Constructive Reverse Mathematics

Habilitationsschrift

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Chapter 0

Introduction

0.1 Constructive Mathematics

Almost all proofs in traditional mathematics invoke the law of excluded middle (LEM) at some point. Sometimes this use is as obvious as starting a proof of φ by "let us assume φ is false" followed by a derivation of a contradiction, and sometimes this use is subtly hidden in details, as it is in the usual interval-halving proof of the intermediate value theorem.[1] One should remember though that *any* application of LEM comes at a price. Sometimes the price to pay is that what is really going on is obfuscated. Sometimes the price is as low as leading to inefficient programs, since a proof might use unbounded search. Often enough, though, the price to pay is that a proof provides *no* algorithmic information at all. We believe that in all these cases it is a price we should not pay.

It is one of the criminally underrated insights of 20th century mathematics that we can capture the idea of constructiveness not by adding layers of notions onto traditional mathematics but simply by removing layers—namely barring the use of LEM. Doing so one might expect to end up stuck in a barren mathematical landscape devoid of any interesting results; or in the words of Hilbert:

> Taking away this law of excluded middle from the mathematician is about the same as taking away the telescope from the astronomer or forbid the boxer the use of his fists.[2]

It was Bishop who showed us that this concern is unfounded, and that mathematical life can thrive in the absence of LEM. It can be, arguably, even a much richer existence than traditional mathematics, extending our horizons to places previously hidden by LEM. Continuing in Hilbert's own analogy: Bishop discovered unknown galaxies without *any* astronomical equipment.

0.2 Constructive Reverse Mathematics

The focus of reverse mathematics—as opposed to normal, everyday mathematics—is *not* to find what theorems we can prove from certain axioms, but to ask which axioms are also *necessary* to prove certain theorems. As such the idea is neither new nor revolutionary. However, applying this approach systematically to some fragment of mathematics is a fairly recent development. Maybe the best known and most developed of these reverse approaches is "Simpson style" reverse mathematics [102] (initiated by H. Friedmann [55]), whose goal is to examine which set existence axioms need to be added to classical second order arithmetic to prove theorems in mathematics (where objects are coded by natural numbers).

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[1] In each halving step one needs to decide whether at the midpoint c we have $f(c) \geqslant 0$ or not.

[2] In the original German: "Dieses Tertium non datur dem Mathematiker zu nehmen, wäre etwa, wie wenn man dem Astronomen das Fernrohr oder dem Boxer den Gebrauch der Fäuste untersagen wollte."

Approaching from a very different point of view, but resulting in a somewhat similar hierarchy, is the theory of Weihrauch reducibility [21]. The question there is: "which theorems can be computably transformed into others?"

The aim of Constructive Reverse Mathematics is, to borrow Ishihara's description from [68],

> "to classify [over intuitionistic logic] various theorems in intuitionistic, constructive recursive and classical mathematics by logical principles, function existence axioms and their combinations."

Interestingly enough, all three of these schools of reverse mathematics share common themes. For example, in all of these approaches one can identify a hierarchical level corresponding to LPO and one to WKL, which have a remarkable number of theorems being categorised into the same levels.

We will describe some of the foundational aspects of Constructive Reverse Mathematics in the next section, but before doing so, we would like to approach the question of why one would be interested in pursuing it. Naturally, most of these remarks also apply to other the kinds of reverse mathematics mentioned above.

1. Probably the foremost aim of reverse mathematics is to gain insight into various theorems and principles, and in particular see whether certain assumptions are really necessary. Mathematicians generally try to be as economical with their assumptions as possible and dispense with unused ones.[3] Reverse results show that a theorem is optimal in this sense.

2. Naturally, as with any other form of mathematics, the intrinsic challenge can be reason enough for pursuing it. In our opinion, the proofs and techniques in CRM are more diverse and can be more intricate than in standard Bishop style constructive mathematics. One reason for this is that, since the latter is very minimal in its assumptions most proofs are somewhat "synthetic"—that is information gets combined straightforwardly to get some result. The absence of LEM means there are very few general, non-trivial disjunctions,[4] which means that proofs tend to be of a very "linear shape." However, in CRM there are general, non-trivial disjunctions coming from the principles themselves, thus breaking this linear pattern.

3. (Constructive) reverse mathematics allows one to explore the foundations of mathematics by analysing what consequences which axioms have. More importantly it allows us to do so without having to actually accept these principles.

4. As described in the next section, results in constructive reverse mathematics often involve a Brouwerian Counterexample. As such they provide details of why certain statements cannot be proved constructively. Often, this gives us the exact information needed to "constructivise" a theorem. For example if a theorem is shown to be equivalent to the uniform continuity theorem (see Section 3.5) it is most likely to be true constructively if we assume that all functions involved are uniformly continuous. Similarly if an existence statement is equivalent to weak König's Lemma (see Section 1.4), experience has shown that the addition of the assumption that there is at most one solution frequently leads to a fully constructive theorem [98]. Of course, these are only heuristics, but they seem to provide a very natural and fruitful way of constructivising classical results.

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[3]This is very reminiscent of the principle of Chekhov's gun, as explained by the well-known playwright Anton Chekhov himself: "Remove everything that has no relevance to the story. If you say in the first chapter that there is a rifle hanging on the wall, in the second or third chapter it absolutely must go off. If it's not going to be fired, it shouldn't be hanging there."

[4]See [46] for some notable exceptions.

0.3 A very Short History of Constructive Reverse Mathematics

It is generally accepted that the following 1984 result by Julian and Richman is the first result in constructive reverse mathematics [71, Theorem 2.4].[5]

Proposition. *The fan theorem for decidable bars (FAN_Δ) is equivalent to the statement that every uniformly continuous, positively valued function $f : [0,1] \to \mathbb{R}$ has a positive infimum.*

Julian and Richman noticed that in Russian recursive mathematics ([75, 1], see also Section 7.1) one could actually construct a positively valued, uniformly continuous function $f : [0,1] \to \mathbb{R}$ with $\inf f = 0$ (that is a uniformly continuous function $f : [0,1] \to \mathbb{R}$ which, in particular, does not attain its minimum) and they were wondering what principles *exactly* were responsible for assuring the existence or ruling out the existence of such a strange object. After classifying this statement in a logical form it was easy to settle this question.

Many principles we consider in CRM these days were used long before 1984. Historically, Brouwer started the tradition of using sequences such as

$$\alpha_n = \begin{cases} 0 & a^{n+2} + b^{n+2} = c^{n+2} \text{ has no integer solutions} \\ 1 & a^{n+2} + b^{n+2} = c^{n+2} \text{ has an integer solution,} \end{cases}$$

or

$$\beta_n = \begin{cases} 0 & 2n \text{ is the sum of two primes} \\ 1 & \text{otherwise,} \end{cases}$$

to show that if a theorem T constructively implies, for example,

$$\forall n \in \mathbb{N} : \alpha_n = 0 \ \lor \ \exists n \in \mathbb{N} : \alpha_n = 1 , \tag{1}$$

then there cannot be a constructive proof of T, since otherwise we would have a constructive proof of 1. Since, under the BHK interpretation [39, Chapter 5] of the logical connectives this implies that we have either proved Fermat's last theorem or found a counterexample, Brouwer rejected such theorems T.

Sequences like α_n and β_n might become obsolete for this purpose, since the underlying problem might get solved; indeed, since Andrew Wiles' 1995 proof we know that

$$\forall n \in \mathbb{N} : \alpha_n = 0 .$$

Most mathematicians believe that there will always be important, unsolved problems which we can use in the place of Fermat's last theorem, but that, in general, there is no algorithmic way, given an *arbitrary* binary sequence γ_n, to decide whether

$$\forall n \in \mathbb{N} : \gamma_n = 0 \ \lor \ \exists n \in \mathbb{N} : \gamma_n = 1 .$$

It was Bishop who grandiosely named this principle the "limited principle of omniscience" (LPO).

And indeed, one can see that from the 1970s on Brouwerian "unstable" examples of the kind

$$T \implies (\forall n \in \mathbb{N} : \alpha_n = 0 \ \lor \ \exists n \in \mathbb{N} : \alpha_n = 1) ,$$

were more and more replaced[6] by "stable" ones of the form

$$T \implies \left(\forall \gamma \in 2^{\mathbb{N}} : (\forall n \in \mathbb{N} : \gamma_n = 0 \ \lor \ \exists n \in \mathbb{N} : \gamma_n = 1) \right) ;$$

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[5]A new, alternative, proof can be found in [12].

[6]There seems to be a certain tendency to stick with the sort of specific Brouwerian counterexamples in papers with a general audience (for example the first half of [86]), since paradoxically a classical mathematician will be more suspicious of being able to decide the Goldbach conjecture than of making the arbitrarily phrased decision in LPO.

that is, by

$$T \implies P \,,$$

where P is a general, constructively dubious statement such as LPO. Because of this, some authors have cited Brouwerian counterexamples as the first instances of CRM. Strictly speaking, this is not CRM since the forward direction of the classification is missing. Indeed, as Iris Loeb has argued in [77], Brouwer would have seen no point in proving $P \implies T$, and that these results therefore can not be claimed as the beginning of CRM. While agreeing that this is formally correct, we would like to point out that often the implication $P \implies T$ is very easy or even trivial and that most of the work and the interesting constructions are happening in the proof of $T \implies P$. So a Brouwerian counterexample $T \implies P$ is, with regards to mathematical content, more than half of a full equivalence. We would also like to add that if the point of an Brouwerian counterexample were only to show that some statement T implied an unacceptable statement and is therefore unacceptable itself, then there would be no point in distinguishing between, say, LPO and LLPO and simply work with the weakest one. The fact that researchers were using different taboo-statements hints that they, at least implicitly, were thinking reversely.

The main group of results in Constructive Reverse Mathematics were proved from around 1988 on, with the—in our personal opinion—deepest results and notions (such as BD-N) being due to Hajime Ishihara. He has also authored the only overview of the area [68], which is slightly harder to get hold of, but also contains more results than its predecessor [67].

Apart from Ishihara, and the author of this thesis himself, many people have contributed to the area, and we hope to have cited most of their relevant articles. These people are, in alphabetical order, J. Berger, D.S. Bridges, M. Hendtlass, I. Loeb, M. Mandelkern, J. Moschovakis, T. Nemoto, F. Richman, P. Schuster, and W. Veldman.[7]

0.4   Foundational Aspects

In the tradition of Bishop-style constructive mathematics [19, 20, 37] we will be working informally, in the same sense that most mathematicians work informally. That is not to say that we will be vague or imprecise, but that we are happy to skip details for the sake of readability and clarity of ideas. However, we are as sure as every other mathematician that all our results can be (almost mechanically) formalised in an appropriate system.

In this vein we will abstain from choosing one of the many possible set-theoretic or type-theoretic foundations such as [2], or [87], or one of the more restricted formal systems that could serve as the basis for our endeavours such as W. Veldmann's Basic Intuitionistic Mathematics (BIM) [110], or Heyting arithmetic in all finite types and related systems [73].

We believe that it is not necessary to choose a formal system, but at the same time believe that anybody preferring to work in one of these systems will be able to easily read our results and translate them into their preferred style. A good analogy to justify this approach might be to compare it to the use of pseudo-code over choosing a specific programming language.

Choosing a particular framework also seems to be against the spirit of constructive mathematics: As constructivists, we are very happy to live with the fact that for a real number x we cannot decide whether $x \in [0,1]$ or $x \notin [0,1]$. It seems just as dubious to be able, on the meta-level, to answer the question of what, for example, a function $\mathbb{N} \to \mathbb{N}$ is. Notice that this does not stop us from developing interesting mathematics. Just as there are numbers which are definitely members of $[0,1]$ and numbers which are definitely not, there are things that we definitely believe are functions $\mathbb{N} \to \mathbb{N}$ such as primitive recursive functions, and things that we definitely cannot define a priori, such as a discontinuous function. Notice that, however, we do not rule out the existence of the latter. We would like to call this approach *humble*

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[7]We would like to point out explicitly that W. Veldman, in particular, has many (sometimes unpublished) results in constructive reverse mathematics that have, unfortunately, not made it into this thesis.

foundations. That is, we demand a maximal burden of proof for us when it comes to showing the existence of an object outright, but make minimal assumptions on arbitrary objects.

Even though there is no reason not to take a humble approach with a formal system, we believe that there is a danger of getting distracted, and, continuing the above example, switch from assuming that every primitive recursive function $\mathbb{N} \to \mathbb{N}$ exists to assuming also the reverse, namely that these are the only functions.

Related to this, there is also a danger of getting distracted and studying properties of the chosen formal system rather than doing mathematics. It is one of the enduring strengths of Bishop's approach to skip past foundational details and "do mathematics". It also gave his results a robustness to survive changing foundational fashions.

There is one aspect in this sometimes heated debate about how much formalism is appropriate, where we have to concede that more precision is needed. Even though the full axiom of choice is a definite constructive taboo, since it implies LEM [42, 57], traditionally the use of the axioms of countable and dependent choice has been more or less tacitly accepted in Bishop style constructive mathematics. These are the following principles.

ACC If S is a subset of $\mathbb{N} \times B$, and for each $n \in \mathbb{N}$ there exists $b \in B$ such that $(n,b) \in S$, then there is a function $f : \mathbb{N} \to B$ such that $(n, f(n)) \in S$ for each $n \in \mathbb{N}$.

ADC If $a \in A$ and $S \subset A \times A$, and for each x in A there exists y in A such that $(x,y) \in S$, then there exists a sequence of elements $a_1, a_2, \ldots$ of A such that $a_1 = a$ and $(a_n, a_{n+1}) \in S$ for each $n \in \mathbb{N}$.

We conjecture that the reason why the use of these choice principles is such a divisive topic might be that there are two fundamentally different ways in which they get used in constructive mathematics:

- Often ACC or ADC is applied to get a representation of an object.

  For example, given a real number x, we often need a binary sequence $(a_n)_{n \geqslant 1}$ such that

$$a_n = 0 \implies x < \frac{1}{2^n} \, ,$$
$$a_n = 1 \implies x > \frac{1}{2^{n+1}} \, .$$

  One might assume that real numbers are Cauchy sequences of rationals with a fixed modulus of Cauchyness, or fast converging Cauchy sequences. Let us simply call these reals "represented" and denote the set of them by $\mathbb{R}_r$. For these represented reals we can easily find a binary sequence $(a_n)_{n \geqslant 1}$ as above with minimal (primitive recursive) effort.

  However, if one makes the minimal "humble" axiomatic assumption that the real numbers $\mathbb{R}$ are not of a specific form but merely satisfy

$$\forall x \in \mathbb{R}, \ n \in \mathbb{N} : \exists q \in \mathbb{Q} : |x - q| < \frac{1}{2^n}$$

  then we cannot guarantee the existence a sequence $(a_n)_{n \geqslant 1}$ as above (see Section 7.2). Of course this difference is minute, and under the assumption of ACC we have $\mathbb{R} = \mathbb{R}_r$.

  Now, if we have a result that relies on the existence of sequences $(a_n)_{n \geqslant 1}$ as above we can use ACC to show it. Formally

$$\text{ACC} \vdash \forall x \in \mathbb{R} : \ldots \, ;$$

  But if ACC was only used to ensure that $\mathbb{R} = \mathbb{R}_r$, we could also restate our result as

$$\vdash \forall x \in \mathbb{R}_r : \ldots \, .$$

These kinds of use of ACC can therefore be seen as simply a matter of style and simplicity. Large parts of traditional Bishop style constructive mathematics could be rendered choice-free by switching to represented reals and making similar definitions and arguments for other objects such as point-wise continuous functions and so on.

- However, there are also uses of ACC and ADC of a more structural kind. The proof that LLPO implies WKL needs ACC not because we do not assume that we have a nice representations of binary trees, but because we need to use LLPO countable many times.

To gain more insights into phenomena of the second kind, attempts have been made to simply move choice principles into the list of principle studied [16]. And indeed, the original plan for this thesis was to work choice-sensitive and distinguish, for example, between the sequential version of LPO and the real version. However this quickly turned out to be too ambitious a project. The big picture (Section 6.5) is already very complicated and the number of principles would multiply in the absence of choice. Any attempt to do CRM without the use of ACC or ADC would need to find a way to present results in a way that highlights the interesting issues of the second kind and somehow manages to not give too much prominence to issues of the first kind.

0.5   Overview and Plan

Contrary to Simpson style reverse mathematics, in which most theorems fall into one of the "big five" categories,[8] there is a plethora of principles that have been considered in constructive reverse mathematics, with a quick count totalling about 17 major ones. We believe that the presentation we will give is a sensible way to group them. If we consider the big three varieties CLASS, INT, RUSS (see Section 7.1) there are seven possible combinations of these varieties such that a principle is true in at least one of them and possibly fails to hold in others. Five of these combinations form our first five chapters.

Chapter 1: Omniscience principles which are true classically, but not in INT or RUSS.
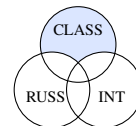
Chapter 2: Markov's principle and its weakenings which are true in CLASS and RUSS. Actually, WMP is true everywhere, but fits better into this chapter than into the chapter about BD-N.
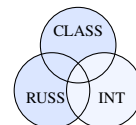
Chapter 3: The Fan theorems, which are true in CLASS and INT.
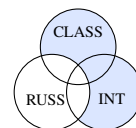
Chapter 4: BD-N which is true in all varieties.

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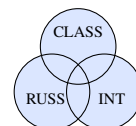
[8]Although recent work has shown that there are more than the big five.

Chapter 5: Recursive principles, which are only true in RUSS, and might be the strangest of all principles considered.
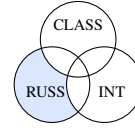
There are two combinations of varieties that are missing: Principles that are only true in INT and principles that are only true in INT and RUSS. The reason is that even though there are principles that fall into these two categories—namely continuous choice (see Section 6.3) in the first and the negation of WLPO (see Section 8.7) in the second one—there are simply not enough equivalences known to warrant their having a chapter by themselves.

After the five principle-chapters, we talk about

relationships between these principles in **Chapter 6** and sketch

techniques to separate them in **Chapter 7**.

We finish with collecting some leftover but intellectually appetizing bits in the last **Chapter 8**.

Chapter 1

Omniscience Principles

1.1 LEM and WLEM

The possibly strongest of all omniscient principles is the *law of excluded middle*[1] itself.

> **(LEM)**   If φ is any syntactically correct statement, then
>
> $$\varphi \vee \neg\varphi \,.$$

Over intuitionistic logic LEM is equivalent[2] to *double negation elimination*, also known as *stability* or *proof by contradiction*, that is that for any syntactically correct statement φ

$$\neg\neg\varphi \implies \varphi \,. \tag{1.1}$$

LEM's slightly weaker sibling is the *weak law of excluded middle*.

> **(WLEM)**   If φ is any syntactically correct statement, then
>
> $$\neg\varphi \vee \neg\neg\varphi \,.$$

We will start with the well-known and simple observation that constructively the notion of a set S being inhabited, that is $\exists x : x \in S$ is stronger than being non-empty, that is $\neg(\emptyset = S)$; in fact the equivalence of both notions is equivalent to the law of excluded middle.

Proposition 1.1.1. *LEM is equivalent to the following statement.*
For all $S \subset \{0\}$
$$\neg(S = \emptyset) \implies S \neq \emptyset \,.$$

Proof. LEM implies that either $0 \in S$ or $0 \notin S$. The second alternative is ruled out, since it implies $S = \emptyset$. Hence the first alternative holds and we are done.

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[1]Often, such as in [107], it is called the *principle of excluded middle*—PEM. Of course it also well known under its Latin name *tertium non datur*.

[2]To be a tiny bit pedantic, this is not an instance-wise equivalence. If $\varphi \vee \neg\varphi$, then for this φ also $\neg\neg\varphi \implies \varphi$. The converse

$$(\neg\neg\varphi \implies \varphi) \implies (\varphi \vee \neg\varphi) \,,$$

however, does not hold in intuitionistic logic. Nevertheless, for an arbitrary formula φ one can easily see that $\neg\neg(\varphi \vee \neg\varphi)$ holds. So if 1.1 holds for *any* formula, we can obtain $\varphi \vee \neg\varphi$.

Conversely let φ be any syntactically correct closed statement and consider the set

$$S = \{x \,|\, x = 0 \wedge (\varphi \vee \neg\varphi)\} \,.$$

Then the assumption that $S = \emptyset$ leads to a contradiction and thus, $S \neq \emptyset$. So there exists $x \in S$ such that $x = 0$ and more importantly $\varphi \vee \neg\varphi$. □

The construction of the set S above is an instance of a very common trick in the toolbox of a constructive (reverse) mathematician. It can be found in many different variations—see for example the next proposition. Interestingly enough, in our experience, mathematicians without a strong background in formal logic are very uncomfortable when they first encounter this construction and will either dispute its validity or at best judge it "pathological". We assure the reader that there is, though, no problem from a set-theoretic viewpoint as long as one has some form of set comprehension at one's disposal, which is the case in the big (constructive) mainstream set theories (ZF, IZF, CZF).

It is also worth highlighting that S is actually a family of sets S_φ, and that the fact that non-emptyness does not imply inhabitedness is not due us actually having a specific counterexample, but to it failing in a uniform fashion. Nevertheless it is common and productive to think of this sort of family of sets as one concrete example of a set, albeit a fairly fuzzy one.

A variation of this construction can also be seen in the next Proposition.

Proposition 1.1.2. *LEM is equivalent to the statement that the supremum[3] of every bounded, inhabited subset of reals exists.*

Proof. One direction is a well-known result (or even an axiom) in basic, classical analysis. For the other direction consider the set

$$S = \{0\} \cup \{x \,|\, x = 1 \wedge (\varphi \vee \neg\varphi)\} \,. \qquad \square$$

Notice that for the set S of the previous proof we can actually show that the supremum cannot be distinct from 1; in other words the supremum of $\neg\neg S$ exists. However, as the part 1 of the next proposition will show, this is not always possible. Part 2 improves upon a result of Mandelkern [82], who showed that it implies WLPO.

Proposition 1.1.3. *The following are equivalent to WLEM*

1. *If S is a bounded, inhabited subset of real number, then* $\sup \neg\neg S$ *exists.*

2. *Whenever two inhabited open subsets U, V of a bounded interval are disjoint there exists a point* $x \notin U \cup V$.

Proof. We will first show that WLEM implies 1. To this end let $S \subset \mathbb{R}$ and $B \in \mathbb{N}$ be such that

$$\forall s \in S : -B \leqslant s \leqslant B \,.$$

By [20, Proposition 4.3, Chapter 1] it suffices to show that $\neg\neg S$ is order located; that is for all $a, b \in \mathbb{R}$ with $a < b$

$$(\neg \forall s \in S : s \leqslant a) \ \vee \ (\forall s \in S : s < b) \,;$$

but this follows easily from WLEM applied to the formula $\forall s \in S : s \leqslant a$.

It is also easy to see that 1 implies 2: if U and V are as stated, then $x = \sup \neg\neg U$ exists, by 1. Since U and V are open, both $x \in U$ as well as $x \in V$ lead to contradictions, and hence $x \notin U \cup V$.

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[3]Notice that we use Bishop's definition of the supremum [20, Chapter 2, Defintion 4.2], which constructively differs from the usual least upper bound one. In fact the set from this proposition's proof shows that both definitions being equivalent is equivalent to LEM.

Finally let φ be any syntactically correct statement. Let

$$U = \left(0, \frac{1}{3}\right) \cup \left\{x \in (0, \tfrac{2}{3}) \mid \varphi \right\}$$

and

$$V = \left\{x \in \left(\tfrac{1}{3}, 1\right) \mid \neg\varphi \right\} \cup \left(\frac{2}{3}, 1\right).$$

Then $0 \in U$ and $1 \in V$ so they are inhabited. They are also easily seen to be open and disjoint. Now assume there exists $x \notin U \cup V$. If $x < \tfrac{2}{3}$, then $\neg\varphi$ leads to the contradiction $x \in V$, so $\neg\neg\varphi$. Similarly $x > \tfrac{1}{3}$ implies $\neg\varphi$. Hence WLEM holds. □

The rules known as *De'Morgan's laws* are

$$\neg(\varphi \wedge \psi) \iff \neg\varphi \vee \neg\psi, \tag{DM1}$$

and

$$\neg(\varphi \vee \psi) \iff \neg\varphi \wedge \neg\psi. \tag{DM2}$$

where φ and ψ are syntactically correct statements. It is easy to see that DM2 is provable in intuitionistic logic. The same is true for the direction from the right to the left of DM1. However, we have

Proposition 1.1.4. *WLEM is equivalent to DM1.*

Proof. First assume DM1 and let φ be arbitrary. Since $\neg(\neg\varphi \wedge \varphi)$ is provable in intuitionistic logic we have $\neg\neg\varphi \vee \neg\varphi$; that is WLEM holds.

Conversely assume that $\neg(\varphi \wedge \psi)$. By WLEM either $\neg\varphi$ or $\neg\neg\varphi$. It is easy to see that in the second case the assumption that ψ holds leads to a contradiction. Hence $\neg\psi$ and we are done. □

Sometimes De Morgan's laws are stated slightly differently—in its "substitution form"—in which case we get an equivalence to LEM:

Proposition 1.1.5. *LEM is equivalent to the following versions of De Morgan's law*

$$\neg(\neg\varphi \wedge \neg\psi) \iff \varphi \vee \psi \tag{DM1$'$}$$

as well as to

$$\neg(\neg\varphi \vee \neg\psi) \iff \varphi \wedge \psi \tag{DM2$'$}$$

Proof. Of course, LEM implies both versions. Conversely let φ be a syntactically correct statement. In intuitionistic logic we have $\neg(\neg\varphi \wedge \neg\neg\varphi)$, so DM1$'$ implies $\varphi \vee \neg\varphi$. We also have

$$\neg(\neg(\varphi \vee \neg\varphi) \vee \neg(\varphi \vee \neg\varphi)),$$

so DM2$'$ also implies $\varphi \vee \neg\varphi$. Thus in both cases LEM holds. □

There are quite a few other basic logical principles equivalent to LEM over intuitionistic logic. (See, however, Section 8.5 for $(\varphi \implies \psi) \vee (\psi \implies \varphi)$)

Proposition 1.1.6. *The following are equivalent to LEM*

1. $((\varphi \implies \psi) \implies \varphi) \implies \varphi$ *(Peirce's law)*

2. $(\varphi \implies \vartheta) \implies (((\varphi \implies \psi) \implies \vartheta) \implies \vartheta)$ *(generalisation of Peirce's law)*

3. $(\neg\varphi \implies \varphi) \implies \varphi$

4. $(\psi \implies \varphi) \vee (\varphi \implies \vartheta)$ *(Linearity)*

5. $(\varphi \implies \psi) \vee \neg\psi$

6. $(\neg(\varphi \implies \psi)) \implies (\varphi \wedge \neg\psi)$ *(the counterexample principle)*

Proof. Clearly all of these can be proved in classical logic, i.e. follow from LEM. Furthermore 3 is a special case of 1 ($\psi \equiv \bot$), which in turn is a special case of 2. To see that 3 implies LEM we will show that it implies 1.1. Clearly $\neg\neg\varphi$ implies $(\neg\varphi \implies \varphi)$, and hence φ. To see that 4 implies LEM, we simply take $\psi \equiv \top$ and $\vartheta \equiv \bot$. Similarly 5 implies LEM by choosing $\varphi \equiv \top$. Lastly, 6 implies 1.1 and therefore LEM if one chooses $\psi \equiv \bot$ (and weakens the consequence). $\qquad\square$

1.2 LPO

The ubiquitousness of the *limited principle of omniscience* (LPO) in analysis might only be rivalled by LLPO's. This is mainly due to the fact, that real numbers and sequences feature prominently in analysis, and LPO tells us everything we want to know about both.

> **(LPO)**   For every binary sequence $(a_n)_{n \geqslant 1}$ we can decide whether
>
> $$\forall n \in \mathbb{N} : a_n = 0 \vee \exists n \in \mathbb{N} : a_n = 1.$$

Basic equivalencies of LPO

Mostly taken directly from [69] are the following equivalences:

Proposition 1.2.1. *The following are equivalent to LPO*

1. $\forall x \in \mathbb{R} : x < 0 \vee x = 0 \vee 0 < x$

2. *For every binary sequence $(a_n)_{n \geqslant 1}$ we can decide whether*

$$\exists N : \forall n \geqslant N : a_n = 0 \vee \exists k_n \in \mathbb{N}^{\mathbb{N}} : \forall n \in \mathbb{N} : a_{k_n} = 1.$$

3. *Every bounded monotone sequence of real numbers converges.*

4. *(Bolzano-Weierstraß theorem) Every sequence in a compact (totally bounded and complete) set has a convergent subsequence.*

5. *Every sequence of closed subsets of a compact metric space with the finite intersection property has nonempty intersection.*

6. *Ascoli's Lemma.*

Proof. The equivalence of LPO to 1 is standard. 2 obviously implies LPO. Conversely we can show 2 by applying LPO countably many times: using LPO (and unique choice) construct a binary sequence b_n such that

$$b_k = 0 \implies \exists n \geqslant k : a_n = 1,$$
$$b_k = 1 \implies \forall n \geqslant k : a_n = 0.$$

Now, using LPO again, either $\exists N : b_N = 1$ or $\forall k : b_k = 0$. In the first case $\forall n \geqslant N : a_n = 0$. In the second case we can use dependent choice to find $k_n \in \mathbb{N}^{\mathbb{N}}$ such that $\forall n \in \mathbb{N} : a_{k_n} = 1$.

The equivalence of LPO to 3 can be found in [85], the one to 4 in [34], the one to 5 in [67], and the one to 6 in [48]. $\qquad\square$

Proposition 1.2.2. *LPO is equivalent to the following statements:*

- *Every real number is either rational or irrational.*

- *Every real number is either algebraic or transcendental.*

- *For all sequences r_n of reals and for all $x \in \mathbb{R}$ either there exists n such that $x = r_n$ or $x \neq r_n$ for all $n \in \mathbb{N}$.*

Proof. First notice that using LPO and countable choice, we can, for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$, decide whether $x = r_n$ or $x \neq r_n$. So using (unique) countable choice there exists a binary sequence $(a_n)_{n \geqslant 1}$ such that $a_n = 0 \iff \neg(x \neq r_n)$. By LPO either $a_n = 0$ for all $n \in \mathbb{N}$ or there exists $n \in \mathbb{N}$ such that $a_n = 1$. That is we can decide whether $x \neq r_n$ for all $n \in \mathbb{N}$ or whether there exists $n \in \mathbb{N}$ such that $x = r_n$.

Conversely let $(a_n)_{n \geqslant 1}$ be a decreasing binary sequence. Define

$$a = \sum_{n \geqslant 1} \frac{a_n}{n!} \ .$$

Notice that if $a_n = 1$ for all $n \in \mathbb{N}$, then $a = e$, and thus transcendental (irrational). If a is algebraic (rational) there exists $\varepsilon > 0$ such that $|e - a| > \varepsilon$. Hence we can easily find n such that $a_n = 0$. $\qquad\square$

Proposition 1.2.3. *LPO is equivalent to the statement that the transitive closure of a decidable relation $R \subset \mathbb{N} \times \mathbb{N}$ is, again, decidable.*

Proof. Let $(a_n)_{n \geqslant 1}$ be a binary sequence. We define a decidable relation R on $\mathbb{N}$ by $(n, m) \in R$ for all $n, m \geqslant 2$ and $(1, n) \in R$ if $a_n = 1$. Now it is easy to see that $(1, 2) \in R^+$ (the transitive closure) if and only if there exists n such that $a_n = 1$.

Conversely let R be a decidable relation on $\mathbb{N}$ and assume that LPO holds. Furthermore let k, ℓ be arbitrary natural numbers. Now define a binary sequence $(a_n)_{n \geqslant 1}$ by

$$a_n = \begin{cases} 1 & \exists k_1, \ldots, k_j \leqslant n : (k, k_1) \in R \wedge (k_1, k_2) \in R \wedge \cdots \wedge (k_j, \ell) \in R \\ 0 & \text{otherwise.} \end{cases}$$

Then, if there exists n such that $a_n = 1$ we have $(k, \ell) \in R^+$ and if $a_n = 0$ for all $n \in \mathbb{N}$ we have $(k, \ell) \notin R^+$. $\qquad\square$

Proposition 1.2.4. *LPO is equivalent to the statement, that if $A \cup B$ is infinite (i.e. there is an injection $\mathbb{N} \rightarrow A \cup B$), then either A or B is infinite.*

Proof. Let $f : \mathbb{N} \rightarrow A \cup B$ an injection. Using countable choice we can assume that there is a binary sequence such that

$$a_n = \begin{cases} 0 & \implies f(n) \in A \\ 1 & \implies f(n) \in B \end{cases}$$

Using LPO and countable choice iteratively we can fix another binary sequence $(b_n)_{n \geqslant 1}$ such that

$$b_n = \begin{cases} 0 & \exists k > n : a_k = 0 \\ 1 & \forall k > n : a_k = 1 \end{cases}$$

One final application of LPO yields that either there is $N \in \mathbb{N}$ such that $b_N = 1$, in which case $n \mapsto f(N+n+1)$ is an injection of $\mathbb{N}$ into B, or $b_n = 0$ for all $n \in \mathbb{N}$, in which case we can find an injection of $\mathbb{N}$ into A. Conversely assume that $(a_n)_{n \geqslant 1}$ is an, without loss of generality increasing, binary sequence. Define the sets $A = \{n \mid a_n = 0\}$ and $B = \{n \mid a_n = 1\}$. Then $A \cup B = \mathbb{N}$ is infinite and A and B are disjoint. Now if A is infinite there cannot be a n with $a_n = 1$, and thus $a_n = 0$ for all $n \in \mathbb{N}$. If B is infinite it is, in particular, inhabited and so we can find $n \in \mathbb{N}$ with $a_n = 1$. $\qquad\square$

The last proposition iterated finitely many times also shows that LPO is equivalent to the *infinite pigeonhole principle*.

> **(IPP)**   For every $f : \mathbb{N} \to \{1,\dots,k\}$ there exists an infinite set $A \subset \mathbb{N}$ and $1 \leqslant K \leqslant k$ such that $f(x) = K$ for all $x \in A$.

Proposition 1.2.5. *LPO is equivalent to the statement that every countable subset of the natural numbers is decidable.*

Proof. It is clear that LPO is enough to prove the statement. Conversely, if $(a_n)_{n \geqslant 1}$ is a binary sequence, then the set $A = \{a_n \mid n \in \mathbb{N}\}$ countable. If this set is decidable, then either $1 \in A$ or $1 \notin A$. In the first case there exists $n \in \mathbb{N}$ with $a_n = 1$. In the second case there cannot be such an n. Hence, in that case, $\forall n \in \mathbb{N} : a_n = 0$. $\qquad\square$

Metastability

In a program suggested by Tao [105], it is proposed to recover the "finite" (constructive) content of theorems by replacing them with logically (using classical logic) equivalent ones that can be proved by "finite methods." For example, since there often is no way to attain the Cauchy condition it is replaced with the following notion of meta-stability. A sequence $(x_n)_{n \geqslant 1}$ in a metric space (X,d) is called *metastable* if

$$\forall \varepsilon > 0, f : \exists m : \forall i, j \in [m, f(m)] : d(x_i, x_j) < \varepsilon ,$$

where $[k,\ell]$ denotes the set $\{k, k+1, \dots, \ell\}$. One can trivially show that a Cauchy sequence is metastable. However, as we will see metastability contains almost no constructive content.

As noted in [3] every non-decreasing sequence of reals bounded by $B \in \mathbb{R}$ is metastable since it is impossible that $d(x_m, x_{f(m)}) > \frac{\varepsilon}{2}$ for all $1 \leqslant m \leqslant \frac{2B}{\varepsilon}$. The converse fails constructively, since we can show:

Proposition 1.2.6. *LPO is equivalent to the statement that every metastable, increasing sequence of rationals is bounded.*

Proof. Assume $(x_n)_{n \geqslant 1}$ is non-decreasing and metastable. Then, using LPO, we can, for every n decide whether n is an upper bound or not. Using LPO again, we can thus either find an upper bound or a function $f : \mathbb{N} \to \mathbb{N}$ such that $x_{f(n)} > x_n + 1$ for all $n \in \mathbb{N}$. Without loss of generality we may assume that f is increasing. Furthermore

$$d(x_n, x_{f(n)}) > 1 ;$$

a contradiction to the metastability, and hence x_n is bounded.

Conversely, let $(a_n)_{n \geqslant 1}$ be a binary sequence that has, without loss of generality, at most one 1. Now consider

$$x_n = \sum_{i=1}^{n} i a_i .$$

It is easy to see that x_n is metastable: if $f : \mathbb{N} \to \mathbb{N}$ is increasing, then either $a_i = 0$ for all $i \in [1, f(1)]$ or $a_i = 0$ for all $i \in [f(2), f(f(2))]$. In both cases x_i is constant.

Now if x_n is bounded, there is $N \in \mathbb{N}$ with $x_n < N$. If there was $i > N$ with $a_i = 1$, then $x_i = i > N$ which is a contradiction. Hence $a_i = 0$ for all $i > N$, that is we only need to check finitely many entries to see if $(a_n)_{n \geqslant 1}$ consists of 0s or whether there is a term equalling 1. $\qquad \square$

One might thus hope that there is maybe a chance that every bounded, metastable sequence converges. However, also this statement is equivalent to LPO.

Proposition 1.2.7. *LPO is equivalent to the statement that every bounded, metastable sequence of reals converges.*

Proof. Assume that LPO holds and that $(x_n)_{n \geqslant 1}$ is a bounded and metastable sequence of reals. Since LPO implies the Bolzano Weierstraß theorem (see Theorem 1.2.1) there exists $x \in \mathbb{R}$ and $k_n \in \mathbb{N}^{\mathbb{N}}$ such that x_{k_n} converges to x. Now let $\varepsilon > 0$ be arbitrary and using LPO fix a binary sequence $(\lambda_n)_{n \geqslant 1}$ such that

$$\lambda_n = 0 \implies |x - x_n| < \varepsilon ,$$
$$\lambda_n = 1 \implies |x - x_n| \geqslant \varepsilon .$$

By Theorem 1.2.1 either there exists N such that $\lambda_n = 0$ for all $n \geqslant N$ or there exists a strictly increasing $\ell_n \in \mathbb{N}^{\mathbb{N}}$ such that $\lambda_{\ell_n} = 1$ for all $n \in \mathbb{N}$. We will show that the second alternative is ruled out by the metastability: fix M such that $|x_{k_n} - x| < \frac{\varepsilon}{2}$ for $n \geqslant M$ and hence

$$|x_{k_n} - x_{\ell_n}| \geqslant \frac{\varepsilon}{2} \text{ for } n \geqslant M . \tag{1.2}$$

Now define $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = \max\{k_{n+M}, \ell_{n+M}\}$. Then f is increasing. Since $(x_n)_{n \geqslant 1}$ is metastable there exists m such that for all $i, j \in [m, f(m)]$ we have $|x_i - x_j| < \frac{\varepsilon}{2}$. Since $k_{m+M}, \ell_{m+M} \in [m+M, f(m)]$ we get the desired contradiction to 1.2.

Conversely let $(a_n)_{n \geqslant 1}$ be a binary sequence with at most one 1. We will show that $(a_n)_{n \geqslant 1}$ is metastable. So let $f : \mathbb{N} \to \mathbb{N}$ an increasing function. Now either there exists $i \in [1, f(1)]$ such that $a_i = 1$ or for all $i \in [1, f(1)]$ we have $a_i = 0$. In the first case, since $(a_n)_{n \geqslant 1}$ has at most one 1, for all $i \in [f(1) + 1, f(f(1) + 1)]$ we have $a_i = 0$. In both cases there exists m such that, regardless of $\varepsilon > 0$, we have

$$\forall i, j \in [m, f(m)] : |a_i - a_j| = 0 < \varepsilon ;$$

that is $(a_n)_{n \geqslant 1}$ is metastable. Now if this sequence converges it must converge to 0. So there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have $a_N = 0$. So we only need to check finitely many indices $n \in \mathbb{N}$ for $a_n = 1$, and hence LPO holds. $\qquad \square$

Hillam's theorem

Hillam's theorem [62] states that if $f : [0, 1] \to [0, 1]$ is a continuous map and one defines a sequence by choosing an arbitrary $x_0 \in [0, 1]$ and then taking $x_{n+1} = f(x_n)$ for all $n > 0$, then

$$|x_n - x_{n+1}| \to 0 \iff (x_n)_{n \geqslant 1} \text{ is Cauchy} .$$

The interesting direction here is the one from the left to the right, since the converse holds trivially. Notice that if the sequence in the theorem converges, it converges to a fix-point. Hillam's theorem is therefore very interesting, since it is a fix-point theorem not equivalent to LLPO (cf. Section 1.4).

Proposition 1.2.8. *Hillam's theorem is equivalent to LPO. (Even for uniformly continuous functions).*

Proof. It is easy to see that the non-constructive steps in original proof [62] hold under the assumption of LPO; and that therefore the latter is enough to prove Hillam's theorem. To prove the converse let $a \in \mathbb{R}$ be a non-negative real number. Without loss of generality assume that $a < 1$. Now define $f : [0,1] \rightarrow [0,1]$ by

$$f(x) = (1-a)x + a \ ;$$

that is f is the linear function through $(0,a)$ and $(1,1)$. Now define a sequence of real numbers by $x_n = f^n(0)$. It is easily shown by induction that $x_n = 1 - (1-a)^n$ and that $x_n \leqslant x_{n+1}$. We want to show that $|x_n - x_{n+1}| \rightarrow 0$ as $n \rightarrow \infty$. For an arbitrary $\varepsilon > 0$ either $a < \varepsilon$ or $a > \frac{\varepsilon}{2}$. In the first case, since for any $x \in [0,1]$

$$|x - f(x)| \leqslant a \ ,$$

for any $n \in \mathbb{N}$ also $|x_n - x_{n+1}| < \varepsilon$. In the second case choose N such that $(1-a)^N < \varepsilon$. Now for any $n \geqslant N$ we get

$$|x_n - x_{n+1}| \leqslant x_{n+1} - x_n \leqslant 1 - x_n = (1-a)^n \leqslant (1-a)^N < \varepsilon \ .$$

In both cases there exists $N \in \mathbb{N}$, such that $|x_n - x_{n+1}| < \varepsilon$ for all $n \geqslant N$; therefore, by assumption, $(x_n)_{n \geqslant 1}$ is a Cauchy sequence and hence converges to a point x_∞. Either $x_\infty < 1$, in which case $a = 0$, or $x_\infty > 0$. In the second case we can find N such that $x_N > x_\infty/2$. Then, by Bernoulli's inequality

$$x_\infty/2 < x_N = 1 - (1-a)^N \leqslant 1 - 1 - N(-a) = Na.$$

Hence $a > x_\infty/(2N)$, after dividing by N, and therefore $a > 0$. $\qquad\square$

Cardinality

It is a well known exercise in first year mathematics to prove that there is a bijection between $[0,1)$, and $(0,1)$ and that therefore both sets have the same cardinality.

Proposition 1.2.9. *LPO is equivalent to the existence of a strongly extensional bijection[4] $f : [0,1) \rightarrow (0,1)$.*

Proof. It is easy to see that with the help of LPO

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n} \\ x & \text{else} \end{cases}$$

is a well-defined, strongly extensional bijection from $[0,1) \rightarrow (0,1)$.

Conversely assume that such a function f exist. Then $0 < f(0) < 1$. Since f is surjective there exist a,b such that $0 < f(a) < f(0) < f(b) < 1$. Since f is strongly extensional either $0 < a < b$ or $0 < b < a$. Without loss of generality assume the first. Setting $a_1 = a$ and $b_1 = b$ and using the usual interval halving technique (notice that since f is injective we have $f(x) < f(0) \vee f(x) > f(0)$ for $x \neq 0$), we construct sequences $(a_n)_{n \geqslant 1}$ and $(b_n)_{n \geqslant 1}$ such that

- $0 < a_n \leqslant a_{n+1} < b_{n+1} \leqslant b_n$

- $f(a_n) < f(0) < f(b_n)$

- $|a_n - b_n| < \left(\frac{1}{2}\right)^n$

---

[4]To be precise: by bijection we obviously mean surjective and injective. Furthermore, a function $f : X \rightarrow Y$ is injective, if $f(x) \neq f(y)$ whenever $x \neq y$. This is a stronger requirement than $x = y$ whenever $f(x) = f(y)$. The latter is called *weakly injective* by Bishop [20, p. 63].

Both sequences converge to the same limit $z > 0$. Now, by injectivity, $f(z) \neq f(0)$, so either $f(z) > f(0)$ or $f(z) < f(0)$. Without loss of generality assume the first alternative holds and let $\varepsilon = f(z) - f(0) > 0$. Using Ishihara's tricks (see [46] or the original [64]) either $|f(a_n) - f(z)| < \varepsilon$ infinitely often or LPO holds. The first alternative is ruled out, since

$$|f(a_n) - f(z)| = f(0) - f(a_n) + \varepsilon > \varepsilon \;,$$

and hence LPO holds. $\qquad\square$

1.3   WLPO

Slightly weaker than LPO is the *weak limited principle of omniscience*.

> **(WLPO)**   For every binary sequence $(a_n)_{n \geqslant 1}$ we can decide whether
>
> $$\forall n \in \mathbb{N} : a_n = 0 \lor \neg \forall n \in \mathbb{N} : a_n = 0.$$

WLPO has fewer equivalences than LPO or LLPO. The following are taken straight from [68].

Proposition 1.3.1. *The following are equivalent to WLPO:*

1. *For all $x, y \in \mathbb{R}$ we have $x \geqslant y$ or $\neg(x \geqslant y)$.*

2. *For all $x, y \in \mathbb{R}$ we have $x = y$ or $\neg(x = y)$.*

3. *The existence of a discontinuous function from $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$.*

4. *The existence of a discontinuous function $[0, 1] \to \mathbb{R}$.*

   The following is the analogue of Proposition 1.2.2.

Proposition 1.3.2. *WLPO is equivalent to the statement that every real number is either irrational or not. More general: if r_n is a sequence of real numbers, then for all $x \in \mathbb{R}$ either*

$$\forall n \in \mathbb{N} : \neg(x = r_n) \lor \neg \forall n \in \mathbb{N} : \neg(x = r_n)$$

Proof. The proof of Proposition 1.2.2 easily adapts to this case. $\qquad\square$

   The proof of the next proposition requires a lemma.

Lemma 1.3.3. *A set $A \subset \mathbb{R}$ is located if and only if it is inhabited and for all $a < b$ and $\varepsilon > 0$ we can decide*
$$\forall x \in [a, b] : x \notin A \;\lor\; \exists x \in [a - \varepsilon, b + \varepsilon] : x \in A \;.$$

Proof. Straightforward. $\qquad\square$

Proposition 1.3.4. *The following are equivalent to WLPO.*

1. *The zero-set*
$$Z_f = \{x \in [0, 1] \mid f(x) = 0\}$$

   of a point-wise continuous function $f : [0, 1] \to \mathbb{R}$ is located (and therefore compact since it is closed).

2. *The "weak support"*
$$S_f^w = \{\, x \in [0,1] \mid \neg(f(x) = 0) \,\}$$

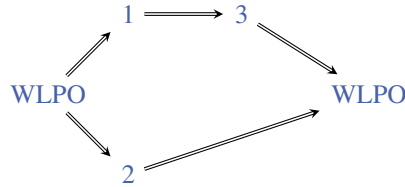*of a point-wise continuous function $f : [0,1] \to \mathbb{R}$ is located whenever it is inhabited.*

3. *(Strong Intermediate Value Theorem) For any point-wise continuous function $f : [0,1] \to \mathbb{R}$ such that $f(0) \leqslant 0$ and $f(1) \geqslant 0$*
$$\inf f^{-1}(\{0\})$$

exists.

Proof. We will show the implications in the following order:

First assume WLPO. Note that since WLPO implies LLPO which in turn implies that f is uniformly continuous (see Proposition 6.1.3). LLPO also implies that a uniformly continuous $|f|$ attains its minimum z and maximum z' (see section 1.4). Now either $|f(z)| = 0$ or $\neg(|f(z)| = 0)$. In the second case, since z is minimal, there cannot be an $x \in [a,b]$ such that $|f(z)| = 0$. In particular, we can therefore decide

$$\forall x \in [a,b] : x \notin Z_f \vee \exists x \in [a,b] : x \in Z_f \;.$$

By the previous lemma Z_f is located.

Similarly either $|f(z')| = 0$ or $\neg(|f(z')| = 0)$ and therefore, again by the previous Lemma S_f^w is located. Altogether we have shown that WLPO implies 1 as well as 2.

Since the infimum of a totally bounded set of real numbers exists [37, Proposition 2.2.5] we have $1 \implies 3$.

To see that $3 \implies$ WLPO holds, consider an arbitrary real number a. Construct the piecewise linear function f by

$$f(x) = \begin{cases} (3 - 3|a|)x - 1 & x \in [0, \frac{1}{3}] \\ -|a| & x \in [\frac{1}{3}, \frac{2}{3}] \\ 3(1 + |a|)x - 2 - 3|a| & x \in [\frac{2}{3}, 1] \end{cases}$$

By our assumption there exists x such that $f(x) = 0$ and for all y with $y < x$ we have $\neg(f(y) = 0)$. Now either $x < \frac{2}{3}$ or $x > \frac{1}{3}$. In the first case a must be 0 as the assumption $|a| > 0$ implies that $f(x) < -|a| < 0$, which is a contradiction to $f(x) = 0$. In the second case assume that $a = 0$, but then $f(\frac{1}{3}) = 0$ which contradicts $x > \frac{1}{3}$. Thus we have decided whether $a = 0$ or $\neg(a = 0)$.

Finally, to see that $2 \implies$ WLPO we can use the same construction as before. Since S_f^w is located the distance $r = d(1, S_f^w)$ exists. If $r < \frac{1}{3}$ we cannot have $a = 0$, which would imply that $r = \frac{2}{3}$. If $r > 0$, then $a = 0$, since $|a| > 0$ implies $r = 0$. $\qquad \square$

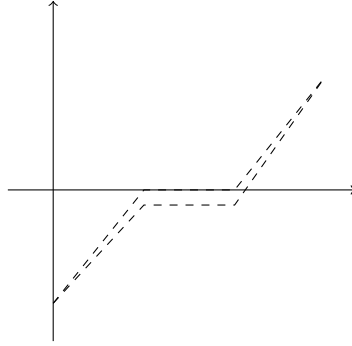Similarly, we can get the following equivalence.

Figure 1.1: We think of $a \geqslant 0$ as being very small. The function f is a variation on the standard example that shows that the intermediate value theorem implies LLPO.

