# Probing quantum steering through incompatible measurements

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## Abstract

This thesis investigates classical models of correlation experiments from a quantum measurement theoretical point of view. Of special interest are the concept of measurement incompatibility and the phenomenon of quantum steering.

As the main result, we establish a one-to-one connection between non-joint measurability, i.e. the impossibility of measuring two or more observables simultaneously, and quantum steering, i.e. the possibility of one party to affect a space-like separated party's quantum state by the means of local actions and classical communication. The result can be used to translate various results between the relatively new research field of quantum steering and the older field of incompatibility. As examples, we use steering inequalities as incompatibility criteria and map joint observables to local hidden state models.

The main result comes with some possible generalisations. The generalisations discussed here are strongly motivated by quantum measurement theory and they concentrate on continuous variable and channel versions of steering. The resulting formalism not only extends the aforementioned one-to-one connection, but it also has natural applications to Gaussian steering and to temporal correlations.

Whereas the main result focuses on the connection between non-joint measurability and steering-like phenomena, in the process we also derive steering witnesses and bounds on noise tolerance of incompatible observables. As examples, we map certain entropic uncertainty relations to steering inequalities and use known steering techniques to prove the tightness of the aforementioned noise bounds on incompatibility.

On top of the measurement theoretical work, we introduce a technique for witnessing steering in scenarios with one completely uncharacterised and one dimension-bounded observer. The resulting witnesses are motivated by former works on entanglement theory and, despite being more general, they don't weaken the detection strength of the known steering criteria in typical symmetric scenarios.

## Zusammenfassung

Diese Arbeit untersucht klassische Modelle von Korrelationsexperimenten wie Makrorealismus und lokalen Realismus aus einer messtheoretischen Perspektive. Von besonderem Interesse sind das Konzept der Inkompatibilität und das Phänomen der Quantensteuerung.

Als Hauptergebnis stellen wir eine Eins-zu-eins-Verbindung zwischen der nicht gemeinsamen Messbarkeit her, d. h. der Unmöglichkeit, zwei oder mehr Observablen gleichzeitig zu messen, und der Quantensteuerung. Eine solche Verbindung erlaubt es, verschiedene Ergebnisse zwischen dem relativ neuen Forschungsgebiet der Quantensteuerung und dem älteren Bereich der Inkompatibilität zu übersetzen. Als Beispiele verwenden wir Ungleichungen als Inkompatibilitätskriterien und bilden gemeinsame Observablen auf lokale Modelle versteckter Zustände ab.

Das Hauptergebnis eröffnet einige Möglichkeiten für Verallgemeinerungen. Die hier betrachteten Verallgemeinerungen sind stark durch die Quantenmesstheorie motiviert und konzentrieren sich auf kontinuierliche Variablen und Quantensteuerung für Kanäle. Die Tragweite unseres Ansatzes zeigt sich nicht nur in der Erweiterung der oben genannten Eins-zu-eins-Verbindung auf allgemeinere Szenarien, sondern auch in Anwendungen zur Steuerung mit kanonischen Quadraturen und dem Nachweis einer strengen Hierarchie zwischen zeitlicher Steuerung von Quantenzuständen und Makrorealismus.

Während sich das Hauptergebnis auf den Zusammenhang zwischen nicht-gemeinsamer Messbarkeit und steuerungsähnlichen Phänomenen konzentriert, nutzen wir auch den messtheoretischen Ansatz zur Ableitung von Steuerungskriterien und gemeinsamen Messunsicherheitsbeziehungen. Als Beispiele zeigen wir, wie bestimmte entropische Unschärferelationen als Steuerungsungleichungen verwendet werden können und wie bekannte Steuerungstechniken verwendet werden können, um die Exaktheit bestimmter Unschärferelationen zu beweisen.

Zusätzlich zur messtheoretischen Arbeit führen wir eine Technik ein, mit der wir Steuerung in einem Szenario beobachten können, in dem eine Partei völlig uncharakterisiert ist und der anderen Partei nur die Dimension ihres Systems bekannt ist. Solche Kriterien werden durch frühere Arbeiten zur Verschränkungstheorie motiviert, und sie schwächen überraschenderweise nicht die Nachweismöglichkeiten bekannter Steuerungskriterien in typischen symmetrischen Szenarien.

# Introduction

Quantum mechanics possesses numerous non-classical properties such as entanglement, non-locality and contextuality. Some of the quantum features are present already on a single system and some appear only in distributed scenarios. For the purposes of this thesis, one property from each category will be of special interest, namely, quantum steering and quantum incompatibility.

Quantum steering describes how actions taking place on one quantum system can affect another space-like separated quantum system in a way not describable by classical mechanics. Originally discussed by Schrödinger [1] and motivated by the work of Einstein, Podolsky and Rosen [2], quantum steering has recently found its modern formulation as a correlation experiment intermediate to entanglement and Bell non-locality [3]. To be more precise, steering is defined as the non-existence of a special type of hidden variable model, namely hidden state model. These models aim to reproduce state assemblages (into which one party is steered) from a local state ensemble through classical data processing. If no such local strategy succeeds, the parties have demonstrated quantum steering.

It is well known from the work of Werner [4] that states allowing steering are a proper subset of entangled states[1]. Answering the question which states are entangled is known to be extremely challenging and the same question posed on steering hasn't appeared any easier[2]. Whereas the set of steerable states remains unknown, one can ask if the set of measurements allowing steering would be easier to characterise. It turns out that this question can be answered, and the answer forms the core of this thesis. Namely, quantum measurements allowing steering are exactly the ones which don't allow a simultaneous measurement (see article I).

Simultaneous measurability is a specific type of measurement compatibility. Typically in text-book quantum mechanics compatibility of observables (i.e. Hermitian operators) is captured by commutativity. However, Hermitian operators have proven to be insufficient to cover all possible measurement scenarios. Consequently, more general concepts have been proposed, such as positive operator valued measures (or POVMs for short), quantum instruments and measurement models. To be clear, all of these concepts can be traced back to Hermitian operators (or unitary time evolution) on a larger quantum system through Naimark and Stinespring dilations, but many times it is more convenient to deal with only one quantum system. These general concepts lead to different notions of compatibility such as non-disturbance, coexistence and joint measurability. All of these notions coincide with

[1]To be more precise, Werner showed that there exists states with a local hidden variable model. Later, within the modern formulation of steering, these models were recognised to be actually local hidden state models.

[2]It should be mentioned that both of these problems have found operationally motivated equivalent formulations [5, 6].

commutativity (of eigenprojections) for text-book observables, but for POVMs they all differ from one another and from commutativity. Indeed, there exist POVMs which do not commute, but still allow a simultaneous or a non-disturbing sequential implementation.

As mentioned above, the core result of this thesis is proving that measurements not allowing a simultaneous implementation characterise the measurement resource of steering. This result opens up a connection between the rather new research field of quantum steering and a way older field of incompatibility. However, the result lets one to translate between these fields only for a limited set of shared states, namely states with a full Schmidt rank. Hence, further techniques have been developed to cover also the scenarios with an arbitrary shared state (see articles II and III). These techniques show a deep connection between the two fields and have shed light to, for example, steering problems with position and momentum observables. Moreover, the techniques have proven useful in deriving incompatibility criteria and steering witnesses.

The core result is not limited to spatial steering. Indeed, a generalisation of the main result to the level of temporal and channel analogues of steering is also discussed in this thesis. The generalisation shows that all three types of steering can be mapped into joint measurability, hence pointing out the theory around incompatibility as a useful framework for all three steering scenarios. The power of this framework is exemplified by proving an equivalence between temporal and spatial steering, and a hierarchy between temporal unsteerability and macrorealism.

On top of the core result, this thesis provides methods for deriving steering criteria through another incompatibility related topic, namely entropic uncertainty relations, and through entanglement detection techniques.

The thesis consists of the original research articles listed below and of an introduction to the topic and the results. The introductory part is organised as follows. In chapters one and two the basics about quantum correlations in space and time are introduced together with the rudiments of quantum measurement theory with special focus on incompatibility. In the third chapter, the core result of the thesis is explained together with its generalisations. The generalisations lead to two alternative formalisms for steering both motivated by incompatibility. The fourth chapter focuses on deriving steering criteria from entropic uncertainty relations and from entanglement detection techniques. The fifth chapter explains how to derive bounds on the noise tolerance of incompatible quantum measurements and how to prove the tightness of these bounds using the core result.

# List of articles

**I Joint measurability of generalised measurements implies classicality**
Roope Uola, Tobias Moroder, Otfried Gühne
Phys. Rev. Lett. 113, 160403 (2014)

**II One-to-one mapping between steering and joint measurability problems**
Roope Uola, Costantino Budroni, Otfried Gühne, Juha-Pekka Pellonpää
Phys. Rev. Lett. 115, 230402 (2015)

**III Continuous variable steering and incompatibility via state-channel duality**
Jukka Kiukas, Costantino Budroni, Roope Uola, Juha-Pekka Pellonpää
Phys. Rev. A 96, 042331 (2017)

**IV Unified picture for spatial, temporal and channel steering**
Roope Uola, Fabiano Lever, Otfried Gühne, Juha-Pekka Pellonpää
Phys. Rev. A 97, 032301 (2018)

**V Steering criteria from general entropic uncertainty relations**
Ana C. S. Costa, Roope Uola, Otfried Gühne
arXiv:1710.04541 (pre-print)

**VI Steering maps and their application to dimension-bounded steering**
Tobias Moroder, Oleg Gittsovich, Marcus Huber, Roope Uola, Otfried Gühne
Phys. Rev. Lett. 116, 090403 (2016)

**VII Adaptive strategy for joint measurements**
Roope Uola, Kimmo Luoma, Tobias Moroder, Teiko Heinosaari
Phys. Rev. A 94, 022109 (2016)

# Contents

# Quantum correlations

A typical way to see the non-classical nature of quantum mechanics is to prepare many identical distributed (resp. single party) systems and to measure sets of observables on them (resp. on one object at different times), and try to violate some (semi-)classical model for the observed probabilities. In this section four such models are discussed, two of which (non-locality and macrorealism) rely on a fully classical strategy and the two others (spatial and temporal steering) rely on a hybrid of classical and non-classical strategies. Moreover, a quantum-quantum version of these strategies leads to the celebrated quantum property called entanglement.

## 1.1 The spatial case

Let us start with a simple correlation experiment where two parties, called Alice and Bob, make local measurements of quantities of their choice in their respective laboratories. The measurement choices are labelled by $x_i$ for Alice and by $y_j$ for Bob. At this point it is not crucial how many measurements are allowed for each party and, hence, we assume that $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$. We also use a shorter notation $x_i = x \in \{1, ..., n\}$ and $y_j = y \in \{1, ..., m\}$ when there is no risk of confusion. Each measurement is assumed for now to have a discrete set of outcomes. For Alice the outcomes are labelled by $a_x$ and for Bob by $b_y$. When there is no risk of confusion, we write simply $a_x = a \in \{1, ..., k\}$ and $b_y = b \in \{1, ..., l\}$.

### 1.1.1 Local realism

To see quantum effects with the above setup, one has to check if the scenario possesses any classical limits. Classically, a preparation of a system includes the knowledge about all subsequent measurements, i.e. the outcomes of the measurements performed by Alice and Bob are encoded in the initial state of the system. This is typically called realism and it is reflected by the fact that there exist hidden variables $\lambda \in \Lambda$ (i.e. classical preparations) which predict (with probabilistic certainty) the outcomes of each possible measurement. In general $\Lambda$ can be a continuous set of parameters, e.g. the real numbers, but for a discrete set of measurements with discrete outcome sets $\Lambda$ can be chosen to be discrete, e.g. the natural numbers. The hidden variables can be drawn randomly from the set $\Lambda$ (which is assumed to be discrete) according to a probability distribution $p : \Sigma_\Lambda \to [0, 1]$, where $\Sigma_\Lambda$ is a $\sigma$-algebra generated by the singletons in $\Lambda$. In this case the probability of

Alice and Bob getting the outcomes $a$ and $b$ from measurements $x$ and $y$ is

$$p(a, b|x, y) = \sum_{\lambda \in \Lambda} p(\lambda) f(a, b|x, y, \lambda), \tag{1.1.1.1}$$

where $f$ is a response function, i.e. a mapping from the set of outcomes to the set $\{0, 1\}$.

