}_{(M(A_1),...,M(A_n))}(D_1 \times \cdots \times D_n) &= \rho_{A_1,...A_n}(D_1 \times \cdots \times D_n) \\
&= \int\limits_{\mathbb{R}^d} \cdots \int\limits_{\mathbb{R}^d} \prod_{j=1}^{n} \mathbb{1}_{D_j}\left(x_j\right) \mu_{A_n}(dx_n) \ldots \mu_{A_1}(dx_1) \\
&= \int\limits_{\mathbb{R}^d \times \cdots \times \mathbb{R}^d} \mathbb{1}_{D_1 \times \cdots \times D_n}(x_1, ..., x_n) \bigotimes_{j=1}^{n} \mu_{A_j}(dx_1, ..., dx_n) \\
&= \bigotimes_{j=1}^{n} \mu_{A_j}(D_1 \times \cdots \times D_n) \\
&= \bigotimes_{j=1}^{n} \mathbb{P}_{M(A_j)}(D_1 \times \cdots \times D_n)
\end{aligned}$$

with $D_1, ..., D_n \in \mathcal{B}(\mathbb{R}^d)$. The last equality is due to (3.1.12), which will subsequently be established in Part 4. Since $\{D_1 \times \cdots \times D_n : D_1, ..., D_n \in \mathcal{B}(\mathbb{R}^d)\}$ is a π-system generating the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)^{\otimes n} = \mathcal{B}(\mathbb{R}^d \times \cdots \times \mathbb{R}^d)$, (3.1.22) follows.

Part 4: (f-implicit α-Fréchet)
Property (3.1.12) is an easy consequence of the fact that we have $\mathbb{P}_{M(A)} = \rho_A$ for all $A \in \mathcal{E}_0$. Indeed, the simple choice $B = A$ being a suitable representation of A in terms of disjoint sets yields immediately

$$\mathbb{P}_{M(A)}(D) = \rho_A(D) = \int\limits_{\mathbb{R}^d} \mathbb{1}_D(x)\, \mu_A(dx) = \mu_A(D)$$

for all $D \in \mathcal{B}(\mathbb{R}^d)$. This is precisely property (3.1.12).

Part 5: (f-max σ-sup-additive)
Undoubtedly, the proof of property (3.1.13) requires considerable efforts. Thus, we start to outline the further course of action. First, we prove that

$$M(A_1 \cup A_2) = M(A_1) \vee_f M(A_2) \tag{3.1.23}$$

almost surely for any collection of two disjoint sets $A_1, A_2 \in \mathcal{E}_0$ and consequently, by induction, that

$$M\left(\bigcup_{j=1}^{n} A_j\right) = \bigvee_{j=1}^{n}{}_f\, M(A_j) \tag{3.1.24}$$

almost surely for any $n \geq 1$ and any collection of disjoint sets $A_1, ..., A_n \in \mathcal{E}_0$. To this end, we need only show that

$$\mathbb{P}(M(A_1 \cup A_2) \in D) = \mathbb{P}(M(A_1) \vee_f M(A_2) \in D)$$

$$= \mathbb{P}(M(A_1 \cup A_2) \in D, M(A_1) \vee_f M(A_2) \in D) \tag{3.1.25}$$

for all $D \in \mathcal{B}(\mathbb{R}^d)$ and arbitrary disjoint sets $A_1, A_2 \in \mathcal{E}_0$ as shown in Lemma 3.1.13. Having established (3.1.23) and as a consequence thereof (3.1.24), we secondly deduce (3.1.13) from this property by applying the considerations of Section 1.1 and 1.3.
Now, we begin to prove (3.1.23) by verifying (3.1.25). Let A_1, A_2 denote arbitrary disjoint subsets of $\mathcal{E}_0$. The first equality of (3.1.25) is an easy consequence of Part 3 and 4. Indeed, $M(A_1), M(A_2)$ being independent and having the distinctive f-implicit α-Fréchet distributions μ_{A_1} and μ_{A_2}, respectively, gives

$$\mathbb{P}_{M(A_1)\vee_f M(A_2)} = \mathbb{P}_{M(A_1)} *_f \mathbb{P}_{M(A_2)} = \mu_{A_1} *_f \mu_{A_2} = \mu_{A_1 \cup A_2} = \mathbb{P}_{M(A_1 \cup A_2)}.$$

Note that the first equality is due to Lemma 1.2.4 (b), the third due to (3.1.14) and the last once again due to Part 4. Thus, the proof of (3.1.25) is completed by showing that

$$\mathbb{P}(M(A_1) \vee_f M(A_2) \in D) = \mathbb{P}(M(A_1 \cup A_2) \in D, M(A_1) \vee_f M(A_2) \in D)$$

for all $D \in \mathcal{B}(\mathbb{R}^d)$. For this purpose, it turns out to be helpful to evaluate the measure

$$\mathbb{P}_{(M(A_1), M(A_2), M(A_1 \cup A_2))} = \rho_{A_1, A_2, A_1 \cup A_2}.$$

Applying $B_1 = A_1$ and $B_2 = A_2$ as the most reasonable representation of the sets $A_1, A_2, A_1 \cup A_2$, we obtain

$$\rho_{A_1, A_2, A_1 \cup A_2}(\tilde{D}) = \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \mathbb{1}_{\tilde{D}} \left(x_1, x_2, x_1 \vee_f x_2 \right) \mu_{A_2}(dx_2)\, \mu_{A_1}(dx_1)$$

for $\tilde{D} \in \mathcal{B}(\mathbb{R}^d)^{\otimes 3}$. This is due to the definition of $\rho_{A_1, A_2, A_1 \cup A_2}$ and its independency of the particular representation of $A_1, A_2, A_1 \cup A_2$ in terms of disjoint sets. The fact that the latter equation can be rewritten as

$$\rho_{A_1, A_2, A_1 \cup A_2}(\tilde{D}) = \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \left(\varepsilon_{x_1} \otimes \varepsilon_{x_2} \otimes \varepsilon_{x_1 \vee_f x_2} \right) (\tilde{D})\, \mu_{A_2}(dx_2)\, \mu_{A_1}(dx_1)$$

implies instantly

$$\begin{aligned}
&\mathbb{P}(M(A_1 \cup A_2) \in D, M(A_1) \vee_f M(A_2) \in D) \\
&\quad = \mathbb{P}_{(M(A_1), M(A_2), M(A_1 \cup A_2))}(\{(x, y, z) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d : x \vee_f y \in D, z \in D\}) \\
&\quad = \rho_{A_1, A_2, A_1 \cup A_2}(\{(x, y, z) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d : x \vee_f y \in D, z \in D\}) \\
&\quad = \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \left(\varepsilon_{x_1} \otimes \varepsilon_{x_2} \otimes \varepsilon_{x_1 \vee_f x_2} \right) \left(\left(T^{(2)}\right)^{-1} (D) \times D \right) \mu_{A_2}(dx_2)\, \mu_{A_1}(dx_1) \\
&\quad = \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \left(\varepsilon_{x_1} \otimes \varepsilon_{x_2} \right) \left(\left(T^{(2)}\right)^{-1} (D) \right) \cdot \varepsilon_{x_1 \vee_f x_2}(D)\, \mu_{A_2}(dx_2)\, \mu_{A_1}(dx_1)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{1}_D\left(x_1 \vee_f x_2\right) \cdot \mathbb{1}_D\left(x_1 \vee_f x_2\right) \mu_{A_2}(dx_2)\, \mu_{A_1}(dx_1) \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{1}_D\left(x_1 \vee_f x_2\right) \mu_{A_2}(dx_2)\, \mu_{A_1}(dx_1) \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{1}_{\left(T^{(2)}\right)^{-1}(D)}\,(x_1, x_2)\, \mu_{A_2}(dx_2)\, \mu_{A_1}(dx_1) \\
&= \rho_{A_1,A_2}\left(\left(T^{(2)}\right)^{-1}(D)\right) \\
&= \mathbb{P}_{(M(A_1),M(A_2))}\left(\left(T^{(2)}\right)^{-1}(D)\right) \\
&= \mathbb{P}(M(A_1) \vee_f M(A_2) \in D)
\end{aligned}
$$

for all $D \in \mathcal{B}(\mathbb{R}^d)$. Note that $T^{(2)} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is defined as usual. Observe further that we benefited from the definition of ρ_{A_1,A_2} and its independency of the particular representation of A_1, A_2 in terms of disjoint sets. In conclusion, the proof of (3.1.25) and hence of (3.1.23) is completed.

Now, we proceed to prove (3.1.13). Accordingly, we have to show, for any collection $A_n, n \geq 1$, of disjoint subsets of $\mathcal{E}_0$ with $\bigcup_{n\geq 1} A_n \in \mathcal{E}_0$, that

$$
M\left(\bigcup_{n\geq 1} A_n\right) = \bigvee_{n=1}^{\infty}{}_f\, M(A_n) = M\left(A_{n_0}\right)
$$

almost surely, where n_0 is a random index. To this end, we shall also refer to the considerations of Remark 3.1.10 and Lemma 3.1.11. To start with, fix any collection $A_n, n \geq 1$, of disjoint subsets of $\mathcal{E}_0$ with $A := \bigcup_{n\geq 1} A_n \in \mathcal{E}_0$. If $m(A_n) = 0$ for all $n \geq 1$, we have $M(A_n) = 0$ almost surely for all $n \geq 1$ as well as $M(A) = 0$ almost surely. This observation and the fact that countable unions of null sets are null sets yield

$$
M\left(\bigcup_{n\geq 1} A_n\right) = 0 = M(A_1) = \bigvee_{n=1}^{\infty}{}_f\, M(A_n)
$$

almost surely and as a result the desired conclusion. Therefore, there is no loss of generality in assuming that there exists an $n_1 \geq 1$ with $m(A_{n_1}) > 0$. Fix $F_1, F_2 \in \mathcal{E}_0$ such that $F_1 \subset F_2$. Since $F_2 = F_1 \cup (F_2 \setminus F_1)$, (1.3.3) and (3.1.23) give

$$
M(F_1) \leq_f M(F_1) \vee_f M(F_2 \setminus F_1) = M(F_1 \cup (F_2 \setminus F_1)) = M(F_2)
$$

almost surely. Consequently, we have proved that

$$
\forall\, F_1, F_2 \in \mathcal{E}_0 \text{ such that } F_1 \subset F_2 \quad \Rightarrow \quad M(F_1) \leq_f M(F_2) \text{ almost surely.} \tag{3.1.26}
$$

To complete the proof of Part 5, we now establish diverse properties resulting from (3.1.23), (3.1.24) and (3.1.26). Applying (3.1.23), we obtain

$$
M(A) = M\left(\left(\bigcup_{j=1}^{n} A_j\right) \cup \left(\bigcup_{j=n+1}^{\infty} A_j\right)\right) = M\left(\bigcup_{j=1}^{n} A_j\right) \vee_f M\left(\bigcup_{j=n+1}^{\infty} A_j\right) := \xi_n \vee_f \eta_n
$$

almost surely for all $n \geq 1$. From (3.1.26) it may further be concluded that

$$\xi_n = M\left(\bigcup_{j=1}^{n} A_j\right) \leq_f M\left(\bigcup_{j=1}^{n+1} A_j\right) = \xi_{n+1}$$

and

$$\eta_{n+1} = M\left(\bigcup_{j=n+2}^{\infty} A_j\right) \leq_f M\left(\bigcup_{j=n+1}^{\infty} A_j\right) = \eta_n$$

almost surely for all $n \geq 1$. In addition to this, we have

$$\xi_n = M\left(\bigcup_{j=1}^{n} A_j\right) = \bigvee_{j=1}^{n}{}_f\, M(A_j)$$

almost surely for all $n \geq 1$ being due to (3.1.24). By assumption, we can finally also find an $n_1 \geq 1$ with $m(A_{n_1}) > 0$. Since

$$M(A_{n_1}) \overset{d}{=} m(A_{n_1})^{\frac{1}{\alpha}} Z\Theta$$

with Z and Θ defined as in Definition 3.1.2, Remark 3.1.3 (ii) implies

$$\mathbb{P}\left(M(A_{n_1}) = 0\right) = 0.$$

Accordingly, we have $M(A_{n_1}) \neq 0$ almost surely. Summarizing the previous observations and taking into account that countable unions of null sets are null sets, we conclude that there exists a null set $N_1 \in \mathfrak{A}$ such that for every ω in $\Omega \setminus N_1$

(1.) $M(A)(\omega) = \xi_n(\omega) \vee_f \eta_n(\omega) \qquad \forall\, n \geq 1$ (3.1.27)

(2.) $(\xi_n(\omega))_{n\geq 1}$ is $\leq_f$-increasing (3.1.28)

(3.) $(\eta_n(\omega))_{n\geq 1}$ is $\leq_f$-decreasing (3.1.29)

(4.) $\xi_n(\omega) = \bigvee_{j=1}^{n}{}_f\, M(A_j)(\omega) \qquad \forall\, n \geq 1$ (3.1.30)

(5.) $M(A_{n_1})(\omega) \neq 0$ (3.1.31)

(6.) $f\left(M(A_n)(\omega)\right) \xrightarrow[(n\to\infty)]{} 0.$ (3.1.32)

Here, (3.1.32) is nothing but the assertion of Lemma 3.1.11. This list of properties finally enables us to complete the proof of (3.1.13). Indeed, fix $\omega \in \Omega \setminus N_1$. For convenience, we both revert to the previous notation excluding the letter ω and refrain from including the term *almost surely* into our argumentation. From (3.1.28) and Lemma 1.3.5 it follows that $(f(\xi_n))_{n\geq 1}$ is an increasing sequence of non-negative real numbers. By applying f to both sides of (3.1.30), we further get

$$f(\xi_n) = \max\Big(f\left(M(A_1)\right), \ldots, f\left(M(A_n)\right)\Big) \tag{3.1.33}$$

for all $n \geq 1$ showing again that $(f(\xi_n))_{n\geq 1}$ is increasing. Moreover, (3.1.31) yields the existence of a positive real number δ with

$$f(M(A_{n_1})) = \delta > 0,$$

where δ actually depends on $\omega \in \Omega \setminus N_1$. Taking into account (3.1.32), we eventually obtain the existence of an integer $N \geq n_1$ depending on $\delta > 0$ such that for all $n > N$ we have

$$f(M(A_n)) < \delta.$$

As a result, we conclude that

$$\begin{aligned} f(\xi_N) &= \max\Big(f(M(A_1)), ..., f(M(A_N))\Big) \\ &= \max\Big(f(M(A_1)), ..., f(M(A_N)), f(M(A_{N+1})), ..., f(M(A_{N+\ell}))\Big) \\ &= f(\xi_{N+\ell}) \end{aligned} \tag{3.1.34}$$

for any $\ell \geq 1$, the first and third equality being consequences of (3.1.33). Consequently, the fact that $(\xi_n)_{n\geq 1}$ is $\leq_f$-increasing gives

$$\xi_N = \xi_{N+\ell}$$

for all $\ell \geq 1$, for if not, we would have $f(\xi_N) < f(\xi_{N+l})$ for at least one $\ell \geq 1$ contradicting the latter observation. Including the well-defined integer

$$n_0 := \min\Big\{i \in \{1, ..., N\} : \ f(\xi_i) = \max\Big(f(\xi_1), ..., f(\xi_N)\Big)\Big\} \tag{3.1.35}$$

into our considerations and applying the fact that $(f(\xi_n))_{n\geq 1}$ is an increasing sequence, we deduce that

$$f(\xi_1) \leq f(\xi_2) \leq ... \leq f(\xi_{n_0-1}) < f(\xi_{n_0}) = f(\xi_{n_0+1}) = ... = f(\xi_N). \tag{3.1.36}$$