Proposition 1.3.5. *LPO is equivalent to the statement that if $f : [0,1] \to \mathbb{R}$ is a point-wise continuous function, then its support*

$$S_f = \{x \in [0,1] \mid f(x) \neq 0\}$$

is located, provided it is inhabited.

Proof. One direction is clear by Proposition 1.3.4, the fact that LPO $\iff$ WLPO $\wedge$ MP (see Section 2.1 for details on MP), and the fact that under the assumption of MP

$$S_f = \{x \in [0,1] \mid \neg(f(x) = 0)\} .$$

For the other direction it is enough to show that the statement implies MP. To this end let $a \in \mathbb{R}$ be a real number such that $\neg(a = 0)$. Now consider $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} a & \text{for } x \in \left[0, \frac{1}{2}\right] \\ (1-a)2x - 1 + 2a & \text{for } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Then $f(1) = 1$, so $1 \in S_f$. If the support is located that means that $\delta = d(0, S_f)$ exists. Now either $\delta > 0$ or $\delta < \frac{1}{2}$. In the first case we must have $f(0) = 0$, since $f(0) \neq 0$ implies $\delta = 0$; but this cannot happen since $f(0) = a$. So we must have $\delta < \frac{1}{2}$. That means, by the definition of the infimum, that there exists $x \in [0, \frac{1}{2}]$ with $x \in S_f$; and hence $a = f(x) \neq 0$. $\qquad\square$

A result by Richman from [95] characterises a well know classical result about functions of bounded variation.

Proposition 1.3.6. *If every uniformly continuous function on $[0,1]$ of bounded variation[5] is the difference of two increasing functions, then WLPO holds. The converse holds in the presence of countable choice.*

As the last result of this Section we have the following, which is the obvious analogue of Proposition 1.2.9. What is interesting though is that the function f is injective in the sense that $f(x) \neq f(y) \to x \neq y$, and one would not expect to see that $x \neq y$ without the help of the MP-part of LPO.

---

[5]Here a function $f : [a,b] \to \mathbb{R}$ has bounded variation if the set

$$S = \left\{ \textstyle\sum_{i=1}^{n-1} |f(x_i) - f(x_{i+1})| \,\middle|\, a = x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n = b \right\}$$

is bounded. A result by Bridges [24] shows that if one assumes that the supremum (in Bishop's sense) of the set S exists and that the function f is strongly extensional (no continuity assumption), then it can be written as the sum of two increasing functions. This result was improved upon by Richman, who showed that one can work without the assumption of strong extensionality [95].

Proposition 1.3.7. *WLPO is equivalent to the existence of a bijection[6] $f : [0,1) \to (0,1)$.*

Proof. The proof will be omitted, since it is basically the same as the one of Proposition 1.2.9. One technical point worth pointing out is that one needs to use a variation of Ishihara's tricks [46]. $\square$

As a direct consequence of this we also have that the Cantor–Bernstein–Schröder theorem implies WLPO, even when restricted to functions between subsets of $\mathbb{R}$, since $x \mapsto \frac{1+x}{2}$ and the identity are injections between $[0,1)$ and $(0,1)$.

Question 1. *Which principle is the Cantor–Bernstein–Schröder theorem equivalent to? It seems likely that the answer heavily depends on the precise formulation.*

The Rising Sun Lemma, and LPO and WLPO

We can use Proposition 1.3.4 to show two more equivalences of LPO and WLPO; namely (versions of) the Rising Sun Lemma, which can be used to prove the Hardy-Littlewood maximal inequality [106, pp. 143].

Proposition 1.3.8. *LPO is equivalent to the following statement.*
Consider a continuous function $f : [0,1] \to \mathbb{R}$, and let E be the set

$$E = \{ x \in [0,1] \,|\, \exists y > x : f(y) > f(x) \} \,.$$

Then one can find a, at most countable, family of disjoint, open, non-empty intervals I_n in $[0,1]$ such that $E \cap (0,1) = \bigcup I_n$.

The name "Rising Sun Lemma" is due to imagining the graph of f as a mountain range which the rising sun shines on to from the right. Viewed this way, the set E consists of the areas which are in the shade.

Before we prove the above proposition, we will make a couple of comments.

- Since we assume LPO (and therefore MP) in the forward direction it doesn't make a difference whether we assume that our intervals are non-empty or inhabited. In fact, if (a,b) is nonempty, then $\neg(b \leqslant a)$, and hence because of MP $a < b$. Thus $\frac{b-a}{2} \in (a,b)$.

- Unlike in many other results in Constructive Reverse Mathematics the Brouwerian counterexample part (i.e. the proof that the Rising Sun Lemma implies LPO) is fairly straightforward, but the reverse, classical direction takes some effort. This is due to the fact that the standard proof (like the one in [106]) uses the fact that classically "any open subset U of $\mathbb{R}$ can be written as the union of at most countably many disjoint non-empty open intervals, whose endpoints lie outside of U." However, this statement is equivalent to LEM: Consider the set

$$U = \left\{ x \in (0,1) \,\middle|\, x < \tfrac{1}{2} \vee \varphi \right\} \,,$$

for a syntactically correct statement φ. Now let $U = \bigcup_{n \in I} I_n$, as above, where I is either finite or countable. It is easy to see that I must have (exactly) one element, say i and $I_i = (a,b)$. If $b > \frac{1}{2}$ we must have φ if $b < 1$ the assumption that φ holds leads to a contradiction.

- Notice that even classically we cannot assume that the I_n are increasing or decreasing. To see this consider the piece-wise linear function $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} -2x+1 & x \in [0, \tfrac{1}{3}] \\ x & x \in [\tfrac{1}{3}, \tfrac{2}{3}] \\ -2x+2 & x \in [\tfrac{2}{3}, 1] \,. \end{cases}$$

---

[6]See the footnote in Proposition 1.2.9 for the exact definition of bijection.

For this function the set E from the Rising Sun Lemma is $\left[\frac{1}{6}, \frac{2}{3}\right]$. Now define a functions $g, h :$ $[0,1] \to [0,1]$ by placing a copy of f in each of the squares G_n and H_n indicated in Figure 1.2. It is
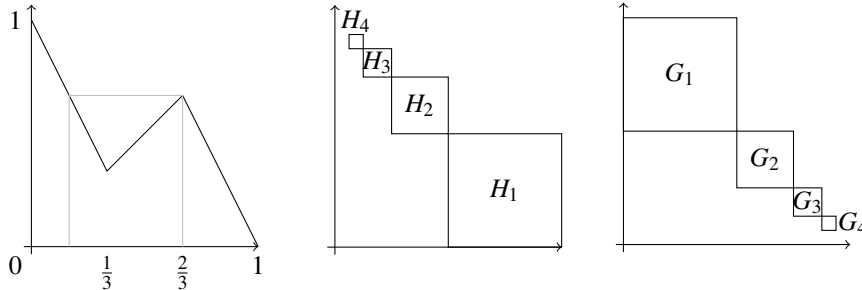
Figure 1.2: An iterated construction.

clear that g's E-set cannot be written as a sequence of open disjoint increasing intervals, and the same holds for h and decreasing intervals. Obviously, we could iterate this construction to see that we cannot write E as a decreasing union of increasing disjoint, open, non-empty intervals, or even further.

Finally, it is time to prove Proposition 1.3.8.

Proof. Assume the Rising Sun Lemma holds and let $a \in \mathbb{R}$ be a real number. Now define a piece-wise linear function $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} (2 - 2|a|)x & x \in [0, \frac{1}{2}] \\ 2|a|x + 1 - 2|a| & x \in [\frac{1}{2}, 1] \end{cases}$$
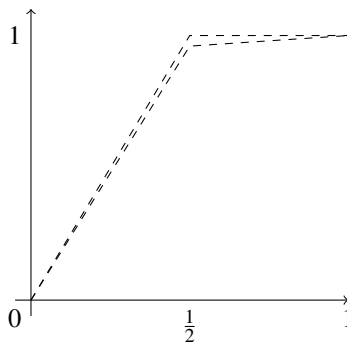
Figure 1.3: Depending on a the function f is either "in the shadow" or flat on $[1/2, 1]$.

Notice that if $x \in [\frac{1}{2}, 1]$, then $1 - |a| \leqslant f(x) \leqslant 1$. Now let E be the "areas in the shade" as in the Rising Sun Lemma, and assume that $E = \bigcup I_n$, where $I_n = (a_n, b_n)$ are open, nonempty, and disjoint. It is easy to see that the family $(I_n)_{n \geqslant 1}$ must consist of exactly one element, say (a, b). Now either $b > 1/2$ or $b < 1$. In the second case the assumption that $|a| > 0$ leads to the contradiction that $E = [0, \frac{1}{2})$, so in that case $a = 0$. In the first case we have $z = \frac{b + 1/2}{2} \in E$. So by definition there exists $y > z$ such that $f(y) > f(z)$. But that means that

$$1 - |a| \leqslant f(z) < f(y) \leqslant 1 \, ,$$

which implies that $|a| > 0$. With both cases taken together we have decided LPO.

Conversely let $f : [0,1] \to \mathbb{R}$ be a continuous function. First notice that LPO allows us to make the following decision. Since LPO implies that f is uniformly continuous (see 6.1.3) and therefore $S_x = \sup_{[x,1]} f$ exists, we can decide whether $f(x) = S_x$, or $f(x) < S_x$. In the second case, using Proposition 1.3.4.3, we can find $b_x = \inf\{z \geqslant x \,|\, f(z) = S_x\}$. Using Proposition 1.3.4.3 again we can also find $a_x = \sup\{z \leqslant x \,|\, f(z) = S_x\}$.

Now let r_n be an enumeration of all rationals in $[0,1]$ and consider

$$F = \left\{ n \in \mathbb{N} \,|\, f(r_n) < S_{r_n} \wedge \forall i < n : r_n \notin (a_{r_i}, b_{r_i}) \right\}.$$

Using LPO we see that F is decidable and using LPO again we can decide whether F is finite or countable. By construction the intervals $((a_{r_n}, b_{r_n}))_{n \in F}$ are non-empty and disjoint. We also have that if $x \in (a_{r_n}, b_{r_n})$ for some n, then $f(x) < S_{r_n}$, since, by the definition of b_{r_n} we cannot have $f(x) \geqslant S_{r_n}$. Using (an approximate version of) the intermediate value theorem we can find a point y such that $x < y < b_{r_n}$ and $f(x) < f(y) < S_{r_n}$.

It remains to show that if $x \in E$ there exists $n \in F$ such that $x \in (a_{r_n}, b_{r_n})$. If $x \in E$, then there exists $y > x$ with $f(x) < f(y) < S_x$. Now find (again an approximate version of the intermediate value theorem suffices) a rational r_n such that $x < r_n < y$ and $f(x) < f(r_n) < f(y)$. If for all $i < n$ we have $r_n \notin (a_{r_i}, b_{r_i})$, then $x \in (a_{r_n}, b_{r_n})$ and therefore $x \in \bigcup_{n \in F} (a_{r_n}, b_{r_n})$. If there is $j < n$ with $r_n \in (a_{r_j}, b_{r_j})$, then also $x \in (a_{r_j}, b_{r_j})$ and therefore also in this case $x \in \bigcup_{n \in F} (a_{r_n}, b_{r_n})$. $\quad\square$

The classification of the Rising Sun Lemma is very dependent on its precise formulation. If we had followed Tao's formulation [106] more closely we get an equivalence to WLPO. Luckily, we can reuse most of the proof above.

Proposition 1.3.9. *WLPO is equivalent to the following statement.*
Consider a continuous function $f : [0,1] \to \mathbb{R}$. Then one can find a, at most countable[7], family of disjoint, open, non-empty intervals I_n in $[0,1]$ such that

1. *$f(a_n) = f(b_n)$ unless $a_n = 0$, in which case $f(b_n) \geqslant f(a_n)$, and*

2. *if $x \notin \bigcup I_n$ and $x > 0$, then $f(x) \geqslant f(y)$ for all $y \geqslant x$.*

Proof. We will only proof the "counterexample direction", since the converse direction is analogous to the one above.

So let $a \in \mathbb{R}$ be a real number. Now define a piece-wise linear function $h : [0,1] \to \mathbb{R}$ by

$$h(x) = \begin{cases} \frac{3}{2}x & x \in [0, \frac{1}{3}] \\ \frac{1}{2} & x \in [\frac{1}{3}, \frac{2}{3}] \\ \frac{1}{2} + |a|(3x - 2) & x \in [\frac{2}{3}, 1]. \end{cases}$$

Again it is easy to see that the "area in the shade" consists of exactly one interval $(0, b)$. If $b < \frac{2}{3}$, then $|a| > 0$ leads to a contradiction, since then $E = (0, 1)$. So we must have $a = 0$. If $b > \frac{1}{3}$, then $a = 0$ leads to a contradiction, since that in that case $E = (0, \frac{1}{3})$. Together we can decide whether $a = 0$ or $\neg(a = 0)$, which means that WLPO holds. $\quad\square$

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[7]This is one of the places we actually have to be tediously precise. In this case, by a "at most countable" set A we mean that $A = \lambda^{-1}(\{0\})$ for an increasing binary sequence λ.
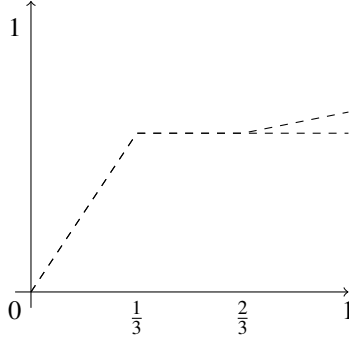
1.4 LLPO and WKL

The final limited omniscience principle is the *lesser limited principle of omniscience*.

> **(LLPO)** If $(a_n)_{n \geqslant 1}$ is a binary sequence with at most one 1, then
>
> $$\forall n \in \mathbb{N} : a_{2n} = 0 \lor \forall n \in \mathbb{N} : a_{2n+1} = 0.$$

LLPO is often presented in the following form which is often actually used as the definition of LLPO; for example in [68].

Proposition 1.4.1. *LLPO is equivalent to one of the De Morgan's laws for simply existential statements. That is for $\alpha, \beta \in 2^{\mathbb{N}}$*

$$\neg(\exists n : \alpha(n) = 1 \land \exists n : \beta(n) = 1) \implies \neg(\exists n : \alpha(n) = 1) \lor \neg(\exists n : \beta(n) = 1)$$

Proof. Straightforward. $\qquad \square$

Proposition 1.4.2. *The following are equivalent to LLPO*

1. $\forall x \in \mathbb{R} : 0 \leqslant x \lor x \leqslant 0$

2. *(The intermediate value theorem) For every sequentially non-discontinuous function $f : [0,1] \to \mathbb{R}$ with $f(0) \leqslant 0 \leqslant f(1)$ there exists $x \in [0,1]$ such that $f(x) = 0$*

3. *Every real number has a binary expansion.*

4. *If x, y are real numbers such that $xy = 0$, then either $x = 0$ or $y = 0$; that is $\mathbb{R}$ is an integral domain.*

5. *If x, y are real numbers, then $\{x, y\}$ is a closed set.*

Proof. All equivalencies are very well known; see for example [36] for the first 4 and [84, Theorem 4.2] for the last one. The equivalence LLPO $\iff$ 4 is also proved in a more general form in Proposition 8.1.2. $\qquad \square$

Completeness of Finite Sets

In Proposition 1.4.2 we cited a paper by Mandelkern [84] that among many other insights shows that LLPO is equivalent to every two-element set of reals being closed. That is constructively we cannot show that for any two reals a, b

$$\overline{\{a,b\}} = \{a,b\} \ .$$

Analysing Mandelkern's proof one might hope that constructively we have at least

$$\overline{\{a,b\}} = \{a, b, \inf\{a,b\}, \sup\{a,b\}\} \ .$$

However, as the next result shows this statement is still equivalent to LLPO.

Proposition 1.4.3. *LLPO is equivalent to the statement that for all $a, b \in \mathbb{R}$ the set*

$$\{a, b, \inf\{a,b\}, \sup\{a,b\}\}$$

is the closure of $\{a,b\}$.

Proof. As mentioned above it is well known [84, Theorem 4.2] that LLPO implies (is equivalent to) that for all a, b the set $\{a,b\}$ is closed.

Conversely let $(a_n)_{n \geqslant 1}$ be a binary sequence with at most one 1. Let $a = \sum \frac{a_n}{2^n}$ and define a sequence $(z_n)_{n \geqslant 1}$ in $\{0, a\}$ by

$$z_n = \begin{cases} 0 & \forall m < n : a_m = 0 \\ 0 & \exists m < n : a_m = 1 \wedge m \text{ is odd} \\ a & \exists m < n : a_m = 1 \wedge m \text{ is even} \ . \end{cases}$$

Using the fact that $(a_n)_{n \geqslant 1}$ has at most one term equal to 1, it is easy to see that z_n is a Cauchy sequence converging to a point z_∞. Now if $z_\infty \in \{a, b, \inf\{a,b\}, \sup\{a,b\}\}$ we can make the following decisions:

$z_\infty = 0$**:** there cannot be an even m such that $a_m = 1$, since in that case $a \neq 0$ and $z_n \to a$.

$z_\infty = a$**:** there cannot be an odd m such that $a_m = 1$, since in that case $a \neq 0$ and $z_n \to 0$.

$z_\infty = \inf\{a,b\}$**:** there cannot be an even m such that $a_m = 1$, since in that case $a \neq 0 = \inf\{a,b\} = 0$ and $z_n \to a$.

$z_\infty = \sup\{a,b\}$**:** there cannot be an odd m such that $a_m = 1$, since in that case $a \neq 0 = \sup\{a,b\} = a$ and $z_n \to 0$.

In any case we can decide whether $\forall n \in \mathbb{N} : a_{2n} = 0 \vee \forall n \in \mathbb{N} : a_{2n+1} = 0$. So we have shown LLPO. $\square$

Actually we can even show the similar result that

Proposition 1.4.4. *LLPO is equivalent to the statement that for all $a, b \in \mathbb{R}$ the closure of $\{a,b\}$ is a finitely enumerable set with 3 elements.*

Proof. Again, as above LLPO implies that $\{a,b\}$ is closed, so the only interesting direction is the converse. Let a_n and a be as above. Since $\{0,a\} \subset \overline{\{0,a\}}$ we may assume that the closure of $\{0,a\}$ is of the form $\{0, a, b\}$. Now define z_n as above, but also z'_n analogously, but with the odd and even case switched. If $z_\infty = 0$ or $z_\infty = a$ we are done as above. Similarly we are done if $z'_\infty = 0$ or $z'_\infty = a$. The only interesting and genuinely new case is that $z_\infty = z'_\infty = b$. But in that case the assumption that there is m with $a_m = 1$ leads to the contradiction $z_\infty \neq z'_\infty$, both in the odd and the even case. Again in all cases we have decided $\forall n \in \mathbb{N} : a_{2n} = 0 \vee \forall n \in \mathbb{N} : a_{2n+1} = 0$. So we have shown LLPO. $\square$

We conjecture that the above proposition can be generalised to any finite number and possibly the countable case. Notice that there exists a realisability model (based on infinite time Turing machines) in which there is a surjection $\mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ [5], which means that the closure of $\{a, b\}$ is countable, but that in that model also LPO and therefore LLPO holds.

WKL, Minima, and Fixed Points

The one principle in Constructive Reverse Mathematics that is possibly best known by non-constructivists is Weak Kőnig's Lemma.

> **(WKL)**   Every infinite and decidable tree admits an infinite path.

A decidable tree is a subset $T \subset 2^*$ of all binary finite sequences which is *closed under restrictions*, that is if $u \in T$ and v is a prefix of u, then also $u \in T$. A tree is infinite, if it is infinite as a set, which in this case is equivalent to containing arbitrarily long finite sequences. Finally, a tree *admits an infinite path*, if there exists an infinite binary sequence $\alpha \in 2^{\mathbb{N}}$ such that every finite prefix of α is in T.

It was, to our knowledge, Ishihara who first pointed out the equivalence of LLPO and WKL over intuitionistic logic and countable choice. If one is thinking in a resource sensitive way, then one can view WKL as countable many versions of LLPO [21]. If one wants to factor out choice, then one can view WKL as LLPO plus a choice principle [16].

Proposition 1.4.5. *WKL and LLPO are equivalent.*

Proof. This proof has been given many times; see for example [16]. It relies on repeatedly using the fact that, given an infinite tree, LLPO allows us to decide whether the left or the right subtree are infinite. We will repeat the proof for completeness sake, and since we are going to reuse the argument in similar situations further on. Let $T \subset 2^*$. For $u \in 2^*$ let T_u denote the set $\{ w \in 2^* \mid u * w \in T \}$; i.e. T_u is the subtree "below" u. Now if T is an infinite decidable tree, then we can define a binary sequence $(a_n)_{n \geqslant 1}$ by

$$a_{2n} = 0 \iff \exists u \in 2^n : 0 * u \in T , \tag{1.3}$$

$$a_{2n+1} = 0 \iff \exists u \in 2^n : 1 * u \in T . \tag{1.4}$$

Notice that, since T is infinite there cannot $n, m \in \mathbb{N}$ such that $a_{2n} = 1$ and $a_{2m+1} = 1$. Hence by 1.4.1 LLPO implies that either all even or all odd terms of a_n are zero. That means that either T_0 or T_1 is infinite. Of course, both are also still decidable trees. Thus we can, using dependent choice, iteratively define a sequence $\alpha \in 2^{\mathbb{N}}$ such that $T_{\overline{\alpha}n}$ is a infinite, decidable tree for all $n \in \mathbb{N}$. In particular $\overline{\alpha}n \in T$. $\square$

Remark 1.4.6. *Analysing the above proof we can see that WLPO implies WKL, only assuming unique choice.*

Proposition 1.4.7. *The following are equivalent to WKL*

1. *Every uniformly continuous function $f : 2^{\mathbb{N}} \to \mathbb{R}$ attains its minimum.*

2. *Every uniformly continuous function $f : X \to \mathbb{R}$ on a compact metric space X attains its minimum.*

3. *Every uniformly continuous function $f : [0, 1] \to \mathbb{R}$ attains its minimum.*

4. *(Brouwer's fix point theorem) Every uniformly continuous function $f : [0, 1]^2 \to [0, 1]^2$ has a fixed point.*

Proof. That 1 implies 2 follows that for every compact space X there exists a surjection $G : 2^{\mathbb{N}} \to X$ [107, Corollary 4.4]. Clearly 2 implies 3. That 3 implies 1 follows from Lemma 3.1.3. Finally a proof of the equivalence of 4 and WKL can be found in [60]. $\square$

Remark 1.4.8. *Using Proposition 6.1.3 we can replace uniform by point-wise continuity in the above proposition.*

This last proposition above also allows the following heuristic.

Heuristic 1.4.9. *Many fix point theorems are equivalent to WKL.*

Of course this is just a heuristic, which actually fails in some situations such as Proposition 1.2.8 or 3.3.4. It does, however, work for many well known ones such as Brouwer's fix point theorem mentioned above and many others [60]. We can even give a "proof" for our heuristic.

Proof. The reason that our heuristic is a good one is that often fix point theorems consider a point-wise continuous function f on a compact space X, and one can construct approximate fix points with constructive, often discrete, arguments. Thus if we consider the function $g : X \to \mathbb{R}$ defined by $g(x) = d(x, f(x))$ we can see that having approximate fix points translates into $\inf g = 0$, and thus the above proposition means that WKL implies the existence of a fix point. Since WKL also implies UCT (see Proposition 6.1.3) and therefore FAN_Δ, this argument is still valid, even if our original function is only point-wise continuous, or if we need a mild version of the fan theorem to prove the existence of approximate fix points.

Conversely most fix point theorems imply the intermediate value theorem as a very simple instance. Thus, by 1.4.2, they imply LLPO and therefore WKL. $\square$

As the "proof" above shows this also implies that if we add the additional assumption that there is uniformly at most one solution to a fix point theorem we often get a purely constructive theorem. This phenomenon is exhaustively treated in [15] and [98].

Space-filling curves

> *The content of this section might well be folklore among some constructivists. We do not claim to be the first to prove these results, but we are also not aware that they have been written down anywhere else.*

One of the stranger objects in (classical) analysis are *space-filling curves*, which are continuous functions $[0, 1] \to [0, 1]^2$ that are surjective. The first such curve was discovered by Peano in 1890, and since then numerous variants have been described. One of these variants which is very similar but slightly easier than Peano's one, is a construction by Hilbert from 1891. We are not going to repeat the well-known ideas here, but simply include a pretty picture of the first 4 steps in the construction and otherwise refer to the exhaustive monograph [97] for details.
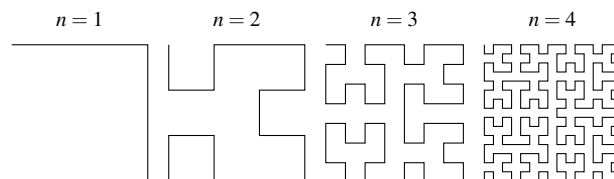

Figure 1.4: The first 4 steps in Hilbert's construction. (Latex code from www.texample.com).

Technically speaking, both the Hilbert curve and the Peano curve are described as the uniform limit of uniformly continuous functions. It is not hard to see that even constructively we can follow Hilbert's or Peano's construction, and obtain uniformly continuous functions $H, P : [0,1] \to [0,1]^2$. One can also easily see that points in $[0,1]^2$ that have a binary expansion have a pre-image. Therefore $P([0,1])$ and $H([0,1])$ are dense in $[0,1]^2$. However, we cannot prove that H and P are surjective. As a matter of fact:

Proposition 1.4.10. *LLPO is equivalent to the statement that Hilbert's space-filling curve or Peano's space-filling curve is surjective.*

Proof. We will only focus on H, since the case of P can be treated similarly. LLPO implies that every real number has a binary expansion (Proposition 1.4.2). Thus, as already remarked above we can actually find the pre-image of any point in $[0,1]^2$.

Conversely note that

$$H\left([0,\nicefrac{1}{4}]\right) \subset [0,\nicefrac{1}{2}] \times [0,\nicefrac{1}{2}] \ ,$$
$$H\left([\nicefrac{1}{4},\nicefrac{3}{4}]\right) \subset [\nicefrac{1}{2},1] \times [0,1] \ ,$$
$$H\left([\nicefrac{3}{4},1]\right) \subset [0,\nicefrac{1}{2}] \times [\nicefrac{1}{2},1] \ ,$$

since these relations are satisfied at every stage of the construction of H and preserved under limits. Now let $a \in \mathbb{R}$ arbitrary. Without loss of generality $|a| < \nicefrac{1}{2}$. Consider the point $p = (0, \nicefrac{1}{2}+a)$. If H is surjective, then there exists t such that $H(t) = p$. Now either $t < \nicefrac{3}{4}$ or $t > \nicefrac{1}{4}$. In the first case it is impossible that $a > 0$, since then $p \notin H([0,\nicefrac{3}{4}])$. Similarly in the second case it is impossible that $a < 0$. Thus we can decide whether $a \leqslant 0$ or $a \geqslant 0$; hence LLPO holds. $\square$

Since a space-filling curve cannot be injective classically it is also not injective constructively. The question of whether we can prove a stronger version of non-injectiveness for an arbitrary space-filling curve constructively appears to be a tricky one.

Question 2. *Is every space-filling curve non-injective? More precise: if f is a space-filling curve, then does there exist $x \neq y$ with $f(x) = f(y)$?*

It seems feasible to alter the constructions of H and P slightly by using overlapping square decompositions to obtain constructively surjective space-filling curves. However, there is a more straightforward and cleaner construction, which follows an idea of Lebesgue.

Lemma 1.4.11. *There exists a space-filling curve.*

Proof. As we will see in a later section (Lemma 3.1.1) there exists a uniformly continuous, surjective function $F : 2^{\mathbb{N}} \to [0,1]$. If we compose this function with the uniformly continuous bijection $\varphi : 2^{\mathbb{N}} \to \left(2^{\mathbb{N}}\right)^2$ defined by

$$\varphi(\alpha_1 \alpha_2 \alpha_3 \dots) = (\alpha_1 \alpha_3 \dots, \alpha_2 \alpha_4 \dots)$$

we get a uniformly continuous surjection $f : 2^{\mathbb{N}} \to [0,1]^2$. By Lemma 3.1.3 we can extend this to a uniformly continuous function $\tilde{f} : [0,1] \to [0,1]^2$. Obviously $\tilde{f}$ is still surjective. $\square$

Lemma 1.4.12. *There exists a uniformly continuous surjection $f : \mathbb{R} \to \mathbb{R}^2$*

Proof. Straightforward with the previous lemma. $\square$

Corollary 1.4.13. *In RUSS there exists a continuous surjection $f : [0,1] \to \mathbb{R}^2$.*

Proof. In RUSS there exists an unbounded point-wise continuous surjective map $h : [0,1] \to \mathbb{R}$ (see Proposition 5.4.4). That map, composed with the one from the previous lemma gives us the desired result. $\square$

We can use the above construction of a space-filling curve to partially answer a question posed by K. Weihrauch, who asked whether in TTE [112] there exist computable curves connecting pairs of diagonally opposite points of the unit square that do not intersect at a computable point. Such functions exist in RUSS, however the original paper (1976 by S.N. Manukyan) is in Russian and hard to obtain, so the details of the construction are not clear. The result below is transferrable to TTE and shows that there cannot be an algorithm taking such curves as inputs and computing the point of intersection.

Proposition 1.4.14. *LLPO is equivalent to the following statement: If $g, h : [0,1] \to [0,1]^2$ are point-wise continuous such that $g(0) = (0,0)$, $g(1) = (1,1)$, $h(0) = (0,1)$, and $h(1) = (1,0)$, then they intersect, that is there exist points t_1, t_2 such that $g(t_1) = h(t_2)$.*

Proof. For one direction we will show that the statement implies the intermediate value theorem (see Proposition 1.4.2.2). So let $f : [0,1] \to \mathbb{R}$ be continuous such that $f(0) < 0$ and $f(1) > 0$. Without loss of generality we may assume that $f([0,1]) \subset [-1,1]$ and that $f(0) = -1$ and $f(1) = 1$. Now define $h, g : [0,1] \to [0,1]^2$ piecewise linearly by

$$h(x) = \begin{cases} -\frac{3}{2}x + 1 & \text{if } 0 \leqslant x \leqslant \frac{1}{3} \\ \frac{1}{2} & \text{if } \frac{1}{3} \leqslant x \leqslant \frac{2}{3} \\ -\frac{3}{2}x + \frac{3}{2} & \text{if } \frac{2}{3} \leqslant x \leqslant 1 \end{cases}$$

and

$$g(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \frac{1}{3} \\ \frac{1}{6}f(3x-1) + \frac{1}{2} & \text{if } \frac{1}{3} \leqslant x \leqslant \frac{2}{3} \\ x & \text{if } \frac{2}{3} \leqslant x \leqslant 1 \ . \end{cases}$$
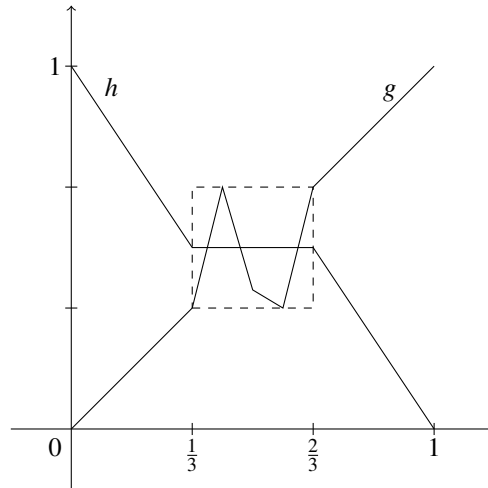
Figure 1.5: A scaled copy of f is placed in the middle third square of the unit square. The roots of f are then linearly related to the points of intersection of h and g.

It is easy to see that if $t \mapsto (t, h(t))$ and $t \mapsto (t, g(t))$ intersect, then they must do so for $t \in [\frac{1}{3}, \frac{2}{3}]$, which implies that $f(3t - 1) = 0$. Thus we have found a root of f.

Conversely assume that LLPO holds, and let h and g be as in the statement of the proposition. Then we can use the space-filling curve $\tilde{f}$ constructed in the previous lemma to consider the function and define

the function $M : [0,1] \to \mathbb{R}$ by

$$M(x) = d(h(\tilde{f}_1(x)), g(\tilde{f}_2(x))) \ .$$

By construction M is uniformly continuous. It is also easy to see that $\inf M = 0$. Thus by 1.4.7 there exists $x \in [0,1]$ such that $M(x) = 0$, and hence the two curves intersect. $\qquad\square$

The Greedy Algorithm

A matroid $(E, \mathscr{F})$ consists of a finite set E together with a collection $\mathscr{F}$ of subsets E, such that

I.1 $\emptyset \in \mathscr{F}$

I.2 $A \in \mathscr{F} \wedge B \subset A \implies B \in \mathscr{F}$

I.3 $A, B \in \mathscr{F} \wedge |B| = |A| + 1 \implies \exists x \in B : A \cup \{x\} \in \mathscr{F}$.

In the following, we will also assume that all the sets in $\mathscr{F}$ are decidable. A nice introduction to matroids is [92]. The standard example of a matroid are the linear independent subsets of a finite set of vectors in a vector space. Property (I.3) then is the Steinitz exchange lemma. This also justifies the terminology of calling a set $A \in \mathscr{F}$ independent. A maximal independent set A is one that is independent and such that there is no $B \in \mathscr{F}$ with $A \subsetneq B$. Matroids are also important since they provide the minimal setting[8] on which a Greedy algorithm yields an optimal solution. A Greedy strategy is the most naïve strategy possible: build up your solution by always choosing the most tantalising looking element next. In the following we assume that there is also a weighting function $w : E \to \mathbb{R}$. The goal is to find a maximal independent set $A \in \mathscr{F}$ that has minimal weight i.e. that is such that the sum

$$\sum_{a \in A} w(a)$$

is smaller than that for any other maximal independent set.

Proposition 1.4.15. *The following are equivalent:*

1. LLPO

2. If $\{x_1, \ldots, x_n\} \subset \mathbb{R}$, then there exists $1 \leqslant i \leqslant n$ with $x_i \leqslant x_j$ for all $1 \leqslant j \leqslant n$.

3. The Greedy algorithm. I.e. if $(E, \mathscr{F})$ is a matroid and $w : E \to \mathbb{R}$ a function, then there exists a maximal independent subset with a minimal weight.

Proof. (1) $\implies$ (2) can be proved using induction and 1.4.2. (2) $\implies$ (3): a maximal independent set with minimal weight is just constructed by the Greedy algorithm:

```
1: Let F₀ = ∅.
2: for i ⩾ 0 do
3:     Let Zᵢ = { zᵢ ∈ E \ Fᵢ | Fᵢ ∪ {zᵢ} ∈ F }.
4:     if Zᵢ = ∅ then terminate and return Fᵢ .
5:     else
6:         choose y ∈ Zᵢ such that w(y) ⩽ w(zᵢ) for all zᵢ ∈ Zᵢ
7:         let Fᵢ₊₁ = Fᵢ ∪ {y}
8:     end if
9: end for
```

---

[8]One can actually do without the property (I2). Such structures are known as greedoids.

Notice, that in step 6 we need (2). For (3) $\implies$ (1) let $x, y \in \mathbb{R}$. Then

$$\mathcal{M} = (\{1,2\}, \{\emptyset, \{1\}, \{2\}\})$$

is a matroid. Now define $w : \{1,2\} \to \mathbb{R}$ by setting $w(1) = x$ and $w(2) = y$. If the Greedy algorithm works, then either $\{1\}$ or $\{2\}$ has minimal weight. But that means that either $x = w(1) \leqslant w(2) = y$ or $w(2) = y \leqslant x = w(1)$. $\qquad\square$

Remark 1.4.16. *It is almost trivial to see that adapting the Greedy algorithm, we can, for every $\varepsilon > 0$ efficiently find a maximal independent set A such that*

$$\sum_{a \in A} w(a) - \varepsilon < \sum_{b \in B} w(b) .$$

Sharkovskii's Theorem

Sharkovskii's[9] Theorem from 1964 is, in our opinion, one of the most entertaining results in mathematics: simple to state yet utterly surprising. It concerns itself with a very rudimentary type of dynamical system, namely a point-wise continuous function f acting on $\mathbb{R}$. We are interested in fix points of f and *(prime) k-periodic points* which are points x such that $f^k(x) = x$ but $\neg(f^i(x) = x)$. Let us now define a total order $\triangleright$ on the natural numbers as indicated below

$$3 \triangleright 5 \triangleright 7 \triangleright 9 \triangleright \cdots \triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \triangleright \ldots 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright 2^2 \cdot 7 \triangleright$$
$$\ldots 2^3 \cdot 3 \triangleright 2^3 \cdot 5 \triangleright 2^3 \cdot 7 \triangleright \ldots 2^{n+1} \triangleright 2^n \triangleright \ldots 8 \triangleright 4 \triangleright 2 \triangleright 1 .$$

Sharkovskii's Theorem states:

> **(ST)**   If f has a k-periodic point and $k \triangleright \ell$, then f has points of prime period ℓ.

In particular, if a function has a 3-periodic point it has points of all periods. The only non-constructive part in the usual proof [41] is the following technicality, which is actually equivalent to LLPO.

Proposition 1.4.17. *LLPO is equivalent to the statement that if $f : [0,1] \to \mathbb{R}$ is a uniformly continuous map and $[a,b]$ an interval such that $f([0,1]) \supset [a,b]$, then there exists $[c,d] \subset [0,1]$ such that $f([c,d]) = [a,b]$.*

Proof. To see that this principle implies LLPO let $(a_n)_{n \geqslant}$ be a binary sequence with at most one 1. Let

$$a = \sum_{n \geq 1} \frac{(-1)^n a_n}{2^n}$$

and define a piecewise linear function $f : [0,1] \to [0,1]$ by

$$f(x) = \begin{cases} (3+3a)x, & x \in [0, \frac{1}{3}]; \\ -(3+6a)x + 2 + 3a, & x \in [\frac{1}{3}, \frac{2}{3}]; \\ (3+3a)x - 2 - 3a, & x \in [\frac{2}{3}, 1]. \end{cases}$$

As $f([0, \frac{1}{3}]) \supset [0, \frac{2}{3}]$ and $f([\frac{2}{3}, 1]) \supset [\frac{1}{3}, 1]$ we see that $f([0,1]) \supset [0,1]$. So assume there exists an interval $[c,d] \subset [0,1]$ such that $f([c,d]) = [0,1]$. Now either $c > 0$ or $c < \frac{1}{6}$. In the first case $a \geq 0$ as the assumption $a < 0$ implies that for all $x \in [c,1]$ $f(x) > 0$. In the second case either $d < 1$ or $d > \frac{5}{6}$. If $d < 1$ we can

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[9]There are various alternative ways of spelling Sharkovskii. Among them Sharkovsky [53], Sarkovskii [41], Sarkovski [16].
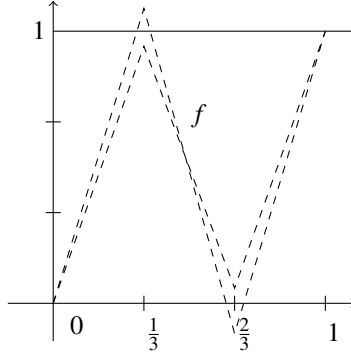
Figure 1.6: We think of a as being very small. Depending on a f crosses the lines $y = 0$ and $y = 1$ or misses both.

conclude as before that $a \geq 0$. In the only case left that $d > \frac{5}{6}$ and $c < \frac{1}{6}$ we must have that $a \leq 0$ as the assumption $a > 0$ would imply that there are $x \in [c, d]$ such that $f(x) > 1$. So we can decide whether $a \leq 0$ or $a \geq 0$. In the first case we must have that for all n $\alpha_{2n} = 0$. In the second case we must have that for all n $\alpha_{2n+1} = 0$.

To see that the converse holds assume LLPO holds. Furthermore let $f : [0,1] \to \mathbb{R}$ be a uniformly continuous map and $[a,b]$ an interval such that $f([0,1]) \supset [a,b]$. Choose a_0 and b_0 such that $f(a_0) = a$ and $f(b_0) = b$. Without loss of generality $a_0 \leq b_0$.

We will define a sequence of nested intervals $([a_n, b_n])$ such that

1. $|a_{n+1} - b_{n+1}| \leq \frac{1}{2}|a_n - b_n|$

2. $\sup\{f([a_0, a_n])\} \leq b$ and $\sup\{f([a_0, b_n])\} \geq b$

3. $f(a_n) \leq b$ and $f(b_n) \geq b$.

So assume we have constructed a_n and b_n. As we assume LLPO either

$$\sup\{f[a_0, \frac{1}{2}(a_n + b_n)]\} \geq b \text{ or } \sup\{f([a_0, \frac{1}{2}(a_n + b_n)])\} \leq b.$$

In the first case set $a_{n+1} = a_n$ and as we assumed LLPO the minimum principle holds [63] and so we can choose $b_{n+1} \in (a_0, \frac{1}{2}(a_n + b_n))$ such that $f(b_{n+1}) \geq b$. In the second case set $b_{n+1} = b_n$ and choose $a_{n+1} \in (a_0, \frac{1}{2}(a_n + b_n))$, such that $f(a_{n+1}) \leq b$. Both sequences converges to the same limit $d = a_\infty = b_\infty$ and furthermore $\sup\{f([a_0, d])\} \leq b$, $\sup\{f([a_0, d])\} \geq b$, $f(d) \leq b$ and $f(d) \geq b$. Hence $\sup\{f([a_0, d])\} = b$ and $f(d) = b$. A similar argument shows that there is a $c \in [a_0, d]$ such that $\inf\{f[c, d]\} = a$ and $f(c) = a$. Hence $f[c, d] = [a, b]$ $\qquad\square$

Of course this still leaves the possibility that there is a fully constructive proof of Sharkovskii's theorem. However, as the next Brouwerian counteraxample shows such a proof does need to use LLPO.

Proposition 1.4.18. *For all natural numbers n the following holds: If every (uniformly) continuous map that has a 3-periodic point has a fixpoint, then LLPO holds.*

Proof. Again let α_n be binary sequence with at most one 1. Let

$$a = \sum_{n \geq 1} \frac{(-1)^n \alpha_n}{2^n}$$

and define a piecewise linear function $f : [0,1] \to [0,1]$ by

$$f(x) = \begin{cases} 3x + \frac{1}{4}, & x \in [0, \frac{1}{4}]; \\ (4a-2)x + \frac{3}{2} - a, & x \in [\frac{1}{4}, \frac{2}{4}]; \\ x + a, & x \in [\frac{2}{4}, \frac{3}{4}]; \\ (-3-4a)x + 3 + 4a, & x \in [\frac{3}{4}, 1]. \end{cases}$$

$f(0) = \frac{1}{4}$, $f(\frac{1}{4}) = 1$ and $f(1) = 0$ so f has a 3–periodic point. If f has a fixed point x, then either $x < \frac{3}{4}$
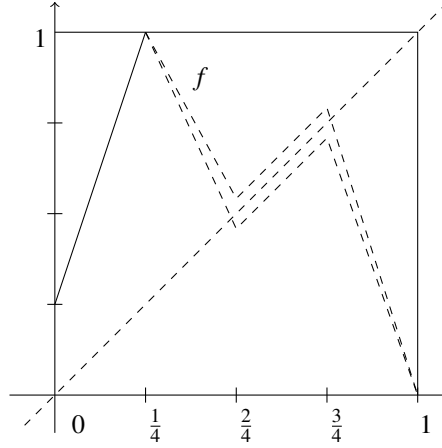
Figure 1.7: We cannot decide whether the graph of f between $\frac{2}{4}$ and $\frac{3}{4}$ lies above or below the diagonal.

or $x > \frac{2}{4}$. In the first case $a \leq 0$, as the assumption $a > 0$ implies that $f(x) > x$ for all $x \in [0, \frac{3}{4}]$. Then for all $\alpha_{2n+1} = 0$ for all n. Similar in the second case $a \geq 0$ and $\alpha_{2n} = 0$ for all n. $\qquad\square$

Putting these two results together we get the following corollary.

Corollary 1.4.19. *Sharkovskii's Theorem is equivalent to LLPO.*

Graph colourings

We assume that the reader is familiar with basic graph theoretic definitions. A nice little theorem by Erdős and Bruijn [52, Theorem 8.1.3], reminiscent of the compactness theorem in logic. is

> **(EB$_k$)**   Let $G = (V,E)$ be a countable graph and $k \in \mathbb{N}$. If every finite subgraph of G is k-colourable, then so is k.

We will show that the countable case is equivalent to WKL.

Proposition 1.4.20. *EB$_k$ is equivalent to WKL for every $k \geqslant 2$.*

Proof. To see that WKL implies EB$_k$ is straightforward. Since we assume countability, $V = \{v_1, v_2, \dots\}$. Every $u \in k^*$ of length m can be seen as a (not necessarily admissible) colouring of the subgraph $G[v_1, \dots, v_m]$ and every $\alpha \in k^{\mathbb{N}}$ as a (not necessarily admissible) colouring of G. Now simply consider

$$T = \{u \in 2^* \mid u \text{ is an admissible colouring of } G\}.$$

T is easily seen to be a tree and the assumption that every finite subset of G has an admissible k-colouring translates into T being infinite. By WKL T has an infinite path α, which represents an admissible colouring of G. Conversely, we will first show that EB$_2$ implies LLPO. So let $(a_n)_{n\geqslant 1}$ be a binary sequence with at most one term equal to 1. Now, let $G = (\mathbb{N}, E)$, with

$$E = \left\{ \{n, n+2\} \mid \forall m \leqslant n : a_m = 0 \right\} \cup \left\{ \{n, n+1\} \mid a_n = 1 \right\}.$$

That is, as long as $a_n = 0$, G can be drawn as two parallel lines. Should we hit $a_n = 1$, we connect the upper and the lower strand. The way this graph is constructed if there is an odd (even) n with $a_n = 1$, then the vertices 1 and 2 are connected by a path of odd (even) length. It is easy to see that every finite
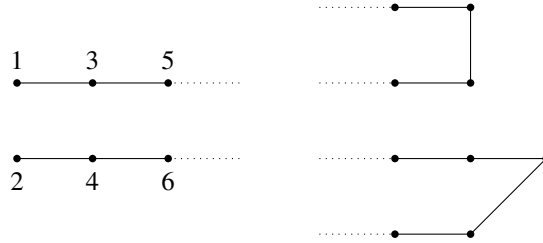
Figure 1.8: G consists of two separate strands as long as $a_n = 0$. If there is n with $a_n = 1$, then G is one strand of even or odd length depending whether n is odd or even.

sub-graph can be 2-coloured. If all of G can be coloured, then if the vertices 1 and 2 get the same colour there cannot be an even n with $a_n = 1$, which means $\forall n : a_{2n} = 0$. Similarly if 1 and 2 are coloured with different colours, then $\forall n : a_{2n+1} = 0$. For an arbitrary $k > 2$ we simply add one copy of the complete graph K_{n-2}, and connect every one of its vertices with every vertex of G. Thus we ensure, that the "original" G still gets coloured with exactly two colours. If one is happy to use countable choice, then LLPO implies WKL and we are done. We can, however, adapt the construction above to show that EB$_k$ implies WKL instance-wise and without countable choice. To this end assume that $T \subset 2^*$ is a decidable infinite tree. We can fix a binary sequence $(b_n^{(u)})_{n\geqslant 1}$ by

$$b_n^{(u)} = 1 \iff \forall w \in 2^n : u * w \notin T.$$

To turn the $b^{(u)}$ into sequences with at most one 1, we consider

$$d_{2n}^{(u)} = b_n^{(u0)} \text{ and } d_{2n+1}^{(u)} = b_n^{(u1)}$$

and

$$a_n^{(u)} = d_n^{(u)} - \max_{i < n} \left\{ d_i^{(u)} \right\}.$$

If $T_u = \left\{ v \in 2^{\mathbb{N}} \mid u * v \in T \right\}$ is infinite, then there cannot be n, m such that $b_n^{(u0)} = 1$ and $b_m^{(u1)} = 1$. Therefore if T_u is infinite and $a_{2n}^{(u)} = 0$ for all $n \in \mathbb{N}$, then $b_n^{(ui)} = 0$ for all $n \in \mathbb{N}$, and if $a_{2n+1}^{(u)} = 0$ for all $n \in \mathbb{N}$, then $b_n^{(ui)} = 0$ for all $n \in \mathbb{N}$.

We can construct a graph $G^{(u)}$ for every $a^{(u)}$ as done above. The formal sum (disjoint union) of these graphs

$$H = \bigoplus_{u \in T} G^{(u)}$$

is also such that every finite subgraph can be coloured with k colours.

If there is an admissible colouring of H, then we can, as above, decide whether $a_n^{(u)}$ is zero for all even or for all odd terms, which means that if T_u is infinite we can decide whether T_{u0} or T_{u1} is infinite.

So we can recursively construct α such that $T_{\overline{\alpha}n}$ is infinite. In particular $\overline{\alpha}n \in T$ for all $n \in \mathbb{N}$, which means we have found a path through T. $\square$

Variations of WKL

Reminiscent of the way WWKL (Section 3.2) is a weakening of FAN_Δ is the following weakening of WKL.[10] Let $k < 1$. We now require that for each n our tree not only have at least one sequence of length n, but for a certain percentage of all sequences of length n to be in the tree.

> (**WKL$'(k)$**)   If T is a binary decidable tree such that
> $$\frac{|\{u \in T \mid |u| = n\}|}{2^n} > k \,,$$
> then T admits an infinite path.

Of course, if $r < k < 1$, then $\text{WKL}'(r) \implies \text{WKL}'(k)$, and $\text{WKL}'(k)$ is implied by WKL. We can prove the following converse.

Proposition 1.4.21. *For any $k \leqslant \frac{1}{2}$, we have* $\text{WKL}'(k) \implies \text{LLPO}$.

Proof. Let $(a_n)_{n \geqslant 1}$ be a binary sequence with at most one term equal to 1. Now define a set $T \subset 2^*$ by

$$T = \{0u \mid \forall m \leqslant |u| : a_{2m} = 0\} \cup \{1u \mid \forall m \leqslant |u| : a_{2m+1} = 0\} \,.$$

Then it is easy to see that T is a tree and that $|\{u \in T \mid |u| = n\}|/2^n \geqslant \frac{1}{2} > k$. Hence, by our assumptions, T admits an infinite path $\alpha \in 2^{\mathbb{N}}$. Now if $\alpha(1) = 0$, then there cannot be an even n with $a_n = 1$. Similarly if $\alpha(1) = 1$ there cannot be an odd n with $a_n = 1$. $\qquad\square$

So under the assumption of countable choice $\text{WKL}'(k)$ is actually equivalent WKL, at least for $k < \frac{1}{2}$. It remains an open question whether the same holds for $k \geqslant \frac{1}{2}$.