To set another classical limitation, we can assume that Alice and Bob perform their measurements at the (approximately) same time. As the velocity of information propagation is limited, the choice of one party's measurement cannot affect the other party's measurement outcome. This assumption is reflected in the hidden variable description as the property of non-signalling, i.e.

$$\sum_a f(a, b|x, y, \lambda) = \sum_a f(a, b|x', y, \lambda) \ \forall b, x, x', y \tag{1.1.1.2}$$

$$\sum_b f(a, b|x, y, \lambda) = \sum_b f(a, b|x, y', \lambda) \ \forall a, x, y, y'. \tag{1.1.1.3}$$

Moreover, a response function is a probability distribution and it can, hence, be written as $f(a, b|x, y, \lambda) = f(a|b, x, y, \lambda) f(b|x, y, \lambda) = f(b|a, x, y, \lambda) f(a|x, y, \lambda)$. As the response functions map to the set $\{0, 1\}$, the non-signalling conditions imply $f(a|b, x, y, \lambda) = f(a|x, \lambda)$ and $f(b|x, y, \lambda) = f(b|y, \lambda)$[1]. Hence,

$$p(a, b|x, y) = \sum_{\lambda \in \Lambda} p(\lambda) f(a|x, \lambda) f(b|y, \lambda), \tag{1.1.1.4}$$

where $f(a|x, \lambda), f(b|y, \lambda)$ are local response functions.

Notice that instead of using deterministic (i.e. $\{0, 1\}$-valued) response functions $f$, one can also use stochastic ones, i.e. general probability distributions $p$[2]. These two descriptions give equivalent predictions. To see this, notice first that trivially deterministic strategies are included in the stochastic ones. For the other direction, from a stochastic hidden variable model

$$p(a, b|x, y) = \sum_{\lambda \in \Lambda} p(\lambda) p(a|x, \lambda) p(b|y, \lambda) \tag{1.1.1.5}$$

we define a set of new hidden variables (corresponding to the set of outcomes of the measurements) $\tilde{\Lambda} = \{(a_1, ..., a_n, b_1, ..., b_m) | a_x = 1, ..., k, b_y = 1, ..., l\}$ with the distribution

$$p(\tilde{\lambda}) = \sum_{\lambda} p(\lambda) \prod_{x,y} p(a_x|x, \lambda) p(b_y|y, \lambda). \tag{1.1.1.6}$$

[1] To see this, write $\sum_a f(a, b|x, y, \lambda) = f(b|x, y, \lambda) = f(b|y, \lambda)$, where the last inequality follows from non-signalling. With a similar argument $\sum_b f(a, b|x, y, \lambda) = f(a|x, y, \lambda) = f(a|x, \lambda)$. Using this, write $\sum_b f(a, b|x, y, \lambda) = \sum_b f(a|b, x, y, \lambda) f(b|x, y, \lambda) = f(a|b', x, y, \lambda) = f(a|x, \lambda)$, where $b'$ is such that $f(b'|x, y, \lambda) = 1$ (note that the cases $f(b|x, y, \lambda) = 0$ are not of interest here as in these cases $f(a, b|x, y, \lambda) = 0$). Hence, $f(a|b, x, y, \lambda)$ is independent of $b$ and $y$.

[2] Note that we use $p$ as a generic label for a probability distribution without singling out which distribution we refer to.

Such distribution of the new hidden variables gives indeed a deterministic model. Namely,

$$p(a_x, b_y|x, y) = \sum_{\tilde{\lambda}} p(\tilde{\lambda}) f(a_x|x, \tilde{\lambda}) f(b_y|y, \tilde{\lambda}), \tag{1.1.1.7}$$

where $f(a_x|x, \tilde{\lambda})$ is one if $a_x$ is present in $\tilde{\lambda}$ and zero otherwise [and similarly for $f(b_y|y, \tilde{\lambda})$]. Throughout this thesis hidden variable models are assumed to be stochastic unless otherwise stated.

Having imposed the classical assumptions of realism and locality on our scenario, we wish to discuss how these assumptions fail in the quantum regime. As a typical example, we take a situation where Alice and Bob both have two measurements (labelled as $x_1, x_2$ for Alice and $y_1, y_2$ for Bob) with values plus and minus one. The aim is to derive an upper bound for the expression

$$|\langle B \rangle| := |\langle x_1 y_1 \rangle + \langle x_1 y_2 \rangle + \langle x_2 y_1 \rangle - \langle x_2 y_2 \rangle| \tag{1.1.1.8}$$

from the deterministic local realistic hidden variable model in Eq. (1.1.1.4)[3]. Here $\langle \cdot \rangle$ refers to expectation value. First, from Eq. (1.1.1.4) one has

$$\begin{aligned} \langle x_i y_j \rangle &= \sum_{\lambda \in \Lambda} \sum_{a,b=\pm 1} p(\lambda) ab f(a|x_i, \lambda) f(b|y_j, \lambda) \\ &= \sum_{\lambda \in \Lambda} p(\lambda) \langle x_i \rangle_\lambda \langle y_j \rangle_\lambda. \end{aligned} \tag{1.1.1.9}$$

Combining Eq. (1.1.1.8) with Eq. (1.1.1.9) we get

$$\begin{aligned} \big|\langle B \rangle\big| &= \Big| \sum_{\lambda \in \Lambda} p(\lambda) \big[ \langle x_1 \rangle_\lambda (\langle y_1 \rangle_\lambda + \langle y_2 \rangle_\lambda) + \langle x_2 \rangle_\lambda (\langle y_1 \rangle_\lambda - \langle y_2 \rangle_\lambda) \big] \Big| \\ &\leq \sum_{\lambda \in \Lambda} p(\lambda) \big( \big| \langle y_1 \rangle_\lambda + \langle y_2 \rangle_\lambda) \big| + \big| \langle y_1 \rangle_\lambda - \langle y_2 \rangle_\lambda) \big| \big) \\ &= 2, \end{aligned} \tag{1.1.1.10}$$

where we have used the triangle inequality together with the fact that $\big|\langle x_i \rangle_\lambda\big| \leq 1$ for all $\lambda$. For the last line, one simply checks both cases $\langle y_1 \rangle_\lambda \leq \langle y_2 \rangle_\lambda$ and $\langle y_1 \rangle_\lambda > \langle y_2 \rangle_\lambda$ and uses the inequality $\big|\langle y_j \rangle_\lambda\big| \leq 1$. Hence, a violation of the inequality $|\langle B \rangle| \leq 2$ would lead to contradictions between our assumptions (locality and realism). Moreover, in the scenario consisting of two observers both with two $\pm 1$ valued measurements, this inequality is known to characterise the existence of a local hidden variable (LHV) model [7]. In general, any inequality possessing limits for LHV models is called a Bell inequality.

### 1.1.2 Entanglement

Before discussing possible violations of local realism, we need the basic ingredients for describing quantum systems. Quantum preparations (or states) are identified as positive trace one (trace-class) operators[4] on the system of interest (i.e. complex finite-dimensional

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[3] As seen above, choosing deterministic response functions over stochastic ones sets no extra limitations.

[4] The set of trace-class operators is defined as $T(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) | \mathrm{tr}[(T^* T)^{1/2}] < \infty\}$, where $L(\mathcal{H}) = \{T : \mathcal{H} \to \mathcal{H} | T \text{ linear and bounded}\}$.

or infinite-dimensional separable Hilbert space $\mathcal{H}$) and the set of preparations is labelled by $\mathcal{S}(\mathcal{H})$. Composite systems are described by tensor products $\mathcal{H}_A \otimes \mathcal{H}_B$ of single system Hilbert spaces. The state space of a composite system can be divided into states which are probabilistic mixtures of local quantum states and to states which are not. Namely, we can ask if a given state $\rho \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ has the decomposition

$$\rho = \sum_{\mu} p(\mu)\rho_{\mu}^{A} \otimes \rho_{\mu}^{B}, \tag{1.1.2.1}$$

where $p(\cdot)$ is a probability distribution, $\rho_{\mu}^{A} \in \mathcal{S}(\mathcal{H}_A)$, and $\rho_{\mu}^{B} \in \mathcal{S}(\mathcal{H}_B)$. States having a decomposition of this form are called separable and states not having such a decomposition are called entangled. Separable states are classical in many ways. For example, these states don't provide any advantage over classical systems in typical quantum protocols such as quantum key distribution or teleportation. Moreover, separable states have always a local hidden variable description. To see this, recall that in quantum mechanics measurements are identified as positive operator valued measures (POVMs), i.e. collections of positive operators $\{A_a\}_a$ summing up to the identity operator[5], whose POVM elements $A_a$ reproduce the measurement outcome probabilities on a state $\rho$ through the formula[6] $p(a|A) = \mathrm{tr}[A_a\rho]$. The joint outcome probabilities for measurements given through POVMs $\{A_{a|x}\}_{a,x}$[7] on Alice's and $\{B_{b|y}\}_{b,y}$ on Bob's system on a separable quantum state $\rho = \sum_{\mu} p(\mu)\rho_{\mu}^{A} \otimes \rho_{\mu}^{B}$ are

$$\begin{aligned} p(a,b|x,y) &= \mathrm{tr}\big[(A_{a|x} \otimes B_{b|y})\rho\big] \\ &= \sum_{\mu} p(\mu)\mathrm{tr}[A_{a|x}\rho_{\mu}^{A}]\mathrm{tr}[B_{b|y}\rho_{\mu}^{B}] \\ &=: \sum_{\mu} p(\mu)p^{Q}(a|i,\mu)p^{Q}(b|j,\mu), \end{aligned} \tag{1.1.2.2}$$

where $p^{Q}(a|i,\mu)$ [resp. $p^{Q}(b|j,\mu)$] refers to the obvious probability distribution arising from the quantum state $\rho_{\mu}^{A}$ (resp. $\rho_{\mu}^{B}$).

Comparing Eq. (1.1.2.2) with Eq. (1.1.1.5) we see that separable states indeed do have a hidden variable description. Moreover, this description arises from quantum mechanics (i.e. from local quantum states) without additional hidden variables.

Hence, in order to see violations of local realism, we need to use non-separable (i.e. entangled) states. To give an example of a scenario which violates Eq. (1.1.1.10), consider Alice measuring the spin observables (given as Hermitian matrices) $(1,0,0)\cdot\vec{\sigma}$ and $(0,0,1)\cdot\vec{\sigma}$ and Bob measuring the spin observables $\frac{1}{\sqrt{2}}(1,0,1)\cdot\vec{\sigma}$ and $\frac{1}{\sqrt{2}}(1,0,-1)\cdot\vec{\sigma}$, where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ with

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[5] Later we also need POVMs with continuous outcome sets. In this case one has to slightly fine-tune the definition.

[6] For readers more common with quantum mechanical measurements being represented as Hermitian operators, the adaptation to the formalism here is rather straight-forward: eigenprojections correspond to POVM elements (although not all POVM elements have to be projections) and eigenvalues (i.e. outcomes) correspond to the index $a$.

[7] In this notation the operators $\{A_{a|x}\}_a$ form a POVM for every $x$.

Now the POVM elements of $\{A_{a|x}\}_{a,x}$ and $\{B_{b|y}\}_{b,y}$ are the eigenprojections of the aforementioned Hermitian matrices. Using the quantum state $\rho = |\psi^+\rangle\langle\psi^+|$ defined through the vector $|\psi^+\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ we get

$$\big|\langle B\rangle\big| = 2\sqrt{2}, \tag{1.1.2.3}$$

which shows that quantum mechanics is, indeed, in contradiction with local hidden variable models.

It is worth noting that whereas separable states have an LHV model for any number of measurements and outcomes, there exists also entangled states with the same property [4]. Hence, entanglement is necessary, but not sufficient for Bell inequality violations.

### 1.1.3 Steering

Local hidden variable models and entanglement are both extensively studied subjects. For in depth reviews, see [8, 9, 10]. What is slightly less studied is a hybrid of these two. Namely, a comparison of Eq. (1.1.1.5) and Eq. (1.1.2.2) raises an obvious question: what kind of correlations do we get by assuming one party's probability distributions to be classical and other party's distributions to be quantum, i.e.

$$\begin{aligned} p(a,b|x,y) &= \sum_{\lambda\in\Lambda} p(\lambda)p(a|x,\lambda)p^Q(b|y,\lambda) \\ &= \sum_{\lambda\in\Lambda} p(\lambda)p(a|x,\lambda)\mathrm{tr}[B_{b|y}\rho_\lambda]. \end{aligned} \tag{1.1.3.1}$$

Such model, when existing, is called a local hidden state (LHS) model. To give a physical interpretation for Eq. (1.1.3.1), we fix a state $\rho$ for the composite system and rewrite the equation using a quantum description on the left-hand-side as

$$\mathrm{tr}_B\Big\{\mathrm{tr}_A\big[(A_{a|x}\otimes\mathbb{I})\rho\big]B_{b|y}\Big\} = \mathrm{tr}_B\Big\{\sum_{\lambda\in\Lambda} p(\lambda)p(a|x,\lambda)\rho_\lambda B_{b|y}\Big\}. \tag{1.1.3.2}$$

Assuming now that Bob can perform tomography on his side, i.e. assuming that the POVM elements $\{B_{b|y}\}_{b,y}$ span the whole operator space $\mathcal{L}(\mathcal{H})$[8], Eq. (1.1.3.2) yields

$$\rho_{a|x} := \mathrm{tr}_A\big[(A_{a|x}\otimes\mathbb{I})\rho\big] = \sum_{\lambda\in\Lambda} p(\lambda)p(a|x,\lambda)\rho_\lambda. \tag{1.1.3.3}$$

The left-hand-side of Eq. (1.1.3.3) has a clear physical interpretation: it represents Bob's side of the (non-normalised) post-measurement state when Alice measures $x$ and gets the outcome $a$. Thus Eq. (1.1.3.3) asks if the post-measurement states $\{\rho_{a|x}\}_{a,x}$ can be obtained from a local ensemble $\{p(\lambda),\rho_\lambda\}_\lambda$ of states on Bob's side by classically data-processing [i.e. implementing $p(a|x,\lambda)$ on] the ensemble according to the classical information $(a,x)$ Alice and Bob are assumed to share. Consequently, correlations without an LHS model are called steerable.