In the event of $n_0 = N$, it is self-evident that (3.1.36) will henceforth be understood as

$$f(\xi_1) \leq f(\xi_2) \leq ... \leq f(\xi_{N-1}) < f(\xi_N).$$

Note that $f(\xi_{n_0}), ..., f(\xi_N)$ being equal follows from $f(\xi_{n_0}) \geq f(\xi_{n_0+\ell})$ for all $\ell \in \{1, ..., N - n_0\}$ and $f(\xi_{n_0}) \leq f(\xi_{n_0+\ell})$ for all $\ell \in \{1, ..., N - n_0\}$. The first assertion is an immediate consequence of (3.1.35), whereas the second one is due to the fact that $(f(\xi_n))_{n\geq 1}$ is an increasing sequence. Taking into account that $\xi_N = \xi_{N+\ell}$ for all $\ell \geq 1$ and applying the fact that $(\xi_n)_{n\geq 1}$ is $\leq_f$-increasing, we get

$$\xi_{n_0} = \xi_{n_0+\ell}$$

for all $\ell \geq 1$. In other words, the sequence $(\xi_n)_{n\geq 1}$ is constant for almost all $n \geq 1$ and hence convergent. In fact, we obtain

$$\lim_{n\to\infty} \xi_n = \xi_{n_0}$$

and may therefore conclude that

$$\xi_n \xrightarrow[(n\to\infty)]{} \xi_{n_0} = \bigvee_{j=1}^{n_0}{}_f\, M(A_j) = M(A_{n_0}) = \bigvee_{j=1}^{\infty}{}_f\, M(A_j) \tag{3.1.37}$$

by first applying (3.1.30), then (3.1.35), and finally (3.1.34) together with (3.1.36). Having established (3.1.37), we proceed to focus on the sequences $(\eta_n)_{n\geq 1}$. Assertion (3.1.29) shows that $(\eta_n)_{n\geq 1}$ is $\leq_f$-decreasing. Thus, by applying Lemma 1.3.5, $(f(\eta_n))_{n\geq 1}$ is a decreasing sequence of non-negative real numbers. Since $(f(\eta_n))_{n\geq 1}$ is in addition bounded below by zero, we deduce that $(f(\eta_n))_{n\geq 1}$ converges. Accordingly, there must exist a non-negative real number η such that

$$f(\eta_n) \xrightarrow[(n\to\infty)]{} \eta.$$

Now, before proceeding with the proof of (3.1.13), we do well to summarize our preliminary findings, thus making our deliberations clearer. Up to this point, we have proved the existence of a null set $N_1 \subset \Omega$ such that for all $\omega \in \Omega \setminus N_1$

$$M\left(\bigcup_{j\geq 1} A_j\right)(\omega) = \xi_n(\omega) \vee_f \eta_n(\omega)$$

for all $n \geq 1$,

$$\xi_n(\omega) \xrightarrow[(n\to\infty)]{} M(A_{n_0(\omega)})(\omega) = \bigvee_{j=1}^{\infty}{}_f\, M(A_j)(\omega)$$

and finally also

$$f(\eta_n(\omega)) \xrightarrow[(n\to\infty)]{} \eta(\omega).$$

In order to complete the proof of (3.1.13), we are consequently left with the task of proving that

$$\eta(\omega) = 0 \tag{3.1.38}$$

for almost all $\omega \in \Omega \setminus N_1$. In fact, provided (3.1.38) were true, Lemma 3.1.14 would yield

$$\eta_n(\omega) \xrightarrow[(n\to\infty)]{} 0$$

for almost all $\omega \in \Omega \setminus N_1$. Referring to Lemma 1.1.9, we could eventually conclude that

$$M\left(\bigcup_{j\geq 1} A_j\right)(\omega) = \lim_{n\to\infty} \xi_n(\omega) \vee_f \eta_n(\omega) = M(A_{n_0(\omega)})(\omega) = \bigvee_{j=1}^{\infty}{}_f\, M(A_j)(\omega)$$

for almost all $\omega \in \Omega \setminus N_1$, and (3.1.13) would be proved. In order to show (3.1.38), we proceed as follows. First note that the assumption $\bigcup_{n\geq 1} A_n \in \mathcal{E}_0$ implies in particular $\bigcup_{j\geq n+1} A_j \in \mathcal{E}_0$ for all $n \geq 1$. Applying Part 4, we hence have

$$\eta_n \stackrel{d}{=} m\left(\bigcup_{j=n+1}^{\infty} A_j\right)^{\frac{1}{\alpha}} Z\Theta$$

for all $n \geq 1$, the random variable Z and the random vector Θ being defined as in Definition 3.1.2. Thus, the 1-homogeneity of f gives

$$f(\eta_n) = f\left(M\left(\bigcup_{j=n+1}^{\infty} A_j\right)\right) \overset{d}{=} f\left(m\left(\bigcup_{j=n+1}^{\infty} A_j\right)^{\frac{1}{\alpha}} Z\Theta\right) = m\left(\bigcup_{j=n+1}^{\infty} A_j\right)^{\frac{1}{\alpha}} Z.$$

Since

$$m\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n=1}^{\infty} m(A_n) < \infty,$$

we obtain

$$\sum_{j=n+1}^{\infty} m(A_j) \xrightarrow[(n\to\infty)]{} 0.$$

Combining these recent findings, we conclude that

$$\mathbb{P}(f(\eta_n) \leq x) \xrightarrow[(n\to\infty)]{} 1$$

for all $x > 0$, that is,

$$\forall\, x > 0\, \forall\, \varepsilon > 0\, \exists\, N_0(x,\varepsilon) := N_0 \geq 1\, \forall\, n \geq N_0 :\ \mathbb{P}(f(\eta_n) \leq x) > 1 - \varepsilon. \tag{3.1.39}$$

Either (3.1.39) follows from the evident fact that

$$\mathbb{P}(f(\eta_n) \leq x) = 1$$

for almost all $n \geq 1$, provided there exists an $\tilde{n} \geq 1$ such that $m\left(\bigcup_{j\geq \tilde{n}+1} A_j\right) = 0$, or it follows from

$$\mathbb{P}(f(\eta_n) \leq x) = \exp\left(-m\left(\bigcup_{j=n+1}^{\infty} A_j\right) x^{-\alpha}\right) \xrightarrow[(n\to\infty)]{} 1.$$

Referring to (3.1.39), we can finally deduce (3.1.38) by contradiction. To this end, let $N^* := \{\eta > 0\} \subset \Omega \setminus N_1$. Suppose, contrary to (3.1.38), that $\mathbb{P}(N^*) := p > 0$. Since N^* may be written as

$$N^* = \bigcup_{n\geq 1} \left\{\eta > \frac{1}{n}\right\},$$

$\{\eta > \frac{1}{n}\}$ here being a subset of $\Omega \setminus N_1$, continuity from below yields

$$\lim_{n\to\infty} \mathbb{P}\left(\eta > \frac{1}{n}\right) = \mathbb{P}(N^*) = p > 0.$$

Consequently, there must exist an integer $\tilde{N}_0 \geq 1$ depending on $\frac{p}{2} > 0$ such that

$$\mathbb{P}\left(\eta > \frac{1}{n}\right) \geq \frac{p}{2} \tag{3.1.40}$$

for all $n \geq \tilde{N}_0$. Evaluating (3.1.39) specifically for $x = \frac{1}{\tilde{N}_0} > 0$ and $\varepsilon = \frac{p}{2} > 0$, we further obtain

$$\mathbb{P}\left(f(\eta_n) \leq \frac{1}{\tilde{N}_0}\right) > 1 - \frac{p}{2}$$

for all $n \geq N_0\left(\frac{1}{\tilde{N}_0}, \frac{p}{2}\right)$. Since $(f(\eta_n))_{n\geq 1}$ is decreasing, we have

$$\left\{f(\eta_n) \leq \frac{1}{\tilde{N}_0}\right\} \subset \left\{\eta \leq \frac{1}{\tilde{N}_0}\right\}$$

for all $n \geq 1$ and hence particularly

$$\mathbb{P}\left(\eta \leq \frac{1}{\tilde{N}_0}\right) > 1 - \frac{p}{2}.$$

This, however, contradicts the claim (3.1.40) that was deduced from the assumption $\mathbb{P}(N^*) := p > 0$, and the proof is complete. □

In order to illustrate the strength and benefit of the notion of f-implicit sup-measures, we proceed with a first insightful example.

Example 3.1.15
Let the measure space $(E, \mathcal{E}, m)$ be specifically chosen as $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \lambda^1)$. Then there exists an f-implicit sup-measure $(M(A))_{A\in\mathcal{E}_0}$ with control measure m defined on an appropriate probability space, where

$$\mathcal{E}_0 = \{B \in \mathcal{B}(\mathbb{R}_+) : \lambda^1(B) < \infty\}.$$

This f-implicit sup-measure actually provides a non-trivial example of an f-implicit max-stable process $\mathbb{X} := (X_t)_{t\geq 0}$ by defining

$$X_t := M([0, t])$$

for all $t \geq 0$. The detailed verification of this claim is skipped for the moment and postponed to Section 3.2, thus connecting the two main concepts of Chapter 3.

Having proved the existence of an f-implicit sup-measure, we proceed with an attractive consequence. Actually, Theorem 3.1.12 guarantees the existence of a random α-Fréchet sup-measure M_α with control measure m according to Definition 2.1 in [StTa05] as an easy byproduct. This indicates that the field of f-implicit extreme value extending parts of the ideas given in [SchSt14] and [StTa05] is worth to be considered. In Section 3.3 we will expand on this by suggesting and motivating further ideas for additional research work concerning f-implicit sup-measures and f-implicit extremal integrals.
Although the concept, and particularly the existence, of a random α-Fréchet sup-measure M_α with control measure m has extensively been studied in [StTa05], we ought to include the next corollary here as it reveals the strength and power of the notion of f-implicit sup-measures. However, it has to be emphasized that the subsequent claim is already well known in literature.

Corollary 3.1.16 ([StTa05, Proposition 2.1])
Let $(E, \mathcal{E}, m)$ be an arbitrary measure space and $\mathcal{E}_0$ as before. For any $\alpha > 0$, there exists a random α-Fréchet sup-measure M_α with control measure m defined over an appropriate probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. In other words, there exists a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ and a mapping

$$M_\alpha : \mathcal{E}_0 \to L_0 := L_0(\Omega, \mathfrak{A}) := \{X : \Omega \to \mathbb{R} \,|\, X \text{ is a random variable}\}$$

such that the following properties are fulfilled:

(i) (*independently scattered*)
For any collection of disjoint sets $A_1, ..., A_n \in \mathcal{E}_0, n \geq 1$, the random variables $M_\alpha(A_1), ..., M_\alpha(A_n)$ are independent.

(ii) (*α-Fréchet*)
For every $A \in \mathcal{E}_0$, we have

$$\mathbb{P}(M_\alpha(A) \leq x) = \begin{cases} \exp\left(-m(A)x^{-\alpha}\right), & x > 0 \\ 0, & x \leq 0. \end{cases}$$

(iii) (*σ-sup-additive*)
For any collection of disjoint sets $A_n \in \mathcal{E}_0, n \geq 1$, such that $\bigcup_{n\geq 1} A_n \in \mathcal{E}_0$, we have

$$M_\alpha\left(\bigcup_{n\geq 1} A_n\right) = \bigvee_{n\geq 1} M_\alpha(A_n)$$

almost surely.

Proof. Let $\alpha > 0$ be fixed. Now, Theorem 3.1.12 ensures the existence of both a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ and a mapping $M : \mathcal{E}_0 \to L_0^d$ having the properties (i)-(iii) in Definition 3.1.8. The mapping $M_\alpha := f \circ M : \mathcal{E}_0 \to L_0$ is then the desired random α-Fréchet sup-measure with control measure m defined over $(\Omega, \mathfrak{A}, \mathbb{P})$. Clearly, the mapping M_α fulfills the first two properties. Being independently scattered follows immediately from the common result that measurable transformations of independent random vectors are again independent, whereas the second property is an easy consequence of Remark 3.1.3 (i). The third and last property can finally be obtained by referring to (3.1.35) and (3.1.37). Indeed, we have proved that

$$\bigvee_{n=1}^{\infty}{}_f\, M(A_n) = M(A_{n_0})$$

almost surely, n_0 being random. Moreover, (3.1.35) and the choice of the integer N specified therein show that

$$\bigvee_{n=1}^{\infty} f(M(A_n)) = f(M(A_{n_0}))$$

almost surely. Combining the latter equations, we conclude

$$M_\alpha\left(\bigcup_{n\geq 1} A_n\right) = f\left(M\left(\bigcup_{n\geq 1} A_n\right)\right) = f\left(\bigvee_{n=1}^{\infty}{}_f\, M(A_n)\right) = \bigvee_{n=1}^{\infty} f(M(A_n)) = \bigvee_{n\geq 1} M_\alpha(A_n)$$

almost surely. □

Our next concern is to establish connections between the aspects of Chapter 2 and the recently introduced notion of f-implicit sup measures, thus making the content of the present thesis self-contained. In doing so, we confine ourselves to only one attractive aspect. Drawing further connections might be an appealing task of future research work.
In Lemma 2.1.5 we saw that every f-implicit max-stable distribution on $\mathbb{R}^d$ is f-implicit max-infinitely divisible. For an arbitrary measure space $(E, \mathcal{E}, m)$ and $\mathcal{E}_0$ as before, each $M(A)$ is therefore f-implicit max-infinitely divisible, provided $(M(A))_{A\in\mathcal{E}_0}$ is an f-implicit sup-measure with control measure m defined on an appropriate probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. Consequently, for all $n \geq 1$, there exist independent and identically distributed random vectors $M(A)_1^{(n)}, ..., M(A)_n^{(n)}$ such that

$$M(A) \overset{d}{=} \bigvee_{j=1}^{n}{}_f\, M(A)_j^{(n)}.$$

We expand this observation by an explicit computation of a suitable nth root

$$\mu_n^{(A)} := \mathbb{P}_{M(A)_1^{(n)}}.$$

To this end, we incorporate our preliminary findings into our argumentation. Here, especially (3.1.7) gains in importance. Moreover, we may restrict our attention to the case $m(A) > 0$, for if not, we have $M(A) = 0$ almost surely. Hence, the particular case $\ell = 0$ in Example 2.1.4 shows that the nth root is equal to zero almost surely for all $n \geq 1$.

Proposition 3.1.17
Let $(E, \mathcal{E}, m)$ be an arbitrary measure space and $\mathcal{E}_0$ as before. Further, suppose that $(M(A))_{A\in\mathcal{E}_0}$ is an f-implicit sup-measure with control measure m defined on an appropriate probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. Then each $M(A)$ is f-implicit max-infinitely divisible. Moreover, for each $n \geq 1$,

$$\mu_n^{(A)}(d\tau, d\theta) = \frac{1}{n} e^{-\frac{1}{n} m(A)\tau^{-\alpha}} \frac{\alpha\, m(A)}{\tau^{\alpha+1}} d\tau\, \sigma(d\theta),$$

serves as an nth root, provided $m(A) > 0$.