Question 3. *To what principle is $\text{WKL}'(k)$ equivalent to for $k > \frac{1}{2}$?*

Notice that this question seems to be related to the question whether LLPO_n (which will be introduced in Section 8.1).

Compactness of Propositional logic

The language of first order logic is, as usual, defined inductively via

1. Every proposition symbol $A_0, A_1, A_2, \ldots$ is a formula.

2. If α, β are formulas, then so is $\neg\alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \to \beta$.

It is straightforward to show that an truth assignment $\mathscr{B}$ from the set of all proposition symbols to the set of truth values $\{\mathbf{f}, \mathbf{t}\}$ can be uniquely extended to one defined on all formulas. As usual we also assume that $\mathscr{B}(A_0) = \mathbf{f}$, and write $\bot$ instead of A_0. A truth assignment is called a *model* of a set of formulas Γ, when $\mathscr{B}(\alpha) = \mathbf{t}$ for all $\alpha \in \Gamma$. In this case we write $\mathscr{B} \vDash \Gamma$. Notice that in this propositional case we have $\mathscr{B} \vDash \varphi \vee \neg\varphi$ for any formula φ and any $\mathscr{B}$.

If a set Γ has a model it is called *satisfiable*. A basic result of logic, known under the name compactness, is that if every finite subset of a set Γ is satisfiable, then so is Γ itself. The proof generally uses König's Lemma. It is thus not surprising that:

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[10]This version was communicated to us by M. Hendtlass by Email. We do not know who deserves credit for proposing it.

Proposition 1.4.22. *WKL is equivalent to the statement that if every finite subset of a set Γ is satisfiable, then so is Γ.*

Proof. One direction is the usual proof, such as the one in [103, Chapter III.1], which apart from using WKL, is actually constructive.

Conversely assume T is an infinite decidable tree, so that there exists $u \in T$ with $|u| = n$ for each n, and define

$$\alpha_n = \bigvee_{\substack{u \in T \\ |u|=n}} \bigwedge_{1 \leqslant i \leqslant n} B_{u,i} \,,$$

where

$$B_{u,i} = \begin{cases} A_i & \text{if } u(i) = 1 \\ \neg A_i & \text{otherwise;} \end{cases}$$

We are going to show that for every $m \in \mathbb{N}$ the subset $\{\alpha_1, \ldots, \alpha_m\}$ of Γ has a model. Since T is infinite there exists $v \in T$ with $|v| = m$. Define $\mathscr{B}'$ by

$$\mathscr{B}'(A_i) = \begin{cases} \mathbf{t} & \text{if } v(i) = 1 \text{ and } i \leqslant m \\ \mathbf{f} & \text{otherwise.} \end{cases}$$

For all $j \leqslant i \leqslant m$, since T is a tree $\bar{u}i \in T$, and therefore we have that $\mathscr{B}'(B_{\bar{u}i,j}) = \mathbf{t}$. Hence $\mathscr{B}'(\alpha_i) = \mathbf{t}$ for all $i \leqslant m$, which means that $\mathscr{B}'$ is a model of $\{\alpha_1, \ldots, \alpha_m\}$.

Now assume that $\mathscr{B}$ is a model of $\Gamma = \{\alpha_1, \alpha_2 \ldots\}$. We claim that $\sigma \in 2^{\mathbb{N}}$ defined by

$$\sigma(i) = 1 \iff \mathscr{B}(A_i) = \mathbf{t}$$

is a path through T. $\qquad\square$

Chapter 2

MP and Below

We will start this chapter by introducing some notation. If a number $x \in \mathbb{R}$ is such that $\neg\neg(0 < x)$ we say that x is *almost positive* and write $0 \lessdot x$. Furthermore, we write $y \lessdot x$ instead of $0 \lessdot x - y$ and $x \gtrdot y$ interchangeably with $y \lessdot x$. Note that since $x \leqslant y$ is, constructively, defined as $\neg(y < x)$ [20], we have that

$$x \lessdot y \iff \neg(y \leqslant x).$$

Real numbers x that satisfy

$$\forall y \in \mathbb{R} : 0 \lessdot y \vee y \lessdot x \tag{2.1}$$

are called *pseudo-positive*. If we apply the pseudo-positiveness property to a number x itself the right disjunct is always ruled out and we get that x is almost positive.

So we have the following notions for a number being positive.

$$\text{positive} \implies \text{pseudo-positive} \implies \text{almost positive} \implies \text{non-negative},$$

or in symbols, for $x \in \mathbb{R}$

$$0 < x \implies (x \text{ pseudo-positive} \implies) \quad 0 \lessdot x \implies 0 \leqslant x.$$

Lemma 2.0.1. *For all $x, y, z \in \mathbb{R}$*

1. *$\neg(\neg(x > y) \wedge \neg(x = y) \wedge \neg(x < y))$.*

2. *if $x \leqslant y$ and $y \lessdot z$, then $x \lessdot z$. (This implies that $\lessdot$ is transitive).*

3. *If $x = \max\{y, z\}$ and $\neg(x = y)$, then $x = z$.*

Proof.   1. If $\neg(y > x)$ and $\neg(y < x)$, then $x = y$, since both the assumption that $x < y$ and $y < x$ lead to contradictions. But this in turn contradicts $\neg(x = y)$.

2. Assume that $x \leqslant y$ and $y \lessdot z$. Furthermore assume that $z \leqslant x$. Then $z \leqslant y$, which is a contradiction. Hence $\neg(z \leqslant x)$, which is what we wanted to show.

3. By definition $x \geqslant z$. Assume that $x > z$. Then $y = x$: since either $y > x$ as well as $y < x$ leads to a contradiction. The first one since $x \geqslant y$, the second one as then we could find r such that $x > r > z$ and $x > r > y$ and therefore $x > r \geqslant \max\{y, z\} = x$. Altogether $x > z$ is a contradiction and therefore $z \leqslant x$, which means that $x = z$. $\qquad\square$

2.1 MP

Markov's principle is a weak form of double negation elimination.

> **(MP)** Every almost positive number is positive: $x \not> 0 \implies x > 0$.

It was also called LPE[1] by Mandelkern. It is generally accepted by recursive schools of constructive mathematics, where it embodies the strategy of an "unbounded search."

Proposition 2.1.1. *The following are equivalent to* MP*:*

1. *If $(a_n)_{n \geqslant 1}$ is a binary sequence such that $\neg(\forall n \in \mathbb{N} : a_n = 0)$, then $\exists n \in \mathbb{N} : a_n = 1$.*

2. $\forall x \in \mathbb{R} : \neg(x = 0) \implies x \neq 0.$

3. *If $A \subset \mathbb{N}$ is countable and so is its complement $\bar{A}$, then A is decidable.*

4. *Every weakly injective[2] map $f : [0,1] \to \mathbb{R}$ is injective.*

5. *Every mapping from a metric space into a metric space is strongly extensional.*

Proof. The equivalences between MP, 1, and 5 are from [32] and [68]. The equivalence between 2 and MP is obvious.

To see that 3 is equivalent to 1 consider a set $A \subset \mathbb{N}$ such that $A = \{a_1, a_2, \dots\}$ and $\bar{A} = \{b_1, b_2, \dots\}$. Given $m \in \mathbb{N}$ we can consider the following binary sequence c_n defined by

$$c_n = \begin{cases} 0 & \text{if } m \notin \{a_1, b_1, \dots, a_n, b_n\} \\ 1 & \text{otherwise .} \end{cases}$$

Now we have $\neg\forall n \in \mathbb{N} : c_n = 0$, since the assumption $\forall n \in \mathbb{N} : c_n = 0$ leads to the contradiction that $m \notin A \wedge m \notin \bar{A}$. So we can use MP to get n such that $c_n = 1$, which immediately tells us that $m \in \{a_1, b_1, \dots, a_n, b_n\}$ which allows us to decide whether $m \in A$ or not. Conversely let a_n be a binary sequence such that $\neg\forall n \in \mathbb{N} : a_n = 0$. We may assume that $a_1 = 0$. Then the set $A = \{2 \,|\, \exists n : a_n = 1\} \cup \{1\}$ is countable, as well as its complement. So we can decide whether $2 \in A$. But the assumption that $2 \notin A$ limplies that $\forall n \in \mathbb{N} : a_n = 0$. Thus we must have $2 \in A$, which means $\exists n : a_n = 1$.

Finally we are going to show that 4 is equivalent to 2. Notice that if f is a weakly injective map and x, y are such that $f(x) \neq f(y)$, then $\neg(x = y)$ which implies that also $\neg(x - y = 0)$. Thus, using 2 we have $x - y \neq 0$, which means that $x \neq y$. So f is injective. Conversely let x be such that $\neg(x = 0)$. Now consider the linear map $f(y) = xy$. It is easy to see that this map is weakly injective, for consider y, z with $\neg(y = z)$. Now assume $f(y) = f(z)$, which means $xy = xz$. But then the assumption that $y \neq z$ implies that $x = 0$, which is a contradiction. So we must have $\neg(y \neq z))$ which is equivalent to $y = z$. This contradicts our initial assumption that $\neg(y = z)$, so we have proved $\neg(f(y) = f(z))$. By 4 f is injective, and thus $0 = f(0) \neq f(1) = x$. $\square$

---

[1] Limited Principle of Existence

[2] A map is called weakly injective if $f(x) = f(y) \implies x = y$, as opposed to injective which means $x \neq y \implies f(x) \neq f(y)$. Notice that a map is weakly injective if and only if $\neg(x = y) \implies \neg(f(x) = f(y))$, by the stability of equality.

2.2 WMP

Weak Markov's Principle states that every pseudo-positive number is positive:

> **(WMP)** For all $x \in \mathbb{R}$, if
>
> $$\forall y \in \mathbb{R} : (y > 0 \ \lor \ y \ll x) , \qquad\qquad (2.2)$$
>
> or equivalently, if
>
> $$\forall y \in \mathbb{R} : \neg(y \leqslant 0) \ \lor \ \neg(y \geqslant x) , \qquad\qquad (2.3)$$
>
> then $0 < x$.

This principle was also called WLPE[3] and ASP[4] by Mandelkern. U. Kohlenbach has shown that it does not hold in a certain intuitionistic formal system [72]. This was improved upon by M. Hendtlass and B. Lubarsky who have recently given a topological model satisfying full IZF as well as dependent choice in which WMP fails (see Corollary 7.2.13).

Proposition 2.2.1. *The following are equivalent to WMP:*

1. *Every pseudo-positive number is positive.*

2. *If $\neg(a = b)$ and $\{a, b\}$ is complete, then $|a - b| > 0$.*

3. *Every mapping from a complete metric space into a metric space is strongly extensional.*

4. *Every real-valued function which is non-decreasing and approximates intermediate values is pointwise continuous.*

Proof. The proofs for these equivalences can be found in [84], [64], [83], respectively. □

The following lemmas provide a stepping stone to characterise WMP in terms of elements of Cantor space.

Lemma 2.2.2. *A non-negative real number x is pseudo-positive if and only if*

$$\forall y \in \mathbb{R} : \neg(x = y) \lor \neg(y = 0) .$$

Proof. Since $a \ll b \implies \neg(a = b)$ one direction is clear. To prove the other direction let us first assume that $y \in [0, x]$, that is $y \geqslant 0$ and $y \leqslant x$. By our assumption either $\neg(y = 0)$ or $\neg(x = y)$. In the first case $y > 0$ by Lemma 2.0.1. In the second case $y \ll x$ as well by Lemma 2.0.1. Next consider $y \in (-\infty, x]$. Then $y' = \max\{0, y\} \in [0, x]$. Hence by what was just proved either $y' > 0$ or $y' \ll x$. In the first case, by part 3 of Lemma 2.0.1, $y' = y$ and therefore $y > 0$. In the second case, since $y \leqslant y'$ also $y \ll x$ by part 2 of Lemma 2.0.1. So we can perform the final step and consider an arbitrary $y \in \mathbb{R}$. Consider a $x' = \min\{x, y\} \in (-\infty, x]$. With the previous step we know that either $x' > 0$ or $x' \ll x$. In the first case, since $y \geqslant x'$, by part 2 of Lemma 2.0.1, we have $y > 0$. In the second case, by part 3 of Lemma 2.0.1, we have $x' = y$ and therefore $y \ll x$. □

Even though it seems most of the work for the next proposition is contained in the last lemma, its proof is not at all that straightforward and, interestingly, we need countable choice in *both* directions.

---

[3]Weak Limited Principle of Existence
[4]Almost Separating Principle

Proposition 2.2.3. *WMP is equivalent to the following statement. For every $\alpha \in 2^{\mathbb{N}}$ such that*

$$\forall \beta \in 2^{\mathbb{N}} : (\neg(\alpha = \beta) \ \vee \ \neg(\beta = 0)) \tag{2.4}$$

there exists n such that $\alpha(n) = 1$.

Proof. Let us assume WMP holds and let $\alpha \in 2^{\mathbb{N}}$ be as described. Without loss of generality we may assume that α is increasing. We claim that $x = \sum_{n \in \mathbb{N}} \frac{\alpha(n)}{2^n}$ is pseudo-positive. To this end let $y \in \mathbb{R}$ be arbitrary. Using countable choice construct a flagging sequence $(\lambda_n)_{n \geqslant 1}$, such that if $\alpha(n) = 1$ and therefore $x > \frac{1}{2^{n+1}}$, then

$$\lambda_n = 0 \implies |x - y| < \frac{1}{2^{n+2}} \ ,$$
$$\lambda_n = 1 \implies |y| > \frac{1}{2^{n+3}} \ .$$

Again, without loss of generality we may assume that λ is increasing. Define $\beta \in 2^{\mathbb{N}}$ by

$$\beta(n) = \min\{\alpha(n), \lambda(n)\}$$

If $y = 0$, then $\beta = 0$ and if $y = x$, then $\beta = \alpha$. So either $\neg(\beta = 0)$ or $\neg(\alpha = \beta)$.

Conversely let x be a pseudo-positive number. Using countable choice construct α such that

$$\alpha(n) = 0 \implies |x| < \frac{1}{2^n} \ ,$$
$$\alpha(n) = 1 \implies |x| > \frac{1}{2^{n+1}} \ .$$

We claim that α has the property 2.4. To see this consider $\beta \in 2^{\mathbb{N}}$. Define a sequence $(y_n)_{n \geqslant 1}$

$$y_n = \begin{cases} x & \text{if } \forall i \leqslant n : \alpha(i) = \beta(i) \\ 0 & \text{otherwise} \end{cases} \ .$$

It is easy to see that $(y_n)_{n \geqslant 1}$ is a Cauchy sequence converging to a limit y. Since x is pseudo-positive by Lemma 2.2.2 either $\neg(0 = y)$ or $\neg(x = y)$. In the first case $\beta = \alpha$, since otherwise $y = 0$. Hence $\neg(\beta = 0)$ since it is easy to see that $\neg(\alpha = 0)$. In the second case $\neg(\alpha = \beta)$, since $\alpha = \beta$ implies that $y_n \to x$ and therefore $x = y$ a contradiction. $\square$

As mentioned above WMP is equivalent to the statement that every map on a complete metric space is strongly extensional. In fact, there are many specific spaces X such that if every map defined on X is strongly extensional, then WMP holds.

Lemma 2.2.4. *Let X be an arbitrary metric space. Every function $f : X \to \mathbb{R}$ is strongly extensional if and only if for every metric space Y we have that every function $f : X \to Y$ is strongly extensional.*

Proof. One direction is trivial. To prove the converse let $f : X \to Y$ be an arbitrary function and let $a, b \in X$ be such that $f(a) \neq f(b)$. Then $g(x) = d(f(a), f(x))$ defines a function $g : X \to \mathbb{R}$. By assumption g is strongly extensional, and $g(a) = 0 \neq g(b) = d(f(a), f(b))$. Hence $a \neq b$. $\square$

Lemma 2.2.5. *Assume $f : \overline{\{a, b\}} \to \mathbb{R}$ is a function such that $f(a) \neq f(b)$.*

  1. *If $0 < p < q < 1$, then we can, for any $x \in \mathbb{R}$, decide whether*

$$x \geqslant pd(a, b) \text{ or } x \leqslant qd(a, b) \ .$$

2. *For any $x \in [0, d(a,b)]$ and for any $n \in \mathbb{N}$ there exists $0 \leqslant i \leqslant 2^n - 2$ such that $x \in [\frac{i}{2^n} d(a,b), \frac{i+2}{2^n} d(a,b)]$.*

3. *There exists a function $g : [0,1] \to \mathbb{R}$ such that $g(0) = f(a)$ and $g(d(a,b)) = f(b)$.*

Proof.   1. Use the axiom of countable choice to get a (increasing) sequence λ_n such that

$$\lambda_n = 0 \implies d(a,b) < \frac{1}{2n} \, ,$$

$$\lambda_n = 1 \implies d(a,b) > \frac{1}{2n+1} \, .$$

Define a sequence $(x_n)_{n \geqslant 1}$ in $\{a,b\}$ by the following algorithm. As long as $\lambda_n = 0$ set $x_n = a$. If $\lambda_n = 1$ for the first time we can decide whether $x > pd(a,b)$ or $x < qd(a,b)$. In the first case we set $x_n = a$ from then on, in the second case we set $x_n = b$ from then on. It is easy to check that $(x_n)_{n \geqslant 1}$ is Cauchy. Considering its limit x_∞ we can check whether $f(x_\infty) \neq f(a)$ or $f(x_\infty) \neq f(b)$. In the first case the assumption that $x < pd(a,b)$ leads to the contradiction that $d(x_\infty, a) = 0$ and therefore $x_\infty = a$. So $x \geqslant pd(a,b)$. Similarly in the second case $x \leqslant qd(a,b)$.

2. This can be decided by a finite iteration of the previous part.

3. Given $x \in [0,1]$ we consider $z_x = \max\{x, d(a,b)\}$. Using countable choice and the preceding part of the lemma we fix a sequence $h : \mathbb{N} \to \mathbb{N}$ such that $z_x \in [\frac{h(n)}{2^n} d(a,b), \frac{h(n)+2}{2^n} d(a,b)]$. It is easy to see that the limit

$$r_x = \lim_{n \to \infty} \frac{h(n)}{2^n} \in [0,1]$$

is independent of the choice of h. In this way we can define a function $g(x) = r_x f(b) + (1 - r_x) f(a)$. Furthermore, it is easy to check that $r_0 = 0$ and $r_{d(a,b)} = 1$ and therefore $g(0) = f(a)$ and $g(d(a,b)) = f(b)$.

$\square$

As a corollary of the first part we also get the following insight.

Corollary 2.2.6. *Assume $f : \overline{\{a,b\}} \to \mathbb{R}$ is a function such that $f(a) \neq f(b)$, then $d(a,b)$ is pseudo-positive.*

This has already been proved by Ishihara [64, Lemma 4]. In the same paper he has also shown that WMP is equivalent to the statement that every every real-valued function on a complete metric space is strongly extensional. In the following we are going to extend this result.

Proposition 2.2.7. *The following are equivalent to WMP.*

1. *Every real-valued function on a complete metric space is strongly extensional.*

2. *Every real-valued function on $\mathbb{N}^{\mathbb{N}}$ is strongly extensional.*

3. *Every real-valued function on $2^{\mathbb{N}}$ is strongly extensional.*

4. *Every real-valued function on $\mathbb{N}_*$ is strongly extensional; where $\mathbb{N}_*$ denotes the space of all increasing binary sequences.*

5. *Every real-valued function on $[0,1]$ is strongly extensional.*

6. *If $X = \overline{\{a,b\}}$, then any function $f : X \to \mathbb{R}$ is strongly extensional.*

Proof. Obviously 1 implies 2.

Assume $f : 2^{\mathbb{N}} \to \mathbb{R}$ is a function with $f(\alpha) \neq f(\beta)$ If $\gamma \in \mathbb{N}^{\mathbb{N}}$, then let $\gamma^{01} \in CS$ be the sequence $\gamma(n) = \min\{\gamma(n), 1\}$. Then $\gamma^{01} = \gamma$ for any $\gamma \in 2^{\mathbb{N}}$. So define $\tilde{f}(\gamma) = f(\gamma^{01})$. Altogether $\tilde{f}(\alpha) \neq \tilde{f}(\beta)$ and if we assume 2 we get $\alpha \neq \beta$ and thus in this case also 3 holds.

Now assume 3 and let $f : \mathbb{N}_* \to \mathbb{R}$ be such that $f(\alpha) \neq f(\beta)$ for some α and β. For any $\gamma \in 2^{\mathbb{N}}$ let $\gamma^{\uparrow}$ denote the binary sequence defined by $\gamma^{\uparrow}(n) = \max_{i \leqslant n} \gamma(i)$. It is obvious that $\gamma^{\uparrow} \in \mathbb{N}_*$ and that for $\gamma \in \mathbb{N}_*$ we have $\gamma = \gamma^{\uparrow}$. Thus by setting $g(\gamma) = f(\gamma^{\uparrow})$ we get a function $g : 2^{\mathbb{N}} \to \mathbb{R}$ with $g(\alpha) \neq g(\beta)$. Hence $\alpha \neq \beta$ and 4 holds.

To see that 4 implies 6 we need to use the axiom of countable choice to get a (increasing) sequence λ_n such that

$$\lambda_n = 0 \implies d(a,b) < \frac{1}{2n} \ ,$$

$$\lambda_n = 1 \implies d(a,b) > \frac{1}{2n+1} \ .$$

To a given sequence $\mu \in \mathbb{N}_*$ we associate the sequence $(x_n^{\mu})_{n \geqslant 1}$ in $\{a, b\}$ by

$$\lambda_n = 0 \wedge \mu_n = 0 \implies x_n = a \ ,$$

$$\lambda_n = 1 \vee \mu_n = 1 \implies x_n = b \ .$$

It is easy to see that x_n^{μ} is a Cauchy sequence and thus converges to a limit x_{∞}^{μ}. Now we can consider the function $g : \mathbb{N}_* \to \mathbb{R}$ defined by $g(\alpha) = f(x_{\infty}^{\alpha})$. Furthermore, it is easy to see that $x_{\infty}^{(000...)} = a$ and $x_{\infty}^{\lambda} = b$. Hence $\lambda \neq 000...$ and thus $a \neq b$.

Next, we will show that 6 implies 1. So let $f : Y \to \mathbb{R}$ be a function and $a, b \in Y$ such that $f(a) \neq f(b)$. Considering the restriction $f \restriction_{\overline{\{a,b\}}}$ we can therefore conclude that $a \neq b$.

Of course, 1 implies 5, so if we can prove that 5 implies 6 we are done. Unfortunately this is the trickiest part of the implications, however most of the work has been done in the preceding lemma. Given $f : X \to \mathbb{R}$ with $f(a) \neq f(b)$ we can find a g as in that lemma. By our assumption $g(0) \neq g(d(a,b))$, which means that $0 \neq d(a,b)$ which in turn means that $a \neq b$. $\square$

2.3 $\mathrm{MP}^{\vee}$

The *disjunctive version of Markov's principle* $\mathrm{MP}^{\vee}$ can be seen as an instance of de'Morgan's laws (see Section 8.6) as well as another weakening of MP. It states that every almost positive number is pseudo-positive.

> **($\mathrm{MP}^{\vee}$)**   For all $x \in \mathbb{R}$
>
> $$0 < x \implies (\forall y \in \mathbb{R} : 0 < y \vee y < x) \ .$$

This principle was also called LLPE[5] by Mandelkern. Berger, Ishihara, and Schuster have suggested WLLPO as a name [16], since it can be viewed as a weakening of LLPO. Finally in [107] it also appears under the name of SEP.

Notice that $\mathrm{MP}^{\vee}$ bears resemblance to the well-known statement

$$0 < x \implies (\forall y \in \mathbb{R} : 0 < y \vee y < x) \ ,$$

---

[5]Lesser Limited Principle of Existence

which is—in contrast—provable in Bishop style constructive mathematics [20, Corollary 2.17], or included as one of the axioms of constructive real analysis [23]

Before we state some equivalences of $MP^\vee$ we would like to point out the fact that

$$MP \iff MP^\vee + WMP\,,$$

which is obvious considering the following diagram illustrating which versions of Markov's principle allow one to "upgrade" between the different notions of positiveness of a real number.
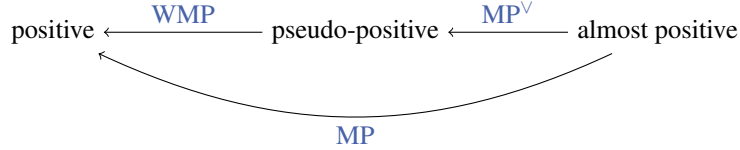
Proposition 2.3.1. *The following are equivalent to $MP^\vee$:*

1. *For any binary sequence $(a_n)_{n\geqslant 1}$ with at most one 1 and $\neg\forall n : a_n = 0$ either*

$$\forall n \in \mathbb{N} : a_{2n} = 0 \vee \forall n \in \mathbb{N} : a_{2n+1} = 0\,.$$

2. *If α is a binary sequence such that $\neg\forall n \in \mathbb{N} : \alpha(n) = 0$, then for any β we have*

$$\forall \beta \in 2^{\mathbb{N}} : \neg(\alpha = \beta) \vee \neg(\beta = 0)$$

3. *For $\alpha, \beta \in 2^{\mathbb{N}}$*

$$\neg(\forall n : \alpha_n = 0 \wedge \forall n : \beta_n = 0) \implies \neg\forall n : \alpha_n = 0 \vee \neg\forall n : \beta_n = 0\,.$$

   (That is the De Morgan law for Π^0_1 formulae).

4. *For all $x, y \in \mathbb{R}$ with $\neg(x = y)$ we have $\neg(x \leqslant y) \vee \neg(y \leqslant x)$*

5. *For all $x, y \in \mathbb{R}$ with $\neg(x = y)$ we have $x \leqslant y \vee y \leqslant x$*

6. *For all $x, y, z \in \mathbb{R}$ we have $\neg(x = y) \implies \neg(x = z) \vee \neg(y = z)$.*

7. *For all $x, y \in \mathbb{R}$ with $x > y$, $\{x, y\}$ is a closed subset of $\mathbb{R}$*

8. *For any $x \in \mathbb{R}$ with $|x| > 0$ either $|x| = x$ or $|x| = -x$.*

Proof. Most of these equivalences to $MP^\vee$ (1,7,8) are taken from [84]. The equivalences between 5,6,3 are due to Josef Berger[6]. The proof of $5 \implies 6$, surprisingly, seems to necessitate countable choice. The equivalence that $4 \iff 5$ is simply because $\neg(x \leqslant y) \iff x \geqslant y$ given that $\neg(x = y)$. Clearly, 3 implies 1. To see that $1 \implies 2$ let α be a binary sequence such that $\neg\forall n \in \mathbb{N} : \alpha_n = 0$, and $\beta \in CS$ be arbitrary. Now define $\gamma \in 2^{\mathbb{N}}$ by

$$\gamma_{2n} = 1 \implies \alpha_n \neq \beta_n\,,$$
$$\gamma_{2n+1} = 1 \implies \beta_n = 1\,.$$

---

[6]currently unpublished, talk at CiE2012 in Cambridge

We have that $\neg(\forall n \in \mathbb{N} : \gamma_n = 0)$, since the assumption that $\forall n \in \mathbb{N} : \gamma_n = 0$ implies that $\alpha = \beta$ and $\beta = 000\ldots$, which contradicts the assumption that $\neg(\alpha = 000\ldots)$. Now consider γ' defined by

$$\gamma'(n) = \gamma(n) - \max_{i<n} \{\gamma(i)\} ,$$

so that γ' contains at most one 1. Furthermore we also have $\neg(\gamma' = 000\ldots)$. So applying 1 we get that either $\forall n \in \mathbb{N} : \gamma'_{2n} = 0$ or $\forall n \in \mathbb{N} : \gamma'_{2n+1} = 0$. In the first case the assumption that $\beta = 000\ldots$ leads to the following contradiction. Then $\gamma_{2n+1} = 0$ for all $n \in \mathbb{N}$, which means that we must have $\neg(\forall n \in \mathbb{N} : \gamma_{2n} = 0)$. But this contradicts $\forall n \in \mathbb{N} : \gamma'_{2n} = 0$. In the second case the assumption that $\alpha = \beta$ leads to the following contradiction. Then $\gamma_{2n} = 0$ for all $n \in \mathbb{N}$ which means that we must have $\neg(\forall n \in \mathbb{N} : \gamma_{2n+1} = 0)$. But this contradicts $\forall n \in \mathbb{N} : \gamma'_{2n+1} = 0$. So we have decided

$$\neg(\alpha = \beta) \vee \neg(\beta = 000\ldots) .$$

To see that 2 $\implies$ 3 let α, β be a binary sequences such that

$$\neg(\forall n : \alpha_n = 0 \wedge \forall n : \beta_n = 0) .$$

Now consider the sequences

$$\gamma = \alpha_0 \beta_0 \alpha_1 \beta_1 \alpha_2 \beta_2 \ldots$$

and

$$\mu = 0\beta_0 0\beta_1 0\beta_2 \ldots .$$

Then we can apply 2 to decide whether $\neg(\mu = 000\ldots)$ which means that $\neg(\beta = 000\ldots)$ or $\neg(\gamma = \mu)$ in which case $\neg(\alpha = 000\ldots)$. Thus we have shown 3. □

As we have seen above WKL is equivalent to LLPO. Because of the similarity of LLPO and $\mathrm{MP}^\vee$ it is not surprising that we can find a weakening of WKL that is equivalent to the latter.

> **(WKL!!!)** If T is an infinite decidable tree that
>
> 1. has at most one path in the weak sense that if α and β are paths through the tree, then $\alpha = \beta$,
>
> 2. and for every infinite subtree it is *impossible not to* admit an infinite path,
>
> then it actually has an infinite path.

Notice that our notion of WKL!!! differs only slightly from the one of J. Moschovakis, who first considered it [89]. In her version, named WKL!!, condition (2) is missing. She also showed that

$$\mathrm{WKL!!} \iff \neg\neg\mathrm{WKL} \wedge \mathrm{MP}^\vee .$$

By, vaguely speaking, moving Condition (2) from $\neg\neg$WKL into WKL!!!, we can strengthen this result: in this way we get a version of weak Kőnig's Lemma that holds in recursive models!

Proposition 2.3.2. $MP^\vee$ *is equivalent to WKL!!!.*

Proof. The proof is essentially the same as the one for Proposition 1.4.5. However it is worth noticing that this one only requires unique choice. Consider a tree T satisfying the conditions of WKL!!!. As in the proof of Proposition 1.4.5 define

$$a_{2n} = 0 \iff \exists u \in 2^n : 0 * u \in T ,$$
$$a_{2n+1} = 0 \iff \exists u \in 2^n : 1 * u \in T .$$

Notice that, since T is infinite there cannot $n, m \in \mathbb{N}$ such that $a_{2n} = 1$ and $a_{2m+1} = 1$. However it is also not possible that $a_n = 0$ for all $n \in \mathbb{N}$, since that would imply (second condition) that it is impossible for T_0 and T_1 not to admit infinite paths. Thus it is impossible that there not two infinite and not-equal paths through T, which would contradict the first condition of WKL!!!.

Hence by 2.3.1 $MP^{\vee}$ implies that either all even or all odd terms of a_n are zero. That means that either T_0 or T_1 is infinite. Also both subtrees satisfy the first and second condition of WKL!!! and are decidable and infinite. Thus we can, using dependent choice, iteratively define a sequence $\alpha \in 2^{\mathbb{N}}$ such that $T_{\overline{\alpha}n}$ is a infinite, decidable tree satisfying both conditions of WKL!!! for all $n \in \mathbb{N}$. In particular $\overline{\alpha}n \in T$.

Conversely consider a binary sequence $(a_n)_{n \geqslant 1}$ with at most one 1 and such that $\neg \forall n \in \mathbb{N} : a_n = 0$. Now consider the decidable tree T defined by

$$u \in T \iff \exists n : u = 0^n \wedge \forall i \leqslant n : a_{2i} = 0 \vee \exists n : u = 1^n \wedge \forall i \leqslant n : a_{2i+1} = 0 .$$

It is straightforward to see that T satisfies the conditions of WKL!!!. Thus, by WKL!!!, it admits an infinite path α. Now either $\alpha(1) = 0$, in which case we have $\alpha = 000\ldots$ and therefore $\forall n : a_{2n} = 0$ or $\alpha(1) = 1$, in which case we have $\alpha = 111\ldots$ and therefore $\forall n : a_{2n+1} = 0$. Hence $MP^{\vee}$ holds. $\qquad \square$

Remark 2.3.3. *Condition (2) cannot be weakened by removing "for every infinite subtree".*

Proof. Let us assume that there exists a Kleene tree K (for example in RUSS). So there exists $u_n \in K$ with $|u_n| = n$. Furthermore let $(a_n)_{n \geqslant 1}$ be an increasing binary sequence.

Define a set $T \subset 2^*$ by $u \in T$ if and only iff

$$\left(a_n = 0 \wedge (u = 1^n \vee \exists w \in T : u = 0w)\right) \vee \left(a_n = 1 \wedge \left(u = u_m 1^{n-m} \vee \exists w \in T : u = 1^m w\right)\right)$$

where $|u| = n$ and m is the smallest number $i \leqslant n$ such that $a_i = 1$. The idea is that as long as $a_n = 0$, T looks like a copy of K below 0 and just a single path $111\ldots$ on the right hand side. If we ever hit an m such that $a_m = 1$ we continue only one path on the left hand side and glue a copy of K on the path 1^m on the right hand side of the tree. Now, clearly, T is an infinite tree. It is also impossible not to admit an infinite path (if $\forall n : a_n = 0$ there is an infinite path, and if $\exists n : a_n = 1$ there is an infinite path; furthermore, by the double negation tranlation, constructively we have $\neg\neg(\forall n : a_n = 0 \vee \exists n : a_n = 1)$). Finally, it is also easy to see that T has at most one path. However if T actually admits a path α we can check whether $\alpha(1) = 0$, in which case $\neg \forall n : (a_n = 0)$, or $\alpha(1) = 1$, in which case $\forall n : a_n = 0$. Thus WLPO holds, which contradicts the existence of a Kleene tree. $\qquad \square$
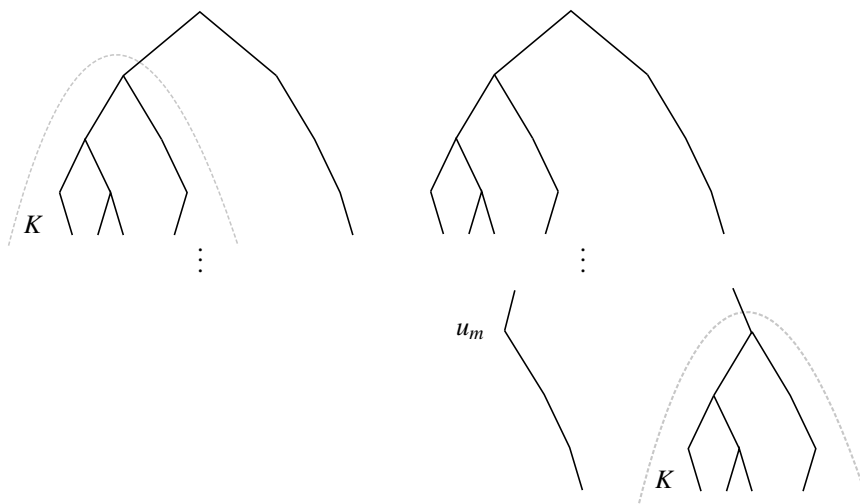
Figure 2.1: As long as $a_n = 0$ the tree consists of a copy of K on the left side and a single path on the right. If there is m with $a_m = 1$ from then on a copy of K is attached to the single path on the right and another single path to one of the elements u_m on the left side.

Chapter 3

The Fan Theorems

As a way to re-capture the unit interval's compactness—that is cover compactness—which was lost when rejecting the law of excluded middle, L.E.J. Brouwer made generous use of the fan theorem. Since he also made free use of the principle of continuous choice the complexity of the sets involved did not make a difference to his mathematics—as we will see in Section 6.3. In the absence of continuous choice we do, however, have to make some careful distinctions. This explains why we talk about the fan theorems in the plural, since we are going to distinguish between different versions, all of which are interesting in constructive (reverse) mathematics. What all fan theorems have in common is that they enable one to conclude that a given bar is uniform. Here a *bar* is a subset $B \subset 2^*$ that "bars" every infinite path in $2^{\mathbb{N}}$, that is

$$\forall \alpha \in 2^{\mathbb{N}} : \exists n \in \mathbb{N} : \overline{\alpha}n \in B.$$

A bar is called *uniform* if this "barring" occurs before a uniform height N, that is if

$$\exists N \in N : \forall \alpha \in 2^{\mathbb{N}} : \exists n \leqslant N : \overline{\alpha}n \in B$$

If a bar B is *closed under extensions*, that is if

$$u \in B \implies \forall w \in 2^* : u * w \in B \, ,$$

then it is uniform if and only if

$$\exists N \in N : \forall \alpha \in 2^{\mathbb{N}} : \overline{\alpha}N \in B \, .$$

The difference between the fan theorems lies in the required complexity of the bar B. This ranges from the very strongest requirement—decidable—to no restriction on the bar at all. Here, a set S is decidable, if, for every x, we can decide whether

$$x \in S \vee x \notin S \, .$$

Of course, in the absence of the law of excluded middle, this might not always be possible. A weaker requirement than decidability is the notion of a *c-bar*. A bar $C \subset 2^*$ is called a *c-bar*, if there exists a decidable set $C' \subset 2^*$ such that

$$u \in C \iff \forall w \in 2^* : \left(u * w \in C' \right) \, .$$

A bar $B \subset 2^*$ is called Π_1^0-*bar*, if there exist a countable family of decidable sets $(B_n)_{n \geqslant 1}$ that are all closed under extension such that

$$B = \bigcap_{n \geqslant 1} B_n \, .$$

The Π_n^0-nomenclature allures to the arithmetical hierarchy in recursion theory. We can now formally state the four versions of the fan theorem, that are going to be of interest to us.

> **(Fan theorems)**
>
> FAN_Δ: Every decidable bar is uniform.
> FAN_c: Every c-bar is uniform.
> $\mathrm{FAN}_{\Pi^0_1}$:   Every Π^0_1-bar is uniform.
> $\mathrm{FAN}_{\mathrm{full}}$:   Every bar is uniform.

It is mostly trivial that

$$\mathrm{FAN}_{\mathrm{full}} \implies \mathrm{FAN}_{\Pi^0_1} \implies \mathrm{FAN}_c \implies \mathrm{FAN}_\Delta \ ,$$

with the exception of the middle implication which does require a modicum of trickery. It follows indirectly from [43, Proposition 4.1.6], but we can also give a direct construction.

Proposition 3.0.1. *For every c-bar C there exists is a Π^0_1-bar B such that C is a uniform bar only if B is.*

Proof. Let $C \subset 2^*$ be a c-bar. So there exists a decidable set $C' \subset 2^*$ such that

$$u \in C \iff \forall w \in 2^* : \left(u * w \in C'\right) \ .$$

Now, for every $n \in \mathbb{N}$ define decidable sets $B_n \subset 2^*$ by

$$B_n = \left\{ u \in 2^{\mathbb{N}} \, \middle| \, \forall w \in 2^* : |w| \leqslant n - |u| \implies w \in C' \right\} \ .$$

The antecedent "$|w| \leqslant n - |u|$" ensures that B_n is closed under extensions (this is the modicum of trickery that we referred to earlier). The set

$$B = \bigcap_{n \geqslant 1} B_n$$

is a bar. For let $\alpha \in 2^{\mathbb{N}}$ be arbitrary. Since C is a bar there exists m such that $\overline{\alpha}m \in C$, which, by definition means that $\overline{\alpha}m * w \in C'$ for all $w \in 2^*$. So this is the case, in particular, for any n and any w such that $|w| \leqslant n - m$. Hence $\overline{\alpha}m \in B_n$ for all n, which means that $\overline{\alpha}m \in B$. Thus B is a Π^0_1-bar.

Now if B is uniform there exists M such that

$$\forall \alpha \in 2^{\mathbb{N}} : \overline{\alpha}M \in B \ ,$$

since Π^0_1-bars are closed under extensions. But now let $\alpha \in CS$ and $w \in 2^*$ be arbitrary. Since $\overline{\alpha}M \in B$, in particular, $\overline{\alpha}M \in B_{M+|w|}$. Since $|w| \leqslant (M + |w|) - M$, we have $\overline{\alpha}M * w \in C'$, and since w was arbitrary that means that $\overline{\alpha}M \in C$. Thus C is a uniform bar. $\qquad\square$

FAN_Δ, and therefore all of the fan theorems, fail in RUSS, since there one can construct a Kleene tree, that is an infinite decidable binary tree that blocks every infinite path [4]. The complement of such a tree is by itself a ready-made counterexample to FAN_Δ. One implication in analysis that the existence of a Kleene tree has (see Section 5.3) is the existence of a Specker sequence, that is a sequence of real numbers in a compact set X that is bounded away from every point of X. To distinguish whether such a strange object exists in a space or not we say that a subspace X of a metric space Y satisfies the *anti-Specker property relative to Y* if

> **(AS^Y_X)**   any sequence in Y that is eventually bounded away from any point in X is eventually bounded away from the entire set X.

Here a sequence $(x_n)_{n \geqslant 1}$ in a metric space (Y,d) is *eventually bounded away* from a point x (a set X), if there exists a natural number N such that $d(x_n,x) > 2^{-N}$ for all $n \geqslant N$ (and for all $x \in X$). It is easily seen that AS_X^Y holds, whenever X is a subspace of Y and satisfies the Heine-Borel property.[1] Notice that for any two one-point extensions Y and Y' of a space X the principles AS_X^Y and $\mathrm{AS}_X^{Y'}$ are equivalent. So in that case we can combine them into a generic principle labeled AS_X^1. Notice that some early papers on the topic have used a more refined notation: what is labeled AS there is what we label $\mathrm{AS}_{[0,1]}^{\mathbb{R}}$, and which is equivalent to $\mathrm{AS}_{[0,1]}^1$.

3.1   Linking $2^{\mathbb{N}}$ and $[0,1]$

In the following we want to establish strong links between $2^{\mathbb{N}}$ and $[0,1]$. We will first adapt Cantor's middle third set construction for our purposes. First consider a fixed $p \in (0,1)$. Let

$$I_u^p = [a_u^p, b_u^p] = \left[(1-p) \sum_{n \leqslant |u|} p^{n-1} u(n), (1-p) \sum_{n \leqslant |u|} p^{n-1} u(n) + p^n \right] .$$

Furthermore, let $I_{()}^p$ be the unit interval. It is easy to see that I_{u0}^p is the left p of I_u^p, and I_{u1}^p is the right p of I_u^p. With this notation the Cantor middle third set is

$$\mathscr{C}^p = \bigcap_{n \geqslant 0} \bigcup_{u \in 2^n} I_u^p ,$$

where $p = \frac{1}{3}$. It is easy to see that $F^p : 2^{\mathbb{N}} \to [0,1]$ defined by

$$F^p(\alpha) = (1-p) \sum_{n \geqslant 1} p^{n-1} \alpha(n)$$

is well defined. It has many nice properties.

Lemma 3.1.1.

1. *For $\alpha, \beta \in 2^{\mathbb{N}}$ and $n \geqslant 1$ we have*

$$\overline{\alpha}n = \overline{\beta}n \implies |F^p(\alpha) - F^p(\beta)| \leqslant p^n .$$

   In particular, F^p is uniformly continuous.

2. *(Variation on Bishop's Lemma)*
   Assume $0 < p < \frac{1}{2}$. For all $x \in [0,1]$ there exists $\alpha \in 2^{\mathbb{N}}$ such that

$$\forall \beta < \alpha : F^p(\beta) < x \quad and \quad \forall \alpha < \beta : x < F^p(\beta) . \tag{3.1}$$

   In particular, F^p is injective and

$$d(x, F^p(\alpha)) > 0 \implies d(x, F^p(2^{\mathbb{N}})) > 0 . \tag{3.2}$$

3. *Assume $\frac{1}{2} < p < 1$. Then for every $u \in 2^*$ and $y \in I_u$ there exists $\alpha \in 2^{\mathbb{N}}$ such that $\overline{\alpha}|u| = u$ and $F^p(\alpha) = y$; in particular F^p is surjective.*

---

[1] Depending on the variety of constructive mathematics, the class of spaces satisfying the Heine-Borel property may, of course, be drastically smaller than in classical mathematics. Nevertheless, maybe surprising, even in BISH there are infinite spaces that satisfy the Heine-Borel property: a prototypical example of such a space is the closure of $\{2^{-n} \mid n \in \mathbb{N}\}$

Proof.   1. Assume $\alpha, \beta \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$ such that $\overline{\alpha}n = \overline{\beta}n$. Then

$$
\begin{aligned}
|F^p(\alpha) - F^p(\beta)| &\leqslant (1-p)\sum_{i \geqslant 1} p^{i-1}|\alpha(i) - \beta(i)| \\
&\leqslant (1-p)\sum_{i > n} p^i \\
&= (1-p)p^n \sum_{i \geqslant 0} p^i \\
&= \frac{1-p}{1-p}p^n = p^n \ .
\end{aligned}
$$

2. Consider $0 < p < \frac{1}{2}$. We will define subsets J_u^p and positive numbers ε_n^p such that

$$
J_u^p = J_{u0}^p \cup J_{u1}^p \ , \tag{3.3}
$$

and

$$
I_u^p \subset J_u^p \quad \text{as well as} \quad I_u^p + \varepsilon_n^p < J_v^p < I_w^p - \varepsilon_n^p \ ,
$$

for $u < v < w$ of length n; according to the lexicographic order. First we will introduce (partial) functions $\text{next} : 2^* \to 2^*$ and $\text{prev} : 2^* \to 2^*$, mapping $u \in 2^*$ to the element of the same length succeeding and preceding it, again in the lexicographical order. Next, set

- $c_{0^{|u|}}^p = 0$, $d_{1^{|u|}}^p = 1$,
- $c_u^p = a_u - \frac{2}{3}(a_u^p - b_{\text{prev}(u)}^p)$, and
- $d_u^p = b_u^p + \frac{1}{3}(a_{\text{next}(u)}^p - b_u^p)$ for $0^{|u|} \neq u \neq 1^{|u|}$.

It is easy to see that the intervals $J_u^p = [c_u^p, d_u^p]$ have the desired properties.
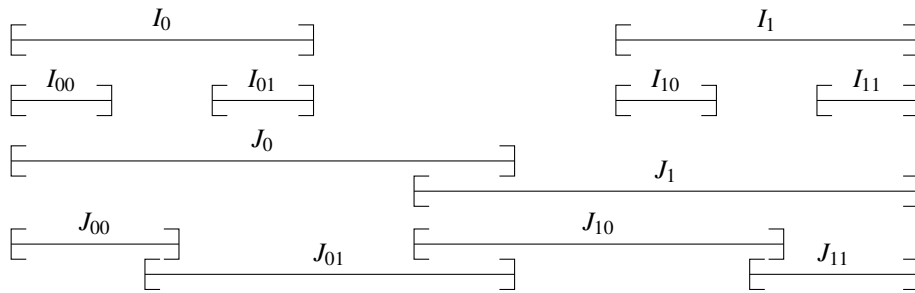
Figure 3.1: Schematic of the first couple of intervals for $p = \frac{1}{3}$.

Given $x \in [0,1]$, we can, using dependent choice and Property 3.3, find $\alpha \in 2^{\mathbb{N}}$ such that

$$
\forall n \in \mathbb{N} : x \in J_{\overline{\alpha}n}^p \ .
$$

Now let $\beta \in 2^{\mathbb{N}}$ be such that $\beta < \alpha$, which means there exists $n \in \mathbb{N}$ such that $\overline{\beta}n < \overline{\alpha}n$. Since $F(\beta) \in I_{\overline{\beta}n}^p$ and $x \in I_{\overline{\alpha}n}^p$, by 3.1 we have $F(\beta) < x$. The case $\alpha < \beta$ can be treated analogously. To see that F^p is injective let $\gamma \neq \beta$. Now let α be as constructed above for $x = F^p(\beta)$. Then

the assumption that $\alpha \neq \beta$ leads to a contradiction: if $\alpha < \beta$ we get $F^p(\beta) < x = F^p(\beta)$ and if $\beta < \alpha$ we get $F^p(\beta) = x < F^p(\beta)$. Therefore $F^p(\alpha) = F^p(\beta)$, and therefore, again by 3.1, also $F^p(\gamma) \neq F^p(\beta)$. Finally assume that $d(x, F^p(\alpha)) > 0$. Since F^p is uniformly continuous by the first part of the proposition, there exists n such that

$$\overline{\beta}n = \overline{\alpha}n \implies d(F^p(\beta), F^p(\alpha)) < \frac{d(x, F^p(\alpha))}{2} \ .$$

Now for any $\gamma \in 2^{\mathbb{N}}$ either $\overline{\gamma}n \neq \overline{\alpha}n$, or $\overline{\gamma}n = \overline{\alpha}n$. In the first case either $\overline{\gamma}n < \overline{\alpha}n$ or $\overline{\gamma}n > \overline{\alpha}n$, but in both cases $d(x, F^p(\beta)) > \varepsilon_n^p$. In the second case

$$d(x, F^p(\beta)) > |d(F^p(\beta), F^p(\alpha)) - d(x, F^p(\alpha))| > \frac{d(x, F^p(\alpha))}{2} \ .$$

In any case

$$d(x, F^p(\beta)) > \min \left\{ \frac{d(x, F^p(\alpha))}{2}, \varepsilon_n^p \right\} > 0 \ .$$

3. Let $\frac{1}{2} < p < 1$. Notice that

$$I_u^p = I_{u0}^p \cup I_{u1}^p \ . \tag{3.4}$$

Similar to above, given $x \in [0, 1]$, we can, using dependent choice and Property 3.4, find $\alpha \in 2^{\mathbb{N}}$ such that

$$\forall n \in \mathbb{N} : x \in I_{\overline{\alpha}n}^p \ .$$

It is easy to see that therefore $F^p(\alpha) = x$, which means we have shown surjectivity. $\qquad \square$

Remark 3.1.2. *Notice that for $p = \frac{1}{2}$ the function F^p cannot be shown to be surjective, constructively, since that would be a restatement of the fact that every real number $x \in [0, 1]$ has a binary expansion and therefore equivalent to LLPO (see Proposition 1.4.2).*

We are now in the position to prove a technical but utterly central lemma.