The interpretation of steering is two-fold. First, clearly steering as a type of correlation is in between entanglement and non-locality. These inclusions are strict [3, 4, 11] (see

[8] In the infinite-dimensional case informationally complete POVMs are exactly the ones whose range's (i.e. range of a measure, not range of an operator) linear span's ultraweak closure is $\mathcal{L}(\mathcal{H})$ [38].

example below). Second, steering can be interpreted as a spooky action at a distance in the sense that Bob can not explain the changes in his system by only classically post-processing a local state ensemble according to the classical information gained from Alice. Motivated by these interpretations, one sees often the more precise term Einstein-Podolsky-Rosen (EPR) steering used instead of steering. In this thesis we talk simply about steering, but we keep in mind that the definition has a strong resemblance to hidden variable models.

At this point, it is the writer's personal experience that the language used around steering and non-locality may cause confusion. To avoid confusion within this thesis, we divide the correlation experiments into three classes.

First, we can fix the measurements and the state used in a correlation experiment, and ask if the obtained probability distributions have an LHS or an LHV model. In these cases we talk about the steerability (or non-locality) of the assemblage $\{\rho_{a|x}\}_{a,x}$ or the steerability (or non-locality) of the setup. For small numbers of measurements and outcomes these questions can be solved with numerical methods [6, 12, 13]. For more complex cases, the numerical methods can get time-consuming and several (sub)optimal analytical methods are known [14, 15, 16], see also articles I-IV.

Second, if we don't manage to prove steering or non-locality for a given set of measurements on a given state, we can add measurements and see if this changes the situation. If the setup remains unsteerable (resp. local) for all possible measurements, then the state is called unsteerable (resp. local). If this is not the case, the state is steerable (resp. non-local). It is noteworthy that whereas steerability or non-locality of a give setup might be fairly simple to prove, the same question posed on the level of states has turned out to be a difficult problem.

Finally, one can invert the second scenario by asking which measurements allow steering when all possible states are considered. In this case we could talk about steerability of measurements, but as we will see later in this work, these measurements turn out to be exactly the non-jointly measurable ones.

To clarify the terminology and the hierarchy between entanglement, steering and non-locality, we present two typical examples of steerable and one-way steerable states. For further examples of steering, we refer to [3, 14] and the articles I-VI.

As our first example, we consider one of the most commonly discussed states in the steering community, i.e. the two-qubit Werner state (see also [17])

$$\rho_p := p|\psi^-\rangle\langle\psi^-| + \frac{1-p}{4}\mathbb{I}, \tag{1.1.3.4}$$

where $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ and $p \in [0,1]$ is called visibility or a noise parameter. It is easy to check with the partial transpose criterion [9] that the state $\rho_p$ is separable if and only if $p \leq 1/3$. It is known from the work of Werner [4] that the states with $0 \leq p \leq 1/2$ have a local hidden state model for all possible projective measurements on Alice's side and from the work of Barrett [18] that the same holds for POVMs for the range $0 \leq p \leq 5/12$. Moreover, the threshold $p = 1/2$ is known to be optimal in the sense that above this value, there are projective measurements which lead to a steering [3]. Later Acín et al. [19] showed that the state is local with projective measurements for visibilities $p \leq 1/K_G(3)$, where $1/K_G(3) \geq 0.68$ is the Grothendieck constant of order three[9]. Hence, with $1/2 \leq p \leq 0.68$ the two-qubit Werner state is local and steerable for

[9] Notice that the exact value of the Grothendieck constant is not known. The approximation used here is from [20].

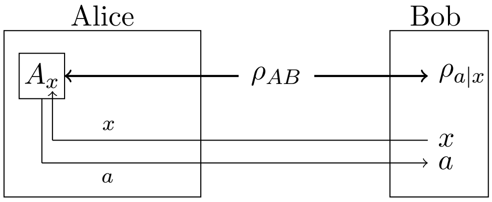


Figure 1.1: Alice and Bob share a bipartite state $\rho_{AB}$, Bob asks Alice to measure $A_x$ and to report the result $a$. The (non-normalised) post-measurement state assemblage Bob receives is given as $\rho_{a|x} = \mathrm{tr}_A[(A_{a|x} \otimes \mathbb{I})\rho_{AB}]$.

all projective measurements. Lastly, Vértesi [21] showed that above the threshold 0.7056 the state is non-local.

At this point it is worth to mention that the above hierarchy has an additional physical insight in it. Namely, steering and Bell non-locality can be both seen as entanglement detection methods. From this point of view, the difference between them is simply the description of the measurement devices: in local hidden state models a quantum description is only assumed for Bob and in local hidden variable models neither party is assumed to be quantum. Hence, steering is sometimes called semi-device-independent entanglement verification and Bell inequalities can be seen as device-independent entanglement witnesses.

As our second example, we consider one-way steerable states. Unlike non-locality or entanglement, steering is an asymmetric type of correlation. This means that there exists states, which allow a local hidden state model for steering attempts with all possible measurements on one side, but there exists measurements on the other side, which result in a steerable state assemblage. To give an example of such a state, we recall the one from [11] which reads

$$\rho^{POVM}_{\text{one-way}} := \frac{1}{3}\rho^{PVM}_{\text{one-way}} + \frac{2}{3}|2\rangle\langle 2| \otimes \mathrm{tr}_A[\rho^{PVM}_{\text{one-way}}], \tag{1.1.3.5}$$

where

$$\rho^{PVM}_{\text{one-way}} := \frac{1}{2}\Big[|\psi^-\rangle\langle\psi^-| + \frac{3}{10}|1\rangle\langle 1| \otimes \mathbb{I} + \frac{2}{10}\mathbb{I} \otimes |0\rangle\langle 0|\Big], \tag{1.1.3.6}$$

$\mathbb{I}$ is the identity operator in the subspace spanned by the vectors $\{|0\rangle, |1\rangle\}$ and $|\psi^-\rangle$ is defined as in the previous example. The states $\rho^{PVM}_{\text{one-way}}$ and $\rho^{POVM}_{\text{one-way}}$ are known to be steerable from Bob to Alice with well-chosen sets of measurements, but $\rho^{PVM}_{\text{one-way}}$ is not steerable from Alice to Bob with projective measurements and $\rho^{POVM}_{\text{one-way}}$ is not steerable from Alice to Bob with POVMs.

To further motivate the concept of steering, we recall an entanglement verification protocol first introduced in [3], see also Fig. 1.1. The idea is that Alice prepares a bipartite state and sends one half of it to Bob[10]. Alice claims that the state she prepares is entangled, but Bob wishes to find some way of verifying this. For this purpose, Bob asks Alice to steer him into an ensemble $\{\rho_{a|x}\}_a$ (from a set of ensembles $\{\rho_{a|x}\}_{a,x}$ Alice has announced prior to the experiment) by making the measurement $x$ on her part of the system and

[10]To be more precise, the protocol is not a single shot one. Namely, Alice prepares various copies of the same state and in each round of the protocol, she sends one particle (i.e. half of a bipartite state) to Bob.

reporting the outcome $a$. Assuming that Bob can do tomography, after many rounds of the protocol Bob possesses the state assemblage $\{\rho_{a|x}\}_{a,x}$, whose steerability he can check with a steering witness of his choice. In case of steering Bob is convinced that the state indeed is entangled.

To mention a possible cheating strategy in the above protocol, Alice could in principle try to fool Bob by sending him states from an ensemble $\{p(\lambda), \rho_\lambda\}_\lambda$ and give Bob an output $a$ according to some probability distribution $p(a|x,\lambda)$ conditioned on Bob's question $x$ and Alice's knowledge on $\lambda$. This way the state assemblage would have a local hidden state model and Bob would not be convinced that Alice actually prepared an entangled state.

## 1.2 The temporal case

Similarly to local realism on composite systems, one can ask if hidden variable models can be posed on single quantum systems. There are two main categories of such models: macrorealism and contextuality. Macrorealism can be easily modified to correspond to a temporal analogue of steering, but for the case of contextuality such a modification is not known. As our main focus is on steering, we will concentrate on macrorealism. For readers interested in contextuality, we refer to a recent work on the topic [22].

### 1.2.1 Macrorealism

Macrorealism is a model for a single system measured at different times. It is built on two assumptions [23]: macrorealism per se and non-invasive measurability. Macrorealism per se refers to a macroscopic object having (distinguishable) macrostates available to it and the object being in one of these states at any given time. Mathematically these states correspond to hidden variables just like in the case of local realism. To write a macrorealistic hidden variable model, assume that we have a set of measurements $x \in \{1, ..., n\}$ with outcomes $a \in \{1, ..., k\}$ on the first time step and a set of measurement $y \in \{1, ..., m\}$ with outcomes $b \in \{1, ..., l\}$ on the second time step (of course we could have more time steps and even continuous observables, but for introducing the idea this simple scenario is sufficient). Hidden variable model for the scenario reads similarly to Eq. (1.1.1.1)

$$p(a,b|x,y) = \sum_{\lambda\in\Lambda} p(\lambda) f(a,b|x,y,\lambda). \tag{1.2.1.1}$$

Note that the distinct macrostates refer to a full catalogue of properties of the system and, hence, deterministic response functions are used here.

So far we have only used the assumption of macrorealism per se. Analogously to locality in local realism, the non-invasiveness measurability (and the fact that future measurements can not affect the past) imply the factorisability of the response functions. Hence, a macrorealistic hidden variable model reads[11]

$$p(a,b|x,y) = \sum_{\lambda\in\Lambda} p(\lambda) p(a|x,\lambda) p(b|y,\lambda), \tag{1.2.1.2}$$

[11] One should notice that macrorealism is typically formulated for a single observable at different times. Here we use a slightly more general formulation as this is better suited for the temporal analogue of steering. The formulation is due to [24].

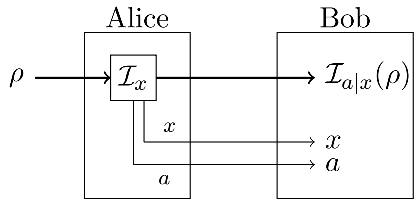


Figure 1.2: Alice applies an instrument (see the next section) $\mathcal{I}_x$ on a single system state $\rho$ and reports the measurement setting $x$ and result $a$ to Bob. Bob is left with the (non-normalised) output state $\mathcal{I}_{a|x}(\rho)$.

where we have changed from deterministic to a stochastic description.

One notices that Eq. (1.1.1.5) and Eq. (1.2.1.2) are identical. Hence, to witness non-macrorealistic behaviour one can use the inequality in Eq. (1.1.1.10). To mention another example of a non-macrorealism witness, consider a scenario with three measurement times and only one ($\pm 1$ valued) observable per time step. The model in Eq. (1.2.1.2) generalises straight-forwardly to this scenario, and by labelling the measurements at different times by $A_1, A_2$ and $A_3$ one gets

$$\langle A_1 A_2 \rangle + \langle A_2 A_3 \rangle - \langle A_1 A_3 \rangle \leq 1. \tag{1.2.1.3}$$

To see this, one can check every possible assignment of values provided by the deterministic response functions. For quantum mechanical scenarios violating this inequality, see [25, 26].

### 1.2.2 Temporal steering

As in the case of local realism, one can introduce modifications of macrorealistic hidden variable models. Namely, one can ask if the classical probability distributions can be replaced with distributions of quantum origin. Models with the second time step having a quantum description have received some attention lately [27, 28, 29], and they will be shortly discussed here, see also Fig. 1.2[12]. The impossibility of the resulting hidden state models is called temporal steering. Namely, a temporal scenario consisting of two measurement steps is called temporally unsteerable if

$$p(a, b|x, y) = \sum_{\lambda \in \Lambda} p(\lambda) p(a|x, \lambda) p^Q(b|y, \lambda), \tag{1.2.2.1}$$

where $p^Q(b|y, \lambda) = \mathrm{tr}[\rho_\lambda B_{b|y}]$ for some states $\{\rho_\lambda\}_\lambda$.

Temporal steering has an analogous definition to that of spatial steering and, as we will see in the forthcoming sections, it has an analogous hierarchy with macrorealism as spatial steering has with local realism. Although temporal steering seems to lack a clear physical interpretation at the moment, the concept has found connections to, for example, non-Markovianity [32]. As such, we consider the temporal version of steering merely as a mathematical concept with possible future applications to, for example, probing macrorealism, and we don't aim to seek for a further physical interpretation of it.