Proof. Fix $A \in \mathcal{E}_0$ such that $m(A) > 0$ and note that $f(M(A)) \sim \Phi_\alpha\left(m(A)^{1/\alpha}\right)$. Consequently, the distribution of $f(M(A))$ is continuous on $(0, \infty)$. Since zero is actually the left end point of $f(M(A))$ with $\mathbb{P}(f(M(A)) = 0) = 0$, we can apply Theorem 2.2.12 (a) to deduce that

$$\mathbb{P}_{M(A)}(dx) = \mu_A(dx) = e^{-f(\nu)(f(x),\infty)}\nu(dx)$$

with ν being the measure on $\mathbb{R}^d \setminus \{0\}$ which is explicitly given by

$$\nu(dx) = \mathbb{P}(f(M(A)) \leq f(x))^{-1}\mu_A(dx). \tag{3.1.41}$$

Referring to Corollary 2.2.21, we further see that (2.2.26) yields a suitable representation for an nth root of $M(A)$. More precisely, we have

$$\mu_n^{(A)}(dx) = \frac{1}{n}e^{-\frac{1}{n}f(\nu)(f(x),\infty)}\nu(dx) \tag{3.1.42}$$

for each $n \geq 1$. The two latter equations combined are finally the basis to complete the proof. Indeed, (3.1.7) and (3.1.41) first give

$$\begin{aligned}
\nu(d\tau, d\theta) &= \mathbb{P}(f(M(A)) \leq f(\tau\theta))^{-1}\mu_A(d\tau, d\theta) \\
&= \mathbb{P}\left(Z \leq m(A)^{-\frac{1}{\alpha}}\tau\right)^{-1} \cdot \frac{\alpha\, m(A)}{\tau^{\alpha+1}}e^{-m(A)\tau^{-\alpha}}d\tau\, \sigma(d\theta) \\
&= \left(e^{-m(A)\tau^{-\alpha}}\right)^{-1} \cdot \frac{\alpha\, m(A)}{\tau^{\alpha+1}}e^{-m(A)\tau^{-\alpha}}d\tau\, \sigma(d\theta) \\
&= \frac{\alpha\, m(A)}{\tau^{\alpha+1}}d\tau\, \sigma(d\theta).
\end{aligned}$$

Then we can further conclude that

$$\begin{aligned}
\mu_n^{(A)}(d\tau, d\theta) &= \frac{1}{n}e^{-\frac{1}{n}f(\nu)(f(\tau\theta),\infty)}\nu(d\tau, d\theta) \\
&= \frac{1}{n}e^{-\frac{1}{n}f(\nu)(f(\tau\theta),\infty)}\frac{\alpha\, m(A)}{\tau^{\alpha+1}}d\tau\, \sigma(d\theta) \\
&= \frac{1}{n}e^{-\frac{1}{n}\nu\left(f^{-1}(\tau,\infty)\right)}\frac{\alpha\, m(A)}{\tau^{\alpha+1}}d\tau\, \sigma(d\theta) \\
&= \frac{1}{n}\exp\left(-\frac{1}{n}\int\limits_{f^{-1}(\tau,\infty)}\nu(dx)\right)\frac{\alpha\, m(A)}{\tau^{\alpha+1}}d\tau\, \sigma(d\theta) \\
&= \frac{1}{n}\exp\left(-\frac{1}{n}\int\limits_S\int\limits_\tau^\infty \frac{\alpha\, m(A)}{r^{\alpha+1}}\, dr\, \sigma(d\theta)\right)\frac{\alpha\, m(A)}{\tau^{\alpha+1}}d\tau\, \sigma(d\theta) \\
&= \frac{1}{n}e^{-\frac{1}{n}m(A)\tau^{-\alpha}}\frac{\alpha\, m(A)}{\tau^{\alpha+1}}d\tau\, \sigma(d\theta),
\end{aligned}$$

and this is precisely the assertion of the proposition. □

Before we proceed with the notion of f-implicit extremal stochastic integrals in Section 3.2, we address one last aspect concerning the recently introduced concept of f-implicit sup-measures. In particular, the rest of this section deals with a more detailed consideration of Definition 3.1.8. More precisely, we are concerned with the legitimate question whether there exist possible generalizations of Definition 3.1.8. The yet unanswered question is whether our assumptions on the mapping $M : \mathcal{E}_0 \to L_0^d$ can be relaxed without endangering the existence of this more general f-implicit sup-measure. Apart from many other attractive possibilities, we focus on only two potential ways of a generalization. On the one hand, we consider the case in which Z and Θ occurring both in Definition 3.1.2 and in Definition 3.1.8 are not assumed to be independent. On the other hand, we attend to the case in which we extend condition (ii) in Definition 3.1.8 to the effect that the angular part Θ is allowed to depend on $A \in \mathcal{E}_0$. From an application-oriented point of view this might prove itself valuable as much more problems could be handled with these enhancements. Of course, other manifold opportunities of generalizations are still waiting to be investigated. Studying these might be an exciting task for the future.

We start to point out that we elaborate on the second mentioned extension only. The case in which Z and Θ are allowed to be dependent is skipped for the time being. We will come back to this issue in Section 3.3. Hence, in the remainder of this section we devote ourselves to the question whether our assumption on Θ of being independent of $A \in \mathcal{E}_0$ can be relaxed in an appropriate way, thus obtaining a more flexible notion of an f-implicit sup-measure. To this end, we first point out that this question is not completely solved yet and therefore constitutes an attractive starting point for future research work. However, under reasonable and quite mild assumptions we can show the most surprising claim that Θ must necessarily be independent of $\mathcal{E}_0$, thus revealing that our construction is in some sense the most general one. This is very satisfying and stresses out the strength of our notion. In order to prove this illuminating assertion, we do well to do some preparatory work first.

To start with, recall that $(E, \mathcal{E}, m)$ denotes some arbitrary measure space. As before, let $\alpha > 0$ be fixed.

Definition 3.1.18

A *modified f-implicit sup-measure* is an $\mathbb{R}^d$-valued stochastic process $(\tilde{M}^f_\alpha(A))_{A\in\mathcal{E}_0} := (\tilde{M}(A))_{A\in\mathcal{E}_0}$ with the subsequent properties:

(i) For any collection of disjoint sets $A_1, ..., A_n \in \mathcal{E}_0, n \geq 1$, the random vectors $\tilde{M}(A_1), ..., \tilde{M}(A_n)$ are independent.

(ii) For every $A \in \mathcal{E}_0$, we have

$$\tilde{M}(A) \stackrel{d}{=} m(A)^{\frac{1}{\alpha}} Z\Theta(A), \tag{3.1.42}$$

where $Z \sim \Phi_\alpha$ and the random vector $\Theta(A) \sim \sigma_A \in M^1(S)$ are independent.

(iii) For any two disjoint sets $A_1, A_2 \in \mathcal{E}_0$, we have

$$\tilde{M}(A_1 \cup A_2) \stackrel{d}{=} \tilde{M}(A_1) \vee_f \tilde{M}(A_2). \tag{3.1.43}$$

Remark 3.1.19. (i) Note, the concepts introduced in Definition 3.1.8 and Definition 3.1.18 are quite similar. There are only two significant differences. Equation (3.1.42) expresses that $\tilde{M}(A)$ has an f-implicit α-Fréchet distribution with scale $m(A)^{1/\alpha}$ and angular part $\sigma_A \in M^1(S)$. The crucial difference to an f-implicit sup-measure is therefore that we assume $\tilde{M}(A)$ to have a distribution according to Definition 3.1.2, where the angular part can depend on the set $A \in \mathcal{E}_0$. However,

this dependency is only relevant in the case $m(A) > 0$ since otherwise we have $\tilde{M}(A) = 0$ almost surely and the random vector $\tilde{M}(A)$ does therefore not know anything about a radial part or an angular part.
Concerning the third property there also exists a difference between Definition 3.1.8 and Definition 3.1.18. Obviously, (3.1.13) implies (3.1.43).

(ii) We do not investigate the legitimate question whether a modified f-implicit sup-measure exists. For our purpose, this is not needed. Indeed, Definition 3.1.18 is just meant to be some kind of auxiliary definition providing the option of a more convenient presentation of the subsequent proposition and its principal consequence.

In deviation from the former situation it is now stipulated that the underlying measure space $(E, \mathcal{E}, m)$ is specifically chosen as $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \lambda^1)$. Hence, $\mathcal{E}_0$ consists of all Borel sets $B \subset \mathbb{R}_+$ having finite (Lebesgue-) measure, that is, $\mathcal{E}_0 = \{B \in \mathcal{B}(\mathbb{R}_+) : \lambda^1(B) < \infty\}$. Applying Definition 3.1.18, we may now establish the desired proposition. Note, however, that we are actually more interested in its immediate consequence given in Remark 3.1.21 (ii).

Proposition 3.1.20
Suppose that $(\tilde{M}(A))_{A\in\mathcal{E}_0}$ is a modified f-implicit sup-measure such that

$$\mathbb{P}_{\Theta(A)} = \sigma_{\lambda^1(A)}$$

for all $A \in \mathcal{E}_0$. For the resulting family of probability measures $\{\sigma_a : a \geq 0\}$ on S we further assume that $a \mapsto \sigma_a$ is (weakly) continuous. Then there exists a probability measure σ on S such that

$$\sigma_a = \sigma$$

for all $a \geq 0$. That is, the distribution of $\Theta(A)$ does not depend on A.

Remark 3.1.21. (i) Recall from Definition 2.2.17 that the mapping $\iota : [0, \infty) \to M^1(S)$, defined by $\iota(a) = \sigma_a$, is said to be (weakly) continuous if

$$a_n \xrightarrow[(n\to\infty)]{} a \qquad \Rightarrow \qquad \int_S h(\theta)\, \sigma_{a_n}(d\theta) \xrightarrow[(n\to\infty)]{} \int_S h(\theta)\, \sigma_a(d\theta)$$

for all bounded and continuous functions h on S.

(ii) Proposition 3.1.20 is ultimately the answer to our previously formulated question. Indeed, it shows that under mild and reasonable assumptions there cannot exist an f-implicit sup-measure such that the distribution of Θ depends on $A \in \mathcal{E}_0$ and thus reveals that our notion of f-implicit sup-measures is in some sense the most general one.

Proof of Proposition 3.1.20. We start to exclude all sets $A \in \mathcal{E}_0$ with $\lambda^1(A) = 0$ for the time being. Since

$$\tilde{M}(A) \sim \Phi^f_{\alpha,\sigma_{\lambda^1(A)}}\left(\lambda^1(A)^{\frac{1}{\alpha}}\right),$$

we can apply Lemma 3.1.4 (b) to deduce that

$$\mathbb{P}\left(\tilde{M}(A) \in D(r,F)\right) = e^{-\lambda^1(A)r^{-\alpha}} \cdot \sigma_{\lambda^1(A)}(F)$$

for all $r > 0$, $F \in \mathcal{B}(S)$ and $A \in \mathcal{E}_0$ with $\lambda^1(A) > 0$. Next, fix any collection of two disjoint sets $B_1, B_2 \in \mathcal{E}_0$ with $\lambda^1(B_1) > 0$ and $\lambda^1(B_2) > 0$. By assumption, we have

$$\tilde{M}(B_1) \stackrel{d}{=} \lambda^1(B_1)^{\frac{1}{\alpha}} Z_1 \Theta(B_1) \qquad \text{and} \qquad \tilde{M}(B_2) \stackrel{d}{=} \lambda^1(B_2)^{\frac{1}{\alpha}} Z_2 \Theta(B_2),$$

where $Z_1 \sim \Phi_\alpha$ and $\Theta(B_1) \sim \sigma_{\lambda^1(B_1)} \in M^1(S)$ as well as $Z_2 \sim \Phi_\alpha$ and $\Theta(B_2) \sim \sigma_{\lambda^1(B_2)} \in M^1(S)$ are independent. Referring to the first property of Definition 3.1.17, we conclude that Z_1, Z_2, $\Theta(B_1)$ and $\Theta(B_2)$ are independent since

$$\begin{aligned}
&\mathbb{P}(Z_1 \le r_1, Z_2 \le r_2, \Theta(B_1) \in F_1, \Theta(B_2) \in F_2) \\
&\quad = \mathbb{P}\left(\tilde{M}(B_1) \in D\left(\lambda^1(B_1)^{\frac{1}{\alpha}} r_1, F_1\right), \tilde{M}(B_2) \in D\left(\lambda^1(B_2)^{\frac{1}{\alpha}} r_2, F_2\right)\right) \\
&\quad = \mathbb{P}\left(\tilde{M}(B_1) \in D\left(\lambda^1(B_1)^{\frac{1}{\alpha}} r_1, F_1\right)\right) \cdot \mathbb{P}\left(\tilde{M}(B_2) \in D\left(\lambda^1(B_2)^{\frac{1}{\alpha}} r_2, F_2\right)\right) \\
&\quad = \mathbb{P}(Z_1 \le r_1, \Theta(B_1) \in F_1) \cdot \mathbb{P}(Z_2 \le r_2, \Theta(B_2) \in F_2) \\
&\quad = \mathbb{P}(Z_1 \le r_1) \cdot \mathbb{P}(\Theta(B_1) \in F_1) \cdot \mathbb{P}(Z_2 \le r_2) \cdot \mathbb{P}(\Theta(B_2) \in F_2)
\end{aligned}$$

for all $r_1, r_2 > 0$ and $F_1, F_2 \in \mathcal{B}(S)$. Combining the last deliberations and applying our assumptions, we get

$$\begin{aligned}
&e^{-(\lambda^1(B_1)+\lambda^1(B_2))r^{-\alpha}} \cdot \sigma_{\lambda^1(B_1)+\lambda^1(B_2)}(F) \\
&\quad = e^{-\lambda^1(B_1 \cup B_2)r^{-\alpha}} \cdot \sigma_{\lambda^1(B_1 \cup B_2)}(F) \\
&\quad = \mathbb{P}(\tilde{M}(B_1 \cup B_2) \in D(r,F)) \\
&\quad = \mathbb{P}(\tilde{M}(B_1) \vee_f \tilde{M}(B_2) \in D(r,F)) \\
&\quad = \mathbb{P}(\tilde{M}(B_1) \vee_f \tilde{M}(B_2) \in D(r,F), f(\tilde{M}(B_1)) \ge f(\tilde{M}(B_2))) \\
&\qquad\quad + \mathbb{P}(\tilde{M}(B_1) \vee_f \tilde{M}(B_2) \in D(r,F), f(\tilde{M}(B_1)) < f(\tilde{M}(B_2))) \\
&\quad = \mathbb{P}(\tilde{M}(B_1) \in D(r,F), f(\tilde{M}(B_1)) \ge f(\tilde{M}(B_2))) \\
&\qquad\quad + \mathbb{P}(\tilde{M}(B_2) \in D(r,F), f(\tilde{M}(B_1)) < f(\tilde{M}(B_2))) \\
&\quad = \mathbb{P}\left(\lambda^1(B_1)^{\frac{1}{\alpha}} Z_1 \le r, \Theta(B_1) \in F, \lambda^1(B_1)^{\frac{1}{\alpha}} Z_1 \ge \lambda^1(B_2)^{\frac{1}{\alpha}} Z_2\right) \\
&\qquad\quad + \mathbb{P}\left(\lambda^1(B_2)^{\frac{1}{\alpha}} Z_2 \le r, \Theta(B_2) \in F, \lambda^1(B_1)^{\frac{1}{\alpha}} Z_1 < \lambda^1(B_2)^{\frac{1}{\alpha}} Z_2\right) \\
&\quad = \mathbb{P}(\Theta(B_1) \in F) \cdot \mathbb{P}\left(\lambda^1(B_1)^{\frac{1}{\alpha}} Z_1 \le r, \lambda^1(B_1)^{\frac{1}{\alpha}} Z_1 \ge \lambda^1(B_2)^{\frac{1}{\alpha}} Z_2\right) \\
&\qquad\quad + \mathbb{P}(\Theta(B_2) \in F) \cdot \mathbb{P}\left(\lambda^1(B_2)^{\frac{1}{\alpha}} Z_2 \le r, \lambda^1(B_1)^{\frac{1}{\alpha}} Z_1 < \lambda^1(B_2)^{\frac{1}{\alpha}} Z_2\right) \\
&\quad = \sigma_{\lambda^1(B_1)}(F) \cdot \int_{(0,r]} \underbrace{\mathbb{P}\left(\lambda^1(B_2)^{\frac{1}{\alpha}} Z_2 \le u\right)}_{=e^{-\lambda^1(B_2)u^{-\alpha}}} \underbrace{\mathbb{P}_{\lambda^1(B_1)^{\frac{1}{\alpha}} Z_1}}_{=\Phi_\alpha\left(\lambda^1(B_1)^{\frac{1}{\alpha}}\right)} (du)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sigma_{\lambda^1(B_2)}(F) \cdot \int\limits_{(0,r]} \underbrace{\mathbb{P}\left(\lambda^1(B_1)^{\frac{1}{\alpha}} Z_1 < u\right)}_{=e^{-\lambda^1(B_1)u^{-\alpha}}} \underbrace{\mathbb{P}_{\lambda^1(B_2)^{\frac{1}{\alpha}} Z_2}}_{=\Phi_\alpha\left(\lambda^1(B_2)^{\frac{1}{\alpha}}\right)} (du) \\
&= \sigma_{\lambda^1(B_1)}(F) \cdot \int\limits_0^r e^{-\lambda^1(B_2)u^{-\alpha}} \frac{\alpha\lambda^1(B_1)}{u^{\alpha+1}} e^{-\lambda^1(B_1)u^{-\alpha}}\, du \\
&\quad + \sigma_{\lambda^1(B_2)}(F) \cdot \int\limits_0^r e^{-\lambda^1(B_1)u^{-\alpha}} \frac{\alpha\lambda^1(B_2)}{u^{\alpha+1}} e^{-\lambda^1(B_2)u^{-\alpha}}\, du \\
&= \sigma_{\lambda^1(B_1)}(F) \cdot \int\limits_0^r \frac{\alpha\lambda^1(B_1)}{u^{\alpha+1}} e^{-(\lambda^1(B_1)+\lambda^1(B_2))u^{-\alpha}}\, du \\
&\quad + \sigma_{\lambda^1(B_2)}(F) \cdot \int\limits_0^r \frac{\alpha\lambda^1(B_2)}{u^{\alpha+1}} e^{-(\lambda^1(B_1)+\lambda^1(B_2))u^{-\alpha}}\, du \\
&= \sigma_{\lambda^1(B_1)}(F) \cdot \frac{\lambda^1(B_1)}{\lambda^1(B_1)+\lambda^1(B_2)} e^{-(\lambda^1(B_1)+\lambda^1(B_2))r^{-\alpha}} \\
&\quad + \sigma_{\lambda^1(B_2)}(F) \cdot \frac{\lambda^1(B_2)}{\lambda^1(B_1)+\lambda^1(B_2)} e^{-(\lambda^1(B_1)+\lambda^1(B_2))r^{-\alpha}} \\
&= e^{-(\lambda^1(B_1)+\lambda^1(B_2))r^{-\alpha}} \cdot \left(\sigma_{\lambda^1(B_1)}(F) \cdot \frac{\lambda^1(B_1)}{\lambda^1(B_1)+\lambda^1(B_2)} + \sigma_{\lambda^1(B_2)}(F) \cdot \frac{\lambda^1(B_2)}{\lambda^1(B_1)+\lambda^1(B_2)}\right)
\end{aligned}
$$