Lemma 3.1.3. *Assume $0 < p < \frac{1}{2}$. If $f : 2^{\mathbb{N}} \to \mathbb{R}$ is a point-wise continuous function, then there exists a point-wise continuous function $\widetilde{f} : [0, 1] \to \mathbb{R}$ such that*

$$f = \widetilde{f} \circ F^p$$

Moreover, $\widetilde{f}$ is uniformly continuous, (bounded, has a positive infimum, attains its infimum) if f is (does).

Proof. The idea of the embedding were simple if we assumed classical logic: for elements $F^p(\alpha)$ in the Cantor set $\mathscr{C}^p \subset [0, 1]$ we obviously want to set $\widetilde{f}(F^p(\alpha)) = f(\alpha)$. If an element is not in $\mathscr{C}^p$ there exist unique $\alpha, \beta \in 2^{\mathbb{N}}$ such that $F^p(\alpha) < x < F^p(\beta)$ and $\mathscr{C}^p \cap (F^p(\alpha), F^p(\beta)) = \emptyset$. For such an x we set

$$\widetilde{f}(x) = t_{\alpha,\beta}(x) = f(\alpha) + \frac{x - F^p(\alpha)}{F^p(\beta) - F^p(\alpha)} f(\beta) \ ; \tag{3.5}$$

that is we simply interpolate linearly between $F^p(\alpha)$ and $F^p(\beta)$. However, constructively, we can, of course, not show that $x \in [0, 1]$ is either in the Cantor set or is in its metric complement. Nevertheless one can, surprisingly, give a construction of $\widetilde{f}$ without the use of LEM. The construction can be found in [29].

Since $\mathscr{C}^p \cup -\mathscr{C}^p$ is dense in $[0, 1]$ it easily follows from Equation 3.5 that all the properties listed above are preserved. $\qquad \square$

It is well known ([108, Chapter 7 Corollary 4.4]) that constructively every compact, that is a complete and totally bounded, space is the uniformly continuous image of $2^{\mathbb{N}}$. Below, we need a slightly stronger version of this result.

Definition 3.1.4. A map $g : X \to Y$ between to metric spaces X and Y is called *uniformly surjective* if

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x, y \in Y : d(x,y) < \delta \implies \exists \alpha, \beta \in Y : d(\alpha, \beta) < \varepsilon \wedge x = g(\alpha) \wedge y = g(\beta) \quad (3.6)$$

Lemma 3.1.5. *For $p > \frac{1}{2}$ the function $F^p : 2^{\mathbb{N}} \to [0,1]$ is uniformly surjective.*

Proof. Let $n \in \mathbb{N}$ be arbitrary. Since $p > 1/2$, neighbouring intervals I_u with $u \in 2^n$ are overlapping by $(2p-1)p^n$. Now consider the slightly smaller intervals

$$I'_u = [a_u + \delta, b_u - \delta]$$

where $\delta = {}^{(2p-1)p^n}/3$, as a special case we also set $I'_{0^n} = [0, b_{0^n}] - \delta$ and $I'_{1^n} = [a_{1^n} + \delta, 1]$. By our choice of δ we still have

$$[0,1] = \bigcup_{u \in 2^n} I'_u .$$

Consider $x, y \in Y$ such that $d(x,y) < \delta$. First, find $u \in 2^n$ such that $x \in I'_u$. Since $d(x,y) < \delta$ we have $y \in I'_u$, and also by construction, $x \in I'_u \subset I_u$. So using Lemma 3.1.1.3 we can find α, β with $\overline{\alpha}n = \overline{\beta}n = u$, and $F(\alpha) = x$ and $f(\beta) = y$. $\qquad \square$

More generally we have:

Proposition 3.1.6. *If X is a compact set, then it is the uniform surjective image of a uniformly continuous function $f : 2^{\mathbb{N}} \to X$.*

Proof. This follows from [108, Proposition 7.4.3.vi]. $\qquad \square$

It is worth taking the time for a little detour via constructive analysis here. We remind the reader that even classically the continuous image of complete space is not necessarily complete. However the continuous image of a cover-compact space is again cover-compact. Now constructively we traditionally use totally boundedness + completeness as a useful notion of compactness in metric spaces. Unfortunately neither totally boundedness, nor completeness are preserved under point-wise continuous maps. If we switch to uniformly continuous maps at least totally boundedness is preserved, but completeness is still not (the graph of $\frac{1}{x}$ is complete, but its image under the projection onto the x-axis is not). As the standard recursive counterexample (cf. Proposition 5.3.1) shows we cannot even rule out the existence of a uniformly continuous function $f : [0,1] \to \mathbb{R}$ such that $f([0,1]) = (0,1]$. But the situation is actually much worse since even a function as well behaved and canonical as F^p with $p > 1/2$ doesn't have the property of mapping complete sets to complete sets.

Lemma 3.1.7. *If, for $p > 1/2$, the function $F^p : 2^{\mathbb{N}} \to [0,1]$ maps complete sets to complete sets (i.e. is a closed map), then LLPO holds.*

Proof. Let $x \in \mathbb{R}$ be arbitrary. We may assume, without loss of generality that $|x| < \frac{1}{2}$. Let $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. Consider

$$A = (F^p)^{-1}\left(\left[0, \frac{1}{2} - x^+\right]\right) \cap \{0\} * 2^{\mathbb{N}}$$

and

$$B = (F^p)^{-1}\left(\left[\frac{1}{2} + x^-, 1\right]\right) \cap \{1\} * 2^{\mathbb{N}} .$$

Both sets are closed and by Lemma 3.1.1.3 we have $F^p(A) = [0, \frac{1}{2} - x^+]$ and $F^p(B) = [\frac{1}{2} + x^-, 1]$. It is easy to see that $\frac{1}{2} \in \overline{F^p(A \cup B)}$. However, assume that there exists $\alpha \in A \cup B$ with $F^p(\alpha) = \frac{1}{2}$. If $\alpha(0) = 0$ we cannot have $x > 0$, since then $F(\beta) < \frac{1}{2}$, similarly, if $\alpha(0) = 1$ we cannot have $x < 0$. Thus we can decide $x \leqslant 0$ or $x \geqslant 0$, which means LLPO holds. $\qquad \square$

Question 4. *Is there a general principle that ensures that complete sets are mapped to complete sets? The above example suggests that connectedness may play a role.*

Definition 3.1.8. We call a sequence $(x_n)_{n \geqslant 1}$ *tail-located* if the distances $d(x, \{x_n, x_{n+1}, \dots\})$ exist for every $n \geqslant 1$ and $x \in \mathbb{R}$.

Lemma 3.1.9. *If $(x_n)_{n \geqslant 1}$ is a sequence of real numbers that is eventually bounded away from every point in $[0, 1]$, then there exists a c-bar $B \subset 2^*$ such that*

1. *B is uniform, if $(x_n)_{n \geqslant 1}$ is eventually bounded away from $[0, 1]$,*

2. *B misses arbitrarily long finite sequences, that is*

$$\forall n \in \mathbb{N} : \exists v \in 2^* : v \notin B \wedge |v| \geqslant n \ ,$$

   if $x_n \in [0, 1]$ infinitely often, and

3. *if $(x_n)_{n \geqslant 1}$ is tail-located, then B is decidable.*

Proof. Assume that $(x_n)_{n \geqslant 1}$ is a sequence that is eventually bounded away from every point in $[0, 1]$. Notice first, that since $(x_n)_{n \geqslant 1}$ is eventually bounded away from 0 and 1 we can, for a tail but without loss of generality for the whole sequence, decide whether $x_n \in [0, 1]$ or not. Since $F^{1/2}(2^{\mathbb{N}})$ is dense in $[0, 1]$ we can, using countable choice, find a sequence $w_n \in 2^* \cup \{\omega\}$ such that

$$w_n \in 2^* \implies x_n \in [0, 1] \wedge \left| F^{1/2}(w_n) - x_n \right| < 2^{-n} \ ,$$

and

$$w_n = \omega \implies x_n \notin [0, 1] \ .$$

Define

$$v_n = \begin{cases} \overline{w_n} n & \text{if } w_n \in 2^* \wedge |w_n| \geqslant n \ , \\ w_n 0^{n - |w_n|} & \text{if } w_n \in 2^* \wedge |w_n| < n, \\ \omega & \text{if } w_n = \omega \ . \end{cases}$$

that is if w_n is not ω either chop of w_n after n places or fill it up to length n with zeroes. Then, by part 1 of Lemma 3.1.1

$$w_n \in 2^* \implies \left| F^{1/2}(v_n) - F^{1/2}(w_n) \right| \leqslant 2^{-n}$$

for all $n \in \mathbb{N}$. Hence

$$w_n \in 2^* \implies \left| F^{1/2}(v_n) - x_n \right| < 2^{-n+1} \tag{3.7}$$

for all $n \in \mathbb{N}$. Since $|v_n| = n$ for all n, the set $D = \{v_1, v_2, \dots\} \cap 2^*$ is decidable. Therefore the set

$$B = \{ u \in 2^* \mid \neg \exists w \in 2^* : u * w \in D \}$$

is a c-set. It is also a bar. To see this let $\alpha \in 2^{\mathbb{N}}$ be arbitrary. There exists $M, N \in \mathbb{N}$ such that $|F^{1/2}(\alpha) - x_n| > 2^{-M}$ for all $n \geqslant N$. Let $K = \max\{N, M\} + 2$. We are going to show that for all $w \in 2^*$ we have $\overline{\alpha} K * w \notin D$.

For assume there was such a w with $\overline{\alpha}K * w \in D$ In this case $\overline{\alpha}K * w = v_j$ for some $j \geqslant K$ (for $j = K + |w|$ to be precise). But then

$$
\begin{aligned}
2^{-M} &< |F^{1/2}(\alpha) - x_j| \\
&\leqslant |F^{1/2}(v_j) - x_j| + |F^{1/2}(v_j) - F^{1/2}(\alpha)| \\
&\leqslant 2^{-j+1} + 2^{-K} \\
&\leqslant 2^{-K+1} + 2^{-K} \\
&\leqslant 2^{-K+2} \leqslant 2^{-M} ;
\end{aligned}
$$

a contradiction. Thus $\overline{\alpha}K * w \notin D$ for all $w \in 2^*$ and therefore $\overline{\alpha}K \in B$.

1. Assume now that B is uniform. That is there $N \in \mathbb{N}$ such that $\overline{\alpha}N \in B$ for all $\alpha \in 2^{\mathbb{N}}$. Now there cannot be $n \geqslant N$ such that $x_n \in [0,1]$, since in that case $w_n \in 2^*$ and therefore $v_n \in D$. Since $|v_n| = n \geqslant N$ this would be a contradiction.

2. By construction, if $(x_n)_{n\geqslant 1}$ is in $[0,1]$ infinitely often, then $v_n \in 2^*$ infinitely often. This implies that $v_n \notin B$ infinitely often. Since, furthermore, $|v_n| = n$ we are done.

3. Assume that $(x_n)_{n\geqslant 1}$ is tail-located, and let $u \in 2^*$ be arbitrary. Let a, b be the endpoints of $F^{1/2}(B_u)$ that is $a = F^{1/2}(u)$ and $b = F^{1/2}(u * 1 * 1 * \ldots)$.[2] Choose N and $\delta > 0$ such that x_n is bounded away from a and b by δ, that is such that

$$
\forall n \geqslant N : |x_n - a| > \delta \wedge |x_n - b| > \delta .
$$

Choose M such that $2^{-M+1} < \delta$, and let $K = \max\{M, N\}$ Since

$$
F^{1/2}(B_u) = \left\{ F^{1/2}(u * w) \,\middle|\, w \in 2^* \right\}
$$

is compact (as the uniform image of a compact set) the distance

$$
\rho = d\left(F^{1/2}(B_u), (x_n)_{n\geqslant K} \right)
$$

exists. Now either $\rho > 2^{-(K+1)}$ or $\rho < 2^{-K}$. In the first case there cannot be $n > K$ with $u = \overline{v_n}|u|$, since that would imply that $v_n \in 2^*$, which implies that $w_n \in 2^*$ and therefore

$$
\left| F^{1/2}(v_n) - x_n \right| < 2^{-n+1} < \delta .
$$

This in turn implies that $d(x_n, F^{1/2}(B_u)) < 2^{-(K+1)}$ which is a contradiction. Thus we only need to check $v_1, \ldots, v_K$ to check whether $u \in B$ or not. In the second case there must be n such that $v_n \in cS$ and $F^{1/2}(v_n) \in [a + \delta, b - \delta]$. This ensures that $u = \overline{v_n}|u|$, which means that $u \in B$. Altogether we have shown that B is decidable. $\qquad \square$

Lemma 3.1.10. *If B is a c-bar, then there exists $(x_n)_{n\geqslant 1}$ a sequence of real numbers that is eventually bounded away from every point in $[0,1]$ such that*

1. *if $(x_n)_{n\geqslant 1}$ is eventually bounded away from $[0,1]$, then B is uniform,*

2. *if $x_n \in [0,1]$ infinitely often, then there are arbitrarily large w such that $w \notin B$, and*

---

[2]That is with the notation of Section 3.1 $a = a_u^{1/2}$ and $b = b_u^{1/2}$.

    3. if B is decidable and closed under extensions, then $(x_n)_{n \geqslant 1}$ is tail-located.

Proof. Assume that $B \subset 2^*$ is a c-bar, and C' is a decidable set as in the definition. Let $\eta : \mathbb{N} \to 2^*$ be a bijection. In particular, that means that we have

$$\forall n \in \mathbb{N} : \exists m \in \mathbb{N} : i \geqslant m \implies |\eta(i)| \geqslant n \,. \tag{3.8}$$

We may also assume that $i \leqslant j \implies |\eta(i)| \leqslant |\eta(j)|$.[3] Define

$$x_n = \begin{cases} F^{1/3}(\eta(n)) & \text{if } \eta(n) \notin C' \\ 2 & \text{otherwise.} \end{cases}$$

We want to show that this sequence is eventually bounded away from every $x \in [0,1]$. To this end choose $\alpha \in 2^{\mathbb{N}}$ as in Lemma 3.1.1.2. Since B is a bar there exists $n \in \mathbb{N}$ such that $\overline{\alpha}n \in B$, which means, that for all $w \in 2^*$ we have $\overline{\alpha}n * w \in C'$. Furthermore, let m be as in Equation 3.8. For every $i \geqslant m$ we have $\overline{\alpha}n \neq \overline{\eta(i)}n$ whenever $\eta(i) \notin C'$ (notice that $|\eta(i)| \geqslant n$), which implies that

$$\left| F^{1/3}(\eta(i)) - F^{1/3}(\alpha) \right| > 3^{-(n+1)} \,,$$

for such $\eta(i)$. Since $x_i = 2$ when $w_i = \omega$ we get that

$$\left| x_i - F^{1/3}(\alpha) \right| > 3^{-(n+1)} \,,$$

for all $i \geqslant m$. Now either $\left| F^{1/3}(\alpha) - x \right| < 3^{-(n+2)}$ or $\left| F^{1/3}(\alpha) - x \right| > 3^{-(n+3)}$. In the first case

$$|x_i - x| > 3^{-n+1}$$

for all $i \geqslant m$. In the second case by Lemma 3.1.1.2 we have that $d(x, F^{1/3}(2^{\mathbb{N}})) > \delta$ for some $\delta > 0$. In both cases $|x_i - x| > \min\{\delta, 3^{-(n+2)}\}$ for all $i \geqslant m$, so $(x_n)_{n \geqslant 1}$ is eventually bounded away from every $x \in [0,1]$.

    So let us tackle the three numbered assertions.

1. Assume that $(x_n)_{n \geqslant 1}$ is eventually bounded away from the entire set $[0,1]$; say $x_n \notin [0,1]$ for all $n \geqslant M$ on. Set $K = \max\{|\eta(j)| \mid 1 \leqslant j < M\}$. Then for all $v \in 2^*$ such that $|v| \geqslant K$ we must have $v \notin C'$, since otherwise $w_{\eta^{-1}(v)} \in 2^*$, and therefore $x_{\eta^{-1}(v)} \in [0,1]$. Hence for all $\alpha \in 2^{\mathbb{N}}$ we have $\overline{\alpha}K \in B$; that is B is uniform.

2. If $x_n \in [0,1]$ infinitely often we must have $w_n \in 2^*$ infinitely often. This in turn implies that $w_n \notin C'$ infinitely often. Since $C' \subset B$ we have that there are arbitrarily large w with $w \notin B$.

3. It suffices to show that the sets

$$A_n = \{ \eta(i) \mid i \geqslant n \wedge \eta(i) \notin B \}$$

are either empty or totally bounded as subsets of $2^{\mathbb{N}}$, since

$$\{x_n, x_{n+1}, \ldots\} = F^{1/3}(A_n) \cup \{2\} \,,$$

---

[3]The function mapping a natural number $n \geqslant 1$ to the string of its binary expansion without the leading 1 would be a suitable bijection.

and the uniformly continuous image of a totally bounded set is also totally bounded and therefore located [37, Proposition 2.2.6 and 2.2.9]. Choose

$$M = 1 + \{\,|\eta(i)|\,|\,1 \leqslant i < n\,\}\,,$$

which ensures that

$$|\eta(j)| \geqslant M \implies j \geqslant n\,. \tag{3.9}$$

Furthermore for any m we can choose $k_m \geqslant k$ large enough such that

$$k_m \geqslant \max\{\,i \in \mathbb{N}\,|\,|\eta(i)| = m\,\}\,. \tag{3.10}$$

Set

$$F_{n,m} = \{\,\eta(i)\,|\,k_m \geqslant i \geqslant n \wedge \eta(i) \notin B\,\}\,.$$

The sets $F_{n,m}$ are, for $m \geqslant M$, a finite 2^{-m}-approximation of A_n: for let $i \geqslant n$ with $\eta(i) \notin B$ and let j be such that $\eta(j) = \overline{\eta(i)}m$. By 3.9 and 3.10 we have $n \leqslant j \leqslant k_m$. Since B is closed under extensions $\eta(j) \notin B$. Hence $\eta(j) \in F_{n,M}$ and $d(\eta(j), \eta(i)) \leqslant 2^{-m}$.

The same argument shows that A_n is empty if and only if $F_{n,M}$ is. $\qquad\square$

3.2 WWKL

We start with a principle that is a weakening of FAN$_\Delta$. The so called *weak weak König's lemma* (WWKL) plays a role in Simpson style reverse mathematics [114]. The name is somewhat misleading since it is not resembling weak König's lemma (see Section 1.4) but rather its contrapositive i.e. the fan theorem.[4] WWKL does not assure that a decidable bar is uniform, that is that there is a level at which all sequences of that length have been barred, WWKL only assures that the ratio of all the sequences of a given length that are in the bar over all sequences of a given length tends to 0.

> **(WWKL)** If $B \subset 2^*$ is a decidable bar that is closed under extensions, then
> $$\lim_{n \to \infty} \frac{|\{\,u \notin B\,|\,|u| = n\,\}|}{2^n} = 0.$$

In constructive reverse mathematics two publications have involved WWKL. Their main results are combined in the next proposition.

Proposition 3.2.1. *The following are equivalent to WWKL*

1. *Every positive, uniformly continuous function $f : [0,1] \to \mathbb{R}$ satisfies the following property: For any $\varepsilon > 0$, there exists $\delta > 0$ such that*
$$\mu(\{\,x\,|\,f(x) < \delta\,\})$$
is defined and
$$\mu(\{\,x\,|\,f(x) < \delta\,\}) < \varepsilon\,.$$

2. *Vitali's theorem: Let $\varepsilon > 0$ be arbitrary. If $\mathcal{V}$ is a countable Vitali cover of $[a,b]$, then there exists a finite set $\{I_1, \dots, I_m\}$ of pairwise disjoint intervals of $\mathcal{V}$ such that*
$$\mu\left([a,b] \setminus \bigcup_{i=1}^{m} I_i\right) < \varepsilon\,.^5$$

---

[4]Of course in Simpson style reverse mathematics which is based on classical logic this distinction is unimportant.

[5]Where by μ we denote the usual measure, which we can obviously define, at least for sets that are finite unions of intervals.

Proof. The first equivalence can be found in [91], the second one in [47].   □

It seems reasonable to conjecture that WWKL has more equivalents in measure theory, which at the moment has not received much attention in constructive analysis.

Proposition 3.2.2. *WWKL is equivalent to the following "weaker" version for every $k > 0$*

> **(WWKL (k))**   *If $B \subset 2^*$ is a decidable bar that is closed under extensions, then*
> $$\frac{|\{u \notin B \,|\, |u| = n\}|}{2^n} < k$$
> *eventually.*

Proof. We will show that if WWKL(k) holds, then also WWKL(k^2). This then shows that WWKL(k) is independent of the choice of k, by a suitable number of iterations. So let B be a decidable bar that is closed under extensions. A first application of WWKL(k) yields a natural number N such that

$$\frac{|\{u \notin B \,|\, |u| = N\}|}{2^N} < k \,.$$

Let $u_1, \ldots, u_m \in 2^N$ be all the finite sequences with $u_1, \ldots, u_m \notin B$; so we have $m < k2^n$. It is clear that for all of these $u_1, \ldots, u_m$ the sets

$$B^{(i)} = \{w \in 2^* \,|\, u_i w \in B\}$$

are again decidable bars that are closed under extensions. So applying WWKL(k) m times we can find N_i such that

$$\frac{\left|\left\{u \notin B^{(i)} \,\middle|\, |u| = N_i\right\}\right|}{2^{N_i}} < k \,.$$

Set $M = \max\{N_1, \ldots, N_m\}$. Since all $B^{(i)}$ are closed under extensions this implies that

$$\left|\left\{u \notin B^{(i)} \,\middle|\, |u| = M\right\}\right| < k2^M \,.$$

Now, if $u \in 2^{N+M}$ is such that $u \notin B$, then it must be of the form $u = u_i w$ with $w \notin B$ and $w \in 2^M$. Hence

$$\left|\left\{u \in 2^{N+M} \,\middle|\, u \notin B\right\}\right| = \left|\bigcup_{i=1}^m \left\{u_i w \in 2^{N+M} \,\middle|\, w \notin B^{(i)}\right\}\right|$$

$$= \sum_{i=1}^m \left|\left\{w \in 2^M \,\middle|\, w \notin B^{(i)}\right\}\right|$$

$$< mk2^M < k2^N k2^M = k^2 2^{N+M} \,,$$

and we are done.   □

As conjectured above WWKL is likely to play an important role in measure theory. One could, of course, also consider the principles which are weakenings of FAN$_c$ and FAN$_{\Pi_1^0}$ (and even UCT) in the same way that WWKL is a weakening of FAN$_\Delta$.

Question 5. *What are the following principles equivalent to?*
If $B \subset 2^*$ is a c-bar (or Π_1^0-bar), then

$$\lim_{n \to \infty} \frac{|\{u \notin B \,|\, |u| = n\}|}{2^n} = 0 \,.$$

eventually.

3.3 FAN$_\Delta$

As mentioned in the introduction FAN$_\Delta$ deserves the prominence of being involved in the first "proper" equivalence of constructive reverse mathematics. It is also a fairly robust statement, as the next lemma shows.

Lemma 3.3.1. *If B is decidable bar, then there exists a decidable bar B' that is closed under extensions, such that B is uniform only if B' is.*

Proof. If B is a decidable bar, then it is easy to see that

$$B' = \{ u \in 2^* \, | \, \exists n \leqslant |u| : \bar{u}n \in B \}$$

is also a decidable bar that has the required properties. □

Another interesting result is the following lemma. Surprisingly it has, to our knowledge, not found many uses apart from [36, Chapter 6.2].

Lemma 3.3.2. *If B is a countable bar, then there exists a decidable bar B' which is uniform only if B is.*

Proof. Let $B = \{b_n\}_{n \geqslant 1}$ a countable bar. Then it is easy to see that

$$B' = \{ u \in 2^* \, | \, \exists i \in \mathbb{N} : \exists v \in 2^* : (i \leqslant |u| \wedge u * v = b_i) \}.$$

has the desired properties.[6] □

Proposition 3.3.3. *The following are equivalent to FAN$_\Delta$*

1. *(POS). Every uniformly continuous, positive-valued function $f : [0,1] \to \mathbb{R}^+$ has a positive infimum.*

2. *Every uniformly continuous, positive-valued function $f : 2^\mathbb{N} \to \mathbb{R}^+$ has a positive infimum.*

3. *Two compact subsets A, B of a metric space that are such that $d(a,b) > 0$ for all $a \in A$ and $b \in B$ are a positive distance apart.*

4. *The Heine-Borel theorem for compact metric spaces and countable coverings with open balls.*

5. *Dini's theorem: If $(f_n)_{n \geqslant 1} : [0,1] \to \mathbb{R}$ is an increasing sequence of uniformly continuous functions converging point-wise to a uniformly continuous $f : [0,1] \to \mathbb{R}$, then the convergence is uniform.*

Proof. The equivalence of FAN$_\Delta$ to POS is, as already mentioned, the first result in CRM and is proved in [71] and [36, Section 6.2]. The equivalence of 1 and 2 follows from Lemmas 3.1.3 and 3.1.1. The equivalence of FAN$_\Delta$ to 4 is in [43, Section 4.2].

It is easy to see that 3 implies 1, by applying the latter to the graph of a uniformly continuous, positive-valued function f and the unit interval $[0,1] \times \{0\}$ as subsets of the Euclidean plane. To see that 2 implies 3 let A, B be subsets such that $d(a,b) > 0$ for all $a \in A$ and $b \in B$. By [108, Chapter 7 Corollary 4.4] there exists surjective, uniformly continuous functions $g_1 : 2^\mathbb{N} \to A$ and $g_2 : 2^\mathbb{N} \to B$. Then the function $h : 2^\mathbb{N} \to \mathbb{R}$ defined by

$$h(\alpha) = d(g_1(\alpha^e), g_2(\alpha^o)),$$

where, $\alpha^e = \alpha_0 \alpha_2 \alpha_4 \ldots$ and $\alpha^o = \alpha_1 \alpha_3 \alpha_5 \ldots$. Then h is easily seen to be continuous, and such that $h(\alpha) > 0$ for all $\alpha \in 2^\mathbb{N}$. Applying 2 implies that there exists $\varepsilon > 0$ such that $h(\alpha) > \varepsilon$ for all $\alpha \in 2^\mathbb{N}$. Since g_1 and g_2 are surjective also $d(a,b) > \varepsilon$ for all $a \in A$ and $b \in B$.

The equivalence to Dini's theorem is proved in [17]. □

---

[6]The full proof can be found in [43, Lemma 4.1.1].

Even though UCT has not been introduced yet, this seems to be a good place to ask the following question.

Question 6. *Is Dini's theorem [17] for point-wise continuous functions equivalent to UCT?*

Proposition 3.3.4. *FAN$_\Delta$ is equivalent to the following statement: Consider a uniformly continuous $f : [0,1] \to \mathbb{R}$ such that $0 < f(x) < x$ for all $x \in (0,1)$. Now let $x_0 \in (0,1)$ be arbitrary and consider the sequence defined by $x_{n+1} = f(x_n) = f^n(x_0)$. The sequence $(x_n)_{n \geqslant 1}$ converges to 0.[7]*

Proof. Clearly $(x_n)_{n \geqslant 1}$ is strictly decreasing. Now assume FAN$_\Delta$ and let $\varepsilon > 0$ be arbitrary. We want to show that there is $n \in \mathbb{N}$ such that $x_n < \varepsilon$. The function $g : [\varepsilon, x_1] \to \mathbb{R}$ defined by

$$g(x) = x - f(x)$$

is uniformly continuous. It is also such that $g(x) > 0$ for all $x \in [\varepsilon, x_1]$. So by POS, which as we have seen above is equivalent to FAN$_\Delta$, there exists $\delta > 0$ such that $g(x) > \delta$ for all $x \in [\varepsilon, x_1]$. Now choose n such that $1 - n\delta < \varepsilon$. Then clearly $x_n < \varepsilon$ and we are done.

Conversely consider a decidable bar $B \subset 2^{\mathbb{N}}$. Let u_n be an enumeration of all elements of B such that $u_n[: -1] = \overline{u_n}(|u_n| - 1) \notin B$. Notice that either $u_n = 0 * \cdots * 0$ or we can find its immediate "left" neighbour $u_{L(n)}$ that is there exists w such that $u_n = w * 1 * 0^k$ and $u_{L(n)} = w * 0 * 1^\ell$ for some $k, \ell \geqslant 0$. We may assume that $u_1 = 1 * \cdots * 1$. Now define a function $f(0,1) \to \mathbb{R}$ by interpolating linearly between the points

$$(F(u_n), F(u_{L(n)})) \,,$$

where $F = F^{\frac{1}{2}}$ as defined in Section 3.1, which simply maps every binary sequence onto the real with the matching binary expansion. Then f satisfies the condition above. Thus we can consider the sequence x_n, again defined as above, where $x_0 = F(u_1) < 1$. Let k be such that $u_k = 0 * \cdots * 0$. If x_n converges to 0, then there must be N such that $f^N(x_0) < 2^{-(|u_k|+1)}$. It is clear that the sequences $u_1, u_{L(1)} \ldots, u_{L^N(1)} = u_k$ describe a cover of $2^{\mathbb{N}}$. In other words their maximum length is a uniform bound for B. □

Proposition 3.3.5. *FAN$_\Delta$ is equivalent to the statement that every tail-located sequence in $\mathbb{R}$ that is eventually bounded away from every point in $[0,1]$ is eventually bounded away from the entire set.*

Proof. Immediate consequence of Lemmas 3.1.9 and 3.1.10. □

Next, we extend the results of Berger in [7]. First we need a lemma, whose proof idea is based on the proof of the main result in the paper mentioned.

Lemma 3.3.6. *For every $f : 2^{\mathbb{N}} \to \mathbb{R}$ that has a continuous (functional) modulus of continuity and for every $e \in \mathbb{N}$, there exists a decidable bar B that is uniform only if there exists $N \in \mathbb{N}$ such that*

$$\overline{\alpha} N = \overline{\beta} N \implies |f(\alpha) - f(\beta)| < 2^{-e} \,.$$

Proof. Say $f : 2^{\mathbb{N}} \to \mathbb{R}$ has a continuous modulus that is there exists a continuous function $\mu : 2^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ such that

$$\forall \alpha, \beta \in 2^{\mathbb{N}} : \forall e > 0 : \left(\overline{\alpha} \mu(\alpha, e) = \overline{\beta} \mu(\alpha, e) \implies |f(\alpha) - f(\beta)| < 2^{-e} \right) \,.$$

Now let $e \in \mathbb{N}$ be arbitrary and define

$$B = \left\{ u \in 2^* \,|\, \mu(u * 000\ldots, e) \leqslant |u| \right\} \,.$$

---

[7]This is Exercise K8—one of many in a very entertaining classical book on analysis [74].

Clearly B is decidable. It is also a bar: for let $\alpha \in 2^{\mathbb{N}}$ be arbitrary. Since μ is continuous itself there exists M such that $\mu(\alpha, e) = \mu(\beta, e)$ for all $\beta \in 2^{\mathbb{N}}$ with $\overline{\alpha}M = \overline{\beta}M$. For $M' = \max\{M, \mu(\alpha, e)\}$ we must have

$$\mu(\overline{\alpha}M' * 000\ldots, e) = \mu(\alpha, e) \leqslant M' \ ,$$

and therefore $\overline{\alpha}M' \in B$.

Next, assume B is uniform. So there exists $N \in \mathbb{N}$ such that

$$\forall \alpha \in 2^{\mathbb{N}} : \exists n \leqslant N : \overline{\alpha}n \in B \ .$$

So for any $\alpha \in 2^{\mathbb{N}}$ there is $n \leqslant N$ such that $\mu(\overline{\alpha}n * 000\ldots, e) \leqslant n \leqslant N$, and hence for all $\beta \in 2^{\mathbb{N}}$ with $\overline{\alpha}N = \overline{\beta}N$ we have $|f(\alpha) - f(\beta)| < 2^{-e}$. $\square$

Proposition 3.3.7. *The following are equivalent to* FAN$_\Delta$

1. *Every continuous function $f : X \to Y$, with a continuous modulus of continuity is uniformly continuous, where X is a compact and Y an arbitrary metric space.*

2. *Every continuous function $f : 2^{\mathbb{N}} \to \mathbb{R}$ with a continuous modulus of continuity is uniformly continuous.*

3. *Every continuous function $f : 2^{\mathbb{N}} \to \mathbb{N}$ with a continuous modulus of continuity is uniformly continuous.*

Proof. It is clear that 2 implies 3 and that 1 implies 2. In [7] it is shown that if B is a decidable bar, then

$$f(\alpha) = \min\{n \in \mathbb{N} \mid \overline{\alpha}n \in B\}$$

is a continuous function with itself (!) as a continuous modulus of continuity. Furthermore it is shown that if f is uniformly continuous then B is uniform. Hence 3 implies FAN$_\Delta$. Next, with the help of Lemma 3.3.6, we can show that FAN$_\Delta$ implies 2.

So we are done if we can show that 2 implies 1. To this end let $f : X \to Y$ be a uniformly continuous map with a continuous modulus of continuity $\mu : X \times \mathbb{N} \to \mathbb{N}$ such that

$$\forall x, y \in X : \forall e \in \mathbb{N} : d(x, y) < 2^{-\mu(x,e)} \implies d(f(x), f(y)) < 2^{-e} \ .$$

Let $g : 2^{\mathbb{N}} \to X$ the mapping from Proposition 3.1.6. Using countable choice we can therefore construct $\tau : \mathbb{N} \to \mathbb{N}$ such that

$$\forall \alpha, \beta \in 2^{\mathbb{N}} : \forall n \in \mathbb{N} : \overline{\alpha}\tau(n) = \overline{\beta}\tau(n) \implies d(g(\alpha), g(\beta)) < 2^{-n} \ .$$

Furthermore, for $\alpha \in 2^{\mathbb{N}}$ let $\alpha^e = \alpha(0)\alpha(2)\alpha(4)\ldots$ and $\alpha^o = \alpha(1)\alpha(3)\alpha(5)\ldots$. Now consider the function $h : 2^{\mathbb{N}} \to \mathbb{R}$ defined by

$$h(\alpha) = d(f(h(\alpha^e)), f(h(\alpha^o))) \ .$$

It is straightforward to show that $\eta : 2^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ defined by

$$\eta(\alpha, n) = 2\tau(\mu(g(\alpha^e), n))$$

is a continuous modulus of continuity for h. By 2 it is also uniformly continuous. So for an arbitrary $\varepsilon > 0$ there exists n such that $\overline{\alpha}n = \overline{\beta}n$ implies $|h(\alpha) - h(\beta)| < \varepsilon$. By Lemma 3.1.5 there exists $\delta > 0$ such that for $x, y \in X$ with $d(x, y) < \delta$ there exist $\alpha, \beta \in 2^{\mathbb{N}}$ such that

$$\overline{\alpha}n = \overline{\beta}n \wedge x = g(\alpha) \wedge y = g(\beta) \ .$$

Now define γ and γ' by

$$\gamma = \alpha(0)\beta(0)\alpha(1)\beta(1)\dots$$

and

$$\tau = \alpha(0)\alpha(0)\alpha(1)\alpha(1)\dots .$$

That way $\gamma^e = \tau^e = \tau^o = \alpha$ and $\gamma^o = \beta$. Furthermore $\overline{\gamma}n = \overline{\tau}n$, and hence

$$
\begin{aligned}
\varepsilon &> |h(\gamma) - h(\tau)| \\
&= |d(f(h(\gamma^e)), f(h(\gamma^o))) - d(f(h(\tau^e)), f(h(\tau^o)))| \\
&= |d(f(h(\alpha)), f(h(\beta))) - d(f(h(\alpha)), f(h(\alpha)))| \\
&= d(f(h(\alpha)), f(h(\beta))) = d(f(x), f(y)) .
\end{aligned}
$$

That means that δ is a modulus of uniform continuity for ε, and we are done. $\qquad\square$

Definition 3.3.8. We will call a function $f : [0,1] \to \mathbb{R}$ *fully located* if, $f([a,b])$ is located for every $a < b$.

Classically every function is fully located. Constructively every uniformly continuous function $[0,1] \to \mathbb{R}$ is fully located, however the converse does not hold, as is shown in Proposition 5.3.3, which will be a consequence of the following lemmas.

Lemma 3.3.9.

1. *Assume $f : [0,1] \to \mathbb{R}$ is fully located. Then f is point-wise continuous if it is sequentially continuous.*

2. *If $f(x) > 0$ for all $x \in [0,1]$, then f is fully located if and only if $\frac{1}{f}$ is.*

Proof.   1. Let $x \in [0,1]$ and $\varepsilon > 0$ be arbitrary. Using our locatedness assumption, we can decide for every $n \in \mathbb{N}$ whether $\exists y \in B_{2^{-n}}(x) : |f(y) - f(x)| < \varepsilon$ or whether $\forall y \in B_x(2^{-n}) : |f(y) - f(x)| > \frac{\varepsilon}{2}$. So using countable choice we can construct a binary sequence $(\lambda_n)_{n \geqslant 1}$ and a sequence of reals $(x_n)_{n \geqslant 1}$ such that $|x_n - x| < \frac{1}{2^n}$ and

$$\lambda_n = 0 \implies |f(x_n) - f(x)| > \frac{\varepsilon}{2} ,$$

$$\lambda_n = 1 \implies \forall y : |x - y| < \frac{1}{2^n} \implies |f(y) - f(x)| < \varepsilon .$$

Since x_n converges to x and f is sequentially continuous there exists N such that for all $n \geqslant N$ we have $|f(x_n) - f(x)| < \frac{\varepsilon}{2}$. This in turn implies that $\lambda_N = 1$, which immediately yields the desired property. Hence f is point-wise continuous.

2. Straightforward. $\qquad\square$

Lemma 3.3.10. *For every decidable bar B there exists a point-wise continuous, fully located function $f : [0,1] \to \mathbb{R}$ such that f is bounded if and only if B is uniform; and vice versa.*

Proof. First, start with a decidable bar B that is without loss of generality closed under extension. We are going to adapt the construction in [36, Theorem 2.2.7]. Even though, given a bar, we could also define a function $f : 2^\mathbb{N} \to \mathbb{N}$ from it and extend it to a function $\hat{f} : [0,1] \to \mathbb{R}$, proving that if f is fully located, then $\hat{f}$ is seems tedious, so we are going to go a different route. Since B is decidable so is $T = \neg B$, and since B is closed under extensions T is closed under restriction, i.e. a tree. Furthermore T does not admit infinite paths. Now consider the set

$$S = \left\{ (F^{1/3}(u), 2^{-|u|}) \in \mathbb{R}^2 \,\middle|\, u \in T \right\} .$$

It is easy to see that S is totally bounded and therefore located. Hence the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = d((x,0),S)$ exists and is uniformly continuous. So, since $[a,b]$ is totally bounded, its image under f also totally bounded and therefore located. Just as in [36, Theorem 2.2.7] the fact that T does not admit infinite paths implies that f is positively valued. Furthermore if $\inf f > 0$, then B is uniform. By the previous lemma $\frac{1}{f}$ is fully located, and is such that it is bounded if B is uniform.

Conversely assume that $f : [0,1] \to \mathbb{R}$ is fully located, and point-wise continuous. Hence, using countable choice, we can fix a decidable set $B \subset 2^*$ such that

$$u \notin B \implies \exists x \in I_u : f(x) > 2|u| \; ,$$
$$u \in B \implies \forall x \in I_u : f(x) < 2|u| + 1 \; ;$$

where I_u are the intervals $I_u^{2/3}$ defined above. We want to show that B is a bar, so let $\alpha \in 2^{\mathbb{N}}$ be arbitrary. Since $f \circ F^{2/3}$ is point-wise continuous there exists $N \in \mathbb{N}$ such that $|f \circ F^{2/3}(\alpha) - f \circ F^{2/3}(\beta)| < 1$ whenever $\overline{\alpha}N = \overline{\beta}N$. Now choose $M > N$ such that $f \circ F^{2/3}(\alpha) < M$. Now assume that $\overline{\alpha}M \notin B$. Then, by the choice of B there exists $y \in I_{\overline{\alpha}M}$ such that $f(y) > 2M$. Since $F^{2/3}$ is surjective on $I_{\overline{\alpha}M}$ (Lemma 3.1.1) there exists $\beta \in 2^{\mathbb{N}}$ such that $F^{2/3}(\beta) = y$ and $\overline{\beta}M = \overline{\alpha}M$. But that means that

$$f \circ F^{2/3}(\alpha) + |f \circ F^{2/3}(\alpha) - f \circ F^{2/3}(\beta)| > f \circ F^{2/3}(\beta) > 2M$$

which implies that $f \circ F^{2/3}(\alpha) > 2M - 1$; but this is a contradiction to $f \circ F^{2/3}(\alpha) < M \leqslant 2M - 1$.

By definition B is also closed under extensions. Therefore, if B is uniform there exists N such that $\overline{\alpha}N \in B$ for all $\alpha \in 2^{\mathbb{N}}$. Since $[0,1] \bigcup_{u \in 2^N} I_u$ that means that $f(x) < 2N + 1$ for all $x \in [0,1]$. $\qquad\square$

Corollary 3.3.11. *FAN$_\Delta$ is equivalent to every sequentially continuous, fully located function $f : [0,1] \to \mathbb{R}$ being bounded.*

Remark 3.3.12. *In RUSS there exists a point-wise continuous, fully located function $f : [0,1] \to \mathbb{R}$ which fails to be bounded.*

Proof. See Proposition 5.3.3. $\qquad\square$

3.4 FAN$_c$

In [8] in order to answer the question what kind of fan theorem is equivalent to the uniform continuity theorem for functions $2^{\mathbb{N}} \to \mathbb{N}$ Berger introduced the notion of a c-bar. Not much later, the resulting principle FAN$_c$ turned out to be equivalent to the "anti-Specker"-principle, which was conceived independently from Berger's work [11].

Proposition 3.4.1. *The following are equivalent to FAN$_c$.*

1. $AS_{[0,1]}^{\mathbb{R}}$

2. $AS_{[0,1]}^1$

3. $AS_{2^{\mathbb{N}}}^1$

4. *For every compact X and every one-point extension Y AS_X^Y holds.*

5. *Every point-wise continuous function $f : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is uniformly continuous.*

6. *Every point-wise continuous function $f : 2^{\mathbb{N}} \to \mathbb{N}$ is uniformly continuous.*

7. *Every point-wise continuous function $f : 2^{\mathbb{N}} \to 2$ is uniformly continuous.*

8. *Every point-wise equicontinuous sequence of mappings of $[0,1]$ into $\mathbb{R}$ is uniformly sequentially equicontinuous.*

9. *Every point-wise continuous mapping of $[0,1]$ into a metric space is uniformly sequentially continuous.*

Proof. Even though the equivalence $\mathsf{FAN}_c \iff \mathsf{AS}^{\mathbb{R}}_{[0,1]}$ has been shown in [11] it also follows from Lemmas 3.1.9 and 3.1.10. For the equivalence between $\mathsf{AS}^{\mathbb{R}}_{[0,1]}$ and 4 see [44], for the one between $\mathsf{AS}^{\mathbb{R}}_{[0,1]}$ and $\mathsf{AS}^1_{[0,1]}$ see [30], and for the one between FAN_c and $\mathsf{AS}^1_{2^{\mathbb{N}}}$ see [43]. The equivalence between 6 and FAN_c has been shown by Berger in [8]. Clearly $5 \implies 6 \implies 7$. To see that also $7 \implies 5$ let $f : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be a point-wise continuous function and $\varepsilon > 0$ arbitrary. First choose N such that $2^{-N} < \varepsilon$. Now define $g : 2^{\mathbb{N}} \to \{0,1\}$ by

$$g(\alpha) = \begin{cases} 0 & \text{if } \forall i \leqslant N : f(\alpha^e)(i) = f(\alpha^o)(i) \\ 1 & \text{otherwise,} \end{cases}$$

where, as before, $\alpha^e = \alpha_0 \alpha_2 \alpha_4 \ldots$ and $\alpha^o = \alpha_1 \alpha_3 \alpha_5 \ldots$. The function g is continuous, since for an arbitrary α there exists M such that for arbitrary β we have

$$\left| f(\overline{\alpha^e}M * \beta) - f(\alpha^e) \right| < 2^{-N} \text{ and } \left| f(\overline{\alpha^o}M * \beta) - f(\alpha^o) \right| < 2^{-N} ,$$

which means that for all $i \leqslant N$

$$f(\overline{\alpha^e}M * \beta)(i) = f(\alpha^e)(i) \text{ and } f(\overline{\alpha^o}M * \beta)(i) = f(\alpha^o)(i).$$

So either $f(\alpha^o)(i) = f(\alpha^e)(i)$ for all $i \leqslant N$ or not. In the first case $g(\alpha) = 0$ and in the second case $g(\alpha) = 1$. But by the choice of M, also for any β we have $g(\overline{\alpha}(2M) * \beta) = g(\alpha)$, and hence g is point-wise continuous. By our assumption it is therefore uniformly continuous, which means we can find an M as above which is independent of α, which immediately gives us uniform continuity of f.

Finally, the equivalences between $\mathsf{AS}^{\mathbb{R}}_{[0,1]}$, 8, 9 are proved in [25]. $\qquad\square$

Notice that, unlike in the case of variations of UCT, AS and many other principles it is not possible to replace $2^{\mathbb{N}}$ with $[0,1]$ in 6 and 7 above, since a point-wise continuous function $f : [0,1] \to \mathbb{N}$ is necessarily constant.

Proposition 3.4.2 (Variant of POS)**.** *FAN_c is equivalent to every point-wise continuous function $f : 2^{\mathbb{N}} \to \mathbb{N}$ being bounded.*

Proof. Clearly, this statement is implied by Part 6 of the above proposition and therefore follows from FAN_c. Conversely we can directly see that it implies FAN_c. Given a c-bar B with $C \subset 2^*$ such that

$$u \in B \iff \forall w \in 2^* : (u * w \in C) ,$$

we can define $f : 2^{\mathbb{N}} \to \mathbb{N}$ by

$$f(\alpha) = \max \left\{ n \in \mathbb{N} \,|\, \overline{\alpha}n \notin C \right\} .^8$$

The fact that B is a c-bar and C is decidable ensures that this is a well-defined and point-wise continuous function. If f is bounded by N, then this N also gives us a uniform bound for B. $\qquad\square$

---

[8] Notice that $f(\alpha) = \min \left\{ n \in \mathbb{N} \,|\, \overline{\alpha}n \in B \right\}$ would not be well-defined, unless B would actually be a decidable bar.

3.5   UCT

The uniform continuity theorem is the standard first step in numerous theorems of classical analysis. It states:

> **(UCT)**   Every point-wise continuous function $f : [0,1] \to \mathbb{R}$ is uniformly continuous.

Its importance in shaping the development of constructive mathematics cannot be overrated. It is not provable with purely constructive methods (i.e. in BISH), which led Brouwer to the notion of bar induction, in order to be able to prove it. Bishop sneakily avoided such additional assumptions by simply building it into his definition of a continuous function.

It is surprising, that UCT for real-valued functions on $[0,1]$ suffices to show the most general conceivable one, as proved in [29]

Proposition 3.5.1. *UCT is equivalent to the statement that every point-wise continuous map $X \to Y$ on a compact metric space X and into an arbitrary metric space Y is uniformly continuous.*

Also in [29], and maybe even more surprising, it is shown it is also equivalent to some weaker versions

Proposition 3.5.2. *UCT is equivalent to the following*

1. *Every point-wise continuous function $f : [0,1] \to \mathbb{R}$ is bounded. (That is $[0,1]$ is pseudo-compact.)*

2. *Every point-wise continuous function $f : 2^{\mathbb{N}} \to \mathbb{R}$ is bounded.*

3. *Every point-wise continuous mapping of $[0,1]$ into $\mathbb{R}$ is integrable.*

Notice that the construction to prove the last equivalence in the above proposition relies on creating a function that has "very high peaks." An interesting question is therefore:

Question 7. *Is there a version of the fan theorem equivalent to the following statement: every bounded point-wise continuous function $f : [0,1] \to \mathbb{R}$ is integrable.*

It is folklore that the following holds.

Proposition 3.5.3. *UCT is equivalent the statement that every point-wise continuous $f : [0,1] \to \mathbb{R}$ can be uniformly approximated with polynomials.*

Proof. Of course, if f can be uniformly approximated with polynomials, then f is uniformly continuous, since polynomials are. Conversely the only non-constructive step in the normal proof that uses Bernstein polynomials [59, Section 4.2.2] is to conclude that f is uniformly continuous, which is exactly what UCT enables us to do. $\square$

Although the following proposition seems like a harmless variant of the previous one, it is actually very different: Notice that for the converse direction the functions are not assumed to be uniformly continuous. Somehow, just the fact that there are countably many of them suffices.

Proposition 3.5.4. *UCT is equivalent to $C([0,1])$ being separable.*
More precise: UCT is equivalent to the statement that there exists a sequence of point-wise continuous functions $f_n : [0,1] \to \mathbb{R}$ such that for all $\varepsilon > 0$ and all point-wise continuous $f : [0,1] \to \mathbb{R}$ there exists $n \in \mathbb{N}$ with
$$\forall x \in [0,1] : |f_n(x) - f(x)| < \varepsilon \ .$$

Proof. Obviously UCT implies that $C([0,1])$ is separable, just as in the classical proof. The interesting part is the proof of the converse: Consider $(f_n)_{n \geqslant 1}$ dense in $C([0,1])$. By Proposition 3.5.1 it suffices to show that every point-wise continuous $g : [0,1] \to \mathbb{R}$ is bounded. To this end consider the functions g_n defined by

$$g_n(x) = \max \{0, 1 - |g(x) - n|\} \ .$$

These have the property that $g_n(x) > 0 \iff g(x) \in (n-1, n+1)$. Thus the function $G : [0,1] \to \mathbb{R}$ defined by

$$G(x) = \sum_{n \in \mathbb{N}} g_n(x) \left(\max \{f_1(x), \ldots f_n(x)\} + 1 \right)$$

is well defined, and continuous. So there exists $M \in \mathbb{N}$ such that

$$\forall x \in [0,1] : |f_M(x) - G(x)| < 1 \ .$$

Now there cannot be x such that $g(x) > M$, since in that case $g_M(x) = 1$ and $g_i(x) = 0$ for all $i \neq M$. Hence $G(x) = \max \{f_1(x), \ldots f_n(x)\} + 1 \geqslant f_M(x) + 1$, which is a contradiction. Thus $g(x) \leqslant M$ for all $x \in [0,1]$ and we are done. $\square$

Corollary 3.5.5. *In RUSS, $C([0,1])$ is not separable.*

In [49] the following variations were considered in the context of differential equations.