[12] One can ask if demanding both response functions to have a quantum description leads to a temporal analogue of entanglement. This question is out of the current work's scope, but for readers interested in the topic we refer to [29, 30, 31].

## 1.3 Channel steering: a unifying picture

### 1.3.1 Basics on channels and instruments

So far we haven't specified how state transformations (e.g. time evolutions) and state updates (caused by measurements) are modelled in quantum mechanics. A valid state transformation (from the state space of a Hilbert space $\mathcal{H}$ to the state space of a Hilbert space $\mathcal{K}$) is a completely positive trace preserving linear map, i.e. a map $\Lambda : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ whose trivial extensions $\Lambda \otimes \mathbb{I}_k$ are positive (i.e. mapping positive operators to positive operators) for every $k \in \mathbb{N}$. Such maps are called quantum channels.

To justify the definition, note that the trace-preserving property and positivity are desirable in order to map quantum states into valid quantum states, linearity comes as an usual built-in feature for quantum mechanics[13], and complete positivity is regarded so that states which are possibly entangled to an environment will remain positive. Note that every channel (and also instrument, see below) comes with a dual mapping $\Lambda^* : \mathcal{L}(\mathcal{K}) \to \mathcal{L}(\mathcal{H})$ defined through

$$\mathrm{tr}[\Lambda^*(S)T] := \mathrm{tr}[\Lambda(T)S] \; \forall S \in \mathcal{L}(\mathcal{K}), T \in \mathcal{T}(\mathcal{H}). \tag{1.3.1.1}$$

The dual mapping is also called the Heisenberg picture and the non-dual version is called the Schrödinger picture of $\Lambda$. Note that the trace-preserving property translates to identity-preserving property in the Heisenberg picture. This property implies that the Heisenberg picture (which operates on observables instead of states) maps POVMs into POVMs.

State updates due to measurements (here POVMs $\{A_{a|x}\}_{a,x}$) are described as well by linear completely positive maps $\mathcal{I}_{a|x} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$, but instead of preserving the trace, they are required to be trace non-increasing[14] together with the property $\mathrm{tr}[\sum_a \mathcal{I}_{a|x}(T)] = \mathrm{tr}[T] \; \forall T \in \mathcal{T}(\mathcal{H})$. Such collections of mappings are called quantum instruments. An instrument $\{\mathcal{I}_{a|x}\}_a$ is said to be compatible with a POVM $\{A_{a|x}\}_a$ if it encodes the measurement outcome probabilities in the post-measurement state, i.e. if

$$\mathrm{tr}[\mathcal{I}_{a|x}(\rho)] = \mathrm{tr}[A_{a|x}\rho] \; \forall \rho, a, \tag{1.3.1.2}$$

or equivalently if in the Heisenberg picture $\mathcal{I}^*_{a|x}(\mathbb{I}) = A_{a|x}$ for all $a$.

A crucial tool for dealing with completely positive maps is the so-called Kraus decomposition. Namely, a liner map $\Lambda : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ is completely positive if and only if there

[13] See also the reasoning on pp. 21 in [33] about a simple connection between linearity and locality in quantum mechanics.

[14] In order to have linear transformations, we don't require the maps to be trace-preserving. As an example, consider the typical projection postulate. When a projective measurement $\{P_a\}_a$ gives an outcome $a$ on state $\rho$, according to the projection postulate the normalised post-measurement state reads $P_a\rho P_a/\mathrm{tr}[P_a\rho P_a]$. This transformation is trace-preserving, but it is also non-linear in $\rho$. By not normalising the post-measurement state (i.e. giving up the trace-preserving property), we get a linear transformation.

exists operators $\{K_k\}_k \in \mathcal{L}(\mathcal{H},\mathcal{K})$ (called Kraus operators of $\Lambda$) such that[15] [16] [33, 34]

$$\Lambda(T) = \sum_k K_k T K_k^* \ \forall\, T \in \mathcal{T}(\mathcal{H}). \tag{1.3.1.3}$$

Moreover, a completely positive linear map $\Lambda$ with the above decomposition is trace non-increasing if and only if

$$\sum_k K_k^* K_k \leq \mathbb{I}. \tag{1.3.1.4}$$

If one changes the inequality in Eq. (1.3.1.4) into an equality, one gets a condition characterising the trace-preserving property. Here the star in $K_k^*$ refers to the adjoint of the operator $K_k$.

Another important tool for quantum channels is the Stinespring dilation. From textbook quantum mechanics we know that any channel $\Lambda : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ can be written as a unitary channel on a larger system, i.e.

$$\Lambda(\rho) = \mathrm{tr}_{\mathcal{H}_E}[U(\rho \otimes \eta)U^*], \tag{1.3.1.5}$$

where $\mathcal{H}_E$ is the Hilbert space of the environment, $\eta \in S(\mathcal{H}_E)$ is the initial state of the environment, and $U$ is a unitary operator on the composite system $\mathcal{H} \otimes \mathcal{H}_E$.

For our purposes, considering a slightly different form of the Stinespring dilation appears convenient. Namely, for a channel $\Lambda : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ given in the Kraus form $\Lambda(T) = \sum_{k_1}^r K_k T K_k^*$ one can define an isometry $V : \mathcal{H} \to \mathcal{H}_{\mathcal{A}} \otimes \mathcal{K}$ through

$$V|\psi\rangle := \sum_{k=1}^r |\varphi_k\rangle \otimes K_k|\psi\rangle, \tag{1.3.1.6}$$

where $\{|\varphi\rangle_k\}_{k=1}^r$ is an orthonormal basis of the dilation (or dummy) system $\mathcal{H}_{\mathcal{A}}$. From this isometry one can define a dilation of $\Lambda$ as

$$\Lambda(T) = \mathrm{tr}_{\mathcal{H}_A}[VTV^*] \ \forall T \in \mathcal{T}(\mathcal{H}). \tag{1.3.1.7}$$

For the case of linearly independent Kraus operators this dilation is called minimal (as the dimension of the dummy system is minimal). Our focus will mainly be on minimal dilations as they allow a one-to-one mapping between instruments $\mathcal{I}_{a|x} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ with $\Lambda = \sum_a \mathcal{I}_{a|x}$ and POVMs $\{A_{a|x}\}_{a,x}$ on the dummy system through the following link [35, 36]

$$\mathcal{I}_{a|x}(T) = \mathrm{tr}_A[(A_{a|x} \otimes \mathbb{I})VTV^*]. \tag{1.3.1.8}$$

Later in this thesis we show how minimal Stinespring dilations works as a unifying framework for different steering scenarios by mapping all these scenarios into joint measurability.

As the final tool for dealing with quantum instruments and channels (in finite-dimensional systems) we need the Choi-Jamiołkowski isomorphism. It states that one

[15] Here $\mathcal{L}(\mathcal{H},\mathcal{K})$ stands for bounded linear maps from $\mathcal{H}$ to $\mathcal{K}$.

[16] Note that the proof of the Kraus decomposition relies typically on Stinespring dilations. Here we will not present the proof. We go actually the opposing way, i.e. we use the Kraus decomposition to construct a specific dilation needed later in this work.

can assign to any quantum channel (resp. instrument) a quantum state (resp. state assemblage) on a larger system. The mapping for a channel $\Lambda : \mathcal{L}(\mathbb{C}^d) \to \mathcal{L}(\mathbb{C}^{d'})$ reads simply

$$M_\Lambda := \frac{1}{d} \sum_{i,j=1}^{d} |i\rangle\langle j| \otimes \Lambda(|i\rangle\langle j|). \tag{1.3.1.9}$$

The importance of this mapping is that it can be used to check whether or not a linear map $\Lambda : \mathcal{L}(\mathbb{C}^d) \to \mathcal{L}(\mathbb{C}^{d'})$ is completely positive. Namely, $\Lambda$ is completely positive if and only if the corresponding operator $M_\Lambda$ (also called Choi matrix) is positive [34].

In article III a slight modification of the isomorphism is introduced to cover also the infinite-dimensional case. This modification is used to provide a framework for spatial steering within which the steerability of a given state maps to the incompatibility breaking property of the corresponding channel.

### 1.3.2 Channel steering

As the Choi-Jamiołkowski isomorphism maps between states and channels, it is natural to ask if one can build a framework for steering in the channel picture. Here we will give the basic definitions for such a framework originally presented in [37] and show how the resulting channel steering captures both temporal and spatial versions of steering.

The objects of interest in channel steering are instrument assemblages $\{\mathcal{I}_{a|x}\}_{a,x}$ (mappings from Charlie to Bob), which are defined through the formula

$$\mathcal{I}_{a|x}(\rho) = \mathrm{tr}_A[(A_{a|x} \otimes \mathbb{I})\Lambda^{C\to A\otimes B}(\rho)] \; \forall \rho \in \mathcal{S}(\mathcal{H}), \tag{1.3.2.1}$$

where $\Lambda^{C\to A\otimes B}$ is an extension of $\Lambda^{C\to B} := \sum_a \mathcal{I}_{a|x}$ $\left[\text{i.e. } \Lambda^{C\to B}(\rho) = \mathrm{tr}_A[\Lambda^{C\to A\otimes B}(\rho)] \; \forall \rho\right]$ and $\{A_{a|x}\}_{a,x}$ are POVMs on the extension (i.e. on Alice's system), see also Fig. 1.3. Unsteerability of these assemblages is defined through the existence of a common instrument $\{\mathcal{I}_\lambda\}_\lambda$ an post-processings $\{p(\cdot|x,\lambda)\}_{x,\lambda}$ such that

$$\mathcal{I}_{a|x} = \sum_\lambda p(a|x,\lambda)\mathcal{I}_\lambda. \tag{1.3.2.2}$$

When such model doesn't exist, the assemblage $\{\mathcal{I}_{a|x}\}_{a,x}$ is called steerable.

There are two simple ways to connect channel steering to state steering: the use of instruments with a one-dimensional input and the use of the Choi-Jamiołkowski isomorphism. First, instruments with one-dimensional input correspond to state preparators, i.e. mappings of the form $\mathcal{I}_{a|x}(|1\rangle\langle 1|) = \rho_{a|x}$, where $|1\rangle\langle 1|$ is a state on the one-dimensional Hilbert space $\mathbb{C}$. Hence, instrument assemblages with trivial input correspond to state assemblages and Eq. (1.3.2.2) translates to a local hidden state model. Moreover, Eq. (1.3.2.1) translates to the typical way of obtaining state assemblages from a shared state $\Lambda^{C\to A\otimes B}(|1\rangle\langle 1|)$. Similarly, one can see temporal state assemblages arising from channel steering[17].

Second, in [37] the connection between channel and state steering through the Choi-Jamiołkowski isomorphism is discussed. The idea is to map the channel extension $\Lambda^{C\to A\otimes B}$

[17]Note that the temporal assemblages can, in principle, be signalling (i.e. $\sum_a \rho_{a|x} \neq \sum_a \rho_{a|x'}$). Such scenarios are trivially temporally steerable and, hence, the channel protocol provides a framework for the non-trivial occasions of temporal steering.

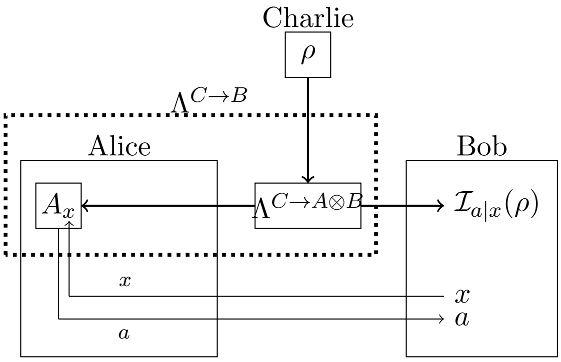


Figure 1.3: The setup is similar to the spatial steering scenario, but in the channel case the shared state is prepared by Charlie via the broadcast channel $\Lambda^{C\to A\otimes B}$. The operations enclosed in the dotted line are then viewed by Bob as instruments which have the total channel $\Lambda^{C\to B}$. The main difference to spatial steering is that here Bob's task is to build a local (instrument) model for all possible input states.

to the corresponding Choi state

$$M_{\Lambda^{C\to A\otimes B}} = \frac{1}{d}\sum_{i,j} |i\rangle\langle j| \otimes \Lambda^{C\to A\otimes B}(|i\rangle\langle j|) \tag{1.3.2.3}$$

and to prove that their respective steerability properties coincide, i.e. the Choi state is steerable if and only if the channel extension allows steerable instruments assemblages. With the isomorphism one can find also further analogies between state steering to channel steering. For example, one can show [37] that incoherent[18] channel extensions lead to unsteerable instrument assemblages and any unsteerable instrument assemblage can be seen as rising from an incoherent extension (cf. every separable state leads to unsteerable state assemblages and every unsteerable state assemblage can be seen as rising from a separable state, see article VI). Note, moreover, that a channel extension is incoherent if and only if the corresponding Choi matrix $M_{\Lambda^{C\to A\otimes B}}$ is separable in the cut $A|BC$ [37].