for all $r > 0$ and $F \in \mathcal{B}(S)$. Equivalently, we have

$$
\sigma_{\lambda^1(B_1)+\lambda^1(B_2)}(F) = \sigma_{\lambda^1(B_1)}(F) \cdot \frac{\lambda^1(B_1)}{\lambda^1(B_1)+\lambda^1(B_2)} + \sigma_{\lambda^1(B_2)}(F) \cdot \frac{\lambda^1(B_2)}{\lambda^1(B_1)+\lambda^1(B_2)}.
$$

In other words, we therefore actually proved that

$$
\sigma_{a+b} = \sigma_a \cdot \frac{a}{a+b} + \sigma_b \cdot \frac{b}{a+b} \tag{3.1.44}
$$

for all $a, b > 0$. Note, the latter conclusion follows from the fact that the underlying measure space is specifically chosen as $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \lambda^1)$. Equation (3.1.44) can now be used to complete the proof. To this end, let $n, m \geq 1$. Iterating (3.1.44), we obtain

$$
\begin{aligned}
&\sigma_{a_1+\ldots+a_n} \\
&\quad = \sigma_{(a_1+\ldots+a_{n-1})+a_n} \\
&\quad = \sigma_{a_1+\ldots+a_{n-1}} \cdot \frac{a_1+\ldots+a_{n-1}}{a_1+\ldots+a_n} + \sigma_{a_n} \cdot \frac{a_n}{a_1+\ldots+a_n} \\
&\quad = \left(\sigma_{a_1+\ldots+a_{n-2}} \cdot \frac{a_1+\ldots+a_{n-2}}{a_1+\ldots+a_{n-1}} + \sigma_{a_{n-1}} \cdot \frac{a_{n-1}}{a_1+\ldots+a_{n-1}}\right) \cdot \frac{a_1+\ldots+a_{n-1}}{a_1+\ldots+a_n} + \sigma_{a_n} \cdot \frac{a_n}{a_1+\ldots+a_n} \\
&\quad = \sigma_{a_1+\ldots+a_{n-2}} \cdot \frac{a_1+\ldots+a_{n-2}}{a_1+\ldots+a_n} + \sigma_{a_{n-1}} \cdot \frac{a_{n-1}}{a_1+\ldots+a_n} + \sigma_{a_n} \cdot \frac{a_n}{a_1+\ldots+a_n}
\end{aligned}
$$

$$\vdots$$
$$= \sum_{i=1}^{n} \frac{a_i}{a_1 + ... + a_n} \sigma_{a_i} \tag{3.1.45}$$

for all $a_1, ..., a_n > 0$. Applying (3.1.45) to the specific choice of positive real numbers $a_1 = ... = a_n = \frac{1}{n}$, we deduce that

$$\sigma_1 = \sigma_{\frac{1}{n}}.$$

Applying (3.1.45) once again - but this time to the specific choice of positive real numbers $a_1 = ... = a_m = \frac{1}{n}$ - further gives

$$\sigma_{\frac{m}{n}} = \sum_{i=1}^{m} \frac{\frac{1}{n}}{\frac{m}{n}} \sigma_{\frac{1}{n}} = \sigma_{\frac{1}{n}}.$$

Hence, the two latter equations combined show that

$$\sigma_1 = \sigma_{\frac{m}{n}}$$

for all $n, m \geq 1$ and therefore actually

$$\sigma_1 = \sigma_q \tag{3.1.46}$$

for all $q \in \mathbb{Q}_{>0} := \{q \in \mathbb{Q} : q > 0\}$. Since for all non-negative real numbers $a \geq 0$ there exists a sequence $(a_n)_{n\geq 1} \subset \mathbb{Q}_{>0}$ such that

$$\lim_{n\to\infty} a_n = a,$$

we conclude, by assumption and (3.1.46), that

$$\int_S h(\theta)\, \sigma_1(d\theta) = \lim_{n\to\infty} \int_S h(\theta)\, \sigma_1(d\theta) = \lim_{n\to\infty} \int_S h(\theta)\, \sigma_{a_n}(d\theta) = \int_S h(\theta)\, \sigma_a(d\theta)$$

for all bounded and continuous functions h on S. Applying common results of probability and measure theory (see for example Corollary 2.6 in [Els10, Chapter 8]), we get $\sigma_a = \sigma_1$ which is ultimately a consequence of Riesz-Markov-Kakutani's representation theorem. At any rate, this completes the proof by defining $\sigma := \sigma_1$. □

Equipped with the concept of f-implicit sup-measures, we may now proceed to establish an exciting integration concept. This idea originates from [SaTa94, Chapter 3] and [StTa05] who first used the notion of random measures in order to provide a new integral of non-random functions with respect to those measures. Although they were actually motivated by different questions, both concepts share striking parallels to each other. Hence, it is not really surprising that our notion of f-implicit extremal stochastic integrals also has many similarities to both of the latter ones.

3.2 The concept of f-implicit extremal stochastic integrals

As mentioned several times and as the heading of this section already reveals, this part of the thesis is devoted to a detailed study of the notion of f-implicit extremal stochastic integrals. The concrete motivation behind this section is the very same as in [SaTa94, Chapter 3] or [StTa05]. Whereas Samorodnitsky and Taqqu tried to generate non-trivial α-stable processes and Stoev and Taqqu non-trivial max-stable processes, we are interested in non-trivial examples of f-implicit max-stable processes. Using the notion of f-implicit extremal stochastic integrals will in fact turn out to be a productive approach and thus, in addition to f-implicit sup-measures, yields a second suitable way to advance towards the construction of f-implicit max-stable processes.
In the introduction of this chapter we stressed out that every f-implicit max-stable process provides an α-Fréchet process. This underlines the signifying benefit of f-implicit max-stable processes and the importance of our purpose. Having justified our intention, we start to introduce the notion of f-implicit extremal stochastic integral. Here, we are guided by the constructions pursued in [SaTa94, Chapter 3] and [StTa05].
Broadly speaking, an f-implicit extremal stochastic integral is an integral of a non-random function $g : E \to \mathbb{R}$ with respect to an f-implicit sup-measure $M := (M(A))_{A\in\mathcal{E}_0}$. Note that this integral being random and depending on f justifies the specific terminology f-implicit extremal stochastic integral. At this point it must be emphasized that we actually give a well-defined definition of such an integral for simple functions only. Of course, this might be unsatisfactorily in some sense. Nevertheless, this first attempt to the notion of f-implicit extremal stochastic integrals is definitely worth it. Suggestions of an appropriate extension will be considered in Section 3.3.
To start with, let $(E, \mathcal{E}, m)$ denote some arbitrary measure space. Furthermore, let $\mathcal{E}_0$ be defined as in the previous section.

Definition 3.2.1
A function $g : E \to \mathbb{R}$ is said to be *simple* if

$$g(u) = \sum_{i=1}^{n} \alpha_i \, \mathbb{1}_{A_i}(u) \tag{3.2.1}$$

for some $n \geq 1$, real numbers $\alpha_1, ..., \alpha_n \geq 0$ and disjoint sets $A_1, ..., A_n \in \mathcal{E}_0$.

Now, let $M := (M(A))_{A\in\mathcal{E}_0}$ be an f-implicit sup-measure with control measure m defined on an appropriate probability space $(\Omega, \mathfrak{A}, \mathbb{P})$.

Definition 3.2.2
Suppose that $g : E \to \mathbb{R}$ is simple. The *f-implicit extremal stochastic integral of g with respect to M* is defined by

$$\int_E^{e,f} g(u)\, M(du) := \int_E g(u)\, M(du) := \bigvee_{i=1}^{n}{}_f \; \alpha_i \, M(A_i). \tag{3.2.2}$$

This integral is henceforth shortly referred to as *f-implicit extremal integral*.

Definition 3.2.2 is the natural analogue of the definitions occurring in [SaTa94, Chapter 3] and [StTa05]. However, we still need to prove that this definition is well-defined, that is, (3.2.2) does not depend on the specific representation of the simple function g. To this end, we use similar arguments and ideas as in the proof of Theorem 3.1.12. After having proved this crucial aspect, we proceed with some first desirable properties of the f-implicit extremal integral which ultimately result in Proposition 3.2.6 and Remark 3.2.7.

Proposition 3.2.3
Suppose that

$$g(u) = \sum_{i=1}^{n_1} \alpha_i \, \mathbb{1}_{A_i}(u) \qquad \text{and} \qquad g(u) = \sum_{\ell=1}^{n_2} \beta_\ell \, \mathbb{1}_{B_\ell}(u)$$

are two different representations for the same simple function g, where $n_1, n_2 \geq 1$ are suitable integers, $\alpha_1, ..., \alpha_{n_1}, \beta_1, ..., \beta_{n_2} \geq 0$ appropriate non-negative real numbers and $A_1, ..., A_{n_1} \in \mathcal{E}_0$ as well as $B_1, ..., B_{n_2} \in \mathcal{E}_0$ collections of disjoint sets. Then we have

$$\bigvee_{i=1}^{n_1}{}_f \, \alpha_i \, M(A_i) = \bigvee_{\ell=1}^{n_2}{}_f \, \beta_\ell \, M(B_\ell)$$

almost surely showing that (3.2.2) is well-defined.

Proof. Without loss of generality, we may assume that $\alpha_1, ..., \alpha_{n_1}, \beta_1, ..., \beta_{n_2} > 0$. Similar to the principal idea in the proof of Theorem 3.1.12 we first construct a third representation for g bridging the gap between the two prespecified representations. Namely, define

$$C_{i,\ell} := A_i \cap B_\ell \in \mathcal{E}_0$$

for all $i = 1, ..., n_1$ and $\ell = 1, ..., n_2$. Obviously, these sets are disjoint and can therefore serve as suitable sets for a representation of g. In fact, we see immediately that

$$g(u) = \sum_{\substack{i=1,...,n_1 \\ \ell=1,...,n_2}} \gamma_{i,\ell} \, \mathbb{1}_{C_{i,\ell}}(u) \, ,$$

the non-negative real numbers $\gamma_{i,\ell}$ being defined as

$$\gamma_{i,\ell} := \begin{cases} \alpha_i, & \text{if } C_{i,\ell} \neq \emptyset \\ 0, & \text{if } C_{i,\ell} = \emptyset \end{cases} = \begin{cases} \beta_\ell, & \text{if } C_{i,\ell} \neq \emptyset \\ 0, & \text{if } C_{i,\ell} = \emptyset. \end{cases}$$

Applying (3.2.2) to this specific representation for g, taking into account the fact that most of the sets $C_{i,\ell}, 1 \leq i \leq n_1, 1 \leq \ell \leq n_2$ are empty, which can therefore be ignored, and using the essential fact that the random vectors $M(C_{i,\ell}), 1 \leq i \leq n_1, 1 \leq \ell \leq n_2$ commute under the f-implicit max-operation, which is due to (3.1.9), we deduce that

$$\int_E g(u) \, M(du) = \bigvee_{\substack{i=1,...,n_1 \\ \ell=1,...,n_2}}{}_f \; \gamma_{i,\ell} \, M(C_{i,\ell})$$

$$
\begin{aligned}
&= \left(\bigvee_{\ell=1}^{n_2}{}_f \underbrace{\gamma_{1,\ell}}_{=\alpha_1} M(C_{1,\ell}) \right) \vee_f \ldots \vee_f \left(\bigvee_{\ell=1}^{n_2}{}_f \underbrace{\gamma_{n_1,\ell}}_{=\alpha_{n_1}} M(C_{n_1,\ell}) \right) \\
&= \alpha_1 \left(\bigvee_{\ell=1}^{n_2}{}_f M(C_{1,\ell}) \right) \vee_f \ldots \vee_f \alpha_{n_1} \left(\bigvee_{\ell=1}^{n_2}{}_f M(C_{n_1,\ell}) \right) \\
&= \alpha_1 M \left(\bigcup_{\ell=1}^{n_2} C_{1,\ell} \right) \vee_f \ldots \vee_f \alpha_{n_1} M \left(\bigcup_{\ell=1}^{n_2} C_{n_1,\ell} \right) \\
&= \alpha_1 M(A_1) \vee_f \ldots \vee_f \alpha_{n_1} M(A_{n_1}) \\
&= \bigvee_{i=1}^{n_1}{}_f \alpha_i M(A_i)
\end{aligned}
$$

almost surely. Note that we gained from both (1.1.4) and property (iii) in Definition 3.1.8. Similarly, we get

$$
\bigvee_{\substack{i=1,\ldots,n_1 \\ \ell=1,\ldots,n_2}}{}_f \gamma_{i,\ell} M(C_{i,\ell}) = \bigvee_{\ell=1}^{n_2}{}_f \beta_\ell M(B_\ell)
$$

almost surely, and this is precisely the assertion of the proposition. □

Now, that we have justified the key aspect of well-definedness of the f-implicit extremal integral we concern ourselves with some first useful properties of this integral. As the next proposition reveals, there are several of them being closely related to the properties of extremal stochastic integrals (see for instance [StTa05, Proposition 2.2 and Proposition 2.3]) and α-stable stochastic integrals (see for example [SaTa94, Section 3.5]).

Proposition 3.2.4
Let g_1 and g_2 be simple functions defined as in (3.2.1). Then the following properties hold.