Proposition 3.5.6. *The following are equivalent to UCT*

1. *(BUCT). Every bounded, continuous function $f : [0,1] \to \mathbb{R}$ is uniformly continuous.*

2. *(LUCT). Every continuous function $f : [0,1] \to \mathbb{R}$ is locally uniformly continuous.*

3.6 FAN$_{\Pi_1^0}$ and FAN$_{stable}$

We will start this subsection with an attempt to remove some confusion about principles running under the name FAN$_{\Pi_1^0}$. Above we defined a Π_1^0-bar to be a bar that is the intersection of decidable sets that are closed under extension. Equivalently one often finds the following definition for a Π_1^0-bar B. There exists a decidable set $S \subset 2^* \times \mathbb{N}$ such that

1. $u \in B \iff \forall n \in \mathbb{N} : (u, n) \in S$

2. If $(u, n) \in S$, then for any $w \in 2^*$ also $(u * w, n) \in S$.

However, some authors[9] have also referred to bars only satisfying the first condition as Π_1^0-bars. In the following we will call these bars *stable bars*. Notice that stable bars are exactly bars that are the complement of a countable set. Of course another complexity of a bar leads to another fan theorem.

> **(FAN$_{stable}$)**   Every stable bar is uniform.

Trivially,

$$\text{FAN}_{full} \implies \text{FAN}_{stable} \implies \text{FAN}_{\Pi_1^0} \ ,$$

but the converses seem unlikely. The unproven (and probably false) assumption that FAN$_{stable}$ and FAN$_{\Pi_1^0}$ are equivalent can be explained by Lemma 3.3.1 taken together with the following observation.

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[9]Including ourself.

Proposition 3.6.1. *If a stable bar B is closed under extensions, then B is a Π_1^0-bar.*

Proof. This proof is based on one by J. Berger.[10] Notice that a similar construction is also used in the proof of Theorem 2 of [48] as well as in the proof of Proposition 2 of [44] and in the proof of Proposition 3.6.2. Assume that the bar B is the intersection of the decidable sets B_n. Now define

$$\widetilde{B}_n = \{ u \in 2^* \mid \forall w : (|w| \leqslant n - |u| \implies u * w \in B_n) \} \ .$$

By construction every $\widetilde{B}_n$ is closed under extensions. We want to show that

$$B = \bigcap_{n \geqslant 1} \widetilde{B}_n \ .$$

To see that "$\subset$" holds let $u \in B$, which means that $u * w \in B$ for all $w \in 2^*$. This implies that for all $w \in 2^*$ and all $n \in \mathbb{N}$ we have $u * w \in B_n$ and in particular that $u * w \in \widetilde{B}_n$. For the other direction "$\supset$" consider u such that $u \in \bigcap_{n \geqslant 1} \widetilde{B}_n$, and consider an arbitrary $m \in \mathbb{N}$. If $m \leqslant |u|$, then write $u = \bar{u}m * w$. For this w we have $|w| \leqslant m - |u|$, and therefore, since $u \in \widehat{B}_m$, also $u \in B_m$. If $m > |u|$, then taking the ε, the empty sequence, as w we get $u * w = u \in B_m$. Altogether $u \in B_m$ for any m and therefore $u \in \bigcap_{n \geqslant 1} B_n = B$. $\square$

With the help of this proposition it seems as if we can show that for every stable bar there exists an equivalent Π_1^0 bar. However, the missing step is to prove that for every stable bar there exists an equivalent stable bar that is closed under extensions. The proof of Lemma 3.3.1 does not generalise to this case, even if it seems on first glance easy to adapt the proof somehow. Notice that it would be enough to show that the closure (under extensions) of a stable bar is stable.

Our first $FAN_{\Pi_1^0}$-equivalence is a sequential version of 3.3.3.1.

Proposition 3.6.2. *$FAN_{\Pi_1^0}$ is equivalent to the following statement:*

Every equi-continuous, and equi-positive,[11] sequence of functions $(f_k)_{k \geqslant 1} : 2^{\mathbb{N}} \to \mathbb{R}$ is such that there exists $\delta > 0$ such that

$$f_k(\alpha) > \delta \text{ for all } \alpha \in 2^{\mathbb{N}} \text{ and } k \in \mathbb{N} \ .$$

Proof. First, assume that $FAN_{\Pi_1^0}$ holds. Using countable choice fix decidable sets $B_{n,k} \subset 2^*$ such that

$$u \notin B_{n,k} \implies \exists w \in 2^* : |w| + |u| \leqslant n \wedge f_k(u * w) < 2^{-|u|} \ , \tag{3.11}$$

$$u \in B_{n,k} \implies \forall w \in 2^* : |w| + |u| \leqslant n \implies f_k(u * w) > 2^{-(|u|+1)} \ . \tag{3.12}$$

We will show that $B = \bigcap_{n,k \in \mathbb{N}} B_{n,k}$ is a bar. To this end let $\alpha \in 2^{\mathbb{N}}$ be arbitrary. Since $(f_k)_{k \geqslant 1}$ is equi-positive there exists N such that $f_k(\alpha) > 2^{-(N-1)}$ for all $k \in \mathbb{N}$. Now choose $M \in \mathbb{N}$ such that

$$\forall \beta : \bar{\beta}M = \bar{\alpha}M \implies |f_k(\beta) - f_k(\alpha)| < 2^{-N} \ .$$

Let $K = \max\{N, M\}$. For any $k \in \mathbb{N}$ and $w \in 2^*$ we get that

$$\begin{aligned} f_k(\bar{\alpha}K * w) &> f_k(\alpha) - |f_k(\bar{\alpha}K * w) - f_k(\alpha)| \ , \\ &> 2^{-(N-1)} - 2^{-N} \\ &> 2^{-N} > 2^{-K} \ . \end{aligned}$$

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[10]Personal Correspondence, 18th Mar 2011.
[11]We call a sequence of functions $(f_k)_{k \geqslant 1} : X \to \mathbb{R}$ *equi-positive* if for every $x \in X$ there exists $\varepsilon > 0$ with $f_k(x) > \varepsilon$ for all $k \in \mathbb{N}$.

Hence $\overline{\alpha}K \in B_{n,k}$, since case 3.11 is ruled out. Therefore B is a bar. By construction it is also closed under extensions and we can apply $\text{FAN}_{\Pi_1^0}$, to get $L \in \mathbb{N}$ such that $u \in B$ for all $u \in 2^L$. So for all $\alpha \in 2^{\mathbb{N}}$ we get, by continuity, $f_k(\alpha) \geqslant 2^{-(L+1)}$, which concludes one direction of the proof.

Conversely, let $B = \bigcap_{k \in \mathbb{N}} B_k$ a stable bar, with decidable sets B_k that are closed under extensions. Define $f_k : 2^{\mathbb{N}} \to \mathbb{R}$ by

$$f_k(\alpha) = 2^{-\min\{\ell \in \mathbb{N} \,|\, \overline{\alpha}\ell \in B_k\}} \, .$$

We will show that $(f_n)_{k \geqslant 1}$ is equi-continuous and equi-positive. To this end let $\alpha \in 2^{\mathbb{N}}$ be arbitrary. Since B is a bar there exists $n \in \mathbb{N}$ such that $\overline{\alpha}n \in B$, which means that for all $k \in \mathbb{N}$ we can find the minimal $n_k \leqslant n$ such that $\overline{\alpha}n_k \in B_k$. This, in turn, implies that if $\overline{\beta}n = \overline{\alpha}n$, then $\overline{\beta}n_k = \overline{\alpha}n_k$ and therefore

$$f_k(\alpha) = f_k(\beta) = 2^{-n_k} \geqslant 2^{-n} > 0 \, .$$

Thus we have both shown equi-positivity and equi-continuity. By our assumptions there exists $N \in \mathbb{N}$ such that $f_k(\alpha) > 2^{-N}$ for all $k \in \mathbb{N}$ and $\alpha \in 2^{\mathbb{N}}$. This translates back, for all $\alpha \in 2^{\mathbb{N}}$, into the existence of $m \leqslant N$ such that $\overline{\alpha}m \in B_k$. Since the sets B_k are closed under extensions also $\overline{\alpha}N \in B_k$ for all $n \in \mathbb{N}$ and $\alpha \in 2^{\mathbb{N}}$. Therefore $\overline{\alpha}N \in B$ for all $\alpha \in 2^{\mathbb{N}}$, which means B is uniform. $\qquad\square$

Proposition 3.6.3. *The following are equivalent to* $FAN_{\Pi_1^0}$

1. *Every locally constant function $f : X \to Y$ is uniformly locally constant; where X is a compact and Y an arbitrary metric space.*

2. *Every locally constant function $f : 2^{\mathbb{N}} \to \mathbb{R}$ is uniformly locally constant.*

Proof. The equivalence of $\text{FAN}_{\Pi_1^0}$ to 2 is shown in [10]. The direction 1 $\implies$ 2 is trivial. To see that the converse is valid let $f : X \to Y$ be a locally constant function. As usual, we can find a surjective and uniformly continuous function $g : 2^{\mathbb{N}} \to X$ [108, Chapter 7 Corollary 4.4]. Now consider the function $h : 2^{\mathbb{N}} \to \mathbb{R}$ defined by:

$$h(\alpha) = d(f(\alpha^e), f(\alpha^o)) \, ,$$

where α^e is the sequence of all even terms of α and α^o the one of all odd ones (compare the proof of Proposition 3.3.7). Since f is locally constant so is h. By our assumption that means that h is globally constant, so there exists N such that $\overline{\alpha}N = \overline{\beta}N$ implies that $h(\alpha) = h(\beta)$. Now given α, β with $\alpha N' = \beta N'$ where $N' \geqslant N/2$ consider $\gamma = \alpha_0 \beta_0 \alpha_1 \beta_1 \dots$ and $\eta = \alpha_0 \alpha_0 \alpha_1 \alpha_1 \dots$. By that construction $\overline{\gamma}N = \overline{\eta}N$, so $h(\gamma) = h(\eta)$. So

$$0 = d(f(\alpha), f(\alpha)) = d(f(\eta^e), f(\eta^o)) = h(\eta) = h(\gamma) = d(f(\gamma^e), f(\gamma^o)) = d(f(\alpha), f(\beta)) \, .$$

In other words $f(\alpha) = f(\beta)$. Hence f is globally constant. $\qquad\square$

Remark 3.6.4. *Notice that for functions $f : 2^{\mathbb{N}} \to \mathbb{N}$ the notion of (uniformly) locally constant coincides with the one for (uniformly) point-wise continuous. For such functions the statement that "every locally constant function is uniformly locally constant" is therefore equivalent to FAN_c.*

Maybe the most interesting equivalence is the following, involving a notion well known from the Arzela-Ascoli-Theorem in analysis. A proof can be found in [48].

Proposition 3.6.5. *$FAN_{\Pi_1^0}$ is equivalent to the statement that every equi-continuous sequence of functions $[0, 1] \to \mathbb{R}$ is uniformly equicontinuous.*

It seems feasible to also prove this theorem for functions functions of type $2^{\mathbb{N}} \to \mathbb{R}$ and for functions $f : X \to Y$, where X is an arbitrary compact and Y is an arbitrary metric space. This is a common theme throughout a lot of the results in this chapter; namely that $2^{\mathbb{N}}$ and $[0,1]$ are often interchangeable without changing the classification. However, with the anti-Specker principle we have to be a bit careful. Here the space that $2^{\mathbb{N}}$ or $[0,1]$ are embedded in makes a difference, as was shown in [44].

Proposition 3.6.6. *The following are equivalent to* $FAN_{\Pi_1^0}$.

1. $AS_{[0,1]}^{\mathbb{R}^2}$

2. AS_X^Y *for any compact subspace X of a metric space Y*

3.7 FAN_{full}

There is, maybe surprisingly, hardly any equivalence to the full, unrestricted, fan theorem. The only one, which is almost a kind of restatement, is that $2^{\mathbb{N}}$ is cover-compact.

Proposition 3.7.1. *The following are equivalent*

1. FAN_{full}

2. *Every compact (totally bounded and complete) metric space is cover compact.*

3. $2^{\mathbb{N}}$ *is cover compact.*

Proof. Trivially 2 implies 3. To see that 3 implies 1 let B be an arbitrary bar. The sets $B_u = \left\{ \alpha \in 2^{\mathbb{N}} \,\middle|\, \overline{\alpha}|u| = u \right\}$ form an open cover of $2^{\mathbb{N}}$. If this cover can be refined to a finite cover $B_{u_1}, \dots, B_{u_n}$, then $\max \{u_1, \dots, u_n\}$ is a uniform bound for B. Finally, assume 1 and let X be totally bounded and complete. We can find a continuous $g : 2^{\mathbb{N}} \to X$ [108, Chapter 7 Corollary 4.4]. Now if $(U_i)_{i \in I}$ is an open cover of X let

$$B = \left\{ u \in 2^{\mathbb{N}} \,\middle|\, \exists i \in I : \forall \beta \in 2^{\mathbb{N}} : u * \beta \in g^{-1}(U_i) \right\} .$$

For every $\alpha \in 2^{\mathbb{N}}$ there exists $i \in I$ such that $g(\alpha) \in U_i$, or equivalently $\alpha \in g^{-1}(U_i)$. Since $g^{-1}(U_i)$ is open there exists n such that $\overline{\alpha}n * \beta \in g^{-1}(U_i)$ for all $\beta \in 2^{\mathbb{N}}$. So B is a bar. Applying FAN_{full} yields n such that $u \in B$ for all $u \in 2^n$. That means we can find finitely many i_u such that $(g^{-1}(U_{i_u}))_{u \in 2^n}$ covers $2^{\mathbb{N}}$. But since g is surjective this means that $(U_{i_u})_{u \in 2^n}$ cover X. $\qquad\square$

Given the similarities between them we might be tempted to try and replace $2^{\mathbb{N}}$ with $[0,1]$ in the above proposition. However, the work in [88] shows that $[0,1]$ being cover compact is strictly weaker than FAN_{full}. This is a very subtle distinction since in the case of a countable cover with basic opens the cover compactness of $[0,1]$ is equivalent to the one of $2^{\mathbb{N}}$ (and both are equivalent to FAN_Δ [43]).

3.8 Comparing the Fan Theorems

The differences between the fan theorems are overall very subtle and often confusingly minute. Since many of the results in this section are variations of each other, we hope that the following table might highlight some of the differences.

	FAN_Δ	FAN_c	UCT	$\text{FAN}_{\Pi_1^0}$
Uniform continuity	Every pwc. fully located $f : [0,1] \to \mathbb{R}$ is uc. Every $f : [0,1] \to \mathbb{R}$ with a pwc. modulus of cont. is uc.	Every pwc. $f : 2^{\mathbb{N}} \to \mathbb{N}$ is uc. Every pwc. $f : [0,1] \to \mathbb{R}$ is usc.	Every pwc. $f : [0,1] \to \mathbb{R}$ is uc. Every pwc. $f : 2^{\mathbb{N}} \to \mathbb{R}$ is uc.	
Anti-Specker	$\text{AS}_{[0,1]}^{\mathbb{R}}$ for tail-located sequences	$\text{AS}_{[0,1]}^{\mathbb{R}}$, $\text{AS}_{[0,1]}^1$, $\text{AS}_{2^{\mathbb{N}}}^1$,		$\text{AS}_{[0,1]}^{\mathbb{R}^2}$
Boundedness statements	POS	Every pwc. $f : 2^{\mathbb{N}} \to \mathbb{N}$ is bounded.	Every pwc. $f : 2^{\mathbb{N}} \to \mathbb{R}$ or $f : [0,1] \to \mathbb{R}$ is bounded.	
Equicontinuity		Every ec. $f_n : [0,1] \to \mathbb{R}$ is usec.		Every ec. $f_n : [0,1] \to \mathbb{R}$ is uec.
constancy		Every locally const. $f : 2^{\mathbb{N}} \to \mathbb{N}$ is globally const.		Every locally const. $f : 2^{\mathbb{N}} \to \mathbb{R}$ is globally const.

The following abbreviations are used:

pwc: point-wise continuous

uc: uniformly continuous

usc: uniformly sequentially continuous

ec: equicontinuous (i.e. point-wise equicontinuous)

uec: uniformly equicontinuous

usec: uniformly sequentially equicontinuous

We can recognise the following themes.

- FAN_c is involved whenever sequential continuity or sequences of real numbers are involved.

- $\text{FAN}_{\Pi_1^0}$ is involved in sequences of functions.

- Introducing locatedness assumptions reduces principles to FAN_Δ.

Chapter 4

BD-N and Below

Together with Weak Markov's principle (Section 2.2), BD-N occupies a special and rare place in constructive mathematics: it is accepted in CLASS, in INT, as well as in RUSS. However, it is not accepted in Bishop style constructive mathematics (BISH). One might think that after reading Proposition 4.1.1, which says that BD-N is equivalent to every sequentially continuous function on a separable metric space being point-wise continuous, one fully understands this principle. However BD-N does appear in some quite unexpected places, and has an entire zoo of weaker principles below it.

4.1 BD-N

A subset S of $\mathbb{N}$ is *pseudobounded* if

$$\lim_{n \to \infty} \frac{s_n}{n} = 0$$

for each sequence $(s_n)_{n \geqslant 1}$ in S. This is equivalent to assuming that for every sequence $(s_n)_{n \geqslant 1}$ in S we have

$$s_n < n$$

eventually.

Trivially, every bounded set is pseudobounded, but the converse is more subtle and the content of our next principle.

> **(BD-N)** Every inhabited, countable, pseudobounded subset of $\mathbb{N}$ is bounded.

One can readily show that when dealing with a decidable set one can establish the consequence:

Proof. Let S be a inhabited, decidable and pseudobounded subset of $\mathbb{N}$. Choose $s_0 \in S$ and define a sequence by

$$s_{n+1} = \begin{cases} n+1, & n+1 \in S; \\ s_0, & n+1 \notin S. \end{cases}$$

As S is pseudobounded there is M such that for all $i \geq M$ $s_i < i$. We claim that $S \subset \{1, \ldots, M\}$. For assume that there is a $k \in S$ such that $k > M$. Then $k = s_k < k$ a contradiction. □

F. Richman has done a wonderful job of showing that there are many alternative equivalent notions for pseudobounded [96]. The most surprising one may be that a subset A of $\mathbb{N}$ is pseudobounded if and only if every nonempty subset of A that is detachable from $\mathbb{N}$ is finite.

The reason for H. Ishihara to investigate BD-N was the question whether constructively there is a difference between sequential and point-wise continuity. The following result is a slight improvement of his original result [65], since there he showed that for every pseudobounded and countable set there exists a separable space and a sequentially continuous function such that if that function is point-wise continuous, then the set is bounded. Our construction BD-N $\iff$ 2 is not only, arguably, simpler it also shows that it suffices to concentrate on Baire space.[1]

Proposition 4.1.1. *The following are equivalent to BD-N.*

1. *Every sequentially continuous map on a separable metric space is point-wise continuous.*

2. *Every sequentially continuous map $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is point-wise continuous.*

3. *Every sequentially continuous map $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is locally bounded.*

Proof. BD-N $\iff$ 1 is shown in [65]. Moreover, it is clear that 1 $\implies$ 2, and that 2 $\implies$ 3, so it remains to show that 3 $\implies$ BD-N. Let $A = \{a_n\}$ be a countable, pseudobounded subset of the natural numbers. We may assume that a_n is non-decreasing. Notice that for all $\alpha \in \mathbb{N}^{\mathbb{N}}$ the sequence $\widehat{\alpha}$ defined by $\widehat{\alpha}(n) = \sum_{i=1}^{n} \alpha(i) + 1$ is strictly increasing. In particular $\widehat{0} = \mathrm{id}$. Conversely, for every strictly increasing $\beta \in \mathbb{N}^{\mathbb{N}}$ there exists γ with $\widehat{\gamma} = \beta$, but we will not use this fact.
Since A is pseudobounded, the function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ defined by

$$f(\alpha) = \max \left\{ n \in \mathbb{N} \,\middle|\, a_{\widehat{\alpha}(n)} > n \right\}$$

is well-defined. It is also easy to see that f is strongly extensional and that LPO implies that it is sequentially continuous. Hence, using a corollary to Ishihara's tricks [46, Corollary 3.4.], we get that f is sequentially continuous. (Alternatively, sequential continuity also follows from the uniform sequential continuity of f as proved directly below.) Our assumptions then imply that f is locally bounded. So there exist N, K such that, if $\overline{\alpha}N = 0$, then $f(\alpha) \leqslant K$. That means that if we take $M = \max\{N, K\}$ we have that if $\overline{\alpha}M = 0$, then $f(\alpha) \leqslant M$.
We will show that there cannot be $i > M$ with $a_i > M + 1$. If there were we could consider $\beta \in 2^{\mathbb{N}}$ defined by

$$\beta = \underbrace{0 \ldots 0}_{M}(i - M)0 \ldots ,$$

so that $\overline{\beta}M = 0$ and $\beta(M + 1) = i - M$. This way $\widehat{\beta}(M + 1) = i$, which means that $a_{\widehat{\beta}(M+1)} = a_i > M + 1$ and therefore $f(\beta) \geqslant M + 1$. But this gives a contradiction, since $\overline{\beta}M = 0$, which means that $f(\beta) \leqslant M$, by our choice of M. That means that $\max\{a_1, \ldots, a_M, M + 1\}$ is an upper bound of A and therefore BD-N holds. $\square$

Proposition 4.1.2. *The following are equivalent to BD-N.*

1. *Every uniformly sequentially continuous mapping of a separable metric space into a metric space is uniformly continuous.*

2. *Every uniformly sequentially continuous map $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is uniformly continuous.*

3. *Every uniformly sequentially continuous map $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is locally bounded.*

---

[1]Since every separable metric space is a continuous image of Baire space one could probably get the same result by lifting the original equivalence. However, that would involve not only showing that every separable metric space is the continuous image of Baire space [108], but also that this embedding has some sort of openness property.

Proof. In [35] it is shown that BD-N is equivalent to 1. Clearly $1 \implies 2$, and $2 \implies 3$, so it remains to show that the latter implies BD-N. To this end let $A = \{a_n\}$ be a countable, pseudobounded subset of the natural numbers, and define f in the same way as in the proof of Proposition 4.1.1. We want to show that f is uniformly sequentially continuous. Let α_n and β_n be sequences such that $d(\alpha_n, \beta_n) \to 0$ as $n \to \infty$. More specificly, with the help of countable choice, we can choose a modulus $\mu : \mathbb{N} \to \mathbb{N}$ such that $\overline{\alpha_i}n = \overline{\beta_i}n$ for all $i \geqslant \mu(n)$. Now, for every n we can decide whether there is an i with

$$\mu(n) \leqslant i < \mu(n+1) \wedge f(\alpha_i) \neq f(\beta_i)$$

or not. In the first case there must be k_n such that $a_{k_n} > n$: for assume there is such i with $f(\alpha_i) \neq f(\beta_i)$ such that $\overline{\alpha_i}n = \overline{\beta_i}n$. Then either $f(\alpha_i) > n$ or $f(\beta_i) > n$, since if both were $\leqslant n$ we would have $f(\alpha_i) \neq f(\beta_i)$. But that means that there is $\ell > n$ such that either

$$a_{\widehat{\alpha_i}(\ell)} > \ell > n \text{ or } a_{\widehat{\beta_i}(\ell)} > \ell > n .$$

In the first case let $k_n = \widehat{\alpha_i}(\ell)$, in the second case let $k_n = \widehat{\beta_i}(\ell)$.

Now define

$$\gamma_n = \begin{cases} k_n & \text{in the first case} \\ n & \text{otherwise.} \end{cases}$$

Since A is pseudobounded there exists N such that for all $n \geqslant N$ we have $a_{\gamma(n)} \leqslant n$. But that means that for all $n \geqslant N$ we cannot be in the first case, which means that we must have $f(\alpha_n) = f(\beta_n)$. So f is uniformly sequentially continuous.

That means that if can apply 3 to get that f is locally bounded, and continue as in the proof of Proposition 4.1.1. $\square$

We can also show that 3 of Proposition 4.1.1 implies 2 of Proposition 4.1.2 directly without referring to BD-N, by using the following lemma. This might seem like a fairly obscure point to make, however the construction itself is quite interesting and the result also works for $2^{\mathbb{N}}$.

Lemma 4.1.3. *If $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ (or $2^{\mathbb{N}} \to \mathbb{N}$) is uniformly sequentially continuous, then there exists a $g : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ (respectively $2^{\mathbb{N}} \to \mathbb{N}$) such that*

- *g is sequentially continuous, and*

- *if g is locally bounded at $000\dots$, then f is uniformly continuous.*

Proof. We will only treat the case $\mathbb{N}^{\mathbb{N}}$; the case of $2^{\mathbb{N}}$ can be treated analogously. We first need to introduce some notation. We assume that we have fixed a bijection $\varphi : \mathbb{N}^2 \to \mathbb{N}$. Given $\gamma \in \mathbb{N}^{\mathbb{N}}$, we can slice up γ into countably many sequences and turn it into a double sequence $(\curlywedge \gamma) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

$$(\curlywedge \gamma)(m,n) = \gamma(\varphi(m,n)) .$$

We can also zip a double indexed sequence $\sigma : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ up into a single indexed sequence $(\curlyvee \sigma) \in \mathbb{N}^{\mathbb{N}}$ by setting

$$(\curlyvee \sigma)(n) = \sigma(\varphi_1^{-1}(n), (\varphi_2^{-1}(n)) .$$

These operations are inverses, so we have

$$\curlywedge\curlyvee \sigma = \sigma \text{ and } \curlyvee\curlywedge \gamma = \gamma .$$

Again, consider an arbitrary $\gamma \in \mathbb{N}^{\mathbb{N}}$, and define two sequences of sequences $\alpha^{(n)}$ and $\beta^{(n)}$ by

$$\alpha^{(n)} = \curlywedge \gamma(2n, \cdot), \quad \text{and} \quad \beta^{(n)} = \overline{\curlywedge \gamma(2n, \cdot)}(n) * \curlywedge \gamma(2n+1, n+1) * \gamma(2n+1, n+2) * \dots .$$

These sequences are constructed such that

$$\overline{\alpha^{(n)}}(n) = \overline{\beta^{(n)}}(n) ,$$

that means $d(\alpha^{(n)}, \beta^{(n)}) \to 0$. Since f is uniformly sequentially continuous $f(\alpha^{(n)}) = f(\beta^{(n)})$ eventually. That means the number

$$\left| \left\{ n \in \mathbb{N} \,\middle|\, f(\alpha^{(n)}) \neq f(\beta^{(n)}) \right\} \right| \tag{4.1}$$

exists. So we can define a function $g : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ that maps γ to that number. It is easy to see that g is strongly extensional and that it is point-wise continuous provided LPO holds. Thus, using a corollary to Ishihara's tricks [46, 64] we can conclude that g is sequentially continuous.

Assume that g is locally bounded at $000\ldots$. So there exists N, K such that for all $\alpha \in \mathbb{N}^{\mathbb{N}}$ we have $g(0^N * \alpha) < K$. Since φ is bijective there exists M such

$$\left\{ \varphi_1^{-1}(i) \,\middle|\, i \leqslant N \right\}$$

is bounded by $2M$.

We claim that $L = M + K$ is a modulus of constancy for f. Let μ, η such that $\overline{\mu}(L) = \overline{\eta}(L)$, but $f(\mu) \neq f(\eta)$. Now consider the following sequence of sequences:

$$\underbrace{000\ldots, 000\ldots, \ldots, 000\ldots}_{2M}, \underbrace{\mu, \eta, \mu, \eta, \ldots, \mu, \eta}_{2K}, 000\ldots, 000\ldots, \ldots .$$

If we combine these with $\curlyvee$ into a sequence ϑ we have the following properties:

1. $\overline{\vartheta}(N) = 0\ldots0$, and

2. $\curlywedge \vartheta(2i, \cdot) = \mu$ for $M < i \leqslant L$

3. $\curlywedge \vartheta(2i+1, \cdot) = \nu$ for $M < i \leqslant L$.

This is set up such that $g(\vartheta) = K$, which is a contradiction to K being a local bound at $000\ldots$. That means that for any $\mu, \eta \in BS$ such that $\overline{\mu}(L) = \overline{\eta}(L)$, we have $f(\mu) = f(\eta)$. $\qquad\square$

From [33]

Proposition 4.1.4. *The following are equivalent to* BD-N

1. *If T is a nonzero bounded linear mapping of a separable Hilbert space H into itself such that T^* exists and $\mathrm{ran}(T)$ is complete, then T is open.*

2. *Every one-one self-adjoint sequentially continuous linear mapping from a Hilbert space onto itself is bounded.*

In [14] a weakened form of the usual Cauchy condition is considered. There a sequence $(x_n)_{n \geqslant 1}$ in a metric space (X, d) is called *almost Cauchy*, if for any strictly increasing $f, g : \mathbb{N} \to \mathbb{N}$

$$d(x_{f(n)}, x_{g(n)}) \to 0$$

as $n \to \infty$. A closer analysis of this and equivalent conditions can be found in [50]. The main theorem proved in [14] is the following.

Proposition 4.1.5. BD-N *is equivalent to the statement that every almost sequence in a semi-metric space is Cauchy.*

Here a semi-metric space is a metric space not necessarily satisfying the triangle-inequality condition. Notice that the statement is not provable in BISH for metric spaces (see [50] referring to [80]). Another equivalence of BD-N concerns a lesser known fact of analysis. It together with its proof can be found in [13]

Proposition 4.1.6. *BD-N is equivalent to the statement that for all conjugate exponents p and q, if a is any sequence of complex numbers such that $\sum_{n=1}^{\infty} a_n x_n$ converges for each $x \in l_p$, then $\sum_{n=1}^{\infty} |a_n|^q$ has bounded partial sums.*

Lastly, we would like to mention that in [70] it is shown that BD-N is equivalent to two important spaces being complete.

4.2 Below BD-N

Between, approximately, 2007 and 2010 a couple of statements were considered by researchers working in CRM for which a proof in BISH could not be found, but that were all implied by BD-N. Naturally a considerable amount of time was spent trying to prove that they were in fact equivalent to BD-N. As it turned out around 2011 for most of them it could be shown that that could not be done, but that also they could not be proved within BISH. These statements, and some additional ones, are the following:

Proposition 4.2.1. *BD-N implies the following statements.*

1. *If X is separable, then AS_X^1 implies AS_X^Y for any $Y \supset X$.*

2. *FAN_c is equivalent to $FAN_{\Pi_1^0}$.*

3. *FAN_c is equivalent to UCT.*

4. *The anti-Specker property is closed under products in the sense that if AS_X^1 and AS_Y^1 holds, then also $AS_{X \times Y}^1$ holds.*

5. *The anti-Specker property implies pseudo-compactness. That is, if AS_X^1 and X is separable, then any point-wise continuous function $f : X \to \mathbb{R}$ is bounded.*

6. *The anti-Specker property implies totally boundedness That is, if AS_X^1 and X is separable, then X is totally bounded.*

7. *A converse of the Riemann permutation theorem: if $(a_n)_{n \geq 1}$ is a sequence of real numbers such that for all σ permutations the series*
$$\sum_{n=1}^{\infty} a_{\sigma(n)}$$
converges, then $(a_n)_{n \geq 1}$ converges absolutely.

8. *Every almost Cauchy sequence in a metric space is Cauchy.*

Furthermore, 1 implies both 2 and 5, 2 implies 3, and 5 and 6 are equivalent and they also imply 3.

Proof. To see that 1 follows from BD-N let $(x_n)_{n \geq 1}$ be a dense sequence in X; without loss of generality we can assume that every point is repeated infinitely often in the sequence $(x_n)_{n \geq 1}$. Furthermore, consider a sequence $(z_n)_{n \geq 1}$ in Y that is bounded away from every point in $X \subset Y$. Now define the set

$$A = \left\{ n \in \mathbb{N} \,\middle|\, \exists i, j \geq n : d(z_i, x_j) < 2^{-n} \right\} \cup \{0\} \ .$$

The set A is easily seen to be countable; and, unsurprisingly, we are going to show that it is also pseudobounded. To this end let a_n be a sequence in A and with the use of countable choice define a sequence $(y_n)_{n\geqslant 1}$ in $X \cup \{\omega\}^2$ such that

$$a_n < n \iff y_n = \omega \,,$$
$$a_n \geqslant n \iff y_n = x_j,\ j \geqslant n,\ \exists i \geqslant n : d(z_i, x_j) < 2^{-n} \,.$$

The sequence $(y_n)_{n\geqslant 1}$ is eventually bounded away from every point in x: for let x in X be arbitrary. Since $(z_n)_{n\geqslant 1}$ is eventually bounded away from x there exists N such that

$$d(x, z_k) > 2^{-N} \text{ for all } k > N \tag{4.2}$$

We claim that $d(x, y_k) > 2^{-(N+2)}$ for all $k > N$. For assume $d(x, y_k) < 2^{-(N+1)}$. Then we must have $a_k \geqslant k$, and therefore $y_k = x_j$ for some $j \geqslant k$ and $d(z_i, x_j) < 2^{-k}$ for some $i \geqslant k$. Hence

$$d(x, z_i) \leqslant d(x, x_j) + d(x_j, z_i) = d(x, y_k) + d(x_j, z_i) < 2^{-(N+1)} + 2^{-k} \leqslant 2^{-N} \,,$$

but this is a contradiction to 4.2. Hence we have shown that the sequence $(y_n)_{n\geqslant 1}$ is eventually bounded away from x. Since we assumed AS^1_X it is therefore bounded away from the entire set X, which means there must exist N such that $y_n = \omega$ for all $n \geqslant N$ and therefore $a_n < n$ for all $n \geqslant N$. Hence A is bounded, say by a number M. Now there cannot be an $x \in X$ and $i \geqslant M$ with $d(x, z_i) < 2^{-(M+2)}$, since otherwise by the density we can find $j \geqslant M$ with $d(x_j, x) < 2^{-(M+2)}$, which would imply that $d(x_i, x_j) < 2^{-(M+1)}$ and therefore $M + 1 \in A$, which is a contradiction. So for all $x \in X$ and $i \geqslant M$ we have $d(x, z_i) \geqslant 2^{-(M+2)}$, which means that $(z_n)_{n\geqslant 1}$ is eventually bounded away from the set X.

That 2 follows from 1 is an immediate consequence of Propositions 3.4.1 and 3.6.6.

To see that 1 implies 5 let $f : X \to \mathbb{R}$ be a point-wise continuous function and $(x_n)_{n\geqslant 1}$ a dense sequence in X. We may assume that $f(x) > 1$ for all $x \in X$. Hence $g(x) = 1/f(x)$ is well defined and also point-wise continuous. Furthermore we have that $g(x) > 0$ for all $x \in X$. Hence, together with the continuity, the sequence $((x_n, g(x_n)))_{n\geqslant 1}$ in $X \times \mathbb{R}$ is eventually bounded away from every point in X (embedded in $X \times \mathbb{R}$ as $X \times \{0\}$). Since we assume AS^1_X and therefore by assumption also AS^Y_X, this sequence is also eventually bounded away from the entire set $X \times \{0\}$. So there exists M such that $d((x, 0), (x_i, g(x_i))) > 2^{-M}$ for all $i \geqslant M$ and $x \in X$. In particular, $|g(x_i)| > 2^{-M}$ for all $i \geqslant M$, which is equivalent to $f(x_i) < 2^M$. Since $(x_n)_{n\geqslant 1}$ is dense and f continuous $f(x) \leqslant 2^M$ for all $x \in X$, and thus f is bounded.

The equivalence of BD-N to the statements 4 and 5 were proved in [26] and [30] respectively.

To see that 5 and 6 are equivalent notice, for one direction, that pseudo-compactness implies totally boundedness [22, Theorem 2]. For the other direction notice that if $f : X \to \mathbb{R}$ is point-wise continuous, then $f(X)$ is also separable and that $AS^1_{f(X)}$ follows from AS^1_X, whence $f(X)$ is totally bounded and therefore, in particular, bounded (also see [30, Theorem 11]).

To see that 1 implies 4 let z_n be a sequence in $X \times Y \cup \{\omega\}$, where AS^1_X and AS^1_Y. Let y_n be the sequence in $Y \cup \{\omega\}$ that is, basically, the projection of z_n onto Y. More precisely:

$$y_n = \begin{cases} \pi_Y(z_n) & \text{if } z_n \in Y \\ \omega & \text{if } z_n = \omega \,. \end{cases}$$

We want to show that y_n is bounded away from every point in Y. So let $y \in Y$ be arbitrary. By 1 we have AS^Z_X, where we think of X being embedded in $X \times Y \cup \{\omega\}$ by $x \mapsto (y, x)$. In other words, we consider $X \times \{y\} \subset X \times Y \cup \{\omega\}$. Since z_n is bounded away from every point in $X \times \{y\}$ there exists $\delta > 0$ and $N \in \mathbb{N}$ such that $d(z_n, (x, y)) > \delta$ for all $x \in X$ and $n \geqslant N$. Hence $d_{Y \cup \{\omega\}}(y_n, y) > \delta$ for all $n \geqslant N$. So y_n

---

^{2}We choose a metric on $X \cup \{\omega\}$ that is equivalent to the one on X and is such that $d(x, \omega) \geqslant 1$.

is bounded away from every point in Y. By $\text{AS}^!_Y$ that means that there exists K such that $y_n = \omega$ for all $n \geqslant K$, but that implies, by construction of y_n, that $z_n = \omega$ for all $n \geqslant K$.

The implications 2 implies 3, and 6 implies 3 are trivial.

Finally, the proof of the equivalence of BD-N and 7 is sketched in [46] and that BD-N implies 8 is simply Proposition 4.1.5 restricted to metric spaces and without the converse. $\qquad\square$

Since in every topological model (Section 7.2) UCT holds (consequence of [54, Theorem 3.2]) and there is a model in which BD-N fails [78], Statement 2 cannot imply BD-N. In [79] it is shown that it cannot be proved in BISH. That 4, 7, and 8 are strength-wise also between BD-N and unadorned BISH was shown in [80]. It is unknown, whether this also holds for 1 and 5, but it seems likely.

Another natural principle which falls into the same category is

> **(wBD-N)**   Every sequentially continuous mapping $f : 2^{\mathbb{N}} \to \mathbb{N}$ is point-wise continuous.

Proposition 4.2.2. *wBD-N is equivalent to the statement that every sequentially continuous mapping $f : 2^{\mathbb{N}} \to \mathbb{N}$ is locally bounded.*

Proof. One direction is clear. Conversely, consider $f : 2^{\mathbb{N}} \to \mathbb{N}$ and let $\alpha \in 2^{\mathbb{N}}$. Notice that on complete spaces sequential continuity implies strong extensionality and so we can, for every $\beta \in 2^{\mathbb{N}}$, if $f(\beta) \neq f(\alpha)$ find the minimal n such that $\alpha(n) \neq \beta(n)$. So define

$$g(\beta) = \begin{cases} n & \text{where } f(\beta) \neq f(\alpha) \text{ and } n \text{ is as above} \\ 0 & \text{otherwise.} \end{cases}$$

Similar to proofs above, it is easy to see that g is strongly extensional and that it is point-wise continuous provided LPO holds. Thus, using a corollary to Ishihara's tricks [46, 64] we can conclude that g is sequentially continuous.

Now if g is locally bounded around α there exists N, K such that $\overline{\alpha}N = \overline{\beta}N$, then $g(\beta) \leqslant K$. If we use $M = \max N, K$ we get that if $\overline{\alpha}M = \overline{\beta}M$, then $g(\beta) \leqslant M$. But that means that there cannot be any β such that $\overline{\alpha}M = \overline{\beta}M$ such that $f(\beta) \neq f(\beta)$, since in that case $g(\beta) > M$. Thus f is point-wise continuous. $\qquad\square$

Remark 4.2.3. *It does not seem to be possible to extend the above result analogously to Proposition 4.1.1 and show that wBD-N is equivalent to sequentially continuous functions $2^{\mathbb{N}} \to \mathbb{R}$ being point-wise continuous. Moreover, it even seems impossible to get an equivalence analogous to the one of Proposition 4.1.2.*

Proposition 4.2.4. *wBD-N implies that every uniformly sequentially continuous mapping $f : 2^{\mathbb{N}} \to \mathbb{N}$ is uniformly continuous.*

Proof. This follows easily from the next lemma, or alternatively, from Lemma 4.1.3. $\qquad\square$

Lemma 4.2.5. *If f is a function $2^{\mathbb{N}} \to \mathbb{N}$, and f is both uniformly sequentially continuous and point-wise continuous, then f is uniformly continuous.*

Proof. Let u_n be an enumeration of all finite binary sequences. Then the set of all sequences $\alpha_n = u_n * 000\ldots$ is dense in $2^{\mathbb{N}}$. We will also consider the related sequences $\beta_n = u_n * 1 * 000\ldots$. Since for all m there exists N such that $|u_n| \geqslant m$ for all $n \geqslant N$ (a fact which holds for any enumeration, but in particular for the obvious one enumerating the sequences by length and then lexicographically), we have that $d(\alpha_n, \beta_n) \to 0$. Since f is uniformly sequentially continuous we also have that $d(f(\alpha_n), f(\beta_n)) \to 0$,

which means that there exists K such that $f(\alpha_n) = f(\beta_n)$ for $n \geqslant K$. Now choose $L = \max\{|u_1|, \ldots, |u_K|\}$. We claim that if $\alpha \in 2^{\mathbb{N}}$ arbitrary, then $f(\overline{\alpha L} * 0 * \ldots) = f(\overline{\alpha L} * \beta)$ for all $\beta \in 2^{\mathbb{N}}$; which means that f is uniformly continuous. For assume there exists $\alpha, \beta \in CS$ such that $f(\overline{\alpha L} * 0 * \ldots) \neq f(\overline{\alpha L} * \beta)$. Since f is point-wise continuous there exists M such that

$$f(\overline{\overline{\alpha L} * \beta} M * 000 \ldots) = f(\overline{\alpha L} * \beta) .$$

We must have $M > L$, so for $M' = M - L$ we have

$$f(\overline{\alpha L} * \overline{\beta} M' * 000 \ldots) = f(\overline{\alpha L} * \beta) \neq f(\overline{\alpha L} * 000 \ldots) .$$

By checking all finitely many prefixes of $\overline{\beta} M'$ we can find $w \in 2^*$ such that

$$f(\overline{\alpha L} * w * 000 \ldots) \neq f(\overline{\alpha L} * w * 1 * 000 \ldots) .$$

Let k be such that $u_k = \overline{\alpha L} * w$. Since $|u_k| \geqslant L$ we must have $k \geqslant K$ and therefore $f(\alpha_k) = f(\beta_k)$. But this is a contradiction, since

$$f(\alpha_k) = f(\overline{\alpha L} * w * 000 \ldots) \neq f(\overline{\alpha L} * w * 1 * 000 \ldots) = f(\beta_k) .$$

Altogether L is a modulus of constancy and therefore for uniform continuity of f. □

It seems worth noting that the above proof does not work for functions defined on $\mathbb{N}^{\mathbb{N}}$.

It is unknown if wBD-N is equivalent to, or implies, or is implied by any of the statements of Proposition 4.2.1.

Question 8. *Are there any other equivalences to wBD-N? Can the issues outlined in Remark 4.2.3 be resolved?*

We finish this section with a diagram summing up all these statements and their known interactions. Here a double headed arrow means that the implication is strict. The numbers are the ones from Proposition 4.2.1.
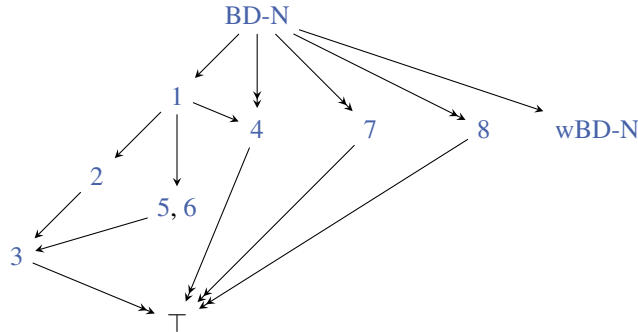
Chapter 5

The recursive side

5.1 Introduction

Let us assume that $K \subset 2^*$ is a decidable tree (i.e. it is closed under restriction) that does not admit infinite paths; that is

$$\forall \alpha \in 2^{\mathbb{N}} : \exists n \in \mathbb{N} : \overline{\alpha}n \notin K .$$

Then the complement $B \subset 2^*$ has the following properties:

- B is decidable, since K is decidable,

- B is a bar, since K does not admit infinite paths, and

- B is closed under extension, since K is a tree and therefore closed under restriction.

In other words B is a decidable bar that is closed under restriction. Conversely, the complement of such a bar is a decidable tree that does not admit infinite paths.[1] Now in RUSS (and other recursive varieties of constructive mathematics) there exists a Kleene tree, which is a tree having the above properties that is also infinite as a set. The latter condition is actually equivalent to

$$\forall n \in \mathbb{N} : \exists u \in T : |u| \geqslant n .$$

With the above observation we see that the complement of a Kleene tree fails to be a uniform bar and therefore provides a counterexample to FAN_Δ. This gives rise to the idea of considering "Anti-Fan"-principles.

> **(Anti-FAN□)**   There exists a □-bar B such that there is a sequence $(u_n)_{n \geqslant 1}$ with
>
> - $|u_n| \geqslant n$ and
> - $u_n \notin B$.

As already pointed out in [43, Proposition 4.5.2.] Anti-FAN_{full} is actually equivalent to Anti-FAN_c: for if B is an arbitrary bar and $(u_n)_{n \geqslant 1}$ as above, then the complement B' of $\{u_1, u_2, \ldots\}$ is actually a c-bar and a superset of B. Furthermore, by construction, $u_n \notin B'$. Therefore there are actually only two Anti-Fan principles to consider: Anti-FAN_Δ and Anti-FAN_c.

In a similar spirit we can also define Anti-WWKL for $k \in (0,1)$:

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[1]Note that the complement of a decidable set is again decidable.

> **(Anti-WWKL (k))** There exists a decidable bar B that is closed under extension such that
> $$\forall n \in \mathbb{N} : \frac{|\{u \notin B \,|\, |u| = n\}|}{2^n} > k$$

As we will see in the following section this principle is actually independent of the choice of k, and we will therefore simply refer to it as Anti-WWKL.

So we have the following hierarchy of recursive principles:

$$\text{Anti-WWKL}(\text{SC}) \implies \text{Anti-FAN}_\Delta(\text{KT}) \implies \text{Anti-FAN}_c(\text{SS}) .$$

In the following sections we are going to show that most recursive counterexamples in analysis are equivalent to one of these three, which will also lead to better names (already bracketed).

Although we are going to cite details later on we would like to point out that W. Veldmann has considered equivalences to the existence of a Kleene tree [110].

5.2 Singular Covers

By slightly extending, in an obvious way, the observation at the start of Section 5.1 we can see that Anti-WWKL (k) is equivalent to

> **(KTs(k))** There exists a decidable binary tree T such that
> $$\forall n \in \mathbb{N} : \frac{|\{u \in T \,|\, |u| = n\}|}{2^n} > k , \qquad (5.1)$$
> and
> $$\forall \alpha \in 2^{\mathbb{N}} : \exists n \in \mathbb{N} : \overline{\alpha}n \notin T . \qquad (5.2)$$

Lemma 5.2.1.
$$KTs(k) \implies KTs((2-k)k)$$

Proof. For the following we are going to introduce some notation for slicing sequences.[2] If $u \in 2^{\mathbb{N}}$, $|u| \geqslant m \geqslant n$, then $u[n:m]$ is the sequence $u(n)u(n+1)\cdots u(m)$. Furthermore, $u[n:] = u[n:|u|]$ and $u[:n] = u[1:n]$. Finally, $u[n:-m] = u[n:|u|-m]$ and in particular $u[:-1]$ is the sequence u cut short by the last element. Naturally, apart from the last two, these notations also make sense for infinite sequences.

Let T be a decidable Kleene Tree with property 5.1. Define
$$K_n = |\{u \in T \,|\, |u| = n\}| \cdot 2^n .$$

By 5.1 we know that $K_n \geqslant k2^n$ for every $n \in \mathbb{N}$. The set B of all sequences $u \in 2^*$ such that $u \notin T$, but $u[:-1] \in T$ is decidable. Notice that for every $u \in 2^*$ either $u \in T$ or there exists at exactly one n such that $u[:n] \in B$, which means that
$$S = T \cup \bigcup_{u \in B} \{u * v \,|\, v \in T\} .$$

is a disjoint union. The idea behind this construction is to attach to every leaf of T another copy of T.

As it contains T this set S is still an infinite tree. It also blocks every path: for let $\alpha \in 2^{\mathbb{N}}$ be arbitrary. Since T blocks α there is $n_1 \in \mathbb{N}$ such that $\alpha[:n_1] \in B$. Since also $\alpha[n_1+1:]$ gets blocked there exists n_2 such that $\alpha[n_1+1:n_2] \in B$, which means that $\alpha[:n_2] \notin S$.

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[2] Basically be the Python programming language list notation.

Last, we need to count the nodes at a level n in S. There are finitely many $u_1, \ldots, u_m \in B$ such that $|u_i| \leqslant n$ and for every $w \in 2^n$ either $w \in T$, or there exists a unique i such that $u_i \sqsubseteq w$. For every $1 \leqslant i \leqslant m$ there are $K_{n-|u_i|}$ Elements $w \in 2^n$ with $u_i \sqsubseteq w$ and $w[|u_i|+1:] \in T$, which means there $k_{n-|u_i|}$ Elements $w \in S$ with $u_i \sqsubseteq w$. Furthermore there are exactly $2^n - K_n$ sequences of length n starting with a sequence from B. Altogether:

$$|\{ u \in S \,|\, |u| = n \}| = |\{ u \in 2^n \,|\, u \in T \}| + \sum_{i=1}^{m} |\{ w \in 2^n \,|\, u_i \sqsubseteq w \wedge w[|u_i|+1:] \in T \}|$$

$$= K_n + \sum_{i=1}^{m} K_{n-|u_i|}$$

$$\geqslant K_n + k \sum_{i=1}^{m} 2^{n-|u_i|}$$

$$= K_n + k(2^n - K_n)$$

$$= (1-k)K_n + k2^n$$

$$\geqslant (1-k)k2^n + k2^n = (2-k)k2^n$$

But that is exactly the inequality we wanted. □

Corollary 5.2.2. *$KTs(k)$ is independent of the choice of $k > 0$.*

Proof. For any $k \in (0,1)$ the sequence defined by $x_0 = k$, and $x_n = (2-k)k$ converges to 1. Therefore for $y_n = 1 - x_n$ we have $y_{n+1} = (y_n)^2$. That shows that y_n converges to 0. □

We can actually view a tree T that does not admit infinite paths as an open cover of Cantor space. If we collect all of the finite sequences w_n that are just barely not in T, i.e. $w_n \notin T$ but $\overline{w_n}(|w_n| - 1) \in T$, then these give us basic open sets $U_n = U_{w_n}$ where $U_w = \{ \alpha \in 2^{\mathbb{N}} \,|\, \overline{\alpha}|w| = w \}$. These form a cover of Baire space because for every $\alpha \in 2^{\mathbb{N}}$ we can find n such that w_n is a prefix of α. Moreover, this cover has a small measure in the sense that

$$\sum_{i=1}^{n} \mu(U_n) < k \, ,$$

where $\mu(U_w) = 2^{-|w|}$.[3] This seems paradoxical, since $\mu(2^{\mathbb{N}}) = 1$. However, such a strange topological behaviour is well-known in RUSS [6, Theorem 6.1]: an α-*singular cover* is a sequence of intervals $(J_n)_{n \geqslant 1}$ with rational endpoints such that

SC1 $\sum_{i=1}^{n} |J_n| < \alpha$ for all n,

SC2 any two J_n are disjoint or have only an endpoint in common,

SC3 for any $x \in [0,1]$ there exists $n, m \in \mathbb{N}$ such that the intervals $J_n = [a_n, b_n]$ and $J_m = [a_m, b_m]$ are such that $a_m = b_n$ and $x \in [a_n, b_m]$.