In article IV channel steering is used to approach all three steering scenarios in one go. Using the techniques introduced in this chapter we map all the scenarios into incompatibility, show how temporal and spatial steering are very closely related, and prove a hierarchy between temporal steering and non-macrorealism.

[18] A channel extension $\Lambda^{C\to A\otimes B}$ is called incoherent if $\Lambda^{C\to A\otimes B} = \sum_\lambda \Lambda_\lambda^{C\to B}\otimes\sigma_\lambda^A$ for some instrument $\{\Lambda_\lambda^{C\to B}\}_\lambda$ and normalised states $\{\sigma_\lambda^A\}_\lambda$, and coherent otherwise.

# Measurement incompatibility in quantum mechanics

One peculiar feature of quantum mechanics is measurement incompatibility. Incompatibility manifests itself in various different forms depending on the level of detail we have in our description of quantum measurements. In this section we discuss three different ways of describing measurements (PVMs[1], POVMs and instruments) and see what sort of fine-tunings of incompatibility they allow.

## 2.1 Commutativity

The text-book description of quantum measurements as Hermitian operators or, equivalently, as projection valued measures comes with a natural notion of incompatibility. Namely, two mutually non-commuting Hermitian operators $A$ and $B$ are typically called incompatible as their measurement statistics have limitations set by a preparation uncertainty relation [see e.g. [38]]:

$$\Delta(A)_\psi \Delta(B)_\psi \geq \frac{1}{2}\big|\langle\psi|[A,B]\psi\rangle\big|, \tag{2.1.0.1}$$

where $[A,B] = AB - BA$ and $\Delta(C)^2_\psi = \langle\psi|C^2\psi\rangle - \langle\psi|C\psi\rangle^2,\ C = A, B$.

Such limitations do not, however, capture the whole story behind measurement incompatibility. Whereas the above inequality is state-dependent, various (operationally motivated) state-independent notions of measurement incompatibility have been introduced. In the following sections we analyse in detail such concepts and show how they reduce to commutativity of POVM elements in the case of PVMs. However, for pairs of POVMs the concepts satisfy a strict hierarchy and, hence, highlight not only operationally but also mathematically different fine-tunings of incompatibility.

## 2.2 Non-disturbance

In a sequential measurement scenario (consisting here of two time-steps) one can ask if there exists a way to measure the first measurement, say $\{A_a\}_a$, without disturbing the statistics of a subsequent measurement, say $\{B_b\}_b$. To answer the question, recall that

[1] The abbreviation PVM refers to projection valued measure. These measures are defined as POVMs whose POVM elements are projections and they correspond to text-book observables, i.e. Hermitian operators.

any instrument $\{\mathcal{I}_a\}_a$ implementing $\{A_a\}_a$ has to fulfil

$$\mathrm{tr}[\mathcal{I}_a(\rho)] = \mathrm{tr}[A_a\rho] \;\forall \rho, a. \tag{2.2.0.2}$$

Now, the joint probability distribution for getting an outcome $a$ from the first measurement and an outcome $b$ from the second measurement in the state $\rho$ reads

$$p(a,b|A,B) = \mathrm{tr}[\mathcal{I}_a(\rho)B_b]. \tag{2.2.0.3}$$

In order not to disturb the statistics of the second measurement, we need $\sum_a p(a,b|A,B) = p(b|B)$ for all $b$, i.e.

$$\sum_a \mathrm{tr}[\mathcal{I}_a(\rho)B_b] = \mathrm{tr}[\rho B_b] \;\forall b. \tag{2.2.0.4}$$

To capture non-disturbance as a property of the measurements, Eq. (2.2.0.4) is required to hold for all input states. In the Heisenberg picture this reads simply $\sum_a \mathcal{I}_a^*(B_b) = B_b$ for all $b$.

To decide the existence of a non-disturbing measurement implementation, numerical methods based on semidefinite programming and analytical methods based on, for example, commutativity have been developed [39]. However, as non-disturbance is not the main focus of this thesis, we will simply state a few basic facts about it without further analysis of the concept.

First, clearly two commuting POVMs allow a non-disturbing sequential implementation through the von-Neumann Lüders instrument. Namely, if $[A_a, B_b] = 0 \;\forall a, b$, we can write $\sum_a \mathcal{I}_a^{L*}(B_b) = \sum_a \sqrt{A_a}B_b\sqrt{A_a} = \sum_a A_aB_b = B_b$ for all $b$, where the superscript $L$ refers to Lüders. However, there exists POVMs which are non-commuting, but nevertheless allow a non-disturbing sequential implementation [39]. To see how the concept of commutativity and non-disturbance become equivalent for PVMs, see subsection 2.4.

Second, non-disturbance has an extra structural property in comparison to the other types of incompatibility discussed here. Namely, non-disturbance can be dependent on the order of measurements, i.e. it is asymmetric [39].

Third, every non-disturbing measurement consists a joint measurement (see below). Moreover, every joint measurement can be implemented through a sequential measurement of possibly different observables [40].

## 2.3 Joint measurability

As our main concept of compatibility we introduce joint measurability. Joint measurability refers to the possibility of inferring the measurement data of various observables from the data of a single observable by the means of classical post-processing. Namely, a set of POVMs $\{A_{a|x}\}_{a,x}$ is called jointly measurable if and only if there exists a POVM $\{G_\lambda\}_\lambda$ (called a joint observable or a joint measurement) together with probability distributions $\{p(\cdot|x,\lambda)\}_{x,\lambda}$ such that

$$\mathrm{tr}[A_{a|x}\rho] = \sum_\lambda p(a|x,\lambda)\mathrm{tr}[G_\lambda\rho] \;\forall \rho, \tag{2.3.0.5}$$

or equivalently $A_{a|x} = \sum_\lambda p(a|x,\lambda)G_\lambda$. Sometimes it is more convenient to use deterministic post-processings, i.e. probability distributions with values 0 and 1. Similarly as in the case of hidden variable models, one can show that this poses no extra restrictions on joint measurability [41]. Namely, the existence of a joint observable with stochastic post-processings is equivalent to the existence of a (possibly different) joint observable with deterministic post-processings.

As in the case of non-disturbance, commuting measurements allow a joint measurement, but the other way around is not always true. To see this, taking the product POVM of two commuting POVMs forms a valid joint observable. For the other case, consider the following non-commuting POVMs

$$S^\mu_{\pm|x} := \frac{1}{2}(I \pm \mu\sigma_x) \tag{2.3.0.6}$$

$$S^\mu_{\pm|z} := \frac{1}{2}(I \pm \mu\sigma_z), \tag{2.3.0.7}$$

where $\mu \in (0,1]$. We can define a joint observable candidate for any $\mu \in (0,1]$ by writing

$$G^\mu_{i,j} := \frac{1}{4}\big[\mathbb{I} + \mu(i\sigma_x + j\sigma_z)\big],\ i,j = \pm 1. \tag{2.3.0.8}$$

Clearly the marginals (i.e. deterministic post-processings) of this candidate are correct, i.e. $\sum_i G^\mu_{i,j} = S^\mu_{j|z}$ and $\sum_j G^\mu_{i,j} = S^\mu_{i|x}$. However, the candidate is a POVM only for the values $0 < \mu \leq \frac{1}{\sqrt{2}}$. Hence, the POVMs $S^\mu_{\pm|x}$ and $S^\mu_{\pm|z}$ are non-commuting but jointly measurable within the parameter range $0 < \mu \leq \frac{1}{\sqrt{2}}$. Moreover, it can be shown that the value $\mu = \frac{1}{\sqrt{2}}$ is critical in the sense that above this threshold the POVMs become not jointly measurable [42] (see also below).

So far we have seen that commutativity implies both non-disturbance and joint measurability, but not the other way around. To build the hierarchy further, we note (as mentioned above) that every non-disturbing measurement works as a joint measurement. Namely, let $\{A_a\}_a$ be a POVM that allows an instrument $\{\mathcal{I}_a\}_a$ not disturbing a POVM $\{B_b\}_b$. Now, the operators $G_{a,b} := \mathcal{I}^*_a(B_b)$ form a POVM with the correct marginals. Hence, non-disturbance implies joint measurability. For an example showing that the inverse implication does not hold, we refer to [39].

To characterise sets of observables that admit joint measurements, various analytical [43, 44, 45, 46, 47, 48] and numerical [13, 49, 50] techniques have been developed. Some of these techniques are discussed and further developed in the articles I,II,III and VII. However, for completeness we wish to reproduce here an analytical characterisation of noise tolerance of non-joint measurability in a simple scenario including two two-valued (unbiased) qubit observables. The characterisation was first found in [42].

To start with, define two POVMs through the operators $A_\pm = \frac{1}{2}(\mathbb{I} \pm \vec{a}\cdot\vec{\sigma})$ and $B_\pm = \frac{1}{2}(\mathbb{I} \pm \vec{b}\cdot\vec{\sigma})$. Assuming that there exists a joint measurement $\{G_{i,j}\}_{i,j=\pm1}$, we can write its POVM elements as

$$G_{+,+} \tag{2.3.0.9}$$

$$G_{+,-} = A_+ - G_{+,+} \tag{2.3.0.10}$$

$$G_{-,+} = B_+ - G_{+,+} \tag{2.3.0.11}$$

$$G_{-,-} = \mathbb{I} - A_+ - B_+ + G_{+,+}. \tag{2.3.0.12}$$

Writing $G_{+,+} = \frac{1}{2}(\gamma\mathbb{I} + \vec{\gamma}\cdot\vec{\sigma})$, the positivity of the operators $\{G_{i,j}\}_{i,j}$ implies

$$\|\vec{\gamma}\| \leq \gamma \tag{2.3.0.13}$$
$$\|\vec{a} - \vec{\gamma}\| \leq 1 - \gamma \tag{2.3.0.14}$$
$$\|\vec{b} - \vec{\gamma}\| \leq 1 - \gamma \tag{2.3.0.15}$$
$$\|\vec{a} + \vec{b} - \vec{\gamma}\| \leq \gamma. \tag{2.3.0.16}$$

Geometrically, the first and the last inequality imply that there exists a point (namely $\vec{\gamma}$) in the intersection of a ball of radius $\gamma$ whose center is the origin and a ball of radius $\gamma$ whose center is the point $\vec{a} + \vec{b}$. This can only be the case if the sum of the radiuses is larger than or equal to the distance between the center points of the balls, i.e. if $\|\vec{a} + \vec{b}\| \leq 2\gamma$. Moreover, a similar argument used on the second and the third inequality gives $\|\vec{a} - \vec{b}\| \leq 2(1-\gamma)$. Summing up these two inequalities gives the following necessary condition for joint measurability

$$\|\vec{a} + \vec{b}\| + \|\vec{a} - \vec{b}\| \leq 2. \tag{2.3.0.17}$$

To prove the sufficiency of this criterion, one can choose $\vec{\gamma} = \frac{1}{2}(\vec{a} + \vec{b})$ and $\gamma = \|\vec{\gamma}\|$. This clearly defines a valid joint observable.

## 2.4 Coexistence

Joint measurability refers to the existence of an observable whose statistics can be used to deduce the statistics of other observables by the means of classical post-processing. One can modify this notion by dropping out the post-processings. Namely, one can ask if the statistics of a set of observables can be included into the statistics of a single observable. More precisely, a set of POVMs $\{A_{a|x}\}_{a,x}$ is called coexistent if there exists a POVM $\{C_\lambda\}_\lambda$ such that

$$\bigcup_x \bigcup_{a\in\mathcal{A}_x} A_{a|x} \subseteq \bigcup_{\lambda\in\mathcal{C}} C_\lambda, \tag{2.4.0.18}$$

where $\mathcal{A}_x$ and $\mathcal{C}$ refer to the sigma-algebras generated by the outcomes of the corresponding observables. Here, for example, $A_{\{1,2\}|x}$ refers to $A_{1|x} + A_{2|x}$.

It is then natural to ask if coexistence is indeed a proper generalisation of joint measurability. First, joint observables (with deterministic post-processings) clearly work as the observable $C$ in Eq. (2.4.0.18). Second, to prove the strictness of this inclusion, we evoke an example from [51]. Namely, define two qutrit POVMs as

$$A_i = \frac{1}{2}(\mathbb{I} - |i\rangle\langle i|),\ i = 0, 1, 2 \tag{2.4.0.19}$$
$$B_1 = \frac{1}{2}|\psi\rangle\langle\psi|,\ B_2 = \mathbb{I} - B_1, \tag{2.4.0.20}$$

where $|\psi\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle)$. To prove coexistence, clearly the following POVM does the job

$$C_i = \frac{1}{2}|i\rangle\langle i|,\ i = 0, 1, 2 \tag{2.4.0.21}$$

$$C_3 = B_1,\ C_4 = \frac{1}{2}\mathbb{I} - B_1. \tag{2.4.0.22}$$

Assuming now that the observables $\{A_a\}_a$ and $\{B_b\}_b$ are jointly measurable with a joint observable $\{G_{a,b}\}_{a,b}$ and deterministic post-processings we get $0 \leq G_{a,1} \leq B_1$ for $a = 0, 1, 2$. As $B_1$ is rank one we have $G_{a,1} = c_a B_1$ for some $0 \leq c_a \leq 1$. Consequently, $A_a = G_{a,1} + G_{a,2} = c_a B_1 + G_{a,2}$ and

$$0 = \langle a|A_a|a\rangle = c_a\langle a|B_1|a\rangle + \langle a|G_{a,2}|a\rangle \geq \frac{c_a}{3}. \tag{2.4.0.23}$$

It follows that $c_a = 0$ for $a = 0, 1, 2$ and $B_1 = \sum_a G_{a,1} = 0$. Hence, the POVMs in Eqs. (2.4.0.19, 2.4.0.20) are coexistent but not jointly measurable.