(i) *(f-implicit max-linearity)*
For any pair of non-negative real numbers $a, b \geq 0$, we have

$$
\int_E (a\, g_1(u) \vee b\, g_2(u))\, M(du) = \left(a \int_E g_1(u)\, M(du) \right) \vee_f \left(b \int_E g_2(u)\, M(du) \right) \tag{3.2.3}
$$

almost surely.

(ii) *(f-implicit α-Fréchet)*
The f-implicit extremal integral is f-implicit α-Fréchet. More precisely, we have

$$
\int_E g_1(u)\, M(du) \stackrel{d}{=} \left(\int_E g_1^\alpha(u)\, m(du) \right)^{\frac{1}{\alpha}} Z\Theta, \tag{3.2.4}
$$

the random variable Z and the random vector Θ being defined as usual (see for instance Definition 3.1.2 or Definition 3.1.8).

(iii) *(f-implicit monotonicity)*
We have $g_1 \leq g_2$ m-almost everywhere if and only if

$$\int_E g_1(u)\, M(du) \leq_f \int_E g_2(u)\, M(du) \tag{3.2.5}$$

almost surely.

(iv) *(f-implicit independence)*
We have $g_1 g_2 = 0$ m-almost everywhere if and only if the random vectors

$$\xi_1 := \int_E g_1(u)\, M(du) \qquad \text{and} \qquad \xi_2 := \int_E g_2(u)\, M(du)$$

are independent.

Proof. The four properties will be proved successively. Here, we gain from some ideas which have already been used in the proof of Proposition 2.2 in [StTa05], from the crucial fact that the random vectors $M(A_1), ..., M(A_n)$ commute under the f-implicit max-operation, provided $A_1, ..., A_n \in \mathcal{E}_0$ are disjoint sets, and finally from Lemma 1.1.5.
(i) Fix $a, b \geq 0$. By rearranging the representations of g_1 and g_2 appropriately, we may assume that

$$g_1(u) = \sum_{i=1}^{n} \alpha_i\, \mathbb{1}_{A_i}(u) \qquad \text{and} \qquad g_2(u) = \sum_{i=1}^{n} \beta_i\, \mathbb{1}_{A_i}(u)$$

for some $n \geq 1$, $\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n \geq 0$ and $A_1, ..., A_n \in \mathcal{E}_0$ (disjoint). Accordingly, we have

$$a\, g_1(u) \vee b\, g_2(u) = \sum_{i=1}^{n} (a\, \alpha_i \vee b\, \beta_i) \mathbb{1}_{A_i}(u)$$

and therefore

$$\begin{aligned}
\int_E (a\, g_1(u) \vee b\, g_2(u))\, M(du) &= \bigvee_{i=1}^{n}{}_f\, (a\, \alpha_i \vee b\, \beta_i)\, M(A_i) \\
&= \bigvee_{i=1}^{n}{}_f\, (a\, \alpha_i\, M(A_i) \vee_f b\, \beta_i\, M(A_i)) \\
&= \left(\bigvee_{i=1}^{n}{}_f\, a\, \alpha_i\, M(A_i) \right) \vee_f \left(\bigvee_{i=1}^{n}{}_f\, b\, \beta_i\, M(A_i) \right) \\
&= \left(a \bigvee_{i=1}^{n}{}_f\, \alpha_i\, M(A_i) \right) \vee_f \left(b \bigvee_{i=1}^{n}{}_f\, \beta_i\, M(A_i) \right)
\end{aligned}$$

$$= \left(a \int_E g_1(u)\, M(du) \right) \vee_f \left(b \int_E g_2(u)\, M(du) \right)$$

almost surely showing (3.2.3).
(ii) Adopting the label of the representation for g_1 from the previous part of the proof, using the second property of an f-implicit sup-measure and finally applying (1.2.7) as well as Lemma 3.1.4 (c), we see immediately that

$$\begin{aligned}
\mathbb{P}\left(\int_E g_1(u)\, M(du) \in A \right) &= \mathbb{P}\left(\bigvee_{i=1}^{n}{}_f\, \alpha_i\, M(A_i) \in A \right) \\
&= \mathbb{P}_{\alpha_1 M(A_1)} *_f \dots *_f \mathbb{P}_{\alpha_n M(A_n)}(A) \\
&= \mathbb{P}_{(\alpha_1^\alpha\, m(A_1))^{\frac{1}{\alpha}} Z\Theta} *_f \dots *_f \mathbb{P}_{(\alpha_n^\alpha\, m(A_n))^{\frac{1}{\alpha}} Z\Theta}(A) \\
&= \mathbb{P}\left(\left(\sum_{i=1}^{n} \alpha_i^\alpha\, m(A_i) \right)^{\frac{1}{\alpha}} Z\Theta \in A \right) \\
&= \mathbb{P}\left(\left(\int_E g_1^\alpha(u)\, m(du) \right)^{\frac{1}{\alpha}} Z\Theta \in A \right)
\end{aligned}$$

for all $A \in \mathcal{B}(\mathbb{R}^d)$, and (3.2.4) is proved. Here, we shall also refer to the considerations in the proof of Lemma 3.1.4 (c) and Remark 3.1.5 (v).
(iii) We start by proving the *only if-part*. Accordingly, there exists a set $E_0 \in \mathcal{E}$ such that $g_1(u) \le g_2(u)$ for all $u \in E_0$ and $m(E \setminus E_0) = 0$. Applying Lemma 1.3.3 (c) and subsequently (3.2.3), we deduce that

$$\int_E g_1(u)\, M(du) \le_f \left(\int_E g_1(u)\, M(du) \right) \vee_f \left(\int_E g_2(u)\, M(du) \right) = \int_E (g_1(u) \vee g_2(u))\, M(du)$$

almost surely. Similar to the proof of (i) we may further assume that

$$g_1(u) = \sum_{i=1}^{n} \alpha_i\, \mathbb{1}_{A_i}(u) \qquad \text{and} \qquad g_2(u) = \sum_{i=1}^{n} \beta_i\, \mathbb{1}_{A_i}(u)$$

for some $n \ge 1$, $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \ge 0$ and $A_1, \dots, A_n \in \mathcal{E}_0$ (disjoint). This assumption proves of use in the following computation. Indeed, we have

$$\begin{aligned}
g_1(u) \vee g_2(u) &= (g_1(u) \vee g_2(u))\mathbb{1}_{E_0}(u) + (g_1(u) \vee g_2(u))\mathbb{1}_{E \setminus E_0}(u) \\
&= g_2(u)\mathbb{1}_{E_0}(u) + g_1(u)\mathbb{1}_{E \setminus E_0}(u) \\
&= \sum_{i=1}^{n} \beta_i\, \mathbb{1}_{A_i \cap E_0}(u) + \sum_{i=1}^{n} \alpha_i\, \mathbb{1}_{A_i \cap (E \setminus E_0)}(u)
\end{aligned}$$

$$
\begin{aligned}
&:= \sum_{i=1}^{2n} \gamma_i \mathbb{1}_{C_i}(u) \\
&:= \tilde{g}(u),
\end{aligned}
$$

where

$$
\gamma_i := \begin{cases} \beta_i, & \text{if } i = 1, ..., n \\ \alpha_{i-n}, & \text{if } i = n+1, ..., 2n \end{cases} \qquad \text{and} \qquad C_i := \begin{cases} A_i \cap E_0, & \text{if } i = 1, ..., n \\ A_{i-n} \cap (E \setminus E_0), & \text{if } i = n+1, ..., 2n. \end{cases}
$$

Consequently, the function $\tilde{g}$ coincides with $g_1 \vee g_2$. The difference is in the representations only. Thus, Proposition 3.2.3, the fact that $M(C_1), ..., M(C_{2n})$ commute under the f-implicit max-operation, the f-max σ-sup-additivity of M and finally the case that $M(C_{n+1}), ..., M(C_{2n})$ are equal to zero almost surely yield

$$
\begin{aligned}
\int_E (g_1(u) \vee g_2(u))\, M(du) &= \int_E \tilde{g}(u)\, M(du) \\
&= \bigvee_{i=1}^{2n}{}_f\, \gamma_i\, M(C_i) \\
&= \left(\bigvee_{i=1}^{n}{}_f\, \beta_i\, M(A_i \cap E_0) \right) \vee_f \left(\bigvee_{i=n+1}^{2n}{}_f\, \alpha_{i-n}\, M(A_{i-n} \cap (E \setminus E_0)) \right) \\
&= \left(\bigvee_{i=1}^{n}{}_f\, \beta_i\, M(A_i \cap E_0) \right) \vee_f \left(\bigvee_{i=n+1}^{2n}{}_f\, \beta_{i-n}\, M(A_{i-n} \cap (E \setminus E_0)) \right) \\
&= \bigvee_{i=1}^{n}{}_f\, \beta_i \left(M(A_i \cap E_0) \vee_f M(A_i \cap (E \setminus E_0)) \right) \\
&= \bigvee_{i=1}^{n}{}_f\, \beta_i\, M(A_i) \\
&= \int_E g_2(u)\, M(du)
\end{aligned}
$$

almost surely. Combing all previous observations, we obtain (3.2.5). Hence, we can proceed with the *if-part* of the proof. By assumption,

$$
\int_E g_1(u)\, M(du) \leq_f \int_E g_2(u)\, M(du)
$$

almost surely. Applying both Lemma (1.3.4) and (3.2.3), we see immediately that

$$
\int_E g_2(u)\, M(du) = \left(\int_E g_1(u)\, M(du) \right) \vee_f \left(\int_E g_2(u)\, M(du) \right) = \int_E (g_1(u) \vee g_2(u))\, M(du)
$$

almost surely. From (3.2.4) it further follows that

$$\exp\left(-\left(\int_E g_2^\alpha(u)\, m(du)\right)\cdot x^{-\alpha}\right) = \exp\left(-\left(\int_E (g_1(u)\vee g_2(u))^\alpha\, m(du)\right)\cdot x^{-\alpha}\right)$$

for all $x > 0$. Thus, we have

$$\int_E g_2^\alpha(u)\, m(du) = \int_E (g_1(u)\vee g_2(u)^\alpha\, m(du). \tag{3.2.6}$$

Since $g_2 \leq g_1 \vee g_2$ and hence $0 \leq (g_1 \vee g_2)^\alpha - g_2^\alpha$, (3.2.6) actually yields

$$(g_1 \vee g_2)^\alpha - g_2^\alpha = 0$$

m-almost everywhere implying $g_1 \leq g_2$ m-almost everywhere, and the proof is complete.
(iv) Once again, we start by proving the *only if-part*. Consequently, there exists a set $E_1 \in \mathcal{E}$ such that $g_1(u)g_2(u) = 0$ for all $u \in E_1$ and $m(E \setminus E_1) = 0$. Following the approach pursued in (i) or (iii), we may assume that

$$g_1(u) = \sum_{i=1}^n \alpha_i\, \mathbb{1}_{A_i}(u) \qquad \text{and} \qquad g_2(u) = \sum_{i=1}^n \beta_i\, \mathbb{1}_{A_i}(u)$$

for some $n \geq 1$, $\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n \geq 0$ and $A_1, ..., A_n \in \mathcal{E}_0$ (disjoint). Therefore, we get

$$g_1(u) = \sum_{i=1}^n \alpha_i\, \mathbb{1}_{A_i\cap E_1}(u) + \sum_{i=1}^n \alpha_i\, \mathbb{1}_{A_i\cap(E\setminus E_1)}(u)$$

and

$$g_2(u) = \sum_{i=1}^n \beta_i\, \mathbb{1}_{A_i\cap E_1}(u) + \sum_{i=1}^n \beta_i\, \mathbb{1}_{A_i\cap(E\setminus E_1)}(u)\,,$$

respectively. Note that the sets $A_i \cap (E \setminus E_1) \in \mathcal{E}_0$ being null sets for all $i = 1, ..., n$ yields

$$\xi_1 = \int_E g_1(u)\, M(du) = {\bigvee_f}_{i=1}^{n} \alpha_i\, M(A_i \cap E_1)$$

and

$$\xi_2 = \int_E g_2(u)\, M(du) = {\bigvee_f}_{i=1}^{n} \beta_i\, M(A_i \cap E_1)$$

almost surely. Since $g_1(u)g_2(u) = 0$ for all $u \in E_1$, we finally conclude that

$$\xi_1 = {\bigvee_f}_{i\in I} \alpha_i\, M(A_i \cap E_1)$$

and

$$\xi_2 = \bigvee_{j=J}{}_f \; \beta_i M(A_i \cap E_1)$$

almost surely, the index sets $I, J \subset \{1, ..., n\}$ being disjoint. Thus, the random vectors ξ_1 and ξ_2 are almost surely composed of weighted f-implicit maxima of random vectors $M(A_i \cap E_1)$ over disjoint sets of indices i. The random vectors $M(A_1 \cap E_1), ..., M(A_n \cap E_1)$ being independent shows, by applying Corollary 1.1.7, that ξ_1 and ξ_2 are independent.
We now focus on proving the *if-part* of (iv). To this end, we apply (i) and (ii) of the present proposition as well as Lemma 3.1.4 (b) and (c). Without loss of generality, we may first exclude the case $\xi_1 = 0$ or $\xi_2 = 0$ almost surely, resulting from (3.2.4). Thus, both ξ_1 and ξ_2 are f-implicit α-Fréchet with positive scales. By assumption, we have

$$\mathbb{P}_{\xi_1 \vee_f \xi_2} = \mathbb{P}_{\xi_1} *_f \mathbb{P}_{\xi_2}.$$

Furthermore, we have

$$\mathbb{P}(\xi_1 \vee_f \xi_2 \in D(r, S)) = \exp\left(-\left(\int_E (g_1(u) \vee g_2(u))^\alpha \, m(du)\right) r^{-\alpha}\right)$$

for all $r > 0$ being a consequence of (i), (ii) and Lemma 3.1.4 (b). Applying Lemma 3.1.4 (c), we additionally obtain

$$\begin{aligned}
\mathbb{P}_{\xi_1} *_f \mathbb{P}_{\xi_2}(D(r, S)) &= \exp\left(-\left(\int_E g_1^\alpha(u) \, m(du) + \int_E g_2^\alpha(u) \, m(du)\right) r^{-\alpha}\right) \\
&= \exp\left(-\left(\int_E (g_1^\alpha(u) + g_2^\alpha(u)) \, m(du)\right) r^{-\alpha}\right)
\end{aligned}$$

for all $r > 0$ showing that

$$\int_E (g_1(u) \vee g_2(u))^\alpha \, m(du) = \int_E (g_1^\alpha(u) + g_2^\alpha(u)) \, m(du). \tag{3.2.7}$$

Since g_1 and g_2 are non-negative, we deduce that

$$g_1^\alpha(u) \vee g_2^\alpha(u) \le g_1^\alpha(u) + g_2^\alpha(u)$$

for all $u \in E$. Consequently, (3.2.7) yields

$$g_1^\alpha(u) \vee g_2^\alpha(u) = g_1^\alpha(u) + g_2^\alpha(u)$$

m-almost surely. This, however, is only possible if $g_1 g_2 = 0$ m-almost surely, and the proof is complete. $\square$

Referring to the third property stated in Proposition 3.2.4, we obtain immediately the following natural and desirable property of the f-implicit extremal integral.

Corollary 3.2.5
For any two simple functions g_1 and g_2 we have $g_1 = g_2$ m-almost surely if and only if

$$\int\limits_E g_1(u)\, M(du) = \int\limits_E g_2(u)\, M(du)$$

almost surely.

Proof. This is an easy consequence of the f-implicit monotonicity. Indeed, the *if-part* of the assertion follows from Definition 1.3.1 and from the *if-part* of Proposition 3.2.4 (iii), whereas the *only if-part* can be deduced from the antisymmetry of the f-implicit max-order and the *only if-part* of Proposition 3.2.4 (iii). □

The conclusion of this short section, which raises no claim to completeness and is rather intended to initiate extensive research projects for the future by providing a solid basis, is devoted to the following proposition.