Remark 5.2.3. *Any cover satisfying SC1 cannot be refined to a finite one.*[4]

After all this build-up the following proposition is no surprise.

Proposition 5.2.4. *The following are equivalent*

---

[3] We are not getting into deep waters of constructive measure theory here and are not saying that μ is a measure in the traditional sense.

[4] Regardless of whether by cover we mean open cover or one in the sense of SC3.

1. *KTs(k) for any $0 \leqslant k < 1$*

2. *For $0 < \alpha < 1$ there exists a sequence of open intervals $(J_n)_{n \geqslant 1}$ covering $[0,1]$ and satisfying SC1*

3. *The existence of an α-singular cover of $[0,1]$ for any $0 < \alpha < 1$*

Proof. Let $1 > \alpha > 0$ be arbitrary. Now let T be a tree as in KTs$(1 - \alpha/2)$, and let $(w_n)_{n \geqslant 1}$ be an enumeration of all the finite sequences such that $w_n \notin T$, but $\overline{w_n}(|w_n - 1|) \in T$.

Define, similarly as in Section 3.1 inductively for every $u \in 2^*$ intervals I_u such that

- $I_u = I_{u0} \cup I_{u1}$

- $|I_u| = \frac{1}{2^{|u|}} + \frac{\alpha}{2^{2|u|+1}}$

We claim that $J_n = I_{w_n}$ is the desired sequence of open intervals.

So let $x \in [0,1]$ be arbitrary. Using dependent choice we can construct a sequence $\alpha \in 2^{\mathbb{N}}$ such that $x \in I_{\overline{\alpha}n}$ for all $n \in \mathbb{N}$. Now, since T does not admit infinite paths, there exists m such that $w_m = \overline{\alpha}M$ where $M = |w_m|$. This means that

$$x \in I_{w_m} \subset \bigcup_{n \in \mathbb{N}} I_{w_n} .$$

Now let $n \in \mathbb{N}$ be arbitrary and choose $N = \max\{|w_1|, |w_2|, \ldots, |w_n|\}$. First notice that

$$\frac{1}{2^{|w_i|}} = \sum_{\substack{u \in 2^N \\ w_i \sqsubseteq u}} \frac{1}{2^N}$$

and that for such u we also have $u \notin T$. We have,

$$\begin{aligned}
\sum_{i=1}^{n} |I_i| &= \sum_{i=1}^{n} \left(\frac{1}{2^{|w_i|}} + \frac{\alpha}{2^{2|w_i|+1}} \right) \\
&\leqslant \sum_{i=1}^{n} \frac{1}{2^{|w_i|}} + \sum_{u \in 2^*} \frac{\alpha}{2^{2|u|+1}} \\
&= \sum_{i=1}^{n} \frac{1}{2^{|w_i|}} + \sum_{n \in \mathbb{N}} 2^n \frac{\alpha}{2^{2n+1}} \\
&= \sum_{i=1}^{n} \frac{1}{2^{|w_i|}} + \alpha \sum_{n \in \mathbb{N}} \frac{1}{2^{n+1}} \\
&\leqslant \sum_{i=1}^{n} \sum_{\substack{u \in 2^N \\ w_i \sqsubseteq u}} \frac{1}{2^N} + \frac{\alpha}{2} \\
&\leqslant \sum_{\substack{u \in 2^N \\ u \notin T}} \frac{1}{2^N} + \frac{\alpha}{2} \\
&\leqslant 2^N (1 - (1 - \alpha/2)) \frac{1}{2^N} + \frac{\alpha}{2} = \alpha .
\end{aligned}$$

Thus we have shown that 1 implies 2. Now assume that 2 holds and let $0 < \alpha < 1$ be arbitrary. We will first show that we may assume that the intervals in 2 have rational endpoints. To start, we may assume, Without loss of generality, that α is rational. Now let $I_n = (a_n, b_n)$ a sequence of open intervals covering $[0,1]$ and

satisfying SC1 for $\alpha/2$. Now choose rationals a'_n and and b'_n such that $a'_n \leqslant a_n$, $b_n \leqslant b'_n$, $|a'_n - a_n| < \frac{\alpha}{2^{n+2}}$, and $|b'_n - b_n| < \frac{\alpha}{2^{n+2}}$. Then $I'_n = (a'_n, b'_n)$ are obviously still a cover of $[0,1]$ and for any $n \in \mathbb{N}$

$$\sum_{i=1}^{n} |I'_n| \leqslant \sum_{i=1}^{n} |I_n| + 2\sum_{i=1}^{n} \frac{\alpha}{2^{i+2}} \leqslant \alpha/2 + \alpha/2 .$$

It is clear that by shrinking the intervals I'_n and possibly cutting up into subintervals we can obtain a singular cover (with the same constant α). Thus 3 follows from 2.

Finally let $1 > k \geqslant 0$ and assume that we have an α-singular cover $(J_n)_{n \geqslant 1}$, where $\alpha = 1 - k$. Let I_u be the intervals defined in Section 3.1 for $p = 1/2$. Now define a set T by

$$T = \left\{ u \in 2^* \,\middle|\, I_u \not\subset \bigcup_{i=1}^{|u|} J_i \right\} .$$

Since J_i have rational endpoints the set T is decidable. It is also closed under restriction, since, for $w \sqsubseteq u$ and $u \in T$ we have $I_u \subset I_w$ and

$$\bigcup_{i=1}^{|w|} J_i \subset \bigcup_{i=1}^{|u|} J_i .$$

Therefore, the assumption that $I_w \subset \bigcup_{i=1}^{|w|} J_i$ implies $I_u \subset \bigcup_{i=1}^{|u|} J_i$; a contradiction. Thus $I_w \not\subset \bigcup_{i=1}^{|w|} J_i$, which means that $w \in T$. Hence T is a tree. Now, consider $\alpha \in CS$ arbitrary. By SC3 there exists n, m such that $b_n = a_m$ and $F^{1/2}(\alpha) \in [a_n, b_m]$. Since all the intervals in a singular cover are proper $|b_m - a_n| > 0$, which implies that there exists N such that $I_{\overline{\alpha}N} \subset [a_n, b_m]$. Thus, for $K = \max\{N, n, m\}$ we have $I_{\overline{\alpha}K} \subset \bigcup_{i=1}^{K} J_i$, which implies $\overline{\alpha}K \notin T$. Hence T does not admit infinite paths.

So the only item left to consider is to show that T satisfies Equation 5.1. First notice that by definition of T for $n \in \mathbb{N}$

$$\sum_{\substack{u \notin T \\ |u|=n}} |I_u| \leqslant \sum_{i=1}^{n} |J_i| .$$

So

$$\sum_{\substack{u \in T \\ |u|=n}} \frac{1}{2^{|u|}} = 1 - \sum_{\substack{u \notin T \\ |u|=n}} \frac{1}{2^{|u|}} = 1 - \sum_{\substack{u \notin T \\ |u|=n}} |I_u| = 1 - \sum_{i=1}^{n} |J_i| \geqslant 1 - \alpha = k .$$

Thus Equation 5.1 is fulfilled and therefore KTs(k). $\square$

Corollary 5.2.5. *If there exists a α-singular cover for a specific $0 < \alpha < 0$, then there exists a α'-singular cover for any $0 < \alpha' < 0$.*

Thus the following principle is independent of the choice of α.

> **(SC)**   There exists a singular cover (for some $0 < \alpha < 0$).

5.3   Kleene Trees

As already mentioned in the introduction of Chapter 3, in some schools of constructive mathematics there exists a Kleene tree. In this section we want to investigate equivalences of the existence of such an object.

> **(KT)**   There exists a decidable binary tree T such that
>
> $$\forall n \in \mathbb{N} : \exists u \in T : |u| \geqslant n$$
>
> and
>
> $$\forall \alpha \in 2^{\mathbb{N}} : \exists n \in \mathbb{N} : \overline{\alpha} n \notin T .$$

In the presence of countable choice[5] we can also assume that there is a sequence $(u_n)_{n \geqslant 1}$ with $|u_n| = n$ and $u_n \in T$. The principle KT was also named Anti-FT$_\Delta$ in [43, Section 4.5] for obvious reasons. There, also, parts of the following proposition are proved.

Proposition 5.3.1. *The following are equivalent to KT.*

1. *There exist two compact subsets A, B of a metric space such that*

   $$\forall a \in A : \forall b \in B : d(a,b) > 0$$

   but

   $$\inf \{ d(x,y) : x \in A, y \in B \} = 0.$$

2. *There is a uniformly continuous map $f : 2^{\mathbb{N}} \to \mathbb{R}^+$ with $\inf f(2^{\mathbb{N}}) = 0$.*

3. *There is a uniformly continuous map $f : [0,1] \to \mathbb{R}^+$ with $\inf f([0,1]) = 0$.*

4. *There exists an countable open cover of $2^{\mathbb{N}}$ of basis sets that does not admit a finite subcover.*

Proof. These equivalences can be proved with the same arguments as in the proof of Proposition 3.3.3. □

Proposition 5.3.2. *KT is equivalent to the existence of a homeomorphism $\varphi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ with a continuous modulus of continuity given by a function $\mu : 2^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$; i.e.*

$$\forall \alpha \in 2^{\mathbb{N}}, n \in \mathbb{N} : \overline{\beta} \mu(\alpha,n) = \overline{\alpha} \mu(\alpha,n) \implies \overline{\varphi(\beta)} n = \overline{\varphi(\alpha)} n .$$

Proof. One direction is well known [6, IV.13], at least for the inverse mapping $\mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$, and will be easy to adapt it to our purposes. Assume that KT holds, so assume that T is a decidable infinite tree that blocks ever infinite path. In particular we can find $\{u_1, u_2, \dots\}$ an injective enumeration of all $u \notin T$ such that $\overline{u}(|u| - 1) \in T$. In particular, for every $\alpha \in 2^{\mathbb{N}}$ there exist unique $k_\alpha, m_\alpha \in \mathbb{N}$ such that $\overline{\alpha} k_\alpha = u_{m_\alpha}$. Now define $(\alpha_n)_{n \geqslant 1}$ recursively by

$$\alpha_0 = \alpha ,$$
$$\alpha_{n+1} = \alpha_n [k_{\alpha_n} :] ;$$

that is we keep chopping off unique u_m prefixes of α. Now $\varphi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ can be succinctly defined by

$$\varphi(\alpha)(n) = m_{\alpha_n} .$$

We still need to show that φ is surjective, injective, and has a continuous modulus if continuity. The easiest is surjectivity: if $\gamma \in \mathbb{N}^{\mathbb{N}}$, then

$$\varphi \left(u_{\gamma(0)} * u_{\gamma(1)} * u_{\gamma(2)} * \dots \right) = \gamma .$$

---

[5]Actually countable choice is too much.

It is easy to see, that by the way it is constructed φ is injective. To see that φ has a continuous modulus of continuity let $e \in \mathbb{N}$ and $\alpha \in 2^{\mathbb{N}}$ be arbitrary. Let $N = \sum_{i=0}^{e} k_{\alpha_i}$. Then

$$\alpha = u_{m_{\alpha_0}} * u_{m_{\alpha_1}} * \cdots * u_{m_{\alpha_e}} * \alpha_{e+1} \ ,$$

and for any β such that $\overline{\alpha}N = \overline{\beta}N$ we have that

$$\beta = u_{m_{\alpha_0}} * u_{m_{\alpha_1}} * \cdots * u_{m_{\alpha_e}} * \beta_{e+1} \ .$$

That means that for $i \leqslant e$ we have that $\varphi(\alpha)(i) = \varphi(\beta)(i)$. Hence $\mu(\alpha, e) = \sum_{i=0}^{e} k_{\alpha_i}$ is a modulus of continuity, which by construction only depends on a finite initial prefix of α and therefore is itself continuous.

Conversely let $\varphi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be a homeomorphism between Cantor and Baire space with μ as stated above. Now define
$$T = \{ u \in 2^* \, | \, \forall i \leqslant |u| : \mu(\overline{u}i * 000\ldots, 1) > |i| \} \ .$$

By definition T is decidable and closed under restriction. We claim that it does not admit infinite paths, but is infinite. To see that T does not admit infinite paths let $\alpha \in 2^{\mathbb{N}}$ arbitrary. Then there exists, since μ is continuous itself, M such that $\mu(\alpha, 1) = \mu(\overline{\alpha}M * \beta, 1)$ for all $\beta \in 2^{\mathbb{N}}$. Now let $N = \max\{M, \mu(\alpha)\}$. Then the assumption that $\overline{\alpha}N \in T$ implies that $\mu(\overline{\alpha}N * 000\ldots, 1) > N$. But

$$N \geqslant \mu(\alpha, 1) = \mu(\overline{\alpha}N * 000\ldots, 1) > N$$

a contradiction and hence $\overline{\alpha}N \notin T$.

Assume that n bounds the height of T that is that $\overline{\alpha}n \notin T$ for all $\alpha \in 2^{\mathbb{N}}$. That means that $\mu(\overline{\alpha}n * 000\ldots) \leqslant n$ for $\alpha \in 2^{\mathbb{N}}$, which means that $\varphi(\overline{\alpha}n * 000\ldots)(1) = \varphi(\alpha)(1)$. This in term implies that $\varphi(\alpha)(1) \leqslant \max_{u \in 2^* : |u| = n} \{\varphi(u * 000\ldots)(1)\}$. So φ cannot be surjective; a contradiction. We conclude that T is infinite. $\qquad\square$

Proposition 5.3.3. *KT is equivalent to the existence of a point-wise continuous, fully located function $f : [0, 1] \to \mathbb{R}$ that fails to be uniformly continuous.*

Proof. This follows from Lemma 3.3.10. $\qquad\square$

5.4 Specker Sequences

In the seminal article [104] E. Specker showed that in recursive mathematics there exists an increasing, computable sequence of rationals $(r_n)_{n \geqslant 1}$ in $[0, 1]$ that does not converge to a computable number. More than that, he showed that it does not converge to a computable number in the strong sense that it is computably eventually bounded away from every computable real. Interpreted in RUSS, this shows that the following principle holds there.

> **(SS)** There exists a sequence $(x_n)_{n \geqslant 1}$ in $[0, 1]$ that is bounded away from every point in $[0, 1]$.

It is worth pointing out that we have not forgotten about the fact that Specker's original sequence is rational, as the next lemma clarifies. The fact that Specker's sequence is increasing is discussed in Section 8.4.

Lemma 5.4.1. *If $(x_n)_{n \geqslant 1}$ is a sequence in $[0, 1]$ that is bounded away from every point in $[0, 1]$, then there exists a sequence of rationals with the same property.*

Proof. Straightforward. □

Proposition 5.4.2. *KT implies that there exists an increasing Specker sequence of rationals.*

Proof. The construction, naturally, is a variation of the one in Lemma 3.1.10. Let T be a Kleene tree. Now construct a sequence $(w_n)_{n \geqslant 1} \in T$ such that for all $n \in \mathbb{N}$

1. $|w_n| = n$, and

2. $(u \in T \wedge |u| = n) \implies w_n \leqslant_{\text{lex}} u$.

Notice that then $w_n 0 \leqslant_{\text{lex}} w_{n+1}$. Define $x_n = F^{1/3}(w_n)$. Then $(x_n)_{n \geqslant 1}$ is an increasing sequence of rationals in $[0,1]$. We want to show that it is eventually bounded away from every $x \in [0,1]$. To this end choose $\alpha \in 2^{\mathbb{N}}$ as in Lemma 3.1.1.2. Since T blocks every infinite path there exists $N \in \mathbb{N}$ such that $\overline{\alpha}N \notin T$. Hence, for every $n \geqslant N$ we must have $\overline{\alpha}N \neq \overline{w_n}N$, which means that

$$\left| F^{1/3}(w_n) - F^{1/3}(\alpha) \right| > 3^{-(N+1)} .$$

Now either $\left| F^{1/3}(\alpha) - x \right| < 3^{-(N+2)}$ or $\left| F^{1/3}(\alpha) - x \right| > 3^{-(N+3)}$. In the first case

$$\left| F^{1/3}(w_n) - x \right| > 3^{-N+1}$$

for all $n \geqslant N$. In the second case by Lemma 3.1.1.2 we have that $d(x, F(2^{\mathbb{N}})) > \delta$ for some $\delta > 0$. In both cases $|x_n - x| > \min\{\delta, 3^{-(N+2)}\}$ for all $n \geqslant N$. □

In the following we want to weaken the requirement of decidability on a Kleene tree. Just as a decidable tree is the complement of a decidable set that is closed under extensions, we define a *c-tree* to be the complement of a *c*-set that is closed under extensions. More formally a tree $T \subset 2^*$ is a *c*-tree, if there exists a decidable set $D \subset 2^*$ such that

$$u \in T \iff \exists w \in 2^* : u * w \in D .$$

A *c*-Kleene tree is then simply a Kleene tree, that is a *c*-tree instead of a decidable one. Obviously every Kleene tree is a *c*-Kleene tree.

Proposition 5.4.3. *The following are equivalent.*

1. *There exists a Specker sequence in $[0,1]$.*

2. *There exists a c-Kleene tree.*

3. *There exists a continuous surjection $\varphi : 2^{\mathbb{N}} \to \mathbb{N}$*

4. *There exists a continuous surjection $\Phi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$.*

Proof. The equivalence 1 $\iff$ 2 follows from Lemmas 3.1.9 and 3.1.10 and the above mentioned fact that a *c*-Kleene tree is just the complement of a *c*-bar and vice-versa.

To see that 2 implies 3 let T be a *c*-Kleene tree. Now consider the function $g : 2^{\mathbb{N}} \to \mathbb{N}$ defined by

$$g(\alpha) = \min \{ n \in \mathbb{N} \mid \overline{\alpha}n \notin T \} .$$

The fact that T does not admit infinite paths ensures that this is well-defined. It is also easily seen to be point-wise continuous. Unfortunately it might not be a surjection. However, the image $R = g(2^{\mathbb{N}})$ is

decidable and infinite, which means we can find a surjection $h : R \to \mathbb{N}$. So the composition $\varphi = h \circ g$ is the desired function.

Next, will now show that 3 implies 2: for assume $\varphi : 2^{\mathbb{N}} \to \mathbb{N}$ is a continuous surjection. Let $\alpha_n = \varphi^{-1}(n)$. Using continuity and (unique) countable choice, we can find $w_n \in 2^*$ such that $\varphi(w_n) = n$. Without loss of generality we may also choose w_n such that $|w_n| \geqslant n$, so that $\{w_1, w_2, \dots\}$ is a decidable subset of 2^*. We want to show that

$$u \in T \iff \exists w \in 2^* : uw \in \{w_1, w_2, \dots\}$$

is a c-Kleene tree. It is clear that it is a c-tree by definition, and that it is infinite, since $w_n \in T$ and $|w_n| \geqslant n$ for all $n \in \mathbb{N}$. So it remains to show that T does not admit any infinite path. To this end let $\alpha \in 2^{\mathbb{N}}$ be arbitrary. Set

$$m = \varphi(\alpha) \, ,$$

and, by continuity, choose M such that $\overline{\alpha}M = \overline{\beta}M$ implies that $\varphi(\beta) = m$; which implies that for all such β and $i > m$ we must have $\overline{\alpha_i}M \neq \overline{\beta}M$. Set $k = \max\{M, |w_1|, \dots, |w_m|\} + 1$. Now assume that $\overline{\alpha}k \in T$; that is that there is $w \in 2^*$ and $j \in \mathbb{N}$ such that $\overline{\alpha}k * w = w_j$. Since $|\overline{\alpha}k * w| \geqslant k$ we must have $j \geqslant k$. But for all $j \geqslant k$ also $j > m, M$ and we have

$$\overline{w_j}M = \overline{\alpha_j}M \neq \overline{\alpha}M = \overline{\overline{\alpha}k * w}M$$

and therefore $w_j \neq \overline{\alpha}k * w$. That means that $\overline{\alpha}k \notin T$.

It is clear that 4 implies 3. Conversely assume $\varphi : 2^{\mathbb{N}} \to \mathbb{N}$ is a continuous surjection. Now consider a bijective pairing function $\pi = (\pi_1, \pi_2) : \mathbb{N} \to \mathbb{N}^2$. Define $\Phi : 2^{\mathbb{N}} \to (\mathbb{N} \to \mathbb{N})$ by

$$\Phi(\alpha)(m) = \varphi(\alpha \circ (\pi^{-1}(m, \cdot))) \, .$$

We want to show that Φ is surjective. So let $\gamma \in \mathbb{N}^{\mathbb{N}}$ be arbitrary. Using countable choice and continuity we can find $w_n \in 2^*$ such $\varphi(\alpha_n) = \gamma(n)$, where $\alpha_n = w_n * 000 \dots$. Now for $\beta \in 2^{\mathbb{N}}$ defined by

$$\beta(i) = \alpha_{\pi_1(i)}(\pi_2(i))$$

we have that $\beta \circ (\pi^{-1}(m, \cdot)) = \alpha_m$ and therefore $\Phi(\beta)(m) = \varphi(\alpha_m) = \gamma(m)$. Thus $\Phi(\beta) = \gamma$, which means Φ is surjective. It is also straightforward to show that it is continuous. $\square$

The last proposition together with Proposition 5.3.2 raises an interesting question.

Question 9. *Is the existence of a bijection $\mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ equivalent to SS or KT (or neither)?*

Analysing the proofs of the two propositions mentioned it seems as if KT is too strong an assumption, whereas SS is just barely not enough.[6]

Proposition 5.4.4. *The following are equivalent*

1. *There exists a Specker sequence in $[0,1]$.*

2. *There exists a point-wise continuous function $f : [0,1] \to \mathbb{R}$ that is unbounded.*

3. *There exists a surjection $f : [0,1] \to \mathbb{R}$*

4. *There exists a point-wise continuous function $f : [0,1] \to [0,1]$ that is not uniformly continuous.*

5. *There exists a point-wise continuous function $f : [0,1] \to \mathbb{R}^+$ with $\inf f = 0$.*

---

[6]A good candidate might actually be iSS as in Section 8.4.

Proof. The equivalences between 1, 2, and 4 are proved in [43, Proposition 4.5.2]. It is also clear that the proof between 1 and 2 there can be extended to show the equivalence of 1, 3. Finally, if $f : [0,1] \to \mathbb{R}^+$ is a function with $\inf f = 0$, then $1/f$ is well defined, point-wise continuous and unbounded. Conversely if $f : [0,1] \to \mathbb{R}$ is point-wise continuous and unbounded, then $h = 1/\max\{|f|,1\}$ is also point-wise continuous and such that $h(x) > 0$ for all $x \in [0,1]$ but $\inf h = 0$. Thus $2 \iff 5$. $\qquad\square$

Chapter 6

Relationships Between the Principles

6.1 Basic Relations

It has long been known that WKL implies FAN_Δ [69]. In [9] Berger has shown that it also implies FAN_c. This result in turn was again slightly improved upon in [45] where we showed that it also implies UCT. For completeness' sake we will include the proof here.

Lemma 6.1.1. *LLPO/ WKL implies that the image of a sequentially continuous map $f : 2^{\mathbb{N}} \to \mathbb{R}$ is order located.*

Proof. Let $f : 2^{\mathbb{N}} \to \mathbb{R}$ be a map and $a < b$ two arbitrary real numbers. Using countable choice and LLPO[1] fix $\mu : 2^* \to \{0,1\}$ such that

$$\mu(u) = 0 \implies f(u * 000\ldots) \leqslant b\,,$$
$$\mu(u) = 1 \implies f(u * 000\ldots) \geqslant b\,.$$

Let $\preceq$ denote the decidable order on 2^* such that

$$u \preceq v \iff |u| < |v| \vee (|u| = |v| \wedge u \leq_{Lex} v),$$

where $\leq_{Lex}$ is the lexicographic order. For convenience and readability's sake we define decidable predicates Λ and F on 2^* by

$$\Lambda(u) \iff \forall v \in 2^* : (v \preceq u \implies \mu(v) = 0)\,,$$
$$F(u) \iff \mu(u) = 1 \wedge (\forall v \in 2^* : \mu(v) = 1 \implies u \preceq v)\,.$$

In words, if we imagine searching for a finite binary sequence u with $\mu(u) = 1$, then as long as we have not been successful Λ holds and F only holds for the first such sequence. Now define a decidable set T by

$$T = \{u \in 2^* \,|\, \Lambda(u)\} \cup \{u \in 2^* \,|\, \exists v : (F(v) \wedge u = v * 0 * \cdots * 0)\}\,.$$

Notice that for every $n \in \mathbb{N}$ either $\mu(u) = 0$ for all $u \in 2^*$ with $|u| \leqslant n$ or there exists $v \in 2^*$ such that $|v| \leqslant n$ and $F(v)$ holds.

Therefore, if $u \in T$ either $\Lambda(u)$ holds or there exists v with $F(v)$ such that $u = v * 0 * \cdots * 0$. In the first case also $\Lambda(\bar{u}(n))$ for all $1 \leq n \leq |u|$, whence $\bar{u}(n) \in T$. In the second case and if $1 \leq n < |v|$, then

---

[1]This application of LLPO is actually not necessary, since the proof still works with a slightly modified definition of μ (namely $\mu(1) \implies f(u) > b - (b-a)2^{-|u|}$). It is, however, convenient and since LLPO is used later in the proof, nothing seems to be gained from dropping it here.

$\overline{v}(n) = \overline{u}(n)$ and therefore $\Lambda(\overline{u}(n))$ holds which in return implies that $\overline{u}n \in T$. Finally in the second case and if $|v| \leq n \leq |u|$, then $\overline{u}n = v * 0 * \cdots * 0$ which means that $\overline{u}n \in T$. This shows that T is closed under restriction and hence is a tree. It also contains elements of arbitrary lengths and therefore, by WKL, this tree admits an infinite path $\alpha \in 2^{\mathbb{N}}$. Now either $f(\alpha) > a$ and we are done, or $f(\alpha) < b$. We claim that in this second case $\mu(u) = 0$ for all $u \in 2^*$. For assuming that there exists $u \in 2^*$ with $\mu(u) = 1$, we can find $v \preceq u$ such that $F(v)$ holds, which means that $\alpha = v * 0 * \ldots$ by the construction of T. But this leads to the contradiction

$$b \leqslant f(v * 000\ldots) = f(\alpha) < b.$$

By sequential continuity, the fact that $f(u * 000\ldots) \leqslant b$ for all $u \in 2^*$ now implies $f(\alpha) \leqslant b$ for all $\alpha \in 2^{\mathbb{N}}$. □

Lemma 6.1.2. *FAN$_c$ implies that an order located image of a point-wise continuous map $f : 2^{\mathbb{N}} \to \mathbb{R}$ is bounded.*

Proof. Define a sequence $(\alpha_n)_{n \geqslant 1}$ in $2^{\mathbb{N}} \cup \{\omega\}$ such that

$$\alpha_n \in 2^{\mathbb{N}} \implies f(\alpha_n) > n - 1 ,$$
$$\alpha_n = \omega \implies \forall \alpha \in 2^{\mathbb{N}} : f(\alpha) \leqslant n .$$

Since f is point-wise continuous it is locally bounded. Hence $(\alpha_n)_{n \geqslant 1}$ is eventually bounded away from every point in $2^{\mathbb{N}}$. In [43] it is shown that FAN$_c$ implies what is know as the anti-Specker property for $2^{\mathbb{N}}$: namely that every sequence in $2^{\mathbb{N}} \cup \{\omega\}$ that is bounded away from every point in $2^{\mathbb{N}}$ is eventually bounded away from the entire set. So there exists N such that $\alpha_N = \omega$, which means that f is bounded. □

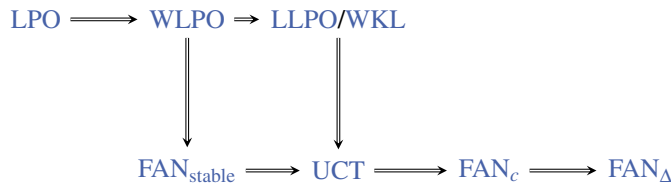**Proposition 6.1.3.** *LLPO/WKL implies UCT .*[2]

Proof. J. Berger's work in [9] shows that LLPO $\implies$ FAN$_c$. So combining the previous lemmas we get that under the assumption of LLPO every point-wise continuous map $f : 2^{\mathbb{N}} \to \mathbb{R}$ is bounded. Moreover, this last statement was shown to be an equivalent of UCT in [29]. □

This result enables us to replace "uniformly continuous" by "point-wise continuous" thereby improving the well known characterisation of WKL [63].

Corollary 6.1.4. *WKL is equivalent to the statement that every point-wise continuous map on a compact space attains its minimum.*

More general, this means that we can replace "uniformly continuous" by "point-wise continuous" in any of the equivalences to LPO, WLPO, and LLPO in Chapter 1.

It is also clear that WLPO implies FAN$_{\text{stable}}$, so we get the following diagram of implications:

$$\text{LPO} \implies \text{WLPO} \Rightarrow \text{LLPO/WKL}$$

$$\text{FAN}_{\text{stable}} \implies \text{UCT} \implies \text{FAN}_c \implies \text{FAN}_\Delta$$

There is a possibility to improve on these results:

Question 10. *Does WKL imply FAN$_{\text{stable}}$ or FAN$_{\Pi^0_1}$?*

---

[2]M. Hendtlass has given a direct proof of this result (unpublished). There is also a generalisation [60, Proposition 30].

6.2   Kripke's Schema and the Principle of Finite Possibility

Kripke's Schema, somewhat accepted by intuitionists, states that

> **(KS)**   for every statement φ there exists a binary sequence $(a_n)_{n\geqslant 1}$ such that
>
> $$\varphi \iff \exists n \in \mathbb{N} : a_n = 1 . \qquad (6.1)$$

This amounts to saying that every statement is simply existential. The philosophy behind it is that, if there is a proof for φ, somebody will eventually find it. Kripke's Schema was introduced by Myhill [90] to capture the idea of Brouwer's creating subject and to formalise the rejection of Markov's principle see Proposition 6.2.1. A weakening of KS which restricts the complexity of the formulas φ is the Principle of Finite Possibility (PFP). It was introduced by Mandelkern in [82] (also named the Principle of Inverse Decision (PID) in one of Mandelkern's later papers).

> **(PFP)**   For every binary sequence $(a_n)_{n\geqslant 1}$ there exists a binary sequence $(b_n)_{n\geqslant 1}$ such that
> $$\forall n \in \mathbb{N} : a_n = 0 \iff \exists n \in \mathbb{N} : b_n = 1 . \qquad (6.2)$$

To reject MP the following weaker version is in fact sufficient

> **(WPFP)**   For every binary sequence $(a_n)_{n\geqslant 1}$ there exists a binary sequence $(b_n)_{n\geqslant 1}$ such that
> $$\forall n \in \mathbb{N} : a_n = 0 \iff \neg\forall n \in \mathbb{N} : b_n = 0 . \qquad (6.3)$$

Trivially

$$\text{KS} \implies \text{PFP} \implies \text{WPFP} .$$

WPFP was also named WPID (weak principle of Inverse Decision) by Mandelkern. The "rejection" of MP using WPFP is of course only convincing if one agrees with the latter and disagrees with LPO.

Proposition 6.2.1. *WPFP $+$ MP $\iff$ LPO*

Note that a proof of this fact has already been sketched in [82, page 258]. It will, however, also follow from Proposition 6.2.3 below.

Proposition 6.2.2. *KS is equivalent to the statement that every inhabited subset of $\mathbb{N}$ is countable.*

Proof. Let φ be any syntactically correct closed statement and let

$$A = \{ x \in \mathbb{N} \,|\, x = 0 \vee (x = 1 \wedge \varphi) \} .$$

Then A is inhabited, since $0 \in A$. If A is countable there exists a, without loss of generality binary, sequence $(a_n)_{n\geqslant 1}$ such that $A = \{ a_n \,|\, n \in \mathbb{N} \}$. In particular, Equation 6.1 holds. Conversely, if A is an inhabited subset of $\mathbb{N}$ and Kripke's schema holds, then for every $m \in \mathbb{N}$ there exists a binary sequence $(a_{m,n})_{n\geqslant 1}$ such that

$$m \in A \iff \exists n \in \mathbb{N} : a_{m,n} = 1 .$$

Using countable choice we can write this as a double sequence. Now, using a standard bijection $\pi : \mathbb{N} \to \mathbb{N}^2$, it is easy to see that the function

$$f(n) = \begin{cases} \pi(n)_1 & \text{if } a_{\pi(n)} = 1 \\ a & \text{otherwise,} \end{cases}$$

where a is an arbitrary fixed element of A, enumerates A. □

Exercise 1.17 [37, page 21] requires one to fill in the (essentially above proposition) easy proof that the statement that "every inhabited subset of $\mathbb{N}$ is countable" implies PFP. As M. Hendtlass has pointed out[3] LLPO + WPFP $\iff$ WLPO. Although on first glance WPFP seems to be the perfect puzzle piece bridging the gap between LLPO and WLPO it is somewhat to strong as the following result shows. We remind the reader that LLPO is strictly stronger than $MP^{\vee}$.

Proposition 6.2.3. *WPFP* + $MP^{\vee} \iff$ *WLPO* .[4]

Proof. Let $(a_n)_{n\geqslant 1}$ be a binary sequence and let $(b_n)_{n\geqslant 1}$ be as in 6.3. Then

$$\neg \forall n \in \mathbb{N} : (a_n = 0 \wedge b_n = 0) .$$

Thus by $MP^{\vee}$ either $a_n = 0$ for all $n \in \mathbb{N}$ or $b_n = 0$ for all $n \in \mathbb{N}$. In the latter case it is impossible that $a_n = 0$ for all $n \in \mathbb{N}$. Thus WLPO holds.

Conversely, it is clear that WLPO implies $MP^{\vee}$. To see that it also implies WPFP let $(a_n)_{n\geqslant 1}$ be a binary sequence. By WLPO either $a_n = 0$ for all $n \in \mathbb{N}$ or there it is impossible that $a_n = 0$ for all $n \in \mathbb{N}$. In the first case set $(b_n)_{n\geqslant 1} \equiv 1$, in the second case set $(b_n)_{n\geqslant 1} \equiv 0$. It is clear that the so defined sequence satisfies Equation 6.3. □

The following can be found in [81].

Proposition 6.2.4. *KS is equivalent to the following*

1. *Every open subspace of a separable metric space is separable.*

2. *Every open subset of a separable metric space is a countable union of open balls.*

6.3 Collapsing the Fan Theorems

As we have seen in Proposition 4.2.1 under the assumption of BD-N we have $\text{FAN}_c \implies \text{FAN}_{\Pi_1^0}$. This result was first proved in [43]. It is still interesting to give a direct proof, in order to understand the subtle difference between FAN_c and $\text{FAN}_{\Pi_1^0}$. J. Berger has given such a proof, however has not published it. The following proof of $\text{FAN}_c \implies \text{FAN}_{\Pi_1^0}$ follows basically Berger's argument.

Proposition 6.3.1. *BD-N implies that* $FAN_c \implies FAN_{\Pi_1^0}$.

Proof. Assume B is a bar and such that there are decidable B_n such that $B = \cap B_n$, and is such that B is closed under extensions. Hence the set

$$A = \left\{ n \in \mathbb{N} \,|\, \exists u \in 2^* : |u| = n \wedge u \notin B \right\}$$

is countable. We want to show that it is also pseudobounded. To this end let $(a_n)_{n\geqslant 1}$ be a sequence in A. By definition there exists a sequence $(w_n)_{n\geqslant 1}$ in 2^* such that $|w_n| = a_n$ and $w_n \notin B$. The complement C of

$$\left\{ w_n \,|\, a_n \geqslant n \right\}$$

is decidable. We claim that

$$D = \left\{ u \in 2^* \,|\, \forall v \in 2^* : u * v \in C \right\}$$

---
[3]Personal correspondence 4. Aug. 2010
[4]This was also pointed out in [60, Proposition 25].

is a bar. But we know that $\overline{C} \subset \overline{B}$, and since C is decidable we can reverse this inclusion to see that $B \subset C$. This immediately tells us that C is a bar that is closed under extensions, which in turn implies that D is a bar. Applying FAN$_c$ we get a uniform bound N for D. Now there cannot be $i \geqslant N$ such that $a_i \geqslant i$: in that case we would have w_i such that $|w_i| \notin C$ and $|w_i| \geqslant N$. But then also $w_i \notin D$, which is a contradiction to N being a uniform bound for D. Altogether we get that A is pseudobounded. Now we can use BD-N to conclude that A is bounded, which immediately translates into B being uniform. $\qquad\square$

Remark 6.3.2. *Notice that the previous proof relies on the assumption that B is closed under extension and therefore does not work for* FAN$_{stable}$.

Since BD-N is a very weak principle already FAN$_{\Pi_1^0}$ and FAN$_c$ are intuitively "close". Since UCT lies between these two, it is, still intuitively speaking, even closer to FAN$_c$. As a matter of fact the author has spend considerable time during his PhD, trying to prove FAN$_c \implies$ UCT and has to admit to sometimes believe to actually having found such a proof. However, all these proofs turned out to be faulty, and often relied on implicitly using the following harmless looking choice principle (majorised choice).

> **(MC)** For every point-wise continuous function $f : 2^{\mathbb{N}} \to \mathbb{R}$ there exists a point-wise continuous function $\tilde{f} : 2^{\mathbb{N}} \to \mathbb{N}$ such that
>
> $$\forall \alpha \in 2^{\mathbb{N}} : f(\alpha) \leqslant \tilde{f}(\alpha) .$$

Proposition 6.3.3. *UCT $\iff$ FAN$_c$ + MC*

Proof. One direction is clear. Conversely it suffices to show (Proposition 3.5.2) that any point-wise continuous $f : 2^{\mathbb{N}} \to \mathbb{R}$ is bounded. So consider such a function f. By MC there exists a point-wise continuous $\tilde{f} : 2^{\mathbb{N}} \to \mathbb{N}$ that bounds f. By FAN$_c$ this function is uniformly continuous (Proposition 3.4.1). That means that $\tilde{f}$ is bounded, and therefore so is f. $\qquad\square$

Question 11. *Is there a (topological) model in which MC fails?*

As already mentioned, Brouwer never distinguished between different version of the fan theorem, since he assumed the principle of continuous choice. Following the notation of [36] continuous choice can be split into two parts.

> **(CC)** Continuous choice.
>
> **CC(1)** Any function from $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is point-wise continuous.
>
> **CC(2)** If $P \subset \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$, and for each $\alpha \in \mathbb{N}^{\mathbb{N}}$ there exists $n \in \mathbb{N}$ such that $(\alpha, n) \in P$, then there is a function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ such that $(\alpha, f(\alpha)) \in P$ for all $\alpha \in \mathbb{N}^{\mathbb{N}}$.

Proposition 6.3.4. *Under the assumption of CC we have FAN$_\Delta \implies$ FAN$_{full}$.*

Proof. Let $B \subseteq 2^*$ be an arbitrary bar. Let $P = \{ (\alpha, n) \,|\, \overline{\alpha}n \in B \}$ and apply CC (2). So there is a function $f : 2^{\mathbb{N}} \to \mathbb{N}$ such that $\overline{\alpha}f(\alpha) \in B$ for all α. Now define

$$B' = \{ u \in 2^* \,|\, f(u * 0 \dots) \leqslant |u| \} .$$

B' is decidable. It is also a bar, since for $\alpha \in 2^{\mathbb{N}}$ we have, by CC (1) that f is continuous at α; that is there exists N such that $\overline{\alpha}N = \overline{\beta}N$ implies that $f(\alpha) = f(\beta)$. Now let $M = \max\{N, f(\alpha)\}$. Then

$$f(\overline{\alpha}M * 000 \dots) = f(\alpha) \leqslant M .$$

So $\overline{\alpha}M \in B'$.

Applying FAN_Δ we get a uniform bound K for B'. It is easy to see that this bound K is also a uniform bound for B. $\square$

Proposition 6.3.5. *Under the assumption of KS we have $FAN_\Delta \implies FAN_{full}$.*

Proof. If B is an arbitrary bar, then, using KS, there exists, for every $u \in 2^*$ a sequence $(a_n^u)_{n \geqslant 1}$ such that

$$u \in B \iff \exists n \in \mathbb{N} : a_n^u = 1 .$$

So B is countable (notice that any bar is inhabited, since $\overline{000\ldots}n$ must be in B for some n). By Lemma 3.3.2 there exists a decidable bar B' that is uniform only if B is. Hence, using FAN_Δ, is enough to give us a uniform bound for B. $\square$

Proposition 6.3.6. *Under the assumption of PFP we have $FAN_\Delta \implies FAN_{stable}$.*

Proof. The proof is similar to the one above. Starting with a stable bar B we can use PFP to conclude that B is countable. Using Lemma 3.3.2 and FAN_Δ again, we can conclude that B is uniform. $\square$

Summarising all these implications in a diagram we get:
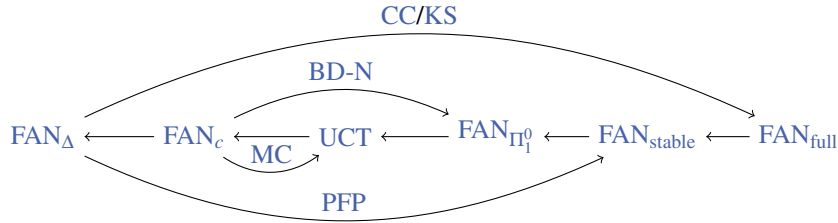
6.4 Other Implications

Proposition 6.4.1. *WMP implies that LPO and WLPO are equivalent.*

Proof. Consider $x \in \mathbb{R}$, such that $x \geqslant 0$. By WLPO we know that either $x = 0$ or $\neg(x = 0)$; or with the notation introduced in Chapter 2 whether $x = 0$ or $0 \ll x$. In the second case we can, using WLPO again, for any $y \in \mathbb{R}$ decide whether

$$z = 0 \vee \neg(z = 0) ,$$

where $z = \max\{x - y, 0\}$. In the first case $z \leqslant 0$, which implies that $x \leqslant y$. Since $0 \ll x$, also $0 \ll y$ (Lemma 2.0.1). In the second case $0 \ll z$, which implies that $y \ll x$. Altogether we can decide, for arbitrary $y \in \mathbb{R}$ whether

$$0 \ll y \vee y \ll x .$$

So, using WMP, we can conclude that $x > 0$; and hence LPO holds. $\square$

Remark 6.4.2. *Therefore WMP also implies that ¬LPO and ¬WLPO are equivalent. Here's an analytical proof for the latter: We only need to show that ¬LPO implies ¬WLPO. So assume ¬LPO and WLPO. Hence there exists a function $f : [0, 1]$ such that $f(0) = 0$ and $f(x) = 1$ for $\neg(x = 1)$. By WMP this function is strongly extensional. Using Ishihara's tricks [46] LPO holds; a contradiction.*

A principle that collapses WLPO and LLPO is PFP.

Proposition 6.4.3. *WPFP implies that WLPO and LLPO are equivalent.*

Proof. Let $(a_n)_{n \geqslant 1}$ be a binary sequence with at most one 1. By WPFP there exists a binary sequence $(b_n)_{n \geqslant 1}$, such that

$$\forall n \in \mathbb{N} : a_n = 0 \iff \neg(\forall n \in \mathbb{N} : b_n = 0) \ .$$

We may also, without loss of generality assume that $(b_n)_{n \geqslant 1}$ contains at most one 1. It is, furthermore easy to see that it cannot be the case that both sequences contain a 1. So we can use LLPO to decide whether $\forall n \in \mathbb{N} : a_n = 0$ or $\forall n \in \mathbb{N} : b_n = 0$. In the second case $\neg(\forall n \in \mathbb{N} : a_n = 0)$ and hence WLPO holds. $\square$

Proposition 6.4.4. *CC (1) implies BD-N.*

Proof. A simple consequence of Proposition 4.1.1. $\square$

6.5 The Big Picture

As a handy overview of the relationship between most of the principles discussed we include the following diagram. Dotted lines indicate contradictions.
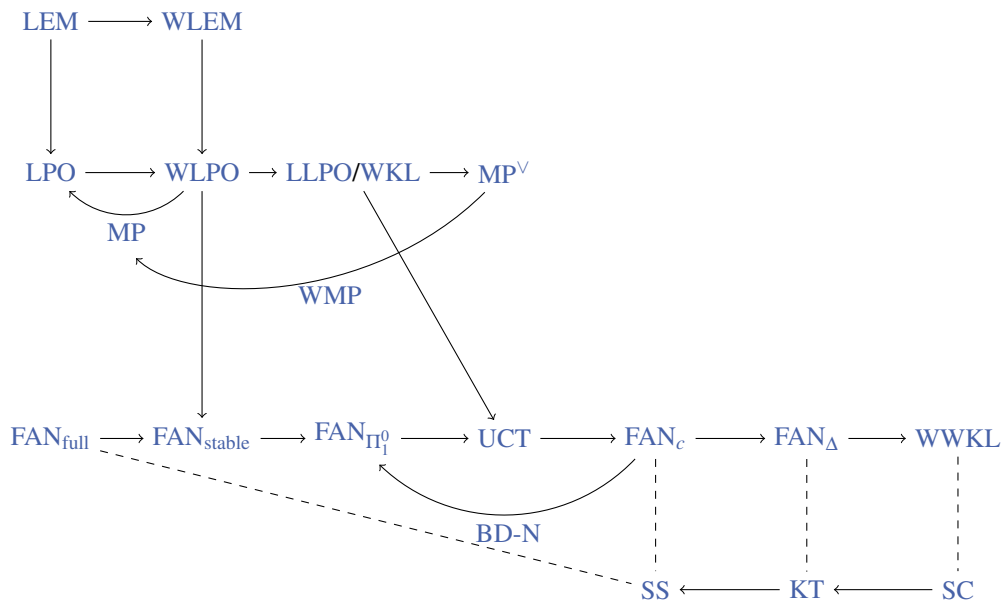
Chapter 7

Separating Principles

7.1 The Big Three

The easiest and most convenient way to see that principle *A* does not imply principle *B*, or more general, that theorem *T* is *not provable* in BISH is to show that theorem *T* is *false* in classical mathematics (CLASS), Brouwer's intuitionism (INT), or in Markov's recursive school of mathematics (RUSS). The view that all three of these varieties of mathematics can be seen as models of BISH has slowly evolved over the years until it was cemented by the publication of [36]. Of course, we use the term model somewhat loosely here, since none of BISH, CLASS, INT, or RUSS are fully formalised systems.[1] Nevertheless, experience has shown that any proof given in BISH can be immediately interpreted and is acceptable by a practitioner of any of the three varieties. We should stress that one should not make the mistake of characterising BISH as the intersection of these three varieties, since there are statements namely BD-N, WMP, and possibly others, that are acceptable in CLASS, INT, and RUSS, but for which there is no compelling reason to accept them in BISH. The three varieties are all well understood so they immediately show many separations between the principles we have discussed so far. For example MP does not imply FAN_Δ, since the first one is true in RUSS, but the second one is false. Many more separations can be read off the diagram below.

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[1] Is countable choice accepted in BISH? Is Kripke's Schema part of INT? Is the continuum hypothesis true in CLASS?
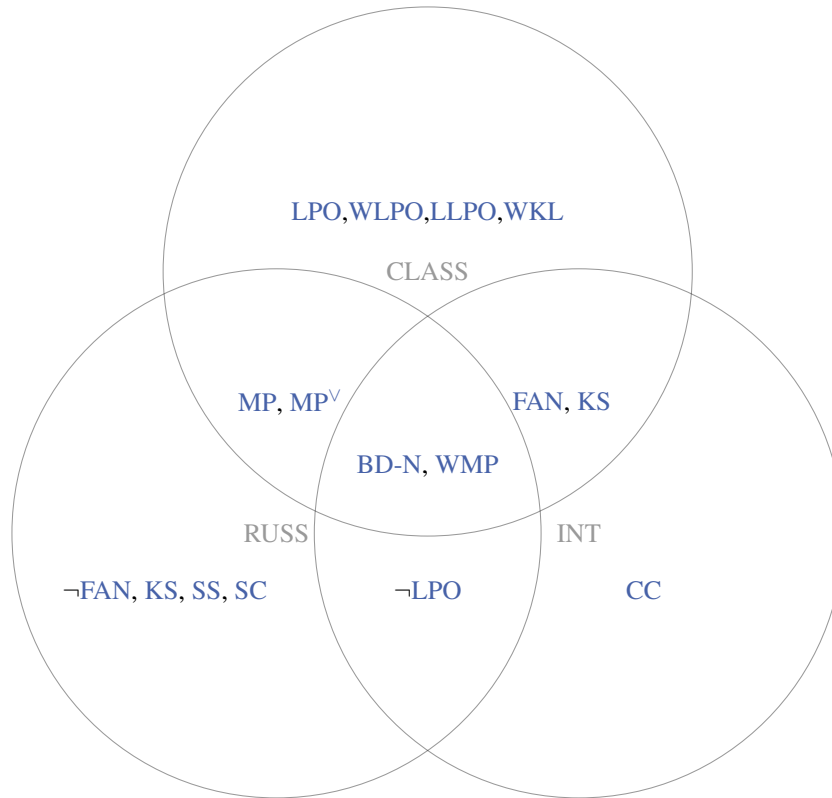
7.2 Topological and Heyting-valued Models

Topological models are a natural setting to interpret formalised intuitionistic theories. By "intuitionistic" we mean theories using intuitionistic logic; it is worth noting though, that topological models also have a distinct intuitionistic flavour a'la Brouwer. For example they all validate FAN$_{full}$ (Proposition 7.2.16). Even though they have a long history starting with several publications around 1970 [100, 101, 58, 54], according to our personal judgement, there is no good introduction available (the subsection in van Dalen's chapter on intuitionistic logic in [56] is probably the most accessible text). This section will not remedy this situation; its aim is to give a decent overview and starting point for researchers working in constructive reverse mathematics. To poach a famous booktitle the theme for this section is "Topological models for the working constructivist."