We are now in the point to show that the strict hierarchy between commutativity, non-disturbance, joint measurability and coexistence does not exist for PVMs. To do this, we recall the proof from [52]. As coexistence deals not only with outcomes of POVMs, but also with the sigma-algebras generated by the outcomes, we will use the set notation $X$ instead of the otherwise used notation $a$ for outcomes[2]. To start with, note that if a POVM $A$ has a projection $A(X)$ in its range, then any other element $A(Y)$ of the range commutes with $A(X)$. To see this, write

$$A(X) = A(X \cap Y) + A(X\backslash(X \cap Y)) \tag{2.4.0.24}$$

$$A(Y) = A(X \cap Y) + A(Y\backslash(X \cap Y)) \tag{2.4.0.25}$$

$$\begin{aligned}\mathbb{I} &\geq A(X \cup Y)\\ &= A(X) + A(Y\backslash(X \cap Y)).\end{aligned} \tag{2.4.0.26}$$

Hence, clearly $A(X \cap Y) \leq A(X)$ and $A(Y\backslash(X \cap Y)) \leq \mathbb{I} - A(X)$. As $A(X)$ and $\mathbb{I} - A(X)$ are both projections, one can write

$$A(X \cap Y) = A(X)A(X \cap Y)A(X) \tag{2.4.0.27}$$

$$A(Y\backslash(X \cap Y)) = (\mathbb{I} - A(X))A(Y\backslash(X \cap Y))(\mathbb{I} - A(X)). \tag{2.4.0.28}$$

The sum of these two operators gives

$$A(Y) = A(X)A(X \cap Y)A(X) + (\mathbb{I} - A(X))A(Y\backslash(X \cap Y))(\mathbb{I} - A(X)). \tag{2.4.0.29}$$

Multiplying this with $A(X)$ either from left or right gives $A(X)A(X \cap Y)A(X)$ and, hence, $[A(X), A(Y)] = 0$.

Now, if a PVM $A : \Sigma_A \to \mathcal{L}(\mathcal{H})$ and a POVM $B : \Sigma_B \to \mathcal{L}(\mathcal{H})$ are coexistent with a common POVM $C : \Sigma_C \to \mathcal{L}(\mathcal{H})$, then $A(X) = C(Z_X)$ and $B(Y) = C(Z_Y)$

[2]Note that the proof works also for continuous POVMs. For this case the former definition of a POVM has to be slightly modified. Namely, a continuous POVM $A : \Sigma_A \to \mathcal{L}(\mathcal{H})$ is a mapping from the (Borel) sigma-algebra $\Sigma_A$ of the outcome space $\Omega_A$ to the set of bounded positive linear mappings satisfying $A(\emptyset) = 0,\ A(\Omega_A) = \mathbb{I},\ A(\cup_i X_i) = \sum_i A(X_i)$ for disjoint sets $X_i \in \Sigma_A$. In the case of infinite-dimensional Hilbert space, the operator sum is assumed to converge in the weak (or equivalently strong [38]) operator topology

for some sets $Z_X, Z_Y \in \Sigma_C$. The above deduction applied to the POVM $C$ shows that $[A(X), B(Y)] = 0\ \forall X \in \Sigma_A,\ Y \in \Sigma_B$. Hence, for pairs of measurements including at least one PVM, all the above notions of compatibility coincide.

## 2.5 Compatibility of state transformations

So far we have concentrated on compatibility of measurements through their statistics. However, compatibility can be also formulated for other measurement descriptions [53, 54, 55]. As examples, one can ask the question about compatibility between an instrument and a channel, instrument and a POVM, channel and a POVM, compatibility of two (or more) channels and compatibility of two (or more) instruments. For our purposes, the compatibility of instruments with channels and other instruments are of most interest, as they provide the needed tools for mapping all steering problems (channel, spatial and temporal) to joint measurability problems.

To start with, consider an assemblage of instruments $\{\mathcal{I}_{a|x}\}_{a,x}$. The assemblage is called compatible if there exists an instrument $\{\mathcal{I}_\lambda\}_\lambda$ and post-processings $p(\cdot|x,\lambda)$ such that

$$\mathcal{I}_{a|x} = \sum_\lambda p(a|x,\lambda)\mathcal{I}_\lambda. \tag{2.5.0.30}$$

Noticing that compatible instruments necessarily have the same total channel, we can reduce the question of instrument compatibility to assemblages $\{\mathcal{I}_{a|x}\}_{a,x}$ with the property $\sum_a \mathcal{I}_{a|x} = \sum_a \mathcal{I}_{a|x'}$ for all $x, x'$. Using the techniques introduced in subsection 1.3.1 (for characterising the instruments compatible with a channel) we can write such instruments through a minimal Stinespring dilation of the total channel as

$$\mathcal{I}_{a|x}(\rho) = \mathrm{tr}_A[(A_{a|x} \otimes \mathbb{I})V\rho V^*]\ \forall \rho \in \mathcal{S}(\mathcal{H}). \tag{2.5.0.31}$$

Using the fact that the instruments $\{\mathcal{I}_{a|x}\}_{a,x}$ correspond one-to-one to the dummy POVMs $\{A_{a|x}\}_{a,x}$ on the dilation space, we can easily characterise compatibility of instruments. Namely, jointly measurable dummy POVMs lead clearly to compatible instrument assemblages. On the other hand, if an instrument assemblage is compatible, then the common instrument $\{\mathcal{I}_\lambda\}_\lambda$ has the same total channel as the instruments $\{\mathcal{I}_{a|x}\}_{a,x}$. Therefore, there exists a unique POVM $\{G_\lambda\}_\lambda$ on the dilation space corresponding to the common instrument. The compatibility of the instruments can be then written as

$$\mathcal{I}_{a|x}(\rho) = \sum_\lambda p(a|x,\lambda)\mathrm{tr}_A[(G_\lambda \otimes \mathbb{I})V\rho V^*] \tag{2.5.0.32}$$

$$= \mathrm{tr}_A[(\sum_\lambda p(a|x,\lambda)G_\lambda \otimes \mathbb{I})V\rho V^*]\ \forall \rho \in \mathcal{S}(\mathcal{H}). \tag{2.5.0.33}$$

As the operators $\sum_\lambda p(a|x,\lambda)G_\lambda$ form POVMs (over the index $a$) for every $x$, and as the POVMs on the dilation space correspond one-to-one to the instruments, we have $A_{a|x} = \sum_\lambda p(a|x,\lambda)G_\lambda$.

# Steering and joint measurability

We are now ready to proceed to the results that led to this thesis. As the detailed results are in the attached articles, we provide an introductory explanation of the ideas and techniques used in the articles. This chapter covers the articles I-IV, the chapter *Steering detection* covers the articles V and VI, and the chapter *Bounding the noise tolerance of incompatibility* is about the article VII.

## 3.1 Spatial steering

In this part, we go through the articles I-III. The results cover the connection between steering and joint measurements on three different levels: steerability of state assemblages originating from pure states, steerability of state assemblages originating from mixed states, and steerability of quantum states (i.e. assemblages originating from a general state and all possible measurements on Alice's side).

### 3.1.1 Non-joint measurability as a measurement resource for spatial steering

In article I a connection between steering and joint measurements in finite dimensional systems is presented. Namely, we show that in spatial steering, when optimisation over all shared states is performed, steerability of Bob's state assemblage is equivalent to non-joint measurability of Alice's observables[1]. The result allows one to translate various results and concepts between the research fields of joint measurability and steering such as incompatibility witnesses and local hidden state models. The work also includes a numerical analysis about the possibility that non-jointly measurable observables don't necessarily lead to Bell non-locality. The analysis shows strong evidence that this would indeed be the case[2].

To be more concrete, we sketch the idea of the main result here. Consider a steering scenario with Alice and Bob both having a $d$-dimensional system. Notice first that jointly measurable observables on Alice's side lead always (i.e. with any shared state) to an unsteerable assemblage on Bob's side. For the opposite direction, label Alice's measurements by $\{A_{a|x}\}_{a,x}$ and let the shared state $\rho_{AB}$ be of Schmidt rank $d$, i.e. of the form $\rho_{AB} = \sum_{i,j=1}^{d} \sqrt{\lambda_i \lambda_j} |ii\rangle\langle jj|$ with $\sum_{i=1}^{d} \lambda_i = 1$, $\lambda_i > 0$ $\forall i$ and $\{|i\rangle\}_{i=1}^{d}$ being an

[1] See also [56] for an independent proof of this result and [57] for a similar connection between non-joint measurability and non-locality in bipartite scenarios with both parties having two binary measurements.

[2] Note that later a proof of this fact was presented in [58, 59].

orthonormal basis, Bob's conditional state assemblage reads

$$\begin{aligned}\rho_{a|x} &= \mathrm{tr}_A[(A_{a|x} \otimes \mathbb{I})\rho_{AB}] \\ &= \mathrm{tr}_A[(A_{a|x} \otimes \mathbb{I})(C \otimes \mathbb{I})|\psi^+\rangle\langle\psi^+|(C \otimes \mathbb{I})],\end{aligned} \tag{3.1.1.1}$$

where $|\psi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |ii\rangle$ and $C = \sum_{i=1}^d \sqrt{d\lambda_i}|i\rangle\langle i|$. Using $\mathrm{tr}[(A \otimes B)|\psi^+\rangle\langle\psi^+|] = \frac{1}{d}\mathrm{tr}[AB^T]\ \forall A, B \in \mathcal{L}(\mathbb{C}^d)$, where $(\cdot)^T$ refers to the transpose (in the basis $\{|i\rangle\}_{i=1}^d$), we get

$$\rho_{a|x} = \frac{1}{d} C A_{a|x}^T C. \tag{3.1.1.2}$$

Now, if this assemblage is unsteerable with a local hidden state model given through an ensemble $\{p(\lambda)\rho_\lambda\}_\lambda$, we can define a joint observable for $\{A_{a|x}\}_{a,x}$ by $G_\lambda = dC^{-1}p(\lambda)\rho_\lambda^T C^{-1}$. Hence, we conclude that a set of POVMs is non-jointly measurable if and only if it can be used for steering.

It follows that any joint measurement criterion on Alice's observables translates into a steering inequality and any local hidden state model translates to a joint observable by the use of a Schmidt rank $d$ state. For example cases, we refer to articles I and VII. Moreover, it is clear that within the hierarchy of different types of incompatibility joint measurability is the only one characterising the task of steering. Interestingly, there exists coexistent observables which can be used for steering, see article I, [51], and subsection 2.4.[3]

### 3.1.2 Mapping between spatial steering and joint measurability problems

In article II we deepen the connection between joint measurability and steering by adding mixed shared states into the picture. The results show that any steering problem can be written as a joint measurability problem and vice versa. This makes the use of joint measurement criteria as steering inequalities more straight-forward and opens up a possibility to translate steering quantifiers into incompatibility quantifiers.

To be more precise, we define a mapping between state assemblages and POVMs. To do this, take a (non-signalling) state assemblage $\{\rho_{a|x}\}_{a,x}$ and define $\rho_B := \sum_a \rho_{a|x}$. By inverting $\rho_B$ (using a pseudo-inverse when necessary), we can define POVMs $\{B_{a|x}\}_{a,x}$ through $B_{a|x} := \rho_B^{-1/2}\rho_{a|x}\rho_B^{-1/2}$. Joint measurability of these POVMs is clearly equivalent to the unsteerability of the assemblage $\{\rho_{a|x}\}_{a,x}$.

Now any incompatibility criterion works as a steering inequality (even without the use of a Schmidt rank $d$ state) for various scenarios as the connection between POVMs and state assemblages is one to many. For examples and for the translation of steering quantifiers into incompatibility quantifiers we refer to article II.[4]

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[3] **Author's contribution**: The author of this thesis contributed to the proofs and examples in article I, but not to the numerical calculations.

[4] **Author's contribution**: The author of this thesis contributed to the proofs and examples in article II, but not to the numerical calculations.