Proposition 3.2.6
Let T denote a non-empty index set. Further, suppose that $(g_t)_{t\in T}$ is a family of simple functions. Then the resultant family $\mathbb{X} := (X_t)_{t\in T}$ of $\mathbb{R}^d$-valued random vectors, defined by

$$X_t := \int\limits_E g_t(u)\, M(du),$$

has the subsequent property. For all $n \geq 1$, $\alpha_1, ..., \alpha_n \geq 0$ and $t_1, ..., t_n \in T$, we have

$$\bigvee_{i=1}^{n}{}_f\; \alpha_i\, X_{t_i} = \int\limits_E \left(\bigvee_{i=1}^{n} \alpha_i\, g_{t_i}(u) \right) M(du) \tag{3.2.8}$$

almost surely.

Proof. The proof is straightforward and an easy consequence of (3.2.3). □

Remark 3.2.7. The principal significance of the preceding proposition is in the resultant deduction that the process $\mathbb{X}$ is actually f-implicit max-stable in accordance with Definition 3.0.1. Indeed, (3.2.4) shows that the right-hand side in (3.2.8) is f-implicit α-Fréchet with scale

$$\kappa = \left(\int\limits_E \bigvee_{i=1}^{n} \alpha_i^{\alpha}\, g_{t_i}^{\alpha}(u)\, m(du) \right)^{\frac{1}{\alpha}} \geq 0$$

and hence particularly f-implicit max-stable.

Combining Proposition 3.2.6 and Remark 3.2.7, we obtain many examples of f-implicit max-stable processes at one go, thus complying with the initially formulated aim of Chapter 3. Since all these examples are essentially based on the concept of f-implicit sup-measures, the importance and great benefit of this notion is once again illustrated explicitly. Besides, Remark 3.2.7 yields the verification of the assertion formulated in Example 3.1.15 as the process $\mathbb{X} := (X_t)_{t\geq 0}$ specified therein can equivalently be written as

$$X_t := M([0,t]) = \int\limits_{\mathbb{R}_+} \mathbb{1}_{[0,t]}(u)\, M(du),\ t \geq 0.$$

Moreover, the latter equation reveals the close connection between the concepts investigated in Section 3.1 and Section 3.2.
With this statement we want to complete Section 3.2. The rest of this chapter, in form of Section 3.3, deals with an extensive outlook concerning several suggestions for improvements and extensions of the aspects investigated and considered in the preceding deliberations. Actually, this is some kind of promotion for the gripping f-implicit extreme value theory.

3.3 Outlook

In Section 3.1 and Section 3.2 we engaged ourselves in basic mathematical research concerning two attractive branches of f-implicit extreme value theory. We introduced both the fundamental notion of f-implicit sup-measures and the concept of f-implicit extremal integrals. This was designed for the construction of f-implicit max-stable processes. Although we have already proved some exciting aspects, such as the existence of f-implicit sup-measures, and although we have established first connections to the common theories of α-stable processes (see [SaTa94]) and max-stable processes (see [DaMi08], [deHaFe06, Chapter 9], [EmKlMi12, Chapter 3 and Chapter 5], [Ka09], [StTa05] or [StWa10]), we point out that there are several further issues that need to be investigated. Therefore, this final section attends to a detailed presentation of attractive themes going beyond the ones considered in Section 3.1 and Section 3.2. Here, we gain intuition from the recently mentioned theories as they provide worthwhile ideas for possible and sensible extensions.
Rather than statistical or computational observations concerning the contents of Section 3.1 and Section 3.2, we are concerned with theoretical ones. Namely, we suggest some ideas pertaining to more general notions of f-implicit sup-measures. Moreover, we propose a possible way of a generalization of f-implicit extremal stochastic integrals being necessary with regard to the construction of further attractive examples of f-implicit max-stable processes. Finally, we consider whether there exist suitable conditions under which f-implicit max-stable processes allow representations in terms of f-implicit extremal stochastic integrals.

1. More general dependence structure in the notion of f-implicit sup-measures. Considering the notion of f-implicit sup-measures introduced in Section 3.1, we need to confess that we have required independence of the radial part Z and the angular part Θ. That is, we have yet to consider conceivable dependence structures that might exist between the radial and the angular part since without the assumption of independence we would have not been able to prove Theorem 3.1.12 in the manner used in this thesis.

From an applied point of view, however, it would be advantageous to get rid of the assumption of independence. This raises the exciting question whether there exists a more general notion of f-implicit sup-measures including specific dependence structures between Z and Θ, or whether the properties (i)-(iii) in Definition 3.1.8 imply that Z and Θ are necessarily independent. To consider this question, we do well to observe an exemplary situation.
As usual, let $\sigma \in M^1(S)$ denote the distribution of the angular part Θ of an arbitrary f-implicit sup-measure. In order to keep it simple, we may assume σ to be independent of the sets $A \in \mathcal{E}_0$, although even more general and complex settings are cogitable. Let Z denote the radial part of the f-implicit sup-measure, where its distribution is not merely α-Fréchet but implicitly given by

$$\mathbb{P}(Z \leq x \,|\, \Theta = \vartheta) := F_{Z\,|\,\Theta=\vartheta}(x) := F_\vartheta(x) := e^{-\varsigma(\vartheta)x^{-\alpha}}, \quad x > 0. \tag{3.3.1}$$

Here, the function $\varsigma : S \to (0, \infty)$ is *nice*. The expression *nice* signifies some unspecified conditions on ς such as continuity, differentiability or other conceivable smoothness properties. In other words, (3.3.1) means that the conditional distribution of Z given the occurrence of the value ϑ of Θ is an α-Fréchet distribution with scale $\varsigma(\vartheta)^{1/\alpha}$. Clearly, Z and Θ are dependent. Note that the setting of Section 3.1 can be recovered by the choice $\varsigma \equiv 1$. To investigate the recently formulated question, it seems wise to compute the distribution of the $\mathbb{R}^d$-valued random vector $X := \kappa^{1/\alpha} Z\Theta$, $\kappa > 0$. We see at once that

$$\mathbb{P}(X \in D(r, F)) = \int\limits_F e^{-\varsigma(\vartheta)\kappa r^{-\alpha}}\, \sigma(d\vartheta)$$

for all $r > 0$ and $F \in \mathcal{B}(S)$. As usual, the distribution of X equals ε_0 if $\kappa = 0$. Having established this formula, we can study the consequences of the condition

$$M(A_1) \vee_f M(A_2) \overset{d}{=} M(A_1 \cup A_2),$$

following from property (iii) of an f-implicit sup-measure as it might imply that ς is constant. This would indicate that Z and Θ must necessarily be independent and that Z has an α-Fréchet distribution, thus revealing that our definition of an f-implicit sup-measure is in some sense the most general one. If it proves otherwise, that is, if ς is not necessarily constant, we could take this as evidence that there exists a more general notion of f-implicit sup-measures. However, this assertion would still be subject to review. As a side note, observe that the random variable $f(X) = \kappa^{1/\alpha} Z$ having a continuous cumulative distribution function on $(0, \infty)$ implies X is f-implicit max-infinitely divisible. Hence, it might be a challenging task to evaluate a possible nth root of X for all $n \geq 1$. We do not address ourselves to this task but proceed with a more detailed consideration of a possible extension of the notion of f-implicit extremal stochastic integrals.

2. Extension of the f-implicit extremal stochastic integrals. Taking the concept of f-implicit sup-measures as a basis, we established the notion of f-implicit extremal stochastic integrals in Section 3.2. For the time being, we confined ourselves to particular non-random integrands. Namely, we considered simple functions as suitable

integrands only. This inevitably leads to the question of whether we can extend the class of possible integrands. Investigating this question is particularly worthwhile with regard to the construction of further examples of f-implicit max-stable processes. A more extensive class of feasible integrands definitely yields further f-implicit max-stable processes by adopting the construction used in Proposition 3.2.6. In addition, an extension of the class of non-random integrands provides a more general notion of f-implicit extremal stochastic integrals being convenient for the study of closer connections between our theory and those introduced in [SaTa94, Chapter 3] and [StTa05], respectively. Finally, it might be beneficial to extend the notion of f-implicit extremal stochastic integrals to establish representations of f-implicit max-stable processes in terms of f-implicit extremal stochastic integrals (see point 3).
Conforming to the respective statements of [SaTa94, Chapter 3] and [StTa05], we sensibly suppose that a possible class of non-random integrands might consist of all measurable, non-negative functions $g : E \to \mathbb{R}$ such that

$$\int_E g^{\alpha}(u)\, m(du) < \infty.$$

However, this is just a conjecture that still needs to be proved. Other classes of possible integrands could also be conceivable. Having an idea of a suitable class of integrands, we proceed with the plan of how to realize the f-implicit extremal stochastic integral of a non-simple function g. Here, we can only conjecture whether it is better to adopt the approach of [SaTa94, Chapter 3] or of [StTa05], or whether it is prudent to contrive a new method. Both [SaTa94] and [StTa05] pursue efficient but quite different ways in their extensions to more general integrands. Unfortunately, both of them have their disadvantages in the f-implicit context. Whereas the approach in [SaTa94], that is, realizing the α-stable stochastic integral as a limit in probability (see [SaTa94, Section 3.4]), tremendously gains from the structure of the corresponding characteristic functions, the approach in [StTa05] essentially takes advantage of a monotonicity structure. Both the concept of characteristic functions and the monotonicity cannot be applied in our setting. Consequently, we cannot adopt one of the approaches one-to-one. Yet, the conspicuous connection between our f-implicit setting and the issues considered in [StTa05] and [SaTa94] suggests that we can profit from at least some ingenious ideas. On the one hand, the notion of α-Fréchet spaces studied in [StTa05] seems promising in this context. As we cannot use Lemma 2.2 of [StTa05], a different strategy has to be developed. Here, the notion of f-implicit α-Fréchet spaces and a clearer comprehension of the f-implicit max-order could be helpful. We refer to a set $\mathcal{M}_{\alpha}^{f} := \mathcal{M} \subset L_0^d$ as *f-implicit α-Fréchet space* if it is closed under taking f-implicit max-linear combinations and consists of jointly f-implicit α-Fréchet random vectors. That is, if:

(i) *(f-implicit max-linear space)*
For all $n \geq 1$, any choice of random vectors $\xi_1, ..., \xi_n \in \mathcal{M}$ and any choice of real numbers $\alpha_1, ..., \alpha_n \geq 0$, the random vector

$$\xi := \bigvee_{i=1}^{n}{}_{f}\; \alpha_i\, \xi_i$$

belongs to $\mathcal{M}$.

(ii) *(jointly f-implicit α-Fréchet)*
Any f-implicit max-linear combination of elements of $\mathcal{M}$ is f-implicit α-Fréchet.

In particular, the space of all f-implicit extremal stochastic integral of simple functions is an f-implicit α-Fréchet space being an easy consequence of Proposition 3.2.6.
On the other hand, the approach pursued in [SaTa94, Chapter 3] could be effective even though we cannot apply the notion of characteristic functions. This is because of the fact that Proposition 3.2.4 yields the explicit distribution of the f-implicit extremal stochastic integrals and thus provides a useful substitute for characteristic functions. Following the approach in [SaTa94, Chapter 3], we may therefore realize the (general) f-implicit extremal stochastic integral as a limit in probability.
Further considerations concerning the recently discussed purpose are skipped for the time being. Anyhow, this seems to be an attractive research project with good prospects.

3. More detailed considerations of f-implicit max-stable processes. Fix an arbitrary and non-empty index set T and let $(X_t)_{t\in T}$ be an f-implicit max-stable process. If we are now interested in sample path properties of such processes, it might be convenient to study them via an integral representation of the type

$$X_t = \int_E g_t(u)\, M(du), \quad t \in T, \tag{3.3.2}$$

the process $(M(A))_{A\in\mathcal{E}_0}$ being some f-implicit sup-measure and the functions $g_t, t \in T$, being suitable non-random integrands. In other words, it seems advantageous to have a representation of an f-implicit max-stable process $(X_t)_{t\in T}$ in terms of an f-implicit extremal stochastic integral in combination with a suitable family of non-random functions $(g_t)_{t\in T}$. Besides, this underlines the tremendous benefit of an appropriate extension of the notion of f-implicit extremal stochastic integrals as described in the previous point.
The idea of studying path properties of f-implicit max-stable processes via (3.3.2) originates from [SaTa94, Chapters 9-12] where sample path properties of α-stable processes, such as separability, continuity, boundedness or oscillation properties, have extensively been investigated by using this method. Moreover, it has been used in [St10, Section 3] in order to facilitate the study of ergodic properties of α-Fréchet processes. To put it straight, the crucial question arises of whether it is possible to realize an f-implicit max-stable process via (3.3.2). This question is not novel but has already been considered both in [SaTa94, Section 3.11 and Chapter 13] for α-stable processes and in [StTa05, Proposition 3.2] for α-Fréchet processes. At this point it should be emphasized that first attempts regarding this question originate from [BrDaKr66], [Schi70], [Schr72], [Kue73] and [Ha84] for the α-stable setting and from [deHa84] for the max-stable one. Some further recent studies concerning this issue are also given in [St08], [St10] and [Ka09]. Instead of going into detail here, we shall only point out that both in the α-stable and in the max-stable case there exists a condition assuring suitable integral representations so that $(E, \mathcal{E}, m)$ can be chosen as $((0,1), \mathcal{B}((0,1)), \lambda^1)$. More precise expositions can be found, for example, in [SaTa94, Theorem 13.2.1] and in [StTa05, Proposition 3.2]. It is

even possible to find integral representations for all symmetric α-stable (see [SaTa94, Theorem 13.2.2]) and all α-Fréchet processes (see [Ka09, Theorem 1]), if the underlying measure space is allowed to be more general. The latter deliberations suggest that it might also be possible to achieve similar results for f-implicit max-stable processes. This, however, is not pursued in the present thesis. Nevertheless, corresponding results could reveal the structure of f-implicit max-stable processes immensely.

To come to a conclusion of this section, we shall finally mention some other suggestions for extensions going beyond the previous ones.
First, it might be beneficial to provide a more flexible notion of f-implicit sup-measures by letting the distribution of the angular part Θ be dependent on the sets $A \in \mathcal{E}_0$. However, this is only possible if we dispense with the assumptions of Proposition 3.1.20 since otherwise $\mathbb{P}_\Theta$ is necessarily independent of $A \in \mathcal{E}_0$. Nevertheless, in more general settings there could exist some intricacies allowing specific dependence structures between the angular part of an f-implicit sup-measure and the sets $A \in \mathcal{E}_0$. Answering the question of whether the recent suggestion can actually occur seems extremely interesting since possible dependences open up further possibilities. In particular, the resultant *angular process* $(\Theta(A))_{A\in\mathcal{E}_0}$ would be a good object of research. Furthermore, mathematical modeling of such processes could be an attractive point as it would be important in statistical investigations.
Second, in consideration of Theorem 3.1.12 we recognize that the existence of an f-implicit sup-measure is a consequence of the seminal Kolmogorov's extension theorem. Therefore, it might be worthwhile to examine whether an f-implicit sup-measure can also be derived constructively. In other words, a more detailed investigation of the structure of f-implicit sup-measures in view of feasible and explicit representations could be an exciting object of study. In this context it seems conceivable to construct an f-implicit sup-measure by combining the notions of $\mathbb{R}^d$-valued α-stable random measures and random α-Fréchet sup-measures. This is reasonable because of the close connections existing between the three theories. A sensible approach is to define $M(A)$ for all $A \in \mathcal{E}_0$ as the product of $M_\alpha(A)$ and a measurable transformation of $\xi(A)$, the real-valued process $(M_\alpha(A))_{A\in\mathcal{E}_0}$ here being a random α-Fréchet sup-measure and $(\xi(A))_{A\in\mathcal{E}_0}$ being an $\mathbb{R}^d$-valued α-stable random measure. Of course, the measurable transformation should depend on the loss function f. In this particular case $(M_\alpha(A))_{A\in\mathcal{E}_0}$ represents the radial part of $(M(A))_{A\in\mathcal{E}_0}$ and the transformation of $(\xi(A))_{A\in\mathcal{E}_0}$ its angular part. The dependence structure of $(M(A))_{A\in\mathcal{E}_0}$ is then heavily related to the one between $(M_\alpha(A))_{A\in\mathcal{E}_0}$ and $(\xi(A))_{A\in\mathcal{E}_0}$. This suggestion is definitely significant as it provides a clearer insight into the theories of $\mathbb{R}^d$-valued α-stable random measures, random α-Fréchet sup-measures and f-implicit sup-measures.
Third and finally, conceivable extensions can be derived by relaxing the assumptions on f as has already been noted in Section 2.4. Instead of itemizing the respective possibilities once again, we conclude with Chapter 3 and proceed with the last chapter.