Notice that, for this section, we work with classical logic in the meta-theory.[2]

Propositional Logic

The basic idea of topological models is to use open sets the truth values. As usual, the propositional case is a lot easier and cleaner to deal with than the predicate case.

A topological model for propositional intuitionistic logic consists of a topological space (T, τ), and a function $[\![\cdot]\!]$ that maps all propositional symbols of the underlying language to elements of τ and is such

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[2]This is almost unavoidable for our purposes. For a lot of omniscience principles we have the rule of thumb that if the principle holds in the model it also holds in the meta-theory. If one was only interested in principles from INT one could potentially develop topological models in BISH (or INT). Notice, however, that then one would have to deal with problems such as Lemma 7.2.1 not being available.

that $[\![\bot]\!] = \emptyset$. We can then extend this function to one defined on all propositional formulas by setting

$$[\![A \wedge B]\!] = [\![A]\!] \cap [\![B]\!] \, ,$$
$$[\![A \vee B]\!] = [\![A]\!] \cup [\![B]\!] \, ,$$
$$[\![A \rightarrow B]\!] = \text{Int}\left(([\![A]\!])' \cup [\![B]\!] \right) \, .$$

We assume, as usual, that negation is defined as $\neg\varphi \equiv \varphi \rightarrow \bot$, so

$$[\![\neg\varphi]\!] = \text{Int}\left(([\![\varphi]\!])' \right) \, .$$

We say that *T models φ*, if $[\![\varphi]\!] = T$. In this case we use the notation $T \Vdash \varphi$. As common, the interpretation $[\![\cdot]\!]$ should be clear from the context and is therefore omitted from the notation.

Lemma 7.2.1. *$T \Vdash \varphi \rightarrow \psi$ if and only if we have $[\![\varphi]\!] \subset [\![\psi]\!]$*

Proof. Straightforward. Notice, however, that we need LEM to prove that $[\![\varphi]\!] \subset [\![\psi]\!]$ implies $T \Vdash \varphi \rightarrow \psi$. $\qquad\qquad\square$

One can easily show that we have soundness, that is if φ is a propositional formula that is derivable in intuitionistic logic from a set of propositional formulae Γ, i.e. $\Gamma \vdash_i \varphi$, then $T \Vdash \varphi$ for any topological space T that models every formula in Γ. Therefore, we can use topological models to show the unprovability of many statements.

Proposition 7.2.2. *LEM and WLEM are not derivable in intuitionistic logic.*

Proof. Let $T = \mathbb{R}$ with the usual topology. Furthermore let $[\![\cdot]\!]$ be such that $[\![P]\!] = (-\infty, 0)$. Then $[\![\neg P]\!] = (0, \infty)$ and therefore $[\![\neg\neg P]\!] = (-\infty, 0)$. However $\mathbb{R} \neq [\![\neg P \vee \neg\neg P]\!] = (-\infty, 0) \cup (0, \infty)$. $\qquad\square$

This space T is actually not the simplest model showing that WLEM is not derivable in intuitionistic logic. Such a model must contain at least three points: in fact, if a space only contains two points there are only three topologies: the trivial one, the discrete one, and the Sierpinski space $\Sigma = (\{0,1\}, \{\emptyset, \{1\}, \{0,1\}\})$. The trivial and the discrete one satisfy LEM, the Sierpinski one validates WLEM but not LEM. Hence Σ shows that WLEM does not imply LEM.

The simplest topology not validating WLEM is $T_2 = (\{1,2,3\}, \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\})$. Then, if P is a propositional symbol such that $[\![P]\!] = \{1\}$ we get that

$$[\![\neg P]\!] = \text{Int}\left((\{1\})' \right) = \{2\}$$

and similarly

$$[\![\neg\neg P]\!] = \text{Int}\left((\{2\})' \right) = \{1\} \, .$$

Thus

$$T_2 \nVdash \neg P \vee \neg\neg P \, .$$

In this sense we can talk about $\mathbb{R}$ or Σ as a topological model being a *counterexample* to LEM. However, we have to be careful that there are two ways a statement can fail to hold. We will say that T is a *weak counterexample* for φ if $T \nVdash \varphi$. It is a *strong counterexample* if $T \Vdash \neg\varphi$. Since we can prove $\vdash_i \neg(\varphi \wedge \neg\varphi)$ we will never be able to find a strong counterexample to LEM or WLEM. Nevertheless for weaker principles such as LPO we can, as we will see, find topological models that are strong counterexamples.

Predicate Logic

To extend the topological interpretation to predicate logic we also need some universe $\mathscr{U}$ to interpret constants and variables. A predicate $[\![P(x_1,\ldots,x_k)]\!]$ should be mapped to a function $\mathscr{U}^k \to \tau$. It is convenient for $d \in \mathscr{U}$ and a formula $\varphi(x)$ to simply write $[\![\varphi(d)]\!]$ for the truth value of $[\![\varphi(x)]\!]$ evaluated at d.

Since we want to interpret equality, we also assume that $[\![x = y]\!]$ maps to a function $\mathscr{U}^2 \to \tau$, such that

$$[\![a = b]\!] = [\![b = a]\!]$$

and

$$[\![a = b]\!] \cap [\![b = c]\!] \subset [\![a = c]\!] \ .$$

Now all predicate symbols $P(x_1,\ldots,x_k)$ should be mapped to a function $[\![P(x_1,\ldots,x_k)]\!] : \mathscr{U}^k \to \tau$ such that equality is respected; that is

$$[\![x = y]\!] \cap [\![P(x)]\!] = [\![x = y]\!] \cap [\![P(y)]\!] \ .$$

Remark 7.2.3. *We can extend every predicate evaluation that is defined on $\mathscr{V} \subset \mathscr{U}$ and that respects the equality.*

Proof. Assume that $k \in \mathbb{N}$ is fixed. Now extend $[\![\varphi(\vec{x})]\!] : \mathscr{V}^k \to \tau$ to a function $\mathscr{U}^k \to \tau$ by setting

$$[\![\varphi(\vec{b})]\!] = \bigcup_{\vec{a} \in \mathscr{V}^k} \left([\![\vec{a} = \vec{b}]\!] \cap [\![\varphi(\vec{a})]\!] \right) \ .$$

It is routine to check that this definition is in fact an extension and that it respects equality. $\square$

Finally we can give the interpretation of the quantifiers:

$$[\![\exists x : \varphi(x)]\!] = \bigcup_{d \in \mathscr{U}} [\![\varphi(d)]\!] \ ,$$

$$[\![\forall x : \varphi(x)]\!] = \mathrm{Int} \left(\bigcap_{d \in \mathscr{U}} [\![\varphi(d)]\!] \right) \ .$$

All of this naturally extends to first order predicate logic with types. *Since it will not affect the following, we will not distinguish between having a logic with different types or having some sort of set theory.*

Again one can easily show soundness, by induction over deductions. Notice that in general we do not have an existence property, that is we do not have that if $T \Vdash \exists x : \varphi(x)$, then there exists $d \in \mathscr{U}$ such that $T \Vdash \varphi(d)$.

Topological Models of Arithmetic and of Analysis

Naturally, we want to consider models that validate, at least, Heyting arithmetic. To be precise, we assume that our language contains a constant 0 and function symbols $s, +, \cdot$ and that a model validates the axioms

H1 $\forall x : \neg(x = s(0))$

H2 $\forall x, y : s(x) = s(y) \to x = y$

H3 $\forall x : x + 0 = x$

H4 $\forall x, y : x + s(y) = s(x + y)$

H5 $\forall x : x \cdot 0 = 0$

H6 $\forall x, y : x \cdot s(y) = x \cdot y + y$

H7 $(\varphi(0) \wedge \forall x : \varphi(x) \to \varphi(x + 1)) \to \forall x : \varphi(x).$

Here the last axiom is, as usual, actually an axiom schema for all statements φ. And, of course, all variables are of a fixed natural number type.

We have a natural embedding of the external natural numbers into the language of the model, via

$$n \mapsto s^n(0) \ .$$

To distinguish the external and the internal natural numbers we will denote the embedded ones by $\check{n}$. Internally there might be more natural numbers than externally: Given any set S equipped with the discrete topology, it is easy to check that the functions $S \to \mathbb{N}$ with the obvious interpretation form a model of Heyting arithmetic. So, for example, in the case that $S = \{0,1\}$ the identity function is a natural number with name, say c, which is not of the form $s^n(0)$. That is there is no n such that $S \Vdash \check{n} = c$. Of course locally we have

$$\{0\} \Vdash c = \check{0} \text{ and } \{1\} \Vdash c = \check{1} \ ,$$

so these new numbers are not really adding any interesting new behaviour. There might, however, also be more internal numbers locally—just think of any non-standard model. We would like to exclude this non-standard case. To be precise, we would like to exclude the case that the internal and the external numbers differ locally—if we assume non-standard numbers in the meta-theory we also have these in the model. One way of ensuring this is by assuming that our model is *full* in the sense that we have have names for all predicates that we can define on the topological space.

Proposition 7.2.4. *Consider* $(T, \tau, [\![\cdot]\!])$ *a model of Heyting arithmetic and assume that there exists a predicate $N(x)$ where x is of type $\mathbb{N}$ such that*

$$[\![N(x)]\!] = \bigcup_{n \in \mathbb{N}} [\![x = \check{n}]\!].$$

Then, if $t \in [\![\varphi(x)]\!]$, where x is of type $\mathbb{N}$, there exists n such that $t \in [\![\varphi(\check{n})]\!]$. In other words, we can export natural numbers locally.

Proof. It is enough to use natural induction (H7) to show that

$$T \Vdash x \in \mathbb{N} \to N(x) \ .$$

The case $x = 0$ is obviously fine. So let x be an arbitrary variable symbol of the natural number type, and let $t \in T$ be arbitrary.

By the induction hypothesis there exists a neighbourhood U of t and an external natural number n such that $U \Vdash x = \check{n}$. So we also have $U \Vdash s(x) = s(\check{n})$. If we can show that $T \Vdash s(\check{n}) = n \check{+} 1$ we are done. But that holds by natural induction (in the metatheory) using H2. $\qquad\square$

From now on we are only going to consider topological models, that are models of HA and that satisfy the conclusion of the previous proposition. The full models introduced in the next subsection are such models.

Let us move on to models of the real numbers. That is our language contains constants $0, 1$ binary function symbols $+, \cdot$, unary function symbols $-, ^{-1}$ and a binary relation $>$ such that the usual axioms are satisfied (for a detailed list of axioms see [23]).

If we have a model T of the real numbers in this sense any internal real can be thought of as a continuous function $T \to \mathbb{R}$.

Proposition 7.2.5. *Consider* $(T, \tau, [\![\cdot]\!])$ *a model of Heyting arithmetic and of real numbers. If x is of type $\mathbb{R}$, then there exists a continuous function $f : T \to \mathbb{R}$ such that*

$$[\![\check{r} < x < \check{s}]\!] = \{t \in T \mid r < f(t) < s\} \ , \tag{7.1}$$

for all $r, s \in \mathbb{Q}$. Therefore also, in particular,

$$\llbracket x > 0 \rrbracket = \{ t \mid f(t) > 0 \} \,,$$
$$\llbracket x = 0 \rrbracket = \mathrm{Int}\,(\{ t \mid f(t) = 0 \}) \,,$$
$$\llbracket x \geqslant 0 \rrbracket = \mathrm{Int}\,(\{ t \mid f(t) \geqslant 0 \}) \,,$$
$$\llbracket \neg(x = 0) \rrbracket = \mathrm{Int}\,\big((\mathrm{Int}\,(\{ t \mid f(t) = 0 \}))'\big) \,.$$

Proof. First, in order to define f, we will show that for $t_0 \in T$ the set

$$D_{t_0} = \{ q \in \mathbb{Q} \mid t_0 \in \llbracket \check{q} < x \rrbracket \} \,,$$

is a Dedekind cut.

- D_{t_0} is bounded and inhabited: since

$$T \Vdash \exists q, p \in \mathbb{Q} : q < x < p \,,$$

  we can use Proposition 7.2.4 to export the rationals (as pairs of natural numbers) locally. That is there exists a neighbourhood U of t_0 and $q, p \in \mathbb{Q}^3$ such that

$$U \Vdash \check{q} < x < \check{p} \,.$$

  Clearly $q \in D_{t_0}$ and $p \notin D_{t_0}$.

- D_{t_0} is downward closed: if $q \in D_{t_0}$ and $p < q$ then $T = \llbracket \check{p} < \check{q} \rrbracket$ and therefore

$$t_0 \in \llbracket \check{q} < x \rrbracket \subset \llbracket \check{p} < x \rrbracket \,.$$

  Thus $p \in D_{t_0}$.

- Even though we are working with a classical meta-theory it seems worth pointing out that, similarly, we can also show that D_{t_0} is order located.

Now consider the function f that maps every $t \in T$ to the real given by the Dedekind cut D_t. We want to show that this function is point-wise continuous. It suffices to show that, given $\varepsilon > 0$, we can find an open neighbourhood U of t_0 such that

$$\forall t, t' \in U, q \in f(t) : \exists p \in f(t') : |q - p| < \varepsilon \,.$$

So let $\varepsilon > 0$ and $t_0 \in T$ be arbitrary. We may, without loss of generality, assume that $\varepsilon \in \mathbb{Q}$. Since

$$T \Vdash \exists r \in \mathbb{Q} : x - \check{\varepsilon} < r < x \,,$$

there exists an open neighbourhood U of t_0 and $r \in \mathbb{Q}$ such that

$$U \Vdash x - \check{\varepsilon} < \check{r} < x \,.$$

Now let $t, t' \in U$ and $q \in f(t)$. The latter implies that $V \Vdash \check{q} < x$ for some open neighbourhood V of t and therefore $V \Vdash \check{q} - \varepsilon < \check{r}$. Since we are dealing with imported reals that means that $q - \varepsilon < r$. Now set $p = q - \varepsilon$. Then $p < r$, which means that $U \Vdash \check{p} < \check{r} < x$, and therefore $p \in f(t')$.

We will show 7.1 by showing that the left hand side set is a subset of the right hand one and vice versa. So let $t \in \llbracket \check{r} < x < \check{s} \rrbracket$. Similar to above, we can find a neighbourhood U of t and $p, q \in \mathbb{Q}$ such that

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[3] Here and below we abuse our notation slightly by reusing the same name for external and internal variables.

$U \Vdash \check{r} < \check{p} < x < \check{q} < s$. Then, in terms of Dedekind cuts $p \in f(t)$ and $q \notin f(t)$ for all $t \in U$ and therefore, seen as reals $r < f(t) < s$.

Conversely, let $t_0 \in T$ such that $r < f(t_0) < s$. By continuity, we may choose a neighbourhood U of t_0 and $q \in \mathbb{Q}$ such that $r < f(t) < q < s$ for all $t \in U$. (The asymmetry in needing q on the right hand side, but no rational on the left comes from the fact that Dedekind cuts are somewhat asymmetric). Since $r < f(t)$ on U that means that for any $t \in U$ we have $r \in D_{t_0}$, which in turn means that $U \subset [\![\check{r} < x]\!]$. Similarly we can show that for any $t \in U$ we have $t \notin [\![x < \check{q}]\!]$. Thus

$$U \subset [\![\neg(\check{q} < x)]\!] \subset ([\![\check{q} < x]\!])' \subset [\![x < \check{s}]\!];$$

because we have assumed that T is a model for the constructive reals and therefore

$$T = [\![\check{q} < x \vee x < \check{s}]\!] = [\![\check{q} < x]\!] \cup [\![x < \check{s}]\!] .$$

Together $t_0 \in [\![\check{r} < x < \check{s}]\!]$. $\qquad\qquad\square$

At this point we have not assumed that we also have the converse, that is that for every continuous real-valued function $f : T \to \mathbb{R}$ we have a constant symbol which has this real as its representative. However, we will do so in the next section. To our knowledge, models that are not full in this sense have not been considered as models of analysis.

The Full Model and Countable Choice

The commonly used models are the "full" ones, whose existence is guaranteed by the following.

Proposition 7.2.6. *For any topological space (T, τ) there exists a model such that*

1. *It is a model of IZF (therefore also of CZF, HA, and the real numbers).*

2. *For every $V \in \tau$ there is a proposition P_V such that $[\![P_V]\!] = V$.*

In particular, that means that we can use Propositions 7.2.4 and 7.2.5, which imply that any internal real number x is represented by a continuous function $f_x : T \to \mathbb{R}$. Fullness also guarantees the converse, namely that for every continuous function $f : T \to \mathbb{R}$ there exists a name for a real number x_f such that $f_{x_f} = f$.

Proof. See the section "Topological Models" in [61]. $\qquad\qquad\square$

We should note that the first point does not cover any form of choice. As a matter of fact, even countable choice often fails in topological models. We therefore need to be a bit careful when talking about, for example, LPO being satisfied, since we need to specify whether we mean the sequential version LPO_σ or the one for the real numbers $\text{LPO}_\mathbb{R}$. We have that $\text{LPO}_\mathbb{R}$ implies LPO_σ, but the converse is only true in the presence of countable choice, as we can see by considering the topological model over the reals. There $\text{LPO}_\mathbb{R}$ fails: Let z be the real given by the identity function, then

$$[\![\neg(z = 0)]\!] = \text{Int}\left(([\![z = 0]\!])'\right) = \text{Int}\left((\text{Int}(\{0\}))'\right) = \text{Int}(\mathbb{R}) = \mathbb{R} ,$$

but

$$[\![z \neq 0]\!] = (-\infty, 0) \cup (0, \infty) .$$

On the other hand if m is an (internal) natural number, then, it is represented by a point-wise continuous function $f_m : \mathbb{R} \to \mathbb{N}$. But since the only such functions are the constant ones we have that for every name of a natural number m there exists an external natural number n such that $\mathbb{R} \Vdash m = \check{n}$. Extending this

argument to binary sequence we get that the internal binary sequences are exactly the external ones. Thus LPO_σ holds, since we are working with a classical metatheory.

Together we have

$$\mathbb{R} \Vdash \text{LPO}_\sigma \text{ but } \mathbb{R} \nVdash \text{LPO}_\mathbb{R} ,$$

and therefore

$$\mathbb{R} \nVdash \text{ACC} .$$

Reverse Reverse Mathematics

In the following we can give some characterisation of properties of topological spaces, such that the full model satisfies certain principles. This is actually quite a natural question to consider, and is helpful to find custom separations of principles. We have—not entirely seriously—named this area "Reverse Reverse Mathematics". The assumption that we are working with full models is essential for this approach.

It is easy to characterise the spaces that satisfy LEM.

Proposition 7.2.7. *A topological space* (X, τ) *validates LEM if and only if every open set is also closed.*

Corollary 7.2.8. *For the Sierpinski space* $\Sigma = (\{0,1\}, \{\emptyset, \{1\}, \{0,1\}\})$, *we have* $\Sigma \Vdash \text{LPO}$[4] *but also* $\Sigma \nVdash \text{LEM}$. *And hence also* $\Sigma \nVdash \text{KS}$.

Proof. To see that Σ satisfies LPO, simply note that all continuous functions $f : \Sigma \to \mathbb{R}$ are constant.[5] $\qquad\square$

Remark 7.2.9. *The Sierpinski space also validates* $\Sigma \Vdash \text{WLEM}$.

Proof. Assume φ is an arbitrary formula. If either $[\![\varphi]\!] = \emptyset$ or $[\![\varphi]\!] = \Sigma$ we have $\Sigma \Vdash \neg\varphi \vee \neg\neg\varphi$. But also in the case that $[\![\varphi]\!] = \{1\}$ we have that $[\![\neg\varphi]\!] = \text{Int}(\{0\}) = \emptyset$ and therefore $X \Vdash \neg\neg\varphi$. $\qquad\square$

Proposition 7.2.10. *Assume* (T, τ) *is a first countable, normal space. Then either* $T \Vdash \text{LEM}$ *or* $T \nVdash \text{MP}^\vee$.

Proof. Either $\{t\}$ is open for all $t \in T$, or there exists $t_0 \in T$ such that for all neighbourhoods U of t_0 there exists $p \in U$ with $p \neq t_0$. In the first case $\{\{t\} \mid t \in T\}$ is an open cover of T and therefore $T \Vdash \text{LEM}$. So let us now concentrate on the second case. We can, inductively, find a sequence t_n and a basis of open neighbourhoods U_n such that $t_n \in U_n$ and $t_n \notin \overline{U_{n+1}}$ [6]. Now let r_n be a sequence of reals such that $\sum_{n \in \mathbb{N}} r_n$ converges absolutely. We claim that there is a continuous function $g : T \to \mathbb{R}$ such that $g(t_n) = r_n$.

Notice that $\{t_n\}$ and $\{t_1, \ldots, t_{n-1}\} \cup \overline{\{t_0, t_{n+1}, t_{n+2}, \ldots\}}$ are disjoint and closed sets, so by Urysohn's Lemma there exists a continuous $g_n : T \to [0,1]$ such that $g_n(t_n) = 1$ and $g_n(t_i) = 0$ for $i \neq n$. Since the series r_n converges absolutely

$$g = \sum_{n \in \mathbb{N}} r_n g_n$$

is well-defined everywhere and continuous. Furthermore $g(t_n) = r_n$.

Now let $r_n = {}^{(-1)^n}/2^n$ and construct g as above, and consider the real x_g given by g. Since every neighbourhood of t_0 can be shrunk to an inhabited one where $g \neq 0$, we have that for every neighbourhood U of t_0

$$U \Vdash \neg(x_g = 0) .$$

However $t_0 \notin \text{Int}(\{t \in T \mid g(t) \geqslant 0\})$ and $t_0 \notin \text{Int}(\{t \in T \mid g(t) \leqslant 0\})$, and therefore

$$T \nVdash \neg(x_g = 0) \to (x_g \leqslant 0 \vee x_g \geqslant 0) .$$ $\qquad\square$

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[4]The real number version of LPO, and hence also the sequential version.

[5]As M. Hendtlass has pointed out, actually, every finite topological model satisfies LPO.

[6]remember that T is assumed to be normal

Fred Richman has shown that weak Markov's principle (WMP) holds in the sheaf model on every metric space [94]—a result which translates to this being true for every topological model over a metric space. To be precise we should also mention that we consider WMP in the following form:

For all $x \in \mathbb{R}$, if

$$\forall y \in \mathbb{R} : \neg\neg(y > 0) \vee \neg\neg(y < x)$$

then $0 < x$.

We can extend Richman's result mentioned above (basically using the same idea).

Proposition 7.2.11. *Assume (T, τ) is a first countable, normal space. Then $T \Vdash$ WMP.*

Proof. Let the real x be given by the continuous function $f : T \to \mathbb{R}$. Now consider a point and a neighbourhood such that

$$t_0 \in [\![\forall y \in \mathbb{R} : \neg\neg(y > 0) \vee \neg\neg(y < x)]\!] .$$

we want to show that $t_0 \in [\![0 < x]\!]$. Now if $f(t_0) > 0$ we are done, so it suffices to concentrate on and exclude the case that $f(t_0) \leqslant 0$. A sub-case that is then easily excluded is that there is an open neighbourhood U of t_0 such that $f(t') \leqslant 0$ for all $t' \in U$, since then $U \Vdash x \leqslant 0$, but for $y = x$ the antecedent of WMP implies $\neg(x \leqslant 0)$. So finally we can assume that there is sequence of points t_n such that $f(t_n) > 0$ and $t_n \in U_n$; where U_n is a base of open neighbourhoods of t_0. With the same construction as above we can find $g : T \to \mathbb{R}$ such that $g(t_{2n}) = f(t_{2n})$ and $g(t_{2n+1}) = 0$. Now let y be given by g. Then neither $t_0 \in [\![\neg\neg(y > 0)]\!]$ nor $t_0 \in [\![\neg\neg(y < x)]\!]$. $\square$

This nicely fits in with a result of Lubarsky and Hendtlass who showed in [61] that there is a topological model satisfying LLPO, but not LPO.

Their space X_U is defined to consist of $\mathbb{N}$ and an added point ω, where a base of the topology is given by

- $\{n\}$ for $n \in \mathbb{N}$;

- $\{\omega\} \cup A$ for $A \in \mathscr{U}$; where $\mathscr{U}$ is an ultrafilter on $\mathbb{N}$.

In particular every set that doesn't contain ω is open.

We also assume that $\mathscr{U}$ is non-principal, i.e. it contains the Fréchet filter of cofinite subsets of $\mathbb{N}$.

Proposition 7.2.12.

1. $X_U \not\Vdash$ MP

2. $X_U \Vdash$ WLEM

Proof.   1. Consider the real x given by $f(n) = \frac{1}{2^n}$ and $f(\omega) = 0$. f is continuous since our filter is non-principal and therefore contains sets $A \in \mathscr{U}$ such that $A \cap \{1, \ldots, n\} = \emptyset$. Since $\text{Int}(\{t \mid f(t) = 0\}) = \text{Int}(\{\omega\}) = \emptyset$ we conclude that $[\![\neg(x = 0)]\!] = X_U$. But $\omega \notin [\![x \neq 0]\!]$, since $f(\omega) = 0$. Hence $X_U \not\Vdash \neg(x = 0) \implies x \neq 0$.

2. Consider an arbitrary φ. We distinguish three cases:

   - $[\![\varphi]\!] \subset \mathbb{N}$. Then either $[\![\varphi]\!] \in \mathscr{U}$ or $\mathbb{N} \setminus [\![\varphi]\!] \in \mathscr{U}$. In the first case $[\![\neg\varphi]\!] = \text{Int}(([\![\varphi]\!])') = \emptyset$ and therefore $[\![\neg\neg\varphi]\!] = X_U$. In the second case $[\![\neg\varphi]\!] = \text{Int}(([\![\varphi]\!])') = \{\omega\} \cup (\mathbb{N} \setminus [\![\neg\varphi]\!])$ and $[\![\neg\neg\varphi]\!] = [\![\varphi]\!]$. In both cases we get $X_U \Vdash \neg\varphi \vee \neg\neg\varphi$.
   - If $[\![\varphi]\!] = \{\omega\} \cup A$, then $A \in \mathscr{U}$. Therefore $[\![\neg\varphi]\!] = \mathbb{N} \setminus A$ and $[\![\neg\neg\varphi]\!] = [\![\varphi]\!]$. So also in this case $X_U \Vdash \neg\varphi \vee \neg\neg\varphi$. $\square$

Corollary 7.2.13. $X_U \nVdash$ *WMP*.

Proof. Since MP $\iff$ MP$^\vee \wedge$ WMP and WLEM $\implies$ MP$^\vee$. □

Corollary 7.2.14. $\nvdash$ *WLPO* $\implies$ *MP*.

Proof. Since WLEM $\implies$ WLPO. □

Remark 7.2.15. *The space X_U is not first countable but normal.*

Proof. Assume that U_n is a base of open neighbourhoods of ω. We may assume that $U_n \supset U_{n+1}$. Since $\mathscr{U}$ contains cofinite, and since $\mathscr{U}$ does not contain the emptyset, sets we can find a strictly increasing sequence y_n of natural numbers and a strictly increasing $f : \mathbb{N} \to \mathbb{N}$, such that $y_n \in U_{f(n)}$ and $y_n \notin U_{f(n+1)}$. Now consider the set $A = \{ y_{2n} \mid n \in \mathbb{N} \}$. Either $A \in \mathscr{U}$ or $A \notin \mathscr{U}$. We will only treat the first case, since the second one can be handled similarly. Then $\{\omega\} \cup A$ is open and should hence contain U_N for some $N \in \mathbb{N}$. Consider M such that $f(2M+1) > N$. Since $y_{2M+1} \in U_{f(2M+1)} \subset U_N \subset A$, we get a contradiction.

Next, consider two disjoint closed subsets $A, B \subset X_U$. If $A, B \subset \mathbb{N}$ there is nothing to show since, then A and B are also open. So let us without loss of generality assume that $\omega \in A$. But B is closed, which implies that $\mathbb{N} \setminus B \in \mathscr{U}$. Hence we have found an open set containing A namely $(\mathbb{N} \setminus B) \cup \{\omega\}$, which is disjoint from B, which it itself open anyway. □

The following is actually a special case of [54, Theorem 3.2], where it is shown that in any spatial topoi $2^{\mathbb{N}}$ is (cover) compact. Nevertheless, in the spirit of this section, a direct proof seems well fitted.

Proposition 7.2.16. *In any topological model* FAN$_{full}$ *holds.*

Proof. Let T be a topological model such that

$$T \Vdash \text{``}B \text{ is a bar''} .$$

Now consider an arbitrary $t \in T$.[7] Define an external set $B_t \subset 2^*$ by

$$u \in B_t \iff t \in [\![u \in B]\!] .$$

B_t is a bar, since we can internalise any $\alpha \in 2^{\mathbb{N}}$ and know that

$$t \in [\![\exists n \in \mathbb{N} : \overline{\alpha}n \in B]\!] .$$

We can externalise this n locally by Proposition 7.2.4, so there exists $n \in \mathbb{N}$ such that

$$t \in [\![\overline{\alpha}n \in B]\!] .$$

Externalising again, we can conclude that $\overline{\alpha}n \in B_t$, which means that B_t is a bar. Using the Fan theorem in the meta-theory we can therefore find a uniform bar N_t for B_t. It is straightforward to show that

$$t \in [\![\text{``}N_t \text{ is a uniform bound for } B\text{''}]\!] .$$

Altogether

$$T = \bigcup_{n \in \mathbb{N}} [\![\text{``}N \text{ is a uniform bound for } B\text{''}]\!] ,$$

which means that

$$T = [\![\text{``}B \text{ is uniform''}]\!] . \qquad \square$$

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[7] This is the exact point, at which this argument fails for certain Heyting valued models; namely for some which are not spatial.

Overview

The following quick overview might be useful to compare the big three varieties and three topological models, discussed above.

	LEM	WLEM	LPO	WLPO	LLPO	MP	WMP	MP$^\vee$
CLASS	✓	✓	✓	✓	✓	✓	✓	✓
INT	✗	✗	✗	✗	✗	✗	✓	✗
RUSS	✗	✗	✗	✗	✗	✓	✓	✓
$[0,1]$	✗	✗	✗	✗	✗	✗	✓	✗
Σ	✗	✓	✓	✓	✓	✓	✓	✓
X_U	✗	✓	✗	✓	✓	✗	✗	✓

We conclude this section by pointing out that topological models have been used to give models that

- do not satisfy BD-N [78],

- separate LPO, WLPO, MP and various variations of these [61],

- separate various principles below BD-N (see Section 6.3).

Analysing the definition of a topological model one can see that the points of the space actually are never mentioned, and that is only the open sets that matter. Therefore we can just as well consider models over algebras that behaves like the open sets of a topological space do. Such algebras are known as *Heyting algebras* and have, indeed, been used for this purpose. Since there are Heyting algebras that cannot be defined in terms of the open of a topological space this is a richer model theory. These non-spatial Heyting valued models can be used to separate FAN_{full} and $\text{FAN}_{\Pi_1^0}$ [79].

7.3   Realizability and Other Methods

In a *realizability model* we extend intuitionistic logic by allowing witnesses of statements to be attached to statements. This is, in particular, interesting in a constructive context, where we want to attach computable objects ("realizers") that describe the computational content of a formula. For example, we would like to express the fact that a formula $\exists m \in \mathbb{N} : P(\vec{x}, m)$ is realized (witnessed) by a computable function f when $P(\vec{x}, f(\vec{x}))$ holds for all $\vec{x}$. There are many different ways to fix details of how the logical connectives and quantifiers are handled, leading to very different interpretations.[8] Most realizability models have a distinct recursive flavour and so we can say that, vaguely speaking, realizability models are to RUSS what topological models are to INT. We will not go into any details and only point the reader to [6, Chapter 7] and [109] for further details.

Realizability models have been used to show that BD-N is not derivable in intuitionistic logic [76], and we would conjecture that they are the natural tool to separate SC, KT, SS.

As in the case of topological models, with realizability models one generally works in a classical meta-theory. This also means that there are strange realizability models, such as one based on infinite time turing machines [5], in which non-constructive principles like LPO are validated. The model mentioned also has the curious property that $\mathbb{N}^\mathbb{N}$ is countable.

Finally, one can also use various proof interpretations [73], which can also be seen as types of realizabilities, to separate principles. This, for example and as was mentioned in the respective chapter, was used to show that WMP is not derivable [72], which implies that WLPO does not imply MP.

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[8]To mention some of the more famous realizability interpretations: there is Kleene's original version, modified realizability, Scott's graph model, models based on relativized computations, . . .

Chapter 8

Bits'n'Pieces

8.1 LLPO_n

In [93] Richman introduced a natural weakening of LLPO.

> **(LLPO_n)**   If $(P_i)_{1\leqslant i\leqslant n}$ is a partition of $\mathbb{N}$ and $(a_n)_{n\geqslant 1}$ is a binary sequence with at most one 1, then there exists m such that $a_i = 0$ for all $i \in P_m$.

In [95] we find the following version

> **(LLPO'_n)**   If $(a_n)_{n\geqslant 1}$ is a binary sequence with at most one 1, then there exists $0 \leqslant m < n$ such that $a_{ni+m} = 0$ for all $i \in \mathbb{N}$.

It is clear that the first version implies the second if one considers the partition

$$P_m = \{ mn + r \,|\, r \in \mathbb{N} \} \,.$$

A proof of the converse is not completely straightforward and is, to our knowledge, nowhere to be found in the literature, so we include one here.

Proposition 8.1.1. *LLPO'_n implies LLPO_n for every $n \in \mathbb{N}$.*

Proof. Let $(P_i)_{1\leqslant i\leqslant n}$ be a partition of $\mathbb{N}$ and $(a_m)_{m\geqslant 1}$ a binary sequence with at most one 1. Define a binary sequence $(b_m)_{m\geqslant 1}$ by

$$b_{rn+i} = \begin{cases} 1 & \text{if } r \in P_i \text{ and } a_r = 1 \\ 0 & \text{otherwise,} \end{cases}$$

for every $r \in \mathbb{N}$ and $i \in \{0,\ldots,n-1\}$. Then $(b_m)_{m\geqslant 1}$ has at most one 1. Thus there exists m such that $b_{ni+m} = 0$ for all $i \in \mathbb{N}$. Hence, by the definition of the sequence, there cannot be $i \in P_m$ with $a_i = 1$. $\square$

The reader has probably already noticed that LLPO_2 is simply LLPO.

Proposition 8.1.2. *LLPO_n is equivalent to the statement that if $x_1,\ldots,x_n$ are real numbers such that $x_i x_j = 0$ for $i \neq j$, then there is m such that $x_m = 0$.*

Proof. Assume LLPO$_n$ and let $x_1, \ldots, x_n$ are real numbers such that $x_i x_j = 0$ for $i \neq j$. Using countable choice we fix binary increasing sequences $(a_m^{(i)})_{m \geqslant 1}$ such that

$$a_m^{(i)} = 0 \implies x_i < \frac{1}{2^m} \, ,$$

$$a_m^{(i)} = 1 \implies x_i > \frac{1}{2^{m+1}} \, .$$

Now let $b_m^{(i)} = a_{m+1}^{(i)} - a_m^{(i)}$, such that for every i the sequence $b^{(i)}$ has at most one 1 and that only if $a^{(i)}$ has one. Finally combine all of these into one binary sequence $(c_m)_{m \geqslant 1}$ by setting

$$c_{in+m} = b_m^{(i)}$$

for every $m \in \mathbb{N}$ and $1 \leqslant i \leqslant n$. Then $(c_m)_{m \geqslant 1}$ has at most one 1: For assume $c_i = c_j = 1$ with $i \neq j$. Furthermore choose $p, p' \in \mathbb{N}$ and $1 \leqslant r, r' \leqslant n$ such that $i = pn + r$ and $j = p'n + r'$. If $p = p'$, then $b_r^{(p)} = b_{r'}^{(p)}$, which means that $r = r'$ and therefore $i = j$; a contradiction. So $p \neq p'$ and $b_r^{(p)} = b_{r'}^{(p')} = 1$. Hence $a_r^{(p)} = a_{r'}^{(p')} = 1$, which means that $x_p > 0$ and $x_{p'} > 0$. So finally $x_p x_{p'} > 0$, contradicting our assumptions. An application of LLPO$_n$ yields $1 \leqslant p \leqslant n$ such that $c_{pn+m} = 0$ for all $m \in \mathbb{N}$. Hence $b_m^{(p)} = 0$ for all $m \in \mathbb{N}$ and therefore $a_m^{(p)} = 0$ for all $m \in \mathbb{N}$.

Conversely let $(a_m)_{m \geqslant 1}$ and for $1 \leqslant i \leqslant n$ define a real number

$$x_i = \sum_{m \geqslant 1} \frac{a_{in+m}}{2^m} \, .$$

It is easy to check that these numbers have the desired property. Now if there is i such that $x_i = 0$, then there cannot be $m \in \mathbb{N}$ such that $a_{in+m} = 1$, since in this case $x_i > 0$. $\square$

Remark 8.1.3. *One might also consider the question whether the following is an interesting statement:*

If $x_1, \ldots, x_n$ are real numbers such that $x_1 x_2 \ldots x_n = 0$, then there is a i such that $x_i = 0$

However this is trivially equivalent to LLPO. If $x_1 x_2 \ldots x_n = 0$, then by LLPO either $x_1 x_2 \ldots x_{n-1}$ or $x_n = 0$. Using LLPO possibly up to $n-1$ times again we get that $x_1 = 0$, $x_2 = 0$, $\ldots$, or $z = 0$. Conversely, if $xy = 0$, then also $x \ldots xy = 0$, so either $x = 0$ or $y = 0$.

A more interesting equivalence is the following:

Proposition 8.1.4. *LLPO$_3$ is equivalent to the statement that whenever x, y, and z are real number such that $xyz = 0$,*
$$xy = 0 \vee yz = 0 \vee xz = 0 \, .$$

Proof. The forward direction is easy: assume that x, y, z are such that $xyz = 0$ and let $a = xy$, $b = yz$, and $c = xz$. Then $ab = bc = ac = xyz = 0$. So we can apply LLPO$_3$ to decide whether $a = 0$, $b = 0$, or $c = 0$; but that is exactly what we wanted to do.

For the converse let x, y, z be such that

$$xy = 0, yz = 0, \text{ and } xz = 0 \, . \tag{8.1}$$

Now let $a = |x - y|$, $b = |y - z|$, and $c = |x - z|$. Now we must have $abc = 0$, since otherwise $x \neq y$, $y \neq z$, and $x \neq z$. And in that case at least two out of x, y, z must be non-zero contradicting the assumption that their pair-wise product is zero. Thus we can decide whether $ab = 0$, $bc = 0$, or $ac = 0$. Assume that w.l.o.g the first one holds. If $ab = 0$, then $y = 0$. Otherwise the assumption that $y \neq 0$ leads to a contradiction: if $y \neq 0$ we must have $x = z = 0$ by Equation 8.1, but then $ab = |y|^2 \neq 0$. $\square$

Question 12. *What about the general principle? Is LLPO$_n$ equivalent to the statement that whenever $x_1, \ldots, x_n$ are reals such that $\prod_{i=1}^n x_i = 0$, there exists j such that*

$$\prod_{i=1,\ i \neq j}^n x_i = 0 \ ?$$

The forward direction is trivial. However the converse does not generalise straightforwardly

The reason that LLPO$_n$ does not feature more prominently in Constructive Reverse Mathematics is, that it has yet failed to have interesting equivalencies. So far it has actually only been used as a tool to show that certain results are non-constructive [93, 31]; that is it has been shown that some statements imply LLPO$_n$, but not the converse. The reason given for LLPO$_n$ to be non-constructive is that it fails in RUSS [93, Theorem 5]. An alternative refutation relies on an argument about stable solutions by Beeson, which one could call "a bit handwavey" [95, p. 112]. Here we will show that it fails in the sheaf model of $\mathbb{N}^{\mathbb{N}}$, which also validates dependent choice.

Proposition 8.1.5. *If a metric space X has an accumulation point, then LLPO$_n$ fails in the topological model over X.*

Proof. The proof builds on the one of Proposition 7.2.10. So let $x_0, x_1, \ldots$ and $r_1, r_2, \ldots$ as there. Furthermore let $f^{(i)} : \mathbb{R}_0^+ \to \mathbb{R}$ be defined to be piecewise linear through the points $(r_{3nj+3i+1}, \frac{1}{2^j})$ and $(r_{i'}, 0)$ for $i' \neq 3nj + 3i + 1$. Furthermore let z_i be the reals in the sheaf model given by $\tilde{f}^{(i)} = f \circ d(x_0, \cdot)$. Then $z_i z_j = 0$ for $i \neq j$, since at every point at most one of the $f^{(i)}$ is non-zero. But also in every neighbourhood U of x_0 and for every i there exists t such that $\tilde{f}(t) > 0$, so

$$U \nVdash \exists i : z_i = 0 \ ,$$

and hence

$$X \nVdash \exists i : z_i = 0 \ .$$

Altogether $X \nVdash$ LLPO$_n$. $\square$

Corollary 8.1.6. *LLPO$_n$ fails in the sheaf model of $\mathbb{R}$ on $\mathbb{N}^{\mathbb{N}}$.*

Proof. Dependent choice is validated by the sheaf model of $\mathbb{R}$ on $\mathbb{N}^{\mathbb{N}}$. Also $\mathbb{N}^{\mathbb{N}}$ has an accumulation point. $\square$

8.2 Open Induction

Coquand [38], U. Berger [18], Schuster [99], and others have all investigated aspects of open induction. This principle is interesting, since, heuristically speaking, invocations of Zorn's Lemma in a classical proof can be replaced by using open induction. Now open induction is constructive at least for some sets, which also depends on the flavour of constructivism one practices. Thus, if open induction instance-wise implies a theorem, one can easily read off its constructive content up to exactly the level that one deems constructive.

We would also like to point out that Veldmann has done some work on open induction, however, the results are, at this stage, only partially published [111].

In classical mathematics, if $A \subset [0,1]$ such that A is open, and *progressive* that is such that

$$\forall x : ((\forall y < x : y \in A) \implies x \in A) \tag{8.2}$$

then A is actually the whole unit interval (notice that necessarily $0 \in A$).

There does not seem to be a good constructive theory of strict orders; or even any treatment of this theory. However, since in this section we will only deal with specific examples namely

- the usual order $<$ on (subsets of) $\mathbb{R}$ and

- the (strict) lexicographic order on (subsets of) $2^{\mathbb{N}}$

we actually do not need to fix a definition to state the principle of open induction for arbitrary strictly ordered sets.

> **(OI$_X$)**   Assume X is a strictly ordered set equipped with the order topology and minimum element $\bot$. If $A \subset X$ is open and progressive, then $A = X$.

Notice that such a set X must necessarily contain all minimal elements $\bot$, since 8.2 is vacuously satisfied.

Veldmann has shown that $\text{OI}_{[0,1]}$ is implied by bar induction and in turn implies the fan theorem (that is the fan theorem for decidable bars.[1])

Proposition 8.2.1. *$OI_{2^{\mathbb{N}}}$ implies FAN_{full}.*

Proof. Let B be a bar. We will write B_u for the set $\{u\} * 2^{\mathbb{N}} \cap B$. Then

$$A = \left\{ \alpha \in 2^{\mathbb{N}} \,\middle|\, \exists u_1, \ldots, u_n \in 2^{\mathbb{N}} : \forall \beta < \alpha : \beta \in \bigcup_{i=1}^{n} B_{u_i} \right\} .$$

We want to show that A is open and progressive. So let $\alpha \in A$. Now there exists m such that $\gamma \in B$, as long as $\overline{\gamma}m = \overline{\alpha}m$. Furthermore there exists $u_1, \ldots, u_n$ such that

$$\forall \beta < \alpha : \beta \in \bigcup_{i=1}^{n} B_{u_i} ,$$

and therefore, with $u_{n+1} = \overline{\gamma}m$ also

$$\forall \beta < \gamma : \beta \in \bigcup_{i=1}^{n+1} B_{u_i} ,$$

that is $\gamma \in A$.

Along similar lines assume that α is such that $\beta \in A$ for all $\beta < \alpha$. Again choose m such that $\overline{\alpha}m \in B$. Now either $\overline{\alpha}m = 0^m$ and we are done, or $\overline{\alpha}m = u10\ldots0$ for some $u \in 2^*$. Then $\gamma < \alpha$ for $\gamma = u011\ldots$. By our assumptions there exists $u_1, \ldots, u_n$ such that

$$\forall \beta < \gamma : \beta \in \bigcup_{i=0}^{n} B_{u_i} .$$

Since, for any $\beta < \alpha$ we can decide whether $\overline{\beta}m = \overline{\alpha}m$ or $\beta \leqslant \gamma$ we have that

$$\forall \beta < \alpha : \beta \in \bigcup_{i=1}^{n+1} B_{u_i} ,$$

for $u_{n+1} = \overline{\alpha}m$. Hence $\alpha \in A$.

Since B is a bar there exists k such that $1^k = \overline{111\ldots}k \in B$.

By $\text{OI}_{2^{\mathbb{N}}}$ we have $A = 2^{\mathbb{N}}$, and in particular $111\ldots \in A$; therefore there exists $u_1, \ldots, u_n$ such that $\beta \in \bigcup_{i \leqslant n} B_{u_i}$ for any $\beta < 111\ldots$. Now for any $\gamma \in 2^{\mathbb{N}}$ we can decide whether $\overline{\gamma}k = 1^k$ or $\gamma < 111\ldots$, and hence

$$2^{\mathbb{N}} = \bigcup_{i=0}^{n} B_{u_i} ,$$

where $u_0 = 1^k$. But this means that B is a uniform bar. $\qquad\square$

---

[1] We are unsure to what degree continuous choice is used in these, to our knowledge unpublished, proofs.

For some sets X we can prove OI_X in pure BISH.

Proposition 8.2.2.

1. *If X is a decidable subset of $\mathbb{N}$, then OI_X holds.*

2. *Let $\mathbb{N}^*$ be the set of all increasing binary sequences (with the order and metric of Cantor space). Then $OI_{\mathbb{N}^*}$ holds.*

Proof.

1. Let $A \subset X$ progressive (since $\mathbb{N}$ has the discrete topology openness is irrelevant here). We need to show that $X \subset A$. So let $x \in X$ be arbitrary. Since X is decidable we can find natural numbers $a_1 < \cdots < a_n$ such that $\{y \in X \,|\, y \leqslant x\} = \{a_1, \ldots, a_n\}$, where $a_1 = \perp_X$ and $a_n = x$. Now $a_1 \in A$ by the observation above. Therefore, since A is progressive also $a_2 \in A$ and similarly after n steps we have $a_n = x \in A$.

2. As an abbreviation we write $\underline{n} = 0^n 111 \ldots$ and $\omega = 000 \ldots$. Now let $A \subset \mathbb{N}^*$ be progressive and open. We can, as above, show that $\underline{n} \in A$ for every $n \in \mathbb{N}$. We can also show that $\omega \in A$, since for every $y < \omega$ we can find $n \in \mathbb{N}$ such that $\underline{n} = y$. It might be surprising that we are not done yet, and in fact we actually need to show that *every* $x \in \mathbb{N}^*$ is also in A, and not just the top element![2] However, since A is open there exists $k \in \mathbb{N}$ such that if $y > \underline{k}$ (that is if y is in the neighbourhood of ω given by k), then $y \in A$. Furthermore for every $y \in \mathbb{N}^*$ we can decide whether $y \leqslant \underline{k}$ and therefore $y = \underline{n}$ for some $n \in \mathbb{N}$ or $y > \underline{k}$. In both cases we already know that $y \in A$. $\qquad\square$

Proposition 8.2.3. *The following are equivalent*

1. *$OI_{2^{\mathbb{N}}}$*

2. *$OI_{[0,1]}$*

3. *$OI_{\mathbb{R}_0^+}$*

Proof. Assume $\mathrm{OI}_{2^{\mathbb{N}}}$ holds and let $A \subset [0,1]$ be open, progressive and such that $0 \in A$. Furthermore let $F^{2/3} : 2^{\mathbb{N}} \to [0,1]$ be the function from Section 3.1. Consider the set

$$E = \left\{\, \alpha \in 2^{\mathbb{N}} \,\big|\, F^{2/3}(\alpha) \in A \,\right\}.$$

Now $0 \in E$, since $F^{2/3}(0) = 0$. Since E is the preimage of an open set under a continuous function it is also open. To see that it is progressive let $\alpha \in 2^{\mathbb{N}}$ be such that $\forall \beta < \alpha : \beta \in E$. It suffices to show that for all $y < F^{2/3}(\alpha)$ we have $y \in A$, since then $F^{2/3}(\alpha) \in A$ by virtue of A being progressive. By Lemma 3.1.1 there is $\gamma \in 2^{\mathbb{N}}$ with $F^{2/3}(\gamma) = y$ and $\gamma < \alpha$. Hence $\gamma \in E$ and therefore $y \in A$. We are now in the position to apply $\mathrm{OI}_{2^{\mathbb{N}}}$ to show that $1 \in E$; and therefore $F(1) = 1 \in A$.

Next, assume $\mathrm{OI}_{[0,1]}$ and let $E \subset 2^{\mathbb{N}}$ be open, progressive, and such that $0 \in E$. We will consider the set

$$A = \left\{\, x \in [0,1] \,\big|\, \forall \alpha \in 2^{\mathbb{N}} : F^{1/3}(\alpha) \leqslant x \implies \alpha \in E \,\right\},$$

where $F^{1/3}$ is the canonical embedding of $2^{\mathbb{N}}$ into the Cantor middle third set from Lemma 3.1.1. It is easy to see that $0 \in A$. We are going to show that if $y \in A$ for all $y < x$, then A contains an open neighbourhood of x. Since A is also downward closed this will show at the same time that A is progressive as well as open.

---

[2]Of course classically we have $\mathbb{N}^* = \{\underline{n} \,|\, n \in \mathbb{N}\} \cup \{\omega\}$, but constructively we cannot prove this; to be precise this equality holds if and only if LPO holds.