### 3.1.3 Incompatibility breaking quantum channels

In article III we use a generalised Choi-Jamiołkowski isomorphism in order to find a continuous variable version of the connection between steering and joint measurements. This results in a formalism where steerability of a state is mapped to incompatibility breaking property of the corresponding quantum channel. The result allows one to solve seemingly different steering problems in one go and to prove that canonical pairs of quadratures are sufficient for steering in the Gaussian regime[5].

To start with, recall that the Choi-Jamiołkowski isomorphism is usually defined only for states which have a totally mixed marginal. This naturally limits the use of the isomorphism to finite-dimensional systems. To write a similar correspondence in the infinite-dimensional case, one can consider states $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ with a fixed full-rank marginal $\sigma = \text{tr}_A[\rho_{AB}]$ (note that here the roles of $A$ and $B$ are changed from the typical Choi-Jamiołkowski isomorphism in order to have steering going from Alice to Bob). Now there is a one-to-one correspondence between such states and channels $T$ from Bob to Alice given as

$$\rho_{AB} = (T \otimes \mathbb{I})|\Omega_\sigma\rangle\langle\Omega_\sigma|, \tag{3.1.3.1}$$

where $|\Omega_\sigma\rangle = \sum_i \sqrt{s_i}|ii\rangle \in \mathcal{S}(\mathcal{H}_B \otimes \mathcal{H}_B)$ and $\{s_i\}_i$ are the eigenvalues of $\sigma$. For the proof, see article III.

A crucial point in this correspondence is that the adjoint of the channel $T$ is given as

$$\sigma^{1/2} T^*(A)\sigma^{1/2} = \text{tr}_A[(A \otimes \mathbb{I})\rho_{AB}]^T, \tag{3.1.3.2}$$

where $(\cdot)^T$ is the transpose in the eigenbasis of $\sigma$. Notice that inputting POVMs to the left hand side results in state assemblages on the right hand side. One can show (article III) that the state $\rho_{AB}$ is steerable with given measurements if and only if the corresponding channel $T$ breaks the incompatibility of these measurements. The connection is also quantitative in that for a given state $\rho_{AB}$ and given measurements $\{A_{a|x}\}_{a,x}$ an incompatibility quantifier called incompatibility robustness of $\{T^*(A_{a|x})\}_{a,x}$ is equal to a steering quantifier called (consistent) steering robustness of the state assemblage originating from $\rho_{AB}$ and $\{A_{a|x}\}_{a,x}$ (see article III).

To further justify the channel formulation, notice that one can encode properties of the state into the channel. As shown in article III, for full Schmidt rank states the channel is unitary (hence generalising the results of article I to the infinite-dimensional setting) and for separable states the channel is entanglement breaking. Moreover, one can show that steering with noisy $NOON$ states and steering from an environment of a non-Markovian system (with amplitude damping dynamics and initial state $|1001\rangle$) to the system have the same steering channel, hence enabling one to solve seemingly different steering problems in one go. Furthermore, applying the formalism to the Gaussian regime allows one to reproduce the known Gaussian steering criterion of [3] and to prove that steering of Gaussian states can be already decided with some canonical pair of quadratures.[6]

[5]Note that as the basics about Gaussian systems are given in the article III, and as Gaussian systems are not the main focus of this thesis, we don't present the Gaussian formalism here.

[6]**Author's contribution**: The author of this thesis contributed to the proof of the connection between steering and joint measurements for Schmidt rank-$d$ states in the infinite-dimensional case (which was independently proven by the first author) and to the search of applications of the main result.

## 3.2 Temporal and channel steering

In article IV we use channel steering as a unifying framework for temporal and spatial steering. This way one can prove results for all three steering scenarios in one go and translate results from one scenario to the other. This is demonstrated by proving that channel steering is equivalent to non-joint measurability of certain observables (see below) and that the known hierarchy between spatial steering and non-locality translates into a hierarchy between temporal steering and non-macrorealism.

In section 1.3, the basic notions and the relation between channel and spatial steering were already discussed. Namely, channel steering with one-dimensional input systems results in spatial steering. The same is also true for temporal steering provided that the assemblages of interest are non-signalling[7]. Hence, channel steering can be used to approach both problems.

Recall that steerability of an instrument assemblage $\{\mathcal{I}_{a|x}\}_{a,x}$ is defined as the non-existence of the model

$$\mathcal{I}_{a|x} = \sum_{\lambda} p(a|x,\lambda)\mathcal{I}_{\lambda}, \tag{3.2.0.3}$$

where $\{\mathcal{I}_{\lambda}\}_{\lambda}$ is an instrument and $\{p(\cdot|x,\lambda)\}_{x,\lambda}$ are probability distributions. Now using the techniques presented in subsection 1.3.1 we know that Eq. (3.2.0.3) is equivalent to the joint measurability of the (dummy) observables $\{A_{a|x}\}_{a,x}$ on a minimal dilation of $\sum_a \mathcal{I}_{a|x}$ that correspond to $\{\mathcal{I}_{a|x}\}_{a,x}$. Applying this result to channels with one-dimensional input space gives a connection between joint measurability and both spatial and temporal steering. One can show (see article IV) that the connection between joint measurability of the dummy observables $\{A_{a|x}\}_{a,x}$ and spatial steering can be used to reproduce the known connection between joint measurability and spatial steering given in article II.

As in the case of spatial steering, such connection allows the use of joint measurability criteria as steering witnesses. In contrast to spatial steering, channel steering extends the applicability of these criteria to all three forms of steering.

In the above discussion, the use of a minimal dilation is crucial[8]. These dilations can be also used to explore the connection between spatial and temporal steering. Namely, it is fairly straight-forward to show that any non-signalling state assemblage has a spatial realisation. Using this fact one can show (see article IV) that in order to produce any non-signalling assemblage in the temporal scenario, it is sufficient to use non-signalling instruments (even ones mapping between two systems of equal dimension are sufficient). Hence, taking a channel whose minimal dilation has a bipartite state with interesting properties (here steerable but local) in the range of the isometry $V$ we can produce temporally steerable correlations which have a macrorealistic model.[9,10]

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[7]Note that signalling assemblages are trivially steerable.

[8]See article IV for an example where the connection between joint measurability and steering fails for a non-minimal dilation.

[9]Note that the hierarchy between temporal steering and non-macrorealism was independently proven in [29].

[10]**Author's contribution**: The author of this thesis contributed to the proofs and examples in article IV.

# Steering detection

In this chapter we concentrate on the articles V and VI in which methods for steering detection are derived. The first work shows how to map entropic uncertainty relations into steering inequalities and the second one builds steering inequalities for dimensions-bounded scenarios from entanglement witnesses.

## 4.1 Steering inequalities from generalised entropies

The idea of article V is to use the convex structure of local hidden state models together with the fact that these models have a quantum description for Bob in order to derive steering criteria. Namely, we show that jointly convex (and additive) functions with a state-independent uncertainty type lower bound translate to steering witnesses.

We will sketch the proof for the special case of Shannon entropy here. First, consider a local hidden state model

$$p(a,b|x,y) = \sum_{\lambda} p(\lambda)p(a|x,\lambda)\mathrm{tr}[\rho_\lambda B_{b|y}]. \tag{4.1.0.1}$$

Here, we further assume all the measurements to have $k$ values. Now, one can define a function $F(X,Y) = -D(X \otimes Y|X \otimes \mathbb{I})$, where

$$D(P|Q) = \sum_i p_i \ln\left(\frac{p_i}{q_i}\right) \tag{4.1.0.2}$$

is the relative entropy and $P = (p_1, ..., p_k)$ and $Q = (q_1, ..., q_k)$ are vectors of probabilities. The notations $X$ and $Y$ refer to the vectors consisting of probabilities of outcomes of the observables $x$ and $y$, and the notation $\mathbb{I}$ refers to the uniform distribution $(1/k, ..., 1/k)$. With $X \otimes Y$ we refer to a vector consisting of joint probabilities.

The idea is to find a lower bound for $F(X,Y)$ given that a local hidden state model exists. Now, for a fixed $\lambda$ we define $p(a,b|x,y,\lambda) := p(a|x,\lambda)\mathrm{tr}[\rho_\lambda B_{b|y}]$. As $D$ is additive for product distributions, we get (for a fixed $\lambda$)

$$F^\lambda(X,Y) := -\sum_{a,b} p(a,b|x,y,\lambda)\ln\left(\frac{p(a,b|x,y,\lambda)}{p(a|x,\lambda)/k}\right) = S^\lambda(Y) - \ln(k), \tag{4.1.0.3}$$

where $S^\lambda(Y) = -\sum_b \mathrm{tr}[\rho_\lambda B_{b|y}]\ln\{\mathrm{tr}[\rho_\lambda B_{b|y}]\}$. As the relative entropy is jointly convex, i.e.

$$D(\lambda P_1 + (1-\lambda)P_2|\lambda Q_1 + (1-\lambda)Q_2) \leq \lambda D(P_1|Q_1) + (1-\lambda)D(P_2|Q_2), \qquad (4.1.0.4)$$

the function $F$ is jointly concave. Hence, we get

$$F(X,Y) \geq \sum_\lambda p(\lambda)F^\lambda(X,Y), \qquad (4.1.0.5)$$

where $\{p(\lambda)\}_\lambda$ is the probability distribution of hidden states. Writing the probability vectors for Alice's measurements $x_1, ..., x_n$ as $X_1, ..., X_n$ and for Bob's measurements $y_1, ..., y_n$ as $Y_1, ..., Y_n$, we get

$$\sum_i F(X_i, Y_i) \geq \sum_i \Big\{ \sum_\lambda \big[p(\lambda)S^\lambda(Y_i)\big] - \ln(k)\Big\}. \qquad (4.1.0.6)$$

An optimisation over all hidden state models translates to an optimisation of the quantity $\sum_i \sum_\lambda p(\lambda)S^\lambda(Y_i)$ over all state ensembles. The result of such an optimisation is an entropic uncertainty relation. As an example of such a relation, in the case Bob's measurements are $y_1 = \sigma_x$ and $y_2 = \sigma_z$ we have [60]

$$S(\sigma_x) + S(\sigma_z) \geq \ln(2). \qquad (4.1.0.7)$$

The inequality in Eq. (4.1.0.6) works already as a steering criterion. However, one can still modify it by noticing that the quantity $F(X_i, Y_i)$ is actually the conditional entropy minus the logarithm of the number of outcomes. Namely, we have

$$F(X_i, Y_i) = S(Y_i|X_i) - \ln(k). \qquad (4.1.0.8)$$

Hence, the Shannon entropy based entropic steering criterion reads

$$\sum_i S(Y_i|X_i) \geq \alpha(Y_1, ..., Y_k), \qquad (4.1.0.9)$$

where $\alpha(Y_1, ..., Y_k)$ is an entropic lower bound. Entropic lower bounds can be found by numerical search, but also tight analytical bounds are known for several scenarios, see [61] and references therein.

To give an example of a generalised entropy that can be mapped into a steering criterion, we consider the Tsallis entropy (see also article V). Tsallis $q$-entropy (where $q > 1$) and the respective relative entropy of a probability distributions $P = (p_1, ..., p_k)$ and $Q = (q_1, ..., q_k)$ are defined as

$$S_q(P) = -\sum_i p_i^q \ln_q p_i, \qquad (4.1.0.10)$$

$$D_q(P|Q) = -\sum_i p_i \ln_q \left(\frac{q_i}{p_i}\right), \qquad (4.1.0.11)$$

where $\ln_q(p_i) = \frac{x^{1-q}-1}{1-q}$. The relative entropy is jointly convex, but it is not additive for product distributions. Namely,

$$D_q(P_1P_2|Q_1Q_2) = D_q(P_1|Q_1) + D_q(P_2|Q_2) + (q-1)D_q(P_1|Q_1)D_q(P_2|Q_2), \quad (4.1.0.12)$$

where $P_j = (p_1^j, ..., p_k^j)$ and $Q_j = (q_1^j, ..., q_k^j)$ for $j = 1, 2$. This, however, does not limit the use of the technique presented above for the Shannon entropy. The only difference in the resulting criterion is an extra term on the left hand side, see article V:

$$\sum_i \big(S_q(Y_i|X_i) + (1-q)C(X_i, Y_i)\big) \geq \alpha^q(Y_1, ..., Y_k) \quad (4.1.0.13)$$

where $\alpha^q(Y_1, ..., Y_k)$ is an entropic bound and

$$C(X_i, Y_i) = \sum_i p_i^q \big(\ln_q(p_i)\big)^2 - \sum_{i,j} p_{ij}^q \ln_q(p_i)\ln_q(p_{ij}). \quad (4.1.0.14)$$

For applications of this criterion we refer to the article V, where the strength of our steering inequalities is compared to known steering witnesses. Interestingly, our technique is optimal or close to optimal in many scenarios.[1]

## 4.2 Steering inequalities from entanglement theory

In article VI we develop a method for deciding the steerability of a state assemblage $\{\rho_{a|x}\}_{a,x}$ by mapping it into an abstract operator $\Sigma_{AB}$ whose entanglement properties are linked to the steering properties of the assemblage. Such a mapping unlocks the use of entanglement witnesses in steering detection and, consequently, opens up the possibility to go beyond the typical steering scenario through the use of dimension-bounded entanglement techniques. These are scenarios where Alice's measurements are uncharacterised and Bob's measurements have only a dimension-bound, but no further description of his measurements is assumed. Moreover, we provide detection thresholds for dimension-bounded steering that are no weaker than the known ones for steering in typical symmetric scenarios. Interestingly, these symmetric scenarios have been implemented in a loophole-free steering experiment [62]. However, at the time of our work the data of the experiments were not available.