4 Concluding remarks

This chapter is designed to bring the present thesis to an appropriate conclusion, two specific goals being pursued in detail. On the one hand, we recapitulate our findings of the present thesis and point out the significance of these results in the context of an f-implicit extreme value theory pioneered by [SchSt14]. In addition, we define our results against the backdrop of research that has already been published. On the other hand, we revisit the open problems that appeared in the course of the preceding chapters and give several suggestions for additional research possibilities exceeding the ones considered in Section 2.4 and 3.3. In this context, we stress the promising future of the f-implicit extreme value theory as there are various open problems and many further interesting issues to be solved.

4.1 Summary

My original interest of research was to extend the seminal findings in [SchSt14]. In particular, my main concern was to achieve proper results on implicit extremes and f-implicit max-stable laws under relaxed assumptions on the loss function f. At the center of my studies was the question whether it would be possible to prove corresponding results if f were assumed to be E-homogeneous instead of 1-homogeneous. Here, E denotes some suitable matrix. At the early stages of this project I realized that this might be a promising field, especially since there existed a supportive body of literature. As to that, see for example [LiReRo14] and [MeSch01] to just name a few. During my research work on this issue I noticed, however, that there were even more promising aspects to be considered, which eventually led to the current form of this thesis.
As regards content, we can divide the present thesis into two main parts. It refers to the notion of f-implicit max-infinitely divisible distributions on the one hand and f-implicit max-stable processes on the other. Both the idea of f-implicit max-infinitely divisible distributions and the idea of f-implicit max-stable processes actually stem from the ambitious goal to extend the theory of f-implicit max-stable distributions considered in [SchSt14].

The first step of an expedient approach to the notion of f-implicit max-infinitely divisible distributions and f-implicit max-stable processes, however, was to establish a profound and theoretical basis. This was for the most part done in the first chapter. In particular, we introduced a new binary operation on $\mathbb{R}^d$ being referred to as f-implicit max-operation. Availing ourselves of this specific operation, we subsequently established both the notion of f-implicit max-convolution and the f-implicit max-order. In addition to these concepts, we finally studied the distribution of the random vector $X_{k(n)}$ in more detail. Here, we supplied many possibilities to compute $\mathbb{P}(X_{k(n)} \in A)$ explicitly.

The idea of considering all these aspects in Chapter 1 arose at the very beginning of our studies on f-implicit max-infinitely divisible distributions and f-implicit max-stable processes. It was motivated by strategies having already been used in extreme value theory as well as by studies on infinitely divisible distributions and α-stable processes. At this point it is worth to underline that Chapter 1 does not only prepare the results of the subsequent chapters, but contains many observations that are of independent interest. Therefore, these topics being novel and having never been considered in earlier research work, except for some parts of Lemma 1.4.1, do not only serve as a suitable basis for our specific purpose, but might rather gain in interest in the context of a much broader theoretical framework.

Equipped with the findings in Chapter 1, we delved into the notion of f-implicit max-infinitely divisible distributions and thus into the first main part of the present thesis. Here, we were strongly guided by the well-established and historically important concepts of infinitely and max-infinitely divisible distributions. In fact, we realized that these concepts are quite similar and depend significantly on the underlying summation and maximum operation. This ultimately led, by replacing the respective operations with the f-implicit max-operation, to the existing notion of f-implicit max-infinitely divisible distributions.
Having established the notion of f-implicit max-infinitely divisible distributions in Definition 2.1.1, we then proceeded to consider the question whether there exists a potential characterization of the class of f-implicit max-infinitely divisible distributions. Regarding this concern, we essentially presented two basic results. We proved that both all random vectors X in $\mathbb{R}^d$ such that $x \mapsto \mathbb{P}(f(X) \leq x)$ is continuous on (ℓ, ∞), the non-negative real number ℓ here being the left endpoint of $\mathbb{P}_{f(X)}$, and all random vectors X in $\mathbb{R}^d$ such that the mass of $\mathbb{P}_{f(X)}$ is concentrated on a countable subset of $[0, \infty)$ are f-implicit max-infinitely divisible. In this context, we gained from the notion of f-implicit max-convolution semigroups as well as from some beneficial and tailor-made substitution rules for Riemann-Stieltjes integrals that had been provided in advance. In addition, we also profited enormously from Lemma 1.4.1.
After having devoted ourselves to the previously mentioned aspects, we gave further examples of f-implicit max-infinitely divisible distributions which are not counted among the classes of distributions considered in Corollary 2.2.21 and Theorem 2.2.24, respectively. Among other indicators, these examples finally gave occasion to the assertion stated in Conjecture 2.4.1. Alleging reasons for the verisimilitude of that conjecture as well as suggesting a possible approach to a proper proof eventually constituted the remainder of Chapter 2. Here, we also broached the notion of f-implicit max-compound Poisson distributions and f-implicit max-compound Poisson processes for the first time as these two closely linked concepts seem to be indispensable for deeper considerations on Conjecture 2.4.1. The idea to use the notion of f-implicit max-compound Poisson distributions and f-implicit max-compound Poisson processes as a promising technique actually originates from [MeSch01, Chapter 3]. In fact, there exists a remarkably close connection between f-implicit max-compound Poisson distributions and generalized Poisson distributions, the latter being an essential tool in the proof of the common Lévy-Khintchine representation for infinitely divisible distributions (see for example

[MeSch01, Chapter 3]). In addition, this close connection indicates an abundance of further research possibilities in this particular field which has already been amplified in Section 2.4.

In the second part we devoted ourselves to another extension of the notion of f-implicit max-stable distributions. We attempted an appropriate approach to the notion of f-implicit max-stable processes by following fundamental ideas and techniques of the theories of max- and α-stable processes, respectively. Here, the theory of max-stable processes was especially conducive to our targeted purpose. Indeed, the main parts of the existing form of Chapter 3 were obtained with the use of [SchSt14] and [StTa05].
Our first step was to establish a clear definition of an f-implicit max-stable process. Motivated by the fact that each f-implicit max-stable process automatically yields a max-stable process with α-Fréchet marginals, that is, an α-Fréchet processes, we eventually formulated our main goal: the construction of non-trivial examples of f-implicit max-stable processes. To achieve this ambitious aim, we availed ourselves of an ingenious technique that had already been applied analogously in [StTa05] in the context of studies on α-Fréchet processes. There is talk of the notion of f-implicit sup-measures and f-implicit extremal integrals bearing strong resemblance to the ones of random α-Fréchet sup-measures and extremal stochastic integrals. Moreover, there also exists an analogy between the latter and the notion of α-stable random measures and α-stable stochastic integrals which occur in [SaTa94] and yield a more profound comprehension of α-stable processes.
Having had the idea of f-implicit sup-measures and f-implicit extremal integrals in mind, we proceeded to put them into concrete terms. In doing so, we focused on the notion of f-implicit sup-measures first. To this end, we began by studying f-implicit max-stable distributions in more detail which in turn induced us to institute the notion of f-implicit α-Fréchet distributions. These distributions constitute a specific subclass of f-implicit max-stable distributions and rate within the framework of f-implicit extreme value theory as some kind of counterpart of α-Fréchet distributions. Equipped with these distributions, we established the notion of f-implicit sup-measures defined on an appropriate probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ and succeeded in proving the existence of such objects. Basically, for any measure space $(E, \mathcal{E}, m)$, the idea was to consider an f-implicit sup-measure $M : \mathcal{E}_0 \to L_0^d$ as a stochastic processes $(M(A))_{\mathcal{E}_0}$ on $(\Omega, \mathfrak{A}, \mathbb{P})$, the sets $\mathcal{E}_0$ and L_0^d being defined by $\mathcal{E}_0 = \{A \in \mathcal{E} : m(A) < \infty\}$ and $L_0^d = \{X : \Omega \to \mathbb{R} : X \text{ is a random vector}\}$, respectively. Without doubt, the latter is one of the basic results in Chapter 3 and should be appreciated particularly. Using the notion of f-implicit sup-measures, we were then actually able to provide a first example of an f-implicit max-stable process. Moreover, we supplied a far less complicated proof of the existence of a random α-Fréchet sup-measure compared to that one proposed in [StTa05, Proof of Proposition 2.1]. Namely, each f-implicit sup-measure $(M(A))_{A \in \mathcal{E}_0}$ yields immediately a random α-Fréchet sup-measure by $(f(M(A)))_{A \in \mathcal{E}_0}$. Despite the latter aspects, our notion of an f-implicit sup-measure was still not entirely satisfactory as we assumed the angular part of $(M(A))_{A \in \mathcal{E}_0}$ to be independent of $A \in \mathcal{E}_0$. It was only when we presented Proposition 3.1.20 that our specific notion of an f-implicit sup-measure turned out to be highly prudent and fairly general. In conclusion, our no-

tion of f-implicit sup-measures did not only prove to be a suitable approach to find examples of f-implicit max-stable processes but is ultimately also a sophisticated construct of independent interest in f-implicit extreme value theory.
After having concerned ourselves with the preceding aspects, we approached the last part of Chapter 3 and thus the last part of the content of the present thesis. We established the notion of f-implicit extremal integrals, that is, integrals of non-random functions with respect to an f-implicit sup-measure. Here, the way of proceeding was considerably guided by [StTa05] and [SaTa94]. To come straight to the point, however, we considered simple integrands only, that is, we introduced the f-implicit extremal integral only for non-random integrands being simple. A generalization to a more extensive class of integrands was postponed and thus remains unsolved for the time being. For all that, we succeeded in providing several desirable properties of our f-implicit extremal integral which in turn resulted in Proposition 3.2.6 and Remark 3.2.7. The latter eventually revealed the real practical usefulness of f-implicit extremal integrals. Namely, similar to [StTa05] and [SaTa94], the notion of f-implicit extremal integrals is likewise perfectly suited to construct f-implicit max-stable processes.

In conclusion, it is to say that we were indeed able to provide a new theoretical framework. This field of f-implicit extreme value theory is based on the pioneering achievements in [SchSt14], whereas the idea of the particular contents examined here does primarily originate from quite similar and already well-established branches of probability theory. Of course, the concepts of f-implicit max-infinitely divisible distributions and f-implicit max-stable processes are just the beginning of an appropriate establishment of f-implicit extreme value theory. As to that aspect, Section 2.4 and Section 3.3 have already demonstrated the wealth of possible issues that still need to be investigated. In addition, Section 4.2 will provide further proposals of interesting topics that might be worth to be considered in more detail.

4.2 Final outlook

The field of f-implicit extreme value theory introduced in this thesis is still in its initial stages. The specific contents based on the pioneering achievements in [SchSt14] are therefore just a small part of a considerably larger theoretical framework that yet needs to be refined in the next years. As regards that matter, we have already supplied many suggestions for conceivable refinements and extensions. Of particular significance are surely two aspects occurring in Section 2.4 and 3.3, respectively. On the one hand, there is talk of fixing the still unsolved assertion stated in Conjecture 2.4.1. On the other hand, there is talk of extending the notion of f-implicit extremal integrals in such a way that even certain non-simple functions can be integrated. But as we have already dwelt upon these topics, we will now rather proceed to expand them by two further ideas that have not been broached so far and can neither be allocated entirely to Chapter 2 nor to Chapter 3.

1. Operator norming limit theorems for implicit extremes. As mentioned previously, my initial concern was to achieve proper results on implicit extremes and f-implicit max-stable laws under relaxed assumptions on the loss function f. Yet, I abandoned this particular objective target and addressed myself to the contents given in this paper. Nevertheless, requiring f to be E-homogeneous instead of just simply 1-homogeneous is still an existing and desirable modification of the assumptions on the loss function f. Actually, the purpose to extend the results stated in [SchSt14] under this more general setting bears a great challenge. To be more accurate, let $E = (e_{ij})_{1\le i,j\le d} \in \mathbb{R}^{d\times d}$ denote some matrix. For convenience, we assume that all eigenvalues of E have positive real part. Then we are particularly interested in the following issues. Which assumptions are sufficient to ensure the existence of a sequences $(a_n)_{n\ge 1} \subset (0,\infty)$ such that

$$a_n^{-E} X_{k(n)} \underset{(n\to\infty)}{\Longrightarrow} Y \tag{4.2.1}$$

for some suitable limit Y? Moreover, what can be said about the limits arising in (4.2.1)? Are there any representations for the distribution of Y? Does there possibly exist a complete characterization of the class of suitable limits? And finally, can the f-implicit (maximum) domain of attraction of a random vector Y be characterized, the latter being understood analogously to Definition 4.3 in [SchSt14]? All these questions come within the field of operator norming limit theorems for implicit extremes and indeed extend the contents in [SchSt14], provided they would be answered.
As already proposed in [SchSt14, Section 3], it is natural to work in the context of regular variation here. However, the classical notion of regular variation is not suitable. Since we are dealing with operator norming limit theorems for implicit extremes, we need to apply the notion of operator regular variation which was developed and used extensively for the study of sums of independent random vectors whose tail index can vary with direction (see [MeSch01, Chapter 5-6]). This notion involves the space $\mathbb{R}^d \setminus \{0\}$ and the topology of $\mathbb{M}^*$-convergence of Borel measures on $\mathbb{R}^d \setminus \{0\}$ which are finite on regions bounded away from the origin. A precise definition of $\mathbb{M}^*$-convergence can be found, for example, in [MeSch01] or in [HuLi06].
On top of that, however, the previously mentioned notion of operator regular variation is still not suitable. Instead, we need to work in the context of an even more general notion involving arbitrary metric spaces. This stems from observations that have already been made in [SchSt14], where the space $\mathbb{R}^d \setminus \{f = 0\}$ evidently looms large. Since f is E-homogeneous, $\{f = 0\}$ is an *E-cone in* $\mathbb{R}^d$ meaning that $\lambda^E\{f = 0\} \subset \{f = 0\}$ for all $\lambda > 0$. Therefore, we conclude that both a more detailed study of closed E-cones in $\mathbb{R}^d$ and the establishment of a sensible notion of *regular variation on E-cones* are necessary in order to be able to approach the above questions. Succeeding with the purpose of providing operator norming limit theorems for implicit extremes thus depends on a profound theoretical groundwork. Here, both [MeSch01] and [LiReRo14] might be quite useful.
In conclusion, the preceding deliberations reveal that the topic of operator norming limit theorems for implicit extremes is indeed a challenging and still unconsidered issue that serves well as an object of study in the context of f-implicit extreme value theory.