So let $x \in [0,1]$ such that $y \in A$ for all $y < x$. Let α be as in Lemma 3.1.1.2. We first want to show that $\beta \in E$ for all $\beta < \alpha$. It is easy to see that if $\beta < \alpha$, then $F^{1/3}(\beta) < F^{1/3}(\alpha)$. Now either $F^{1/3}(\beta) < x$ and therefore $F^{1/3}(\beta) \in A$ which in turn implies that $\beta \in E$, or $x < F^{1/3}(\alpha)$. In the second case Lemma 3.1.1 yields $\forall \beta < \alpha : F^{1/3}(\beta) \leqslant x$, and thus, by the definition of A, $\beta \in E$. So in all cases we have $\beta \in E$ for $\beta < \alpha$, and thus, since E is progressive, $\alpha \in E$. Since E is open there exists $n \in \mathbb{N}$ such that $\gamma \in E$ whenever $\overline{\alpha}n = \overline{\gamma}n$. If $\overline{\alpha}n = 1^n$ we are done, since then $A = [0,1]$ and thus contains an open neighbourhood of x. In the case that $\overline{\alpha}n = u01^m$ for some $u \in 2^*$ and some $m \in \mathbb{N}$ we have that for all $\beta < \gamma = u11^m0\ldots$ we have that $\beta \in E$ and also $x < F^{1/3}(\gamma)$. Hence also in this case A contains an open neighbourhood of x. Thus $\mathrm{OI}_{[0,1]}$ implies $\mathrm{OI}_{2^{\mathbb{N}}}$.

To see that $\mathrm{OI}_{[0,1]}$ implies $\mathrm{OI}_{\mathbb{R}_0^+}$, we can use scaling: assume that $A \subset \mathbb{R}_0^+$ is open and progressive, and let $z \in \mathbb{R}_0^+$ be arbitrary. Then it is easy to see that $A \cap [0,z]$ is progressive and open (in $[0,z]$). Clearly $\mathrm{OI}_{[0,1]}$ implies $\mathrm{OI}_{[0,z]}$, so $A \cap [0,z] = [0,z]$ which means that $[0,z] \subset A$. In particular $z \in A$.

Conversely Let $A \subset [0,1]$ be open, progressive, and such that $0 \in A$. Then

$$A' = (A \cap (0,1)) \cup \left\{ x \in \mathbb{R}_0^+ \,\middle|\, A = [0,1] \right\} .$$

Then trivially A' is open. It is also progressive, since for $x \in A'$ either $x \in A$, in which case we can use the progressiveness of A, or $x \in \left\{ x \in \mathbb{R}_0^+ \,\middle|\, A = [0,1] \right\}$, in which case $A' = \mathbb{R}_0^+$, which is also clearly progressive. This means we can use $\mathrm{OI}_{\mathbb{R}_0^+}$ to conclude that $A' = \mathbb{R}_0^+$. In particular $1 \in A'$, which means that $A = [0,1]$. $\qquad\square$

8.3 The Limited Anti-Specker Property

Douglas Bridges, James Dent, and Maarten McKubre-Jordens [28, 40] have considered various weakenings of the Anti-Specker property. One of them is the so-called *Limited Anti-Specker property*.

> **(AS_X^L)** If $X \cup \{\omega\}$ is a one-point extension of a metric space X and $(x_n)_{n \geqslant 1}$ is a sequence in $X \cup \{\omega\}$ is eventually bounded away from each point of X, then there exists n such that $x_n = \omega$.

Proposition 8.3.1. $AS_{[0,1]}^L \iff AS_{2^{\mathbb{N}}}^L$

Proof. Let $F^{1/3} : 2^{\mathbb{N}} \to [0,1]$ denote the canonical embedding of $2^{\mathbb{N}}$ onto the Cantor middle third set as in Section 3.1. We may assume that $d([0,1],\omega) = 2$. Let $(\alpha_n)_{n \geqslant 1}$ be a sequence in $2^{\mathbb{N}} \cup \{\omega\}$ that is bounded away from every point in $2^{\mathbb{N}}$ and set

$$x_n = \begin{cases} \omega & \text{if } \alpha_n = \omega \\ F^{1/3}(\alpha_n) & \text{if } \alpha_n \in 2^{\mathbb{N}} . \end{cases}$$

We want to show that it is eventually bounded away from every point in $[0,1]$. To this end let $x \in [0,1]$ be arbitrary. By Proposition 3.1.1.2 there exists $\alpha \in 2^{\mathbb{N}}$ such that

$$F^{1/3}(\alpha) \neq x \implies \exists \varepsilon > 0 : d(x, F^{1/3}(2^{\mathbb{N}})) > \varepsilon .$$

Now find $N, M \in \mathbb{N}$ such that for all $n \geqslant N$ we have $\overline{\alpha}M \neq \overline{\alpha_n}M$. It is an easy calculation[3] to see that for all such n we have

$$|F^{1/3}(\alpha) - F^{1/3}(\alpha_n)| \geqslant 3^{-M} = 2\delta .$$

---

[3] If $\overline{\beta}K \neq \overline{\gamma}K$, then $|F^p(\beta) - F^p(\gamma)| \geqslant (1-2p)p^{M-1}$

Now either $|x - F^{1/3}(\alpha)| < \delta$ or $|x - F^{1/3}(\alpha)| > 0$. In the second case there exists $\varepsilon > 0$ such that $d(x, F^{1/3}(2^{\mathbb{N}})) > \varepsilon$, which implies that $|x - x_n| > \varepsilon$ for all $n \in \mathbb{N}$ with $\alpha_n \in 2^{\mathbb{N}}$. In the first case for all $n \geqslant N$ for which $\alpha_n \in 2^{\mathbb{N}}$ we get

$$|x - x_n| = |x - F^{1/3}(\alpha_n)| \geqslant |F^{1/3}(\alpha) - F^{1/3}(\alpha_n)| - |x - F^{1/3}(\alpha)| ,$$
$$\geqslant 2\delta - |x - F^{1/3}(\alpha)| > \delta .$$

Notice that for n with $\alpha_n = \omega$ we have $|x - x_n| \geqslant 1 > \frac{1}{2}$. So in all cases we have the desired $|x_n - x| > \min\{\varepsilon, \delta, \frac{1}{2}\} > 0$ for all $n \geqslant N$. Therefore x_n is bounded away from every point in $[0, 1]$. Furthermore, if there exists m such that $x_m = \omega$, then $\alpha_m = \omega$. Thus $AS^L_{[0,1]} \implies AS^L_{2^{\mathbb{N}}}$.

To see that the converse holds we can use [30, Lemma 0.3] and the fact that there exists a point-wise continuous surjection $2^{\mathbb{N}} \to [0, 1]$ (see Section 3.1). $\qquad\square$

As was shown in [27] AS^L_X is equivalent to the following version of POS (remember that POS is the same with the assumption that f is uniformly continuous, which ensures that $\inf f$ exists).

> **(POS$_X^{\mathrm{pw}}$)** If f is a point-wise continuous, positive-valued mapping on a metric space, and if $\inf_X f$ exists, then $\inf_X f > 0$.

Lemma 8.3.2. *If f is a point-wise continuous, positively valued mapping on a metric space, then $g = \frac{1}{f}$ is also a point-wise continuous, positively valued mapping; furthermore*

- $\inf f$ *exists if and only if $g(X)$ is upper order located, and*

- $\inf f > 0$ *if and only if g is bounded.*

Proof. Straightforward. $\qquad\square$

Corollary 8.3.3. *The following are equivalent:*

1. $AS^L_{[0,1]}$

2. $POS^{pw}_{[0,1]}$

3. *Every point-wise continuous function $f : [0, 1] \to \mathbb{R}$ such that $f([0, 1])$ is located is bounded.*

Remember that in Corollary 3.3.11 it was shown that FAN$_\Delta$ is equivalent to every fully located, point-wise continuous function $[0, 1] \to \mathbb{R}$ being bounded.

Therefore we get the following nice characterisations of POS, POS^X_p and UCT being equivalent to all point-wise continuous functions on $[0, 1]$ with an additional property being bounded.

	additional property X
POS, FAN$_\Delta$	$f([a,b])$ is located for all $a \leqslant b$ (that is f is fully located)
POS$^{pw}_{[0,1]}$, AS$^L_{[0,1]}$	$f([0,1])$ is located
UCT	no assumption on f

8.4   Increasing Specker Sequences

As mentioned above Specker's original sequence is increasing. So it is natural to follow Dent [40] and consider the following principle.

> **(iAS)**   Every *increasing* sequence x_n in $\mathbb{R}$ that is eventually bounded away from every point in $[0, 1]$ is eventually bounded away from the entire interval.

If, for a sequence as in iAS, we ever find $n \in \mathbb{N}$ such that $x_n > 1$, then from then on we have $x_m \geqslant x_n > 1$ for all $m \geqslant n$. Therefore $\mathrm{AS}^L_{[0,1]}$ implies iAS. In turn, we can show the following.

Proposition 8.4.1. *iAS* $\implies$ *FAN$_\Delta$*

Proof. Let B be a decidable bar that is closed under extensions. Notice that we can decide for every $n \in \mathbb{N}$, whether n is a uniform bound of B. Now define a sequence $u_n \in 2^* \cup \{\omega\}$ by

$$u_n = \begin{cases} \omega & \text{if } n \text{ is a uniform bound} \\ u & \text{if } u \in 2^n \text{ is the (lexicographically) smallest element such that } u \notin B \end{cases}$$

It is easy to show that x_n defined by

$$x_n = \begin{cases} 2 & \text{if } u_n = \omega \\ F(u_n) & \text{if } u_n \in 2^* \end{cases}$$

is an increasing sequence, that is eventually bounded away from every point in $[0, 1]$. Thus there exists a n such that $x_n > 1$, which translates back into n being a uniform bound of B. $\qquad\square$

Thus we have shown that

$$\mathrm{FAN}_c \implies \mathrm{AS}^L_{[0,1]} \implies \mathrm{iAS} \implies \mathrm{FAN}_\Delta \,,$$

which of course raises the following question.

Question 13. *Are all of the implications FAN$_c$ $\implies$ AS$^L_{[0,1]}$ $\implies$ iAS $\implies$ FAN$_\Delta$ strict?*

Rather than focussing on the "anti-Specker"-side of things we could, of course, ask the same question about the recursive side and consider the following principle.

> **(iSS)**   There exists an increasing sequence

Obviously we have that iSS $\implies$ SS. So together with Proposition 5.4.2 we have

$$\mathrm{KT} \implies \mathrm{iSS} \implies \mathrm{SS} \,.$$

Question 14. *Does SS imply iSS? And does iSS imply KT?*

Notice that even if we start with a Specker sequence there is no hope to construct a monotone sequence as a subsequence of the original sequence.

Proposition 8.4.2. *If SS holds and every sequence in $[0, 1]$ that is bounded away from every point in $[0, 1]$ contains a monotone subsequence, then LPO holds.*

Proof. Let $(a_n)_{n \geqslant 1}$ be a binary increasing sequence and $(x_n)_{n \geqslant 1}$ be a sequence in $[0,1]$ that is eventually bounded away from every point in $[0,1]$. By our assumption there exists a monotone subsequence $(x_{k_n})_{n \geqslant 1}$. Without loss of generality this sequence is increasing. Using dependent choice we may construct a subsequence $(x_{k_{\ell_n}})_{n \geqslant 1}$ of the subsequence such that

$$x_{k_{\ell_n}} < x_{k_{\ell_{n+1}}} \ .$$

Now define a sequence $(y_n)_{n \geqslant 1}$ by

$$y_n = \begin{cases} x_{k_{\ell_n}} & \text{if } a_n = 0 \\ 1 - x_{k_{\ell_n}} & \text{if } a_n = 1 \ . \end{cases}$$

Given $x \in [0,1]$ there exists N and $\varepsilon > 0$ such that for all $n \geqslant N$

$$|x_{k_{\ell_n}} - x|, |x_{k_{\ell_n}} - (1-x)| > \varepsilon \ ,$$

which means that also $|(1 - x_{k_{\ell_n}}) - x| > \varepsilon$. In both cases $|y_n - x| > \varepsilon$, and so $(y_n)_{n \geqslant 1}$ is eventually bounded away from every point in $[0,1]$. Now assume that there exists a monotone subsequence y_{k_n}. If it is decreasing, then $a_{k_2} = 1$, since otherwise $y_{k_1} < x_{k_2}$. Similarly, if it is increasing, then $a_n = 0$ for all $n \in \mathbb{N}$. In other words LPO holds. □

8.5 Dirk Gently's Principle: $(\varphi \implies \psi) \vee (\psi \implies \varphi)$

One statement that is notably missing from the list of the "paradoxes of material implication" equivalent to LEM (see Proposition 1.1.6) is *Dirk Gently's Principle*[4]

> **(DGP)** If φ and ψ are any syntactically correct statements, then
>
> $$(\varphi \implies \psi) \vee (\psi \implies \varphi) \ .$$

DGP is listed as Proposition 5.13 in Principia Mathematica [113]. As a matter of fact the proof there is actually skipped and Russel and Whitehead "merely indicate the proposition[] used in the proofs." We can be a little bit more precise here.

Proposition 8.5.1. *DGP is implied by LEM and implies WLEM.*

Proof. The proof from LEM is straightforward: either ψ or $\neg\psi$ holds. In the first case $\varphi \implies \psi$ (weakening) and in the second case $\psi \implies \varphi$ (ex falso quodlibet). To see that it implies WLEM apply the principle to φ and $\neg\varphi$. Now either $\varphi \implies \neg\varphi$ or $\neg\varphi \implies \varphi$. In the first case the assumption that φ holds leads to a contradiction, whence $\neg\varphi$. Similarly, in the second case the assumption that $\neg\varphi$ holds leads to a contradiction, and hence $\neg\neg\varphi$. □

We can also show that it is equivalent to yet another "paradox of material implication".

Proposition 8.5.2. *DGP is equivalent to the statement that*

$$((\varphi \wedge \psi) \implies \vartheta) \implies ((\varphi \implies \vartheta) \vee (\psi \implies \vartheta)) \ , \tag{8.3}$$

---

[4]The name is based on the guiding principle of the protagonist of Douglas Adam's novel *Dirk Gently's Holistic Detective Agency* who believes in the "fundamental interconnectedness of all things."

to the statement that

$$((\varphi \implies \vartheta) \wedge (\psi \implies \beta)) \implies ((\varphi \implies \beta) \vee (\psi \implies \vartheta)) , \qquad (8.4)$$

and to the statement that

$$(\varphi \implies \psi \vee \vartheta) \implies (\varphi \implies \psi) \vee (\varphi \implies \vartheta) . \qquad (8.5)$$

Proof. To see the equivalence with the first statement assume that $(\varphi \wedge \psi) \implies \vartheta$. Now if DGP holds, then either $\varphi \implies \psi$ or $\psi \implies \varphi$. In the first case, if φ holds, then also $\varphi \wedge \psi$, and hence ϑ holds. Together that means that in the first case we have $\varphi \implies \vartheta$. In the second case, similarly, we can show that $\psi \implies \vartheta$.

Conversely, apply the statement to $\vartheta \equiv \varphi \wedge \psi$. Then the antecedent is always satisfied, which means that $(\varphi \implies (\varphi \wedge \psi)) \vee (\psi \implies (\varphi \wedge \psi))$. Hence the desired $(\varphi \implies \psi) \vee (\psi \implies \varphi)$ holds.

For the second statement assume that $(\varphi \implies \vartheta) \wedge (\psi \implies \beta)$. By DGP either $\varphi \implies \beta$ and we are done, or $\beta \implies \varphi$. But in that second case if we assume ψ also β holds, which in turn implies φ, which in turn implies ϑ. Together, in the second case, $\psi \implies \vartheta$.

Conversely we can apply the statement to $\vartheta \equiv \varphi$ and $\beta \equiv \psi$, which yields

$$((\varphi \implies \varphi) \wedge (\psi \implies \psi)) \implies ((\varphi \implies \psi) \vee (\psi \implies \varphi)) .$$

But since the antecedent is always satisfied, we get the desired $(\varphi \implies \psi) \vee (\psi \implies \varphi)$.

The last statement is implied by DGP: For either $q \implies r$ or $r \implies q$ .... Conversely we can apply 8.5 to $\varphi \wedge \psi$, φ, and ψ to get:

$$(\varphi \wedge \psi \implies \varphi \vee \psi) \implies ((\varphi \wedge \psi \implies \varphi) \vee (\varphi \wedge \psi \implies \psi)) .$$

Now clearly the antecedent is always satisfied. So we have that

$$(\varphi \wedge \psi \implies \varphi) \vee (\varphi \wedge \psi \implies \psi) ,$$

which is equivalent to the desired

$$(\psi \implies \varphi) \vee (\varphi \implies \psi) .$$

Hence DGP holds.   □

What makes the principle DGP interesting for our point of view is that we can actually show that it lies strictly between LEM and WLEM.

Proposition 8.5.3. *There exist topological models S and T such that*

$$S \nVdash LEM \qquad and \qquad S \Vdash DGP ,$$

and

$$T \nVdash DGP \qquad and \qquad T \Vdash WLEM .$$

Proof. Let $X = \{1, 2, 3, 4\}$ and let the topological space $S = (X, \sigma)$ be given by the base

$$\{\{1\}, \{1, 2\}, \{3\}, \{3, 4\}\} .$$

Then $S \nVdash$ LEM, since if φ is such that $[\![\varphi]\!] = \{1\}$ we have

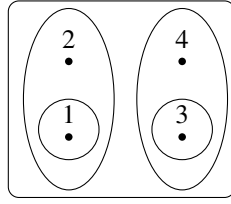$$[\![\neg\varphi]\!] = \text{Int}\left(\left(\{1\}\right)'\right) = \text{Int}\left(\{2, 3, 4\}\right) = \{3, 4\} .$$

Figure 8.1: A base of the space S

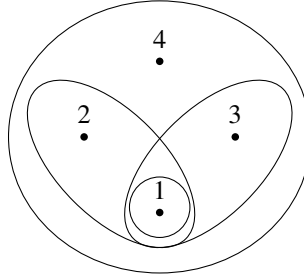Figure 8.2: A base of the space T

Hence $[\![\varphi \vee \neg\varphi]\!] = \{1,3,4\} \neq X$. Conversely we can check by hand, or simply with the aid of a computer (see Appendix B.1) that $S \Vdash$ DGP.

Now consider the topological space $T = (X, \tau)$ be given by the base $\{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3,4\}\}$. Again either through checking by hand, or simply with the aid of a computer (see Appendix B.1) that $T \Vdash$ WLEM. To see that DGP fails consider φ and ψ such that $[\![\varphi]\!] = \{1,2\}$ and $[\![\psi]\!] = \{1,3\}$. Then

$$[\![\varphi \implies \psi]\!] = \text{Int}\left((([\![\varphi]\!])' \cup [\![\psi]\!])\right) = \text{Int}\left(\{3,4\} \cup \{1,3\}\right) = \{1,3\} \;,$$

and similarly $[\![\psi \implies \varphi]\!] = \{1,2\}$. Hence

$$[\![(\varphi \implies \psi) \vee (\psi \implies \varphi)]\!] = \{1,2,3\} \neq X \;. \qquad \square$$

Thus we get the following simple hierarchy:

$$\text{LEM} \implies \text{DGP} \implies \text{WLEM}$$

As a final remark we would like to note that it seems that all simple propositional axiom schemata containing only few propositional symbols, and which are classically true (also see Propositions 1.1.4, 1.1.5, and 1.1.6) fall in one of these three categories (also see [51]).

8.6 Π_1^0 and Σ_1^0

The purpose of this overview is to tie in some of our principles with other presentations such as [107, 108, 66].

In the style of computability theory (of sets) we define a statement φ to be a Σ_1^0 statement if there exists a binary sequence $(a_n)_{n \geqslant 1}$ such that

$$\varphi \iff \exists n \in \mathbb{N} : a_n = 1 \;,$$

and a a Π_1^0 statement if there exists a binary sequence $(b_n)_{n \geqslant 1}$ such that

$$\varphi \iff \forall n \in \mathbb{N} : b_n = 0 \,.$$

We can then identify fragments of logical principles by restricting them to Σ_1^0 statements or Π_1^0 statements respectively. It easy, for example, to see that Σ_1^0–LEM is just LPO. Overall we get:

Proposition 8.6.1. *The following equivalences hold, where $\vdash_i$ means equivalence to $\top$, i.e. provable in intuitionistic logic.*

	Σ_1^0	Π_1^0	Σ_1^0–Π_1^0
LEM $(P \vee \neg P)$	LPO	WLPO	
WLEM $(\neg P \vee \neg\neg P)$	WLPO	WLPO	
Stability $(\neg\neg P \implies P)$	MP	$\vdash_i$	
DM1 $(\neg(P \wedge Q) \implies (\neg P \vee \neg Q))$	LLPO	MP$^\vee$	WLPO
DM2 $(\neg(P \vee Q) \implies (\neg P \wedge \neg Q))$	$\vdash_i$	$\vdash_i$	$\vdash_i$
DM1$'$ $(\neg(\neg P \wedge \neg Q) \implies (P \vee Q))$	MP	LLPO	LPO
DM2$'$ $(\neg(\neg P \vee \neg Q) \implies (P \wedge Q))$	MP	$\vdash_i$	MP
DGP $(P \implies Q) \vee (Q \implies P)$	LLPO	LLPO	WLPO

Proof. Most of these equivalencies are trivial. Notice that Π_1^0 statements are stable, that is we can eliminate preceding double negations.

The equivalencies for the restricted versions of DGP are non-trivial, and the proofs are entertaining: First assume that LLPO holds. And consider two binary sequences a_n and b_n. Now let α be the sequence

$$a_1 b_1 a_2 b_2 \dots \,,$$

and let α' be the sequence

$$\alpha'_n = \begin{cases} 1 & \text{if } \sum_{i=1}^n \alpha_n = 1 \\ 0 & \text{otherwise;} \end{cases}$$

that is α' has at most one term equal to 1. Since we assume LLPO

$$\forall n \in \mathbb{N} : \alpha'_{2n} = 0 \vee \forall n \in \mathbb{N} : \alpha'_{2n+1} = 0 \,. \tag{8.6}$$

In the first case assume that $n \in \mathbb{N}$ is such that $b_n = 1$. Without loss of generality let n be minimal. Then there must be $m \leqslant n$ such that $a_m = 1$, since if $a_m = 0$ for all $m \leqslant n$, and $b_k = 0$ for all $k < n$ we have $\alpha_k = 0$ for all $k < 2n$ and $\alpha_{2n} = b_n = 1$, which means that $\alpha'_{2n} = 1$ contradicting our assumption. Hence in the first case we have $\exists n \in \mathbb{N} : b_n = 1 \implies \exists n \in \mathbb{N} : a_n = 1$. Symmetrically in the second case we have that $\exists n \in \mathbb{N} : a_n = 1 \implies \exists n \in \mathbb{N} : b_n = 1$. Thus we have shown that LLPO implies DGP restricted to Σ_1^0-formulas.

Conversely let a_n be a binary sequence with at most one term equal to 1. By Σ_1^0-DGP we get that

$$(\exists n \in \mathbb{N} : a_{2n} = 1 \implies \exists n \in \mathbb{N} : a_{2n+1} = 1) \vee (\exists n \in \mathbb{N} : a_{2n+1} = 1 \implies \exists n \in \mathbb{N} : a_{2n} = 1) \,.$$

In the first case we see that $\forall n \in \mathbb{N} : a_{2n} = 0$, since the assumption that $\exists n \in \mathbb{N} : a_{2n} = 1$ leads to the contradiction that there is more than one term equalling 1. Similarly in the second case we can show that $\forall n \in \mathbb{N} : a_{2n+1} = 0$. Thus we have shown that Σ_1^0-DGP implies LLPO.

Next, let a_n and b_n be binary sequences and construct α and α' as above. Using LLPO we can, again, make the decision in Equation 8.6. Furthermore, let us assume that the first alternative is the case, and

assume that $\forall n \in \mathbb{N} : a_n = 0$. Also assume that there exists $n \in \mathbb{N}$ such that $b_n = 1$; and without loss of generality n is the minimal such number. Then $\alpha'_{2n} = 1$, contradicting our assumption. Together we have that in case the first alternative holds we have

$$\forall n \in \mathbb{N} : a_n = 0 \implies \forall n \in \mathbb{N} : b_n = 0 .$$

Similarly, in case the second alternative holds we can show that

$$\forall n \in \mathbb{N} : b_n = 0 \implies \forall n \in \mathbb{N} : a_n = 0 .$$

That means that .

Conversely let a_n be a binary sequence with at most one term equal to 1. By Π^0_1-DGP we get that

$$(\forall n \in \mathbb{N} : a_{2n} = 0 \implies \forall n \in \mathbb{N} : a_{2n+1} = 0) \vee (\forall n \in \mathbb{N} : a_{2n+1} = 0 \implies \forall n \in \mathbb{N} : a_{2n} = 0) .$$

In the first case assume that there is n such that $a_{2n+1} = 1$. Then, since there is at most one term equal to 1 we have that $\forall n \in \mathbb{N} : a_{2n} = 0$, which together with the case we are in means that $\forall n \in \mathbb{N} : a_{2n+1} = 0$; a contradiction. Hence in that case we must actually have $\forall n \in \mathbb{N} : a_{2n+1} = 0$. Similarly in case the second alternative holds we can show that $\forall n \in \mathbb{N} : a_{2n} = 0$. Thus Π^0_1-DGP implies LLPO.

Interestingly, the mixed case is equivalent to WLPO: let a_n and b_n be binary sequences (the sequence b_n actually plays no role in what follows). By WLPO either $\forall n \in \mathbb{N} : a_n = 0$ or $\neg\forall n \in \mathbb{N} : a_n = 0$. In the first case, also $\exists n \in \mathbb{N} : b_n = 1 \implies \forall n \in \mathbb{N} : a_n = 0$. In the second case $\forall n \in \mathbb{N} : a_n = 0 \implies \exists n \in \mathbb{N} : b_n = 1$, since the antecedent contradicts our assumption.

Conversely, let a_n be an arbitrary binary sequence, and apply $\Sigma^0_1 - \Pi^0_1$-DGP to this sequence (and itself). This yields

$$\exists n \in \mathbb{N} : a_n = 1 \implies \forall n \in \mathbb{N} : a_n = 0 \vee \forall n \in \mathbb{N} : a_n = 0 \implies \exists n \in \mathbb{N} : a_n = 1 .$$

In case the first alternative holds we must have $a_n = 0$ for all $n \in \mathbb{N}$, since the assumption that there is n with $a_n = 1$ leads to the contradiction that $a_n = 0$ for that same n. In case the second alternative holds the assumption that $\forall n \in \mathbb{N} : a_n = 0$ leads to a contradiction, hence in that case $\neg\forall n \in \mathbb{N} : a_n = 0$. $\qquad\square$

Question 15. *Which principles are $\Sigma^0_1 - \Sigma^0_1$–PP, $\Pi^0_1 - \Pi^0_1$–PP, $\Pi^0_1 - \Sigma^0_1$–PP, and $\Sigma^0_1 - \Pi^0_1$–PP equivalent to, where PP is Peirce's law, as in Proposition 1.1.6.1? (It is easy to see that the Σ^0_1 and Π^0_1 versions of the special case $(\neg\varphi \implies \varphi) \implies \varphi$ of Peirce's law is equivalent to MP and provable in intuitionistic logic respectively.)*

8.7 ¬WLPO and ¬LPO

Proposition 8.7.1. *¬WLPO is equivalent to the statement that every mapping of a complete metric space into a metric space is non-discontinuous.*

Proof. This proof can be found in [65]. $\qquad\square$

The next proposition relies on a property proposed by M. Escardó. A set X is called *searchable*,[5] if we can decide any predicate on it. That is if for any $p : X \to 2$

$$\forall x \in X : p(x) = 0 \vee \exists x \in X : p(x) = 1 .$$

Proposition 8.7.2. *The following are equivalent:*

---

[5]Notice that Escardó's definition is slightly different, but is equivalent to this one in the presence of unique choice.

1. *¬LPO*

2. *Every searchable subset of $\mathbb{N}$ is not unbounded.*

3. *Every searchable subset of $\mathbb{N}$ is bounded.*

Proof. Assume ¬LPO and let $S \subset \mathbb{N}$ be searchable. Assume furthermore that S is unbounded, that is, with a bit of work, there exists an bijection $s : \mathbb{N} \to S$. Then, if a_n is an arbitrary binary sequence consider $p : S \to 2$ defined by

$$p(n) = a_{s^{-1}(n)} \ .$$

Since S is searchable we can therefore decide, whether $\forall n : a_n = 0$ or whether $\exists n : a_n = 1$. In other words LPO holds; a contradiction and hence S is not unbounded. □

Even though ¬LPO and ¬WLPO do not seem to have many natural equivalences they extend the recursive hierarchy nicely.

$$SC \implies KT \implies iSS \implies SS \implies \neg WLPO \implies \neg LPO$$

Appendix A

List of Open Questions

Question 1. *Which principle is the Cantor–Bernstein–Schröder theorem equivalent to? It seems likely that the answer heavily depends on the precise formulation.*

Question 2. *Is every space-filling curve non-injective? More precise: if f is a space-filling curve, then does there exist $x \neq y$ with $f(x) = f(y)$?*

Question 3. *To what principle is $WKL'(k)$ equivalent to for $k > \frac{1}{2}$?*

Question 4. *Is there a general principle that ensures that complete sets are mapped to complete sets? The above example suggests that connectedness may play a role.*

Question 5. *What are the following principles equivalent to?*
If $B \subset 2^*$ is a c-bar (or Π_1^0-bar), then

$$\lim_{n \to \infty} \frac{|\{u \notin B \,|\, |u| = n\}|}{2^n} = 0 \, .$$

eventually.

Question 6. *Is Dini's theorem [17] for point-wise continuous functions equivalent to UCT?*

Question 7. *Is there a version of the fan theorem equivalent to the following statement: every bounded point-wise continuous function $f : [0,1] \to \mathbb{R}$ is integrable.*

Question 8. *Are there any other equivalences to wBD-N? Can the issues outlined in Remark 4.2.3 be resolved?*

Question 9. *Is the existence of a bijection $\mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ equivalent to SS or KT (or neither)?*

Question 10. *Does WKL imply FAN_{stable} or $FAN_{\Pi_1^0}$?*

Question 11. *Is there a (topological) model in which MC fails?*

Question 12. *What about the general principle? Is $LLPO_n$ equivalent to the statement that whenever $x_1, \ldots, x_n$ are reals such that $\prod_{i=1}^n x_i = 0$, there exists j such that*

$$\prod_{i=1, \, i \neq j}^n x_i = 0 \, ?$$

The forward direction is trivial. However the converse does not generalise straightforwardly

Question 13. *Are all of the implications $FAN_c \implies AS^L_{[0,1]} \implies iAS \implies FAN_\Delta$ strict?*

Question 14. *Does SS imply iSS? And does iSS imply KT?*

Question 15. *Which principles are $\Sigma^0_1 - \Sigma^0_1$–PP, $\Pi^0_1 - \Pi^0_1$–PP, $\Pi^0_1 - \Sigma^0_1$–PP, and $\Sigma^0_1 - \Pi^0_1$–PP equivalent to, where PP is Peirce's law, as in Proposition 1.1.6.1? (It is easy to see that the Σ^0_1 and Π^0_1 versions of the special case $(\neg \varphi \implies \varphi) \implies \varphi$ of Peirce's law is equivalent to MP and provable in intuitionistic logic respectively.)*

Appendix B

Source Code

B.1 Topological Models

The Python[1] program used to check all possibilities in the proof of Proposition 8.5.3 is

```python
# finite topological models
class topology:
    def __init__(self,subbase):
        self.X = frozenset.union(*subbase)
        self.base= {i.intersection(j)  for i in subbase for j in subbase}
        top = self.base
        topa, topb = len(top), 0
        while topa > topb:
            top = {i.union(j)  for i in top for j in top}
            topb = topa
            topa = len(top)
        self.top = top

    def interior(self,A):
        L = [U for U in self.top if U <= A]
        if len(L)> 0: return frozenset.union(*L)
        else: return frozenset('')
    def compl(self,A): return frozenset({x for x in self.X if x not in A})
    def neg(self,A): return self.interior(self.compl(A))
    def impl(self,A,B):
        return self.interior( self.compl(A) | B )

    def LEM(self):
        return all([self.X == (U | self.neg(U)) for U in self.top])
    def WLEM(self):
        return all([self.X == (self.neg(U) | (self.neg(self.neg(U)))) for U in
            ↪  self.top])
    def PimpQ(self):
```

---

[1]Version 2.7 or 3.x

```
28          return all([self.X == (self.impl(U,V) |  self.impl(V,U) ) for U in
   ↪   self.top for V in self.top])

30      def testallprop(self):
31          print("")
32          print("LEM: " + str(self.LEM()))
33          print("P->Q v Q->P: " + str(self.PimpQ()))
34          print("WLEM: " + str(self.WLEM()))
35          print("")

37  T = topology({ frozenset('1'), frozenset('12'), frozenset('3'), frozenset('34')
   ↪  })
38  S = topology({ frozenset('12'), frozenset('13'), frozenset('1234')})

40  T.testallprop()
41  S.testallprop()
```

Bibliography

[1]   O. Aberth. *Computable analysis*. McGraw-Hill, New York, 1980.

[2]   P. Aczel and M. Rathjen. Notes on constructive set theory. Technical Report 40, Institut Mittag-Leffler, The Royal Swedish Academy of Sciences, 2001.

[3]   J. Avigad, E. T. Dean, and J. Rute. A metastable dominated convergence theorem. *Journal of Logic and Analysis*, 4, 2012.

[4]   A. Bauer. König's lemma and kleene tree. URL, http://math.andrej.com/2006/04/25/konigs-lemma-and-the-kleene-tree/, 2006. blog entry.

[5]   A. Bauer. An injection from the baire space to natural numbers. *Mathematical Structures in Computer Science*, 25:1484–1489, 10 2015.

[6]   M. J. Beeson. *Foundations of Constructive Mathematics*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer Verlag, 1985.

[7]   J. Berger. The fan theorem and uniform continuity. In *Proceedings of the First international conference on Computability in Europe: new Computational Paradigms*, CiE'05, pages 18–22, Berlin, Heidelberg, 2005. Springer-Verlag.

[8]   J. Berger. The logical strength of the uniform continuity theorem. In A. Beckmann, U. Berger, B. Löwe, and J. V. Tucker, editors, *Logical Approaches to Computational Barriers*, number 3988 in Lecture Notes in Computer Science, pages 35–39. Springer Berlin / Heidelberg, 2006.

[9]   J. Berger. Weak König's lemma implies the fan theorem for c-bars. In *Computation and Logic in the Real World, CiE 2007, Quaderni del Dipartimento di Scienze Matematiche e Informatiche "Roberto Magari"*, pages 86–88. University of Siena, 2007.

[10] J. Berger and D. S. Bridges. A Bizarre Property Equivalent to the Π_1^0-Fan Theorem. *Logic Journal of IGPL*, 14(6):867–871, 2006.

[11] J. Berger and D. S. Bridges. A fan-theoretic equivalent of the antithesis of Specker's theorem. *Indag. Math. (N.S.)*, 18(2):195–202, 2007.

[12] J. Berger and D. S. Bridges. The fan theorem and positive-valued uniformly continuous functions on compact intervals. *New Zealand Journal of Mathematics*, 38:129–135, 2008.

[13] J. Berger and D. S. Bridges. A constructive study of Landau's summability theorem. *Journal of Universal Computer Science*, 16(18):2523–2534, Sep 2010.

[14] J. Berger, D. S. Bridges, and E. Palmgren. Double Sequences, almost Cauchyness and BD-N. *Logic Journal of IGPL*, 20(1):349–354, 2012.

[15] J. Berger, D. S. Bridges, and P. Schuster. The fan theorem and unique existence of maxima. *Journal of Symbolic Logic*, 71(2):713–720, 06 2006.

[16] J. Berger, H. Ishihara, and P. Schuster. The weak könig lemma, brouwer's fan theorem, de morgan's law, and dependent choice. *Reports on Mathematical Logic*, 47:63–86, 2012.

[17] J. Berger and P. Schuster. Classifying dini's theorem. *Notre Dame Journal of Formal Logic*, 47(2):253–262, 2006.

[18] U. Berger. A computational interpretation of open induction. In *Logic in Computer Science, 2004. Proceedings of the 19th Annual IEEE Symposium on*, pages 326–334. IEEE, 2004.

[19] E. Bishop. *Foundations of Constructive Analysis*. McGraw-Hill, 1967.

[20] E. Bishop and D. S. Bridges. *Constructive Analysis*. Springer-Verlag, 1985.

[21] V. Brattka and G. Gherardi. Weihrauch degrees, omniscience principles and weak computability. *Journal of Symbolic Logic*, 76(1):143–176, 2011.

[22] D. S. Bridges. Some notes on continuity in constructive analysis. *Bull London Math. Soc.*, 8(2):179–182, 1976.

[23] D. S. Bridges. Constructive mathematics: a foundation for computable analysis. *Theoretical Computer Science*, 219(1–2):95 – 109, 1999.

[24] D. S. Bridges. A constructive look at functions of bounded variation. *Bulletin of the London Mathematical Society*, 32(3):316, 2000.

[25] D. S. Bridges. Constructive notions of equicontinuity. *Archive for Mathematical Logic*, 48:437–448, 2009. 10.1007/s00153-009-0131-9.

[26] D. S. Bridges. Inheriting the anti-specker property. *Documenta Mathematica*, 15:973–980, 2010.

[27] D. S. Bridges, J. Dent, and M. McKubre-Jordens. Zero-stability in constructive analysis. preprint, March 2013.

[28] D. S. Bridges, J. Dent, and M. McKubre-Jordens. Constructive connections between anti-specker, positivity, and fan-theoretic properties. *New Zealand Journal of Mathematics*, 44:21–33, 2014.

[29] D. S. Bridges and H. Diener. The pseudocompactness of $[0, 1]$ is equivalent to the uniform continuity theorem. *Journal of Symbolic Logic*, 72(4):1379–1384, 2007.

[30] D. S. Bridges and H. Diener. The anti-specker property, positivity, and total boundedness. *Mathematical Logic Quarterly*, 56(4):434–441, 2010.

[31] D. S. Bridges and R. Havea. A constructive analysis of a proof that the numerical range is convex. *LMS Journal of Computation and Mathematics*, 3:191–206, 2000.

[32] D. S. Bridges and H. Ishihara. Linear mappings are fairly well-behaved. *Archiv der Mathematik*, 54(6):558–562, 1990.

[33] D. S. Bridges and H. Ishihara. A definitive constructive open mapping theorem? *Mathematical Logic Quarterly*, 44:545–552, 1998.

[34] D. S. Bridges, H. Ishihara, and P. Schuster. Compactness and continuity, constructively revisited. In *Proceedings of the 16th International Workshop and 11th Annual Conference of the EACSL on Computer Science Logic*, CSL '02, pages 89–102, London, UK, UK, 2002. Springer-Verlag.

[35] D. S. Bridges, H. Ishihara, P. Schuster, and L. S. Vîţă. Strong continuity implies uniform sequential continuity. *Archive for Mathematical Logic*, 44(7):887–895, 2005.

[36] D. S. Bridges and F. Richman. *Varieties of Constructive Mathematics*. Cambridge University Press, 1987.

[37] D. S. Bridges and L. S. Vîţă. *Techniques of constructive analysis*. Universitext. Springer, New York, 2006.

[38] T. Coquand. A note on the open induction principle. Chalmers University, 1997.

[39] D. Dalen. *Logic and Structure*. Universitext (1979). Springer, 2004.

[40] J. Dent. *Anti-Specker Properties in Constructive Reverse Mathematics*. PhD thesis, University of Canterbury, 2013.

[41] R. Devaney. *An Introduction to Chaotic Dynamical Systems*. Addison–Wesley Publishing, 1987.

[42] R. Diaconescu. Axiom of choice and complementation. *Proceedings of the American Mathematical Society*, 51(1):176–178, 1975.

[43] H. Diener. *Compactness under constructive scrutiny*. Phd thesis, University of Canterbury, Christchurch, New Zealand, 2008.

[44] H. Diener. Reclassifying the antithesis of specker's theorem. *Archive for Mathematical Logic*, 51:687–693, 2012.

[45] H. Diener. Weak König's lemma implies the uniform continuity theorem. *Computability*, 2(1):9–13, 2013.

[46] H. Diener. Variations on a theme by Ishihara. *Mathematical Structures in Computer Science*, 25:1569–1577, 10 2015.

[47] H. Diener and A. Hedin. The Vitali covering theorem in constructive mathematics. *Journal of Logic and Analysis*, 4, 2012.

[48] H. Diener and I. Loeb. Sequences of real functions on $[0, 1]$ in constructive reverse mathematics. *Annals of Pure and Applied Logic*, 157:50–61, 2009.

[49] H. Diener and I. Loeb. Constructive reverse investigations into differential equations. *Journal of Logic and Analysis*, 3, 2011.

[50] H. Diener and R. Lubarsky. Notions of cauchyness and metastability. In S. Artemov and A. Nerode, editors, *Logical Foundations of Computer Science*, pages 140–153. Springer International Publishing, 2018.

[51] H. Diener and M. McKubre-Jordens. Classifying material implications over minimal logic. 2016. submitted, May 2016.

[52] R. Diestel. *Graph Theory*. Graduate Texts in Mathematics. Springer, 2005.

[53] S. Elaydi. *Discrete Chaos, Second Edition: With Applications in Science and Engineering*. CRC Press, 2007.

[54] M. Fourman and J. Hyland. Sheaf models for analysis. In M. Fourman, C. Mulvey, and D. Scott, editors, *Applications of Sheaves*, volume 753 of *Lecture Notes in Mathematics*, pages 280–301. Springer Berlin / Heidelberg, 1979. 10.1007/BFb0061823.

[55] H. Friedman. Some systems of second order arithmetic and their use. In *Proceedings of the International Congress of Mathematicians, Vancouver*, volume 1., 1974.

[56] L. Goble. *The Blackwell Guide to Philosophical Logic*. Blackwell Philosophy Guides. Wiley, 2001.

[57] N. Goodman and J. Myhill. Choice implies excluded middle. *Mathematical Logic Quarterly*, 24(25-30):461–461, 1978.

[58] R. J. Grayson. Heyting-valued semantics. In G. L. G. Lolli and A. Marcja, editors, *Logic Colloquium '82*, volume 112 of *Studies in Logic and the Foundations of Mathematics*, pages 181–208. Elsevier, 1984.

[59] G. Hämmerlin and K.-H. Hoffmann. *Numerische Mathematik*. Springer-Verlag Berlin Heidelberg, 4. edition, 1994.

[60] M. Hendtlass. *Constructing fixed points and economic equilibria*. PhD thesis, University of Leeds, 2013.

[61] M. Hendtlass and R. Lubarsky. Separating fragments of WLEM, LPO, and MP. *The Journal of Symbolic Logic*, 81(4):1315–1343, 2016.

[62] B. Hillam. A characterization of the convergence of successive approximations. *The American Mathematical Monthly*, 83(4):273, April 1976.

[63] H. Ishihara. An omniscience principle, the König lemma, and the Hahn–Banach theorem. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 36(3):237–240, 1990.

[64] H. Ishihara. Continuity and nondiscontinuity in constructive mathematics. *Journal of Symbolic Logic*, 56(4):1349–1354, 1991.

[65] H. Ishihara. Continuity properties in constructive mathematics. *Journal of Symbolic Logic*, 57(2):557–565, 1992.

[66] H. Ishihara. Sequential continuity of linear mappings in constructive mathematics. *Journal of Universal Computer Science*, 3(11):1250–1254, 1997.

[67] H. Ishihara. Informal constructive mathematics. Technical Report CDMTCS-229, University of Auckland, Centre for Discrete Mathematics and Theoretical Computer Science, January 2004.

[68] H. Ishihara. Reverse mathematics in bishop's constructive mathematics. *Philosophia Scientiæ*, Cahier spécial 6:43–59, 2006.

[69] H. Ishihara. Weak König's lemma implies Brouwer's fan theorem: A direct proof. *Notre Dame Journal of Formal Logic*, 47(2):249–252, 2006.

[70] H. Ishihara and S. Yoshida. A constructive look at the completeness of the space $\mathscr{D}(\mathbf{R})$. *Journal of Symbolic Logic*, 67(4):1511—1519, 2002.

[71] W. Julian and F. Richman. A uniformly continuous function on $[0,1]$ that is everywhere different from its infimum. *Pacific Journal of Mathematics*, 111(2):333–340, 1984.

[72] U. Kohlenbach. On weak markov's principle. *Mathematical Logic Quarterly*, 48:59–65, 2002.

[73] U. Kohlenbach. *Applied Proof Theory: Proof Interpretations and their Use in Mathematics*. Springer Monographs in Mathematics. Springer Berlin Heidelberg, 2008.

[74] T. Körner. *A Companion to Analysis: A Second First and First Second Course in Analysis*. Graduate studies in mathematics. American Mathematical Society, 2004.

[75] B. Kushner and L. Leffman. *Lectures on Constructive Mathematical Analysis*. Translations of mathematical monographs. American Mathematical Society, 1984.

[76] P. Lietz. *From Constructive Mathematics to Computable Analysis via the Realizability Interpretation*. PhD thesis, TU Darmstadt, 2004.

[77] I. Loeb. Questioning constructive reverse mathematics. *Constructivist Foundations*, 7(2):131–140, 2012.

[78] R. Lubarsky. On the failure of BD-N and BD, and an application to the anti-specker property. *Journal of Symbolic Logic*, 78(1):39–56, 2013.

[79] R. Lubarsky and H. Diener. Separating the fan theorem and its weakenings. In S. Artemov and A. Nerode, editors, *Logical Foundations of Computer Science*, volume 7734 of *Lecture Notes in Computer Science*, pages 280–295. Springer Berlin Heidelberg, 2013.

[80] R. Lubarsky and H. Diener. Principles weaker than BD-N. *Journal of Symbolic Logic*, 78(3):873–885, 2014.

[81] R. Lubarsky, F. Richman, and P. Schuster. The kripke schema in metric topology. *Mathematical Logic Quarterly*, pages n/a–n/a, 2012.

[82] M. Mandelkern. Components of an open set. *Journal of the Australian Mathematical Society (Series A)*, 33(2):249–261, 1982.

[83] M. Mandelkern. *Constructive Continuity*. Memoirs of the American Mathematical Society. American Mathematical Society, 1983.

[84] M. Mandelkern. Constructively complete finite sets. *Mathematical Logic Quarterly*, 34(2):97–103, 1988.

[85] M. Mandelkern. Limited omniscience and the Bolzano-Weierstrass principle. *Bull. London Math. Soc.*, 20:319–320, 1988.

[86] M. Mandelkern. Brouwerian counterexamples. *Mathematics Magazine*, 62(1):3–27, 02 1989.

[87] P. Martin-Löf. An intuitionistic theory of types. In J. M. Smith and G. Sambin, editors, *Twenty-five years of constructive type theory*, volume 36, pages 127–172. Oxford University Press, 1998.

[88] I. Moerdijk. Heine-Borel does not imply the fan theorem. *Journal of Symbolic Logic*, 49(2):514–519, 1984.

[89] J. Moschovakis. *Another Unique Weak König's Lemma*, pages 343–352. Ontos Verlag, 2013.

[90] J. Myhill. Formal systems of intuitionistic analysis i. In B. V. Rootselaar and J. Staal, editors, *Logic, Methodology and Philosophy of Science III*, volume 52 of *Studies in Logic and the Foundations of Mathematics*, pages 161 – 178. Elsevier, 1968.

[91] T. Nemoto. Weak weak könig's lemma in constructive reverse mathematics. In *Proceedings of the 10th Asian Logic Conference (ALC 10), Kobe University in Kobe, Hyogo, Japan, September 1-6, 2008*, pages 263—270, 2008.

[92] J. Oxley. *Matroid Theory*. Oxford Graduate Texts in Mathematics. Oxford University Press, 2006.

[93] F. Richman. Polynomials and linear transformations. *Linear Algebra and its Applications*, Volume 131:Pages 131–137, 1990.

[94] F. Richman. Weak markov's principle, strong extensionality, and countable choice. preprint, 2000.

[95] F. Richman. Omniscience principles and functions of bounded variation. *Mathematical Logic Quarterly*, 48(1):111–116, 2002.

[96] F. Richman. Intuitionistic notions of boundedness in $\mathbb{N}$. *Mathematical Logic Quarterly*, 55(1):31–36, 2009.

[97] H. Sagan. *Space-filling curves*. Universitext Series. Springer-Verlag, 1994.

[98] P. Schuster. Unique solutions. *Mathematical Logic Quarterly*, 52(6):534–539, 2006.

[99] P. Schuster. Induction in algebra: A first case study. In *Proceedings of the 2012 27th Annual IEEE/ACM Symposium on Logic in Computer Science*, LICS '12, pages 581–585, Washington, DC, USA, 2012. IEEE Computer Society.

[100] D. Scott. Extending the topological interpretation to intuitionistic analysis. *Compositio Mathematica*, 20:194–210, 1968.

[101] D. Scott. Extending the topological interpretation to intuitionistic analysis, ii. In J. M. A. Kino and R. Vesley, editors, *Intuitionism and Proof Theory: Proceedings of the Summer Conference at Buffalo N.Y. 1968*, volume 60 of *Studies in Logic and the Foundations of Mathematics*, pages 235 – 255. Elsevier, 1970.

[102] S. Simpson. *Subsystems of second order arithmetic*. Springer-Verlag, Berlin, 1999.

[103] R. Smullyan. *First-order Logic*. Dover Books on Mathematics Series. Dover, 1995.

[104] E. Specker. Nicht konstruktiv beweisbare Sätze der Analysis. *Journal of Symbolic Logic*, 14:145–158, 1949.

[105] T. Tao. Soft analysis, hard analysis, and the finite convergence principle. http://terrytao.wordpress.com/2007/05/23/soft-analysis-hard-analysis-and-the-finite-convergence-principle/, May 2007.

[106] T. Tao. *An Introduction to Measure Theory*. Graduate studies in mathematics. American Mathematical Soc., 2011.

[107] A. S. Troelstra and D. van Dalen. *Constructivism in mathematics. Vol. I*, volume 121 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1988.

[108] A. S. Troelstra and D. van Dalen. *Constructivism in mathematics. Vol. II*, volume 123 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1988.

[109] J. van Oosten. *Realizability: An Introduction to its Categorical Side*. Studies in Logic and the Foundations of Mathematics. Elsevier Science, 2008.

[110] W. Veldman. Brouwer's fan theorem as an axiom and as a contrast to kleene's alternative. *arXiv preprint arXiv:1106.2738*, 2011.

[111] W. Veldman. The Principle of Open Induction on Cantor space and the Approximate-Fan Theorem. *ArXiv e-prints*, Aug. 2014.

[112] K. Weihrauch. *Computable Analysis : An Introduction.* Texts in Theoretical Computer Science. Springer, 2000.

[113] A. Whitehead and B. Russell. *Principia Mathematica.* University Press, 2nd edition, 1968.

[114] X. Yu and S. G. Simpson. Measure theory and weak König's lemma. *Archive for Mathematical Logic*, 30(3):171–180, 1990.

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