2. f-implicit extremal processes. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables in $\mathbb{R}$ with cumulative distribution function F. Suppose further that F is in the (maximum) domain of attraction of a nondegenerate limit function

H, that is, there exist sequences $(a_n)_{n\geq 1} \subset (0,\infty)$ and $(b_n)_{n\geq 1} \subset \mathbb{R}$ such that

$$F^n(a_n x + b_n) \xrightarrow[(n\to\infty)]{} H(x) \tag{4.2.2}$$

for all continuity points x of H. From classical extreme value theory we infer that H is necessarily of the same type of one of the extreme value distributions which are in particular given by the α-Fréchet distribution, the Weibull distribution and the Gumbel distribution. Hence, the convergence in (4.2.2) is nothing but pointwise convergence. Now, define the sequence $(\Upsilon^{(n)})_{n\geq 1}$ of stochastic processes $\Upsilon^{(n)} = \left(Y_t^{(n)}\right)_{t>0}$ by

$$Y_t^{(n)} = \begin{cases} a_n^{-1}(M_{\lfloor nt\rfloor} - b_n), & \text{if } t \geq n^{-1} \\ a_n^{-1}(X_1 - b_n), & \text{if } 0 < t < n^{-1}, \end{cases}$$

where $\lfloor\cdot\rfloor : \mathbb{R} \to \mathbb{R}$ denotes the floor function and M_n the maximum of the random variables $X_1, ..., X_n$ for all $n \geq 1$. Then

$$\Upsilon^{(n)} \xrightarrow[(n\to\infty)]{J_1} \Upsilon, \tag{4.2.3}$$

where the limiting processes $\Upsilon = (Y_t)_{t>0}$ is an *extremal process generated by* H, that is, Υ has the following finite dimensional distributions

$$\mathbb{P}\left(Y_{t_1} \leq x_1, ..., Y_{t_k} \leq x_k\right) = H^{t_1}\left(\bigwedge_{i=1}^{k} x_i\right) H^{t_2-t_1}\left(\bigwedge_{i=2}^{k} x_i\right) \cdots H^{t_k-t_{k-1}}(x_k) \tag{4.2.4}$$

for $k \geq 1$, $0 < t_1 < ... < t_k$ and $x_1, ..., x_k \in \mathbb{R}$. In deference to a common usage, the expression $a \wedge b$ stands for the minimum of $a, b \in \mathbb{R}$. The convergence in (4.2.3) has first been proved in [La64, Theorem 3.2], whereas a thorough treatment of the limiting process Υ can be found in [Dw64]. For an even deeper discussion of the notion of extremal processes and the Skorohod space equipped with the topology of J_1-convergence we refer to [Re07, Chapter 4]. Especially, Proposition 4.20 stated therein plays a decisive role as it refines the assertion given above. Here, the author works in the context of point processes that turn out to be a powerful tool.

As remarked in [Dw64], extremal processes generated by some extremal distribution H bear a natural analogy with stable processes. That is, the latter observations apply similarly if we exchange the maximum operation for the summation operation. Instead of extremal processes generated by some extremal distribution H, however, the limiting processes $\Upsilon = (Y_t)_{t>0}$ are stable ones. Since we recently elaborated on these aspects in the case of the maximum operation, we will not go into detail here once again but refer to [MeSch01] and the references given therein. Nevertheless, the previous deliberations raise the idea to consider the following issues.

Let $X, X_1, X_2, ...$ be independent and identically distributed random vectors in $\mathbb{R}^d$. Define the sequence

$$\left(\Upsilon_f^{(n)}\right)_{n\geq 1} = \left(\left(Y_f^{(n)}(t)\right)_{t>0}\right)_{n\geq 1}$$

of stochastic processes by

$$Y_f^{(n)}(t) = \begin{cases} a_n^{-1} X_{k(\lfloor nt \rfloor)}, & \text{if } t \geq n^{-1} \\ a_n^{-1} X_1, & \text{if } 0 < t < n^{-1}, \end{cases} \tag{4.2.5}$$

$(a_n)_{n\geq 1}$ here being some suitable sequence of positive real numbers. Do there exist sufficient and necessary conditions ensuring the existence of a sequence $(a_n)_{n\geq 1} \subset (0, \infty)$ such that

$$\Upsilon_f^{(n)} \underset{(n\to\infty)}{\overset{\text{fdd}}{\Longrightarrow}} \Upsilon_f \tag{4.2.6}$$

for an appropriate limiting process $\Upsilon_f = (Y_f(t))_{t>0}$, where $\overset{\text{fdd}}{\Longrightarrow}$ means weak convergence of finite dimensional distributions? Do there maybe even exist sufficient and necessary conditions ensuring the existence of a sequence $(a_n)_{n\geq 1} \subset (0, \infty)$ such that

$$\Upsilon_f^{(n)} \xrightarrow[(n\to\infty)]{J_1} \Upsilon_f \tag{4.2.7}$$

for a limiting process $\Upsilon_f = (Y_f(t))_{t>0}$? Moreover, provided (4.2.6) or (4.2.7) were fulfilled, what can be said about the limiting process being referred to as *f-implicit extremal process*? What properties can be derived and what can be said about the finite dimensional distributions of Υ_f?
On account of corresponding results in the context of extremal processes, we may sensibly suggest that X being in the f-implicit (maximum) domain of attraction of a necessarily f-implicit max-stable random vector Y is a reasonable and sufficient assumption for (4.2.6) or (4.2.7) to hold. Finally, as to the structure of an f-implicit extremal process Υ_f it is crucial to consider the question whether the process can be expressed in terms of an appropriate f-implicit sup-measure or an f-implicit extremal integral. All these questions and first suggestions provide starting points for further possible objects of research. In this way, the common notion of extremal processes generated by some extremal distribution H could be translated into the context of f-implicit extreme value theory.

Indeed, there are plenty of further conceivable possibilities to refine the field of f-implicit extreme value theory. Primarily, this concerns statistical and computational extensions which have been disregarded completely in the present thesis. Nevertheless, we will not go into detail here but rather conclude our deliberations.

List of symbols

Numbers

$\mathbb{N}$	set of all positive integers
$\mathbb{N}_0$	set of all non-negative integers
$\mathbb{Q}$	set of all rational numbers
$\mathbb{Q}_{>0}$	set of all positive rational numbers
$\mathbb{R}$	set of all real numbers
$\mathbb{R}_+$	set of all non-negative real numbers
$\mathbb{R}^d$	real coordinate space of $d \geq 1$ dimensions
$\mathbb{C}$	set of all complex numbers

Norms

$\lvert \cdot \rvert$	absolute value of a real number
$\lVert \cdot \rVert$, $\lVert \cdot \rVert_2$	reference norm on $\mathbb{R}^d$, Euclidean norm on $\mathbb{R}^d$

Functions

exp, ln	exponential function, natural logarithm
pr_i	canonical projection on the ith coordinate
$\mathrm{pr}_{1,\ldots,n}$	canonical projection on the first n coordinates
$\mathbb{1}_A$	indicator function of the set A
f	fixed loss function
G	cumulative distribution function of $f(X)$

$T^{(n)} = T_f^{(n)}$	mapping defined by $T^{(n)}(x_1, ..., x_n) = \bigvee_{i=1}^{n}{}_f\, x_i, \;\; n \geq 1$ fixed
V_ρ	tail function of $f(\rho)$, where ρ is a suitable measure
g^{-1}	inverse function of a bijection g

Generalized polar coordinates in $\mathbb{R}^d \setminus \{0\}$

(τ, θ)	generalized polar coordinates in $\mathbb{R}^d \setminus \{0\}$
$S = S_\tau$	unit sphere with respect to τ
$T = T_\tau$	homeomorphism assigning to $x \in \mathbb{R}^d \setminus \{0\}$ its polar coords.

Note that we specifically apply f to τ in this thesis.

Measure and probability theory

$\Omega, \mathfrak{A}, \mathbb{P}$	sample space, σ-algebra, probability law
$(\Omega, \mathfrak{A}, \mathbb{P})$	arbitrary probability space
$(E, \mathcal{E}, m)$	arbitrary measure space
$\mathcal{E}_0$	set of subsets of E having finite mass with respect to m
$\mathfrak{A}(\mathcal{E})$	σ-algebra generated by the family of sets $\mathcal{E}$
$\mathcal{B}(E)$	Borel σ-algebra on a topological space E
$\mathfrak{A}_1 \otimes \mathfrak{A}_2$	product of the σ-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_1$
$\mathfrak{A}^{\otimes n}$	n-fold product of the σ-algebra $\mathfrak{A}$
$M^1(E)$, $M^b(E)$	set of probability and bounded measures on E, respectively
$\mu_1 \otimes \mu_2$	product measure of μ_1 and μ_2
$\bigotimes_{i=1}^{n} \mu_i$	product measure of $\mu_1, ..., \mu_n$
$\mu^{\otimes n}$	n-fold product measure of μ
$\mathbb{P}_X$	distribution of the random vector X

$X \sim \mu$	X has distribution μ
$\mu_1 \ll \mu_2$	μ_1 is absolutely continuous with respect to μ_2
$\mu\big\|_A$	restriction of μ to the set A
$\operatorname{supp} \mu$	support of the measure μ
$\mu_1 *_f \mu_2$	f-implicit max-convolution of $\mu_1, \mu_2 \in M^b(\mathbb{R}^d)$
$\mu^{*_f n}$	n-fold f-implicit max-convolution of $\mu \in M^b(\mathbb{R}^d)$
$\stackrel{d}{=}$	equality in distribution
$\Longrightarrow$, $\xrightarrow{J_1}$	convergence in distribution, J_1-convergence
$\stackrel{\text{fdd}}{\Longrightarrow}$	weak convergence of finite dimensional distributions

Sets

T, I	arbitrary (index) set, non-empty (real) interval
∂A, A°, $\overline{A}$, A^c	boundary, interior, closure and complement of the set A
$g^{-1}(A)$	preimage of the set A under g
S^1	unit sphere in $\mathbb{R}^d$ with respect to $\|\| \cdot \|\|$
$K_\varepsilon(x)$	open ball in $\mathbb{R}^d$ with radius $\varepsilon > 0$ and center x
$D(r, F)$	subset of $\mathbb{R}^d \setminus \{0\}$ defined by $\{x : \tau \le r, \theta \in F\}$, $r > 0, F \in \mathcal{B}(S)$
$D^*(r, F)$	subset of $\mathbb{R}^d \setminus \{0\}$ defined by $\{x : \tau > r, \theta \in F\}$, $r > 0, F \in \mathcal{B}(S)$
L, Γ	$\{f = \ell\}$ and $\{f > \ell\}$, respectively, ℓ left end point of $f(X)$
L_{ℓ_0}, Γ_{ℓ_0}	$\{f = \ell_0\}$ and $\{f > \ell_0\}$, respectively, $\ell_0 \ge 0$ arbitrary
$C(H) = C_f(H)$	set of all $x \in \mathbb{R}^d$ such that H(f(x)-)=H(f(x))
$\mathcal{D}(H) = \mathcal{D}_f(H)$	set of all $x \in \mathbb{R}^d$ such that H(f(x)-)<H(f(x))

Sets of functions, random variables, random vectors and sequences

$L_0 = L_0(\Omega, \mathfrak{A})$	set of all random variables $X : \Omega \to \mathbb{R}$

$L_0^d = L_0^d(\Omega, \mathfrak{A})$	set of all random vectors $X : \Omega \to \mathbb{R}^d$
χ_0^f	set of all sequences $(x_n)_{n\geq 1} \subset \mathbb{R}^d$ with $\lim_{n\to\infty} f(x_n) = 0$
$\mathcal{M} = \mathcal{M}_\alpha^f$	f-implicit α-Fréchet space
$C^k = C^k(\mathbb{R}^n, \mathbb{R}^m)$	differentiability class of order $k \in \mathbb{N}_0 \cup \{\infty\}$ of $f : \mathbb{R}^n \to \mathbb{R}^m$
$\mathcal{R}[a, b]$	set of all regulated functions $g : [a, b] \to \mathbb{R}$

Special distributions

ε_a	Dirac measure centered on $a \in \mathbb{R}^d$
$\Phi_{\alpha,\sigma}^f, \alpha > 0$	standard f-implicit α-Fréchet distribution, $\sigma \in M^1(S)$
$\Phi_{\alpha,\sigma}^f(\kappa), \alpha > 0$	f-implicit α-Fréchet distribution with scale $\kappa \geq 0$, $\sigma \in M^1(S)$
$\Phi_\alpha, \alpha > 0$	standard α-Fréchet distribution
$\Phi_\alpha(\kappa), \alpha > 0$	α-Fréchet distribution with scale $\kappa \geq 0$
$X \sim [\nu]_f$	$\mathbb{P}_X$ comes under the 1st class of distributions with $G(\ell) = 0$
$X \sim [\rho_L, \nu]_f$	$\mathbb{P}_X$ comes under the 1st class of distributions with $G(\ell) > 0$
$\Pi_f(c, \rho_1, \rho_2)$	f-implicit max-compound Poisson distribution

Operations and operators

$\langle \cdot, \cdot \rangle$	standard scalar product on $\mathbb{R}^d$
$\lfloor \cdot \rfloor$	largest previous integer of a real number (floor function)
∇	nabla operator
$x \wedge y = \min(x, y)$	minimum of $x, y \in \mathbb{R}$
$x \vee y = \max(x, y)$	maximum of $x, y \in \mathbb{R}$
$\vee_f$	f-implicit maximum
$\bigvee_{i=1}^n x_i$	maximum of $x_1, ..., x_n \in \mathbb{R}$
$\bigwedge_{i=1}^n x_i$	minimum of $x_1, ..., x_n \in \mathbb{R}$

$x_{k(n)} = \bigvee_{f\, i=1}^{n} x_i$	f-implicit maximum of $x_1, \ldots, x_n \in \mathbb{R}^d$
$\mathcal{F}$	f-implicit max-convolution operator

Stochastic processes and sup-measures

M_α	random α-Fréchet sup-measure
$M = M^f_{\alpha,\sigma}$	f-implicit, α-Fréchet, random sup-measure
$\tilde{M} = \tilde{M}^f_\alpha$	modified f-implicit sup-measure
$\mathbb{X}, \mathbb{Y}$	arbitrary stochastic processes $\mathbb{X} = (X_t)_{t\in T}$, $\mathbb{Y} = (Y_t)_{t\in T}$
$(N_t)_{t\geq 0}$	homogeneous Poisson process with rate $\lambda > 0$

Integrals

$\int_a^b g_1\, dg_2$	Stieltjes integral of g_1 with respect to g_2
$(\mathrm{S}) \int_a^b g_1\, dg_2$	full Stieltjes integral of g_1 with respect to g_2
$\int_E g(u)\, M(du)$	f-implicit extremal stochastic integral of g with respect to M

Miscellaneous

$\inf, \sup$	infimum, supremum
argmax	arguments of the maxima
$h \circ g$	function composition of h and g
$g(t-) = \lim_{s\uparrow t} g(s)$	left-sided limit of g at t
$g(t+) = \lim_{s\downarrow t} g(s)$	right-sided limit of g at t
$(\Delta^- g)(t)$	left jump of g at t

$(\Delta^+ g)(t)$	right jump of g at t
$\mathrm{dist}(x, A)$	distance between the point x and the set A
$\mathrm{dist}(A, B)$	distance between the two sets A and B
$(\mathbb{R}^d, \vee_f)$	non-commutative semigroup with identity element $e = 0$
$\leq_f$	f-implicit max-order on $\mathbb{R}^d$
$x \nleq_f y$	$x \leq_f y$ does not apply
$G_{\leq_f}$	graph of the binary relation $\leq_f$ between $\mathbb{R}^d$ and $\mathbb{R}^d$
$x_n \uparrow_f$	the sequence $(x_n)_{n\geq 1}$ is $\leq_f$-increasing
$x_n \downarrow_f$	the sequence $(x_n)_{n\geq 1}$ is $\leq_f$-decreasing
λ^E	exponential of the matrix $\log \lambda \cdot E$
$k(n) = k_f(n), n \geq 1$	smallest element in $\mathrm{argmax}_{k=1,\dots,n} f(x_k)$, $x_1, \dots, x_n \in \mathbb{R}^d$

Observe the significant relation between $k(n)$ and the f-implicit maximum of $x_1, \dots, x_n \in \mathbb{R}^d$. Note further that in the present thesis $k(n)$ is mainly used for independent and identically distributed random vectors $X_1, \dots, X_n$ in $\mathbb{R}^d$ and is therefore also random.

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