To introduce the main idea of the paper, we sketch the aforementioned operator $\Sigma_{AB}$ in a simple scenario. Namely, consider an unsteerable assemblage $\{\rho_{a|x}\}_{a,x}$, where $a = \pm 1$ and $x = 1, 2$ with a deterministic local hidden state model given by

$$\rho_{+|1} = \sigma_{++} + \sigma_{+-}, \; \rho_{-|1} = \sigma_{-+} + \sigma_{--} \quad (4.2.0.15)$$

$$\rho_{+|2} = \sigma_{++} + \sigma_{-+}, \; \rho_{-|2} = \sigma_{+-} + \sigma_{--}. \quad (4.2.0.16)$$

This state assemblage can be clearly produced by the use of a separable state

$$\eta_{AB} = \sum_{i,j=\pm 1} |ij\rangle\langle ij| \otimes \sigma_{ij}, \quad (4.2.0.17)$$

and measurements $A_{\pm|1} = |\pm\rangle\langle\pm| \otimes \mathbb{I}$, $A_{\pm|2} = \mathbb{I} \otimes |\pm\rangle\langle\pm|$. Hence, if the state $\eta_{AB}$ would

[1] **Author's contribution**: The author of this thesis contributed to miscellaneous calculations concerning the proofs and examples in article V.

be entangled, the underlying assemblage would be steerable. However, the state has two drawbacks that reduce its use as a steering witness. First, the state is not completely determined as the Eqs. (4.2.0.15,4.2.0.16) don't have a unique solution in general. Namely, a possible solution is

$$\sigma_{++} \tag{4.2.0.18}$$
$$\sigma_{+-} = \rho_{+|1} - \sigma_{++} \tag{4.2.0.19}$$
$$\sigma_{-+} = \rho_{+|2} - \sigma_{++} \tag{4.2.0.20}$$
$$\sigma_{--} = \rho_{\Delta} + \sigma_{++}, \tag{4.2.0.21}$$

where $\rho_{\Delta} = \rho_B - \rho_{+|1} - \rho_{+|2}$ and $\rho_B = \sum_a \rho_{a|x}$. However, $\sigma_{++}$ is not determined by the solution. Second, the state $\eta_{AB}$ is not a general separable state, but it is instead a classical-quantum (or zero discord) state. As we aim to use general entanglement detection techniques, we wish to remove this structure. To overcome the drawbacks, we turn our focus to operators of the form

$$\Sigma_{AB} = \sum_{ij} Z_{ij} \otimes \sigma_{ij}, \tag{4.2.0.22}$$

where $Z_{ij}$ are positive semi-definite operators. Now the zero-discord structure is removed and we can simply rewrite the operator in order to eliminate the redundancy that comes with the unknown $\sigma_{++}$ as

$$\Sigma_{AB} = Z_{+-} \otimes \rho_{+|1} + Z_{-+} \otimes \rho_{+|2} + Z_{--} \otimes \rho_{\Delta} + (Z_{++} - Z_{+-} - Z_{-+} + Z_{--}) \otimes \sigma_{++}. \tag{4.2.0.23}$$

Hence, if we use operators $\{Z_{ij}\}_{ij}$ such that $Z_{++} - Z_{+-} - Z_{-+} + Z_{--} = 0$, the operator $\Sigma_{AB}$ is determined by the state assemblage. Moreover, if we pose the normalisation condition $\mathrm{tr}[Z_{+-}]\mathrm{tr}[\rho_{+|1}] + \mathrm{tr}[Z_{-+}]\mathrm{tr}[\rho_{+|2}] + \mathrm{tr}[Z_{--}]\mathrm{tr}[\rho_{\Delta}] = 1$ we see that $\Sigma_{AB}$ is indeed a separable quantum state. Consequently, if $\Sigma_{AB}$ is entangled (or no quantum state), then the underlying state assemblage is steerable.

In article VI we show how the mapping between state assemblages $\{\rho_{a|x}\}_{a,x}$ and quantum states $\Sigma_{AB}$ works in more complicated scenarios. Moreover, we show that for a steerable assemblage and an appropriate choice of the operators $\{Z_{ij}\}_{ij}$ the entanglement of the corresponding state is detected by the swap entanglement witness. For applications of the result, we refer to the article VI.[2]

[2]**Author's contribution**: The author of this thesis contributed to the search of applications of the main result through the connection between steering and joint measurements.

# Bounding the noise tolerance of incompatibility

The article VII introduces a simple technique for building joint observables for a given set of measurements. The technique is based on measuring another set of observables, or more precisely an ensemble of observables, which can be identified as a joint observable of the desired set. The strategy results naturally in sufficient conditions for joint measurability. Interestingly, the connection between steering and joint measurements presented in article I together with the techniques presented in [14] can be used to prove the necessity of these conditions in various symmetric scenarios.

To demonstrate our technique, we recall the simple compatibility problem presented in subsection 2.3. Namely, let

$$A^{\mu}_{\pm|x} = \frac{1}{2}\big(\mathbb{I} \pm \mu\sigma_x\big) \tag{5.0.0.1}$$

$$A^{\mu}_{\pm|z} = \frac{1}{2}\big(\mathbb{I} \pm \mu\sigma_z\big). \tag{5.0.0.2}$$

To build a joint observables for $\{A_{\pm|y}\}_{y=x,z}$, we wish to measure two other observables $\{B_{\pm|y}\}_{y=1,2}$. An educated guess suggests that the other observables have Bloch vectors lying in the $(x,z)-$plane. To adjust these Bloch vectors, one can try directions which give in some sense the most information about the desired observables. One guess is then to use the observables $\{A_{\pm|y}\}_{y=x,z}$ themselves. Indeed, in some cases this gives good results, see article VII. However, as these observables have orthogonal Bloch vectors, measuring one doesn't give any information about the other one. In order to get the most information about $\{A_{\pm|y}\}_{y=x,z}$ with $B_{\pm|1}$ and $B_{\pm|2}$, we choose

$$B_{\pm|1} = \frac{1}{2}\big(\mathbb{I} \pm \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)\big) \tag{5.0.0.3}$$

$$B_{\pm|2} = \frac{1}{2}\big(\mathbb{I} \pm \frac{1}{\sqrt{2}}(\sigma_x - \sigma_z)\big). \tag{5.0.0.4}$$

As we wish to have only one observable (i.e. a joint observable), we put the above measurements into a single ensemble. To get the correct marginals, we choose an ensemble with equal weights. This results in the following POVM

$$G(+,+) = \frac{1}{2}B_{+|1},\ G(+,-) = \frac{1}{2}B_{+|2} \tag{5.0.0.5}$$
$$G(-,+) = \frac{1}{2}B_{-|2},\ G(-,-) = \frac{1}{2}B_{-|1}, \tag{5.0.0.6}$$

which turns out to be an optimal joint measurement, as the marginals are $A^{\mu}_{\pm|x}$ and $A^{\mu}_{\pm|z}$ with $\mu = \frac{1}{\sqrt{2}}$ [42] (see also subsection 2.3).

Summarising the main ingredients of the above calculation, we write an adaptive strategy for finding joint measurements in the qubit case. Assuming that the (unbiased) observables we want to measure are given by the Bloch vectors $\vec{x}_1, \dots \vec{x}_M$, the strategy reads as follows.

1. Fix unit Bloch vectors $\vec{y}_1, \dots, \vec{y}_N$ such that $\vec{x}_l \cdot \vec{y}_k \neq 0$ for all $l = \{1, \dots, M\}$ and $k = \{1, \dots, N\}$.
2. Choose $k \in \{1, \dots, N\}$ with probability $p(k)$.
3. Perform the $\pm 1$ valued measurement $B_{b_k|k}$ corresponding to $\vec{y}_k$.
4. For each $l = \{1, \dots, M\}$ the outcome $a_l$ is $b_k$ if $\vec{x}_l \cdot \vec{y}_k > 0$ and $-b_k$ if $\vec{x}_l \cdot \vec{y}_k < 0$.
5. As a result, one gets a list $(a_1, \dots, a_M)$ of outcomes.
6. Add $p(k)B_{b_k|k}$ to joint observable's corresponding POVM element.
7. If a combination $(a_1, \dots, a_M)$ does not result in the process, the corresponding POVM element of the joint observable is set to be zero, i.e. $G(a_1, \dots, a_M) = 0$.

Note that in the previous example the connection between joint measurements and steering was not needed. To give an example of a situation, where the connection provides an advantage, we apply the above guidelines to a set of $M \geq 2$ unbiased qubit observables $\{A_{\pm|k}\}_{k=1}^{M}$ given by the Bloch vectors

$$\vec{x}_k = (\cos\theta_k, \sin\theta_k, 0),\ \theta_k = (k-1)\pi/M. \tag{5.0.0.7}$$

Following the adaptive strategy, we need to first find suitable guess vectors. Upon trying a few possibilities, one finds out that sets of guess vectors sharing similar symmetries as the original vectors work nicely. Moreover, in many cases equal combinations of the Bloch vectors $\{\pm\vec{x}_k\}_k$ seems to result in an optimal joint observable. Using these intuitive guidelines, let us first divide the example into two cases. Namely, consider first odd values of $M$. For this case, one can choose the guess vectors to be exactly the original Bloch vectors, i.e. $\vec{y}_k = \vec{x}_k$ for all $k$. For the step two, we take a uniform distribution. For steps three and four, we calculate

$$\vec{x}_k \cdot \vec{y}_l = \cos\theta_k\cos\theta_l + \sin\theta_k\sin\theta_l \tag{5.0.0.8}$$
$$= \cos(\theta_k - \theta_l) \tag{5.0.0.9}$$
$$= \cos\left(\frac{(k-l)\pi}{M}\right). \tag{5.0.0.10}$$

Hence,

$$\vec{x}_k \cdot \vec{y}_l > 0 \quad \text{if } |k-l| < M/2, \tag{5.0.0.11}$$
$$\vec{x}_k \cdot \vec{y}_l < 0 \quad \text{if } |k-l| > M/2. \tag{5.0.0.12}$$

The first marginal of $G$ is then

$$\sum_{a_2,\ldots,a_M=\pm 1} G(a_1,\ldots,a_M) = \frac{1}{2}\big(\mathbb{I} + a_1\lambda\vec{x}_1\cdot\vec{\sigma}\big), \tag{5.0.0.13}$$

where

$$\lambda = \frac{1}{M}\left(1 + 2\sum_{k=1}^{(M-1)/2}\cos(\frac{k\pi}{M})\right). \tag{5.0.0.14}$$

By Lagrange's trigonometric identity we have

$$\sum_{k=1}^{(M-1)/2}\cos(\frac{k\pi}{M}) = -\frac{1}{2} + \frac{1}{2\sin(\frac{\pi}{2M})}. \tag{5.0.0.15}$$

Similarly for an odd $M$ one chooses

$$\vec{y}_k = \big(\cos(\theta_k + \frac{\pi}{2M}), \sin(\theta_k + \frac{\pi}{2M}), 0\big), \tag{5.0.0.16}$$
$$\theta_k = \frac{(k-1)\pi}{M}. \tag{5.0.0.17}$$

As in the case of an even $M$ one gets a joint observable with the correct marginals. The amount of noise is equal to the one in Eq. (5.0.0.14), i.e.

$$\lambda = \frac{1}{M\sin(\frac{\pi}{2M})}. \tag{5.0.0.18}$$

To prove that this threshold is indeed optimal we use a steering inequality given in [14] (see below). One first notices that the (white) noise in Alice's measurements can be transferred to the state shared by Alice and Bob in a steering scenario. For our purposes, it is enough to consider the shared state to be the maximally entangled one, in which case the noise transfers as

$$\text{tr}_A[(A_{a|x}\otimes\mathbb{I})|\psi_\lambda^+\rangle\langle\psi_\lambda^+|] = \text{tr}_A[(A_{a|x}^\lambda\otimes\mathbb{I})|\psi^+\rangle\langle\psi^+|], \tag{5.0.0.19}$$

where $|\psi_\lambda^+\rangle\langle\psi_\lambda^+| = \lambda|\psi^+\rangle\langle\psi^+| + \frac{1-\lambda}{4}\mathbb{I}\otimes\mathbb{I}$ and $A_{a|x}^\lambda = \lambda A_{a|x} + \frac{1-\lambda}{2}\text{tr}[A_{a|x}]\mathbb{I}$.

The steering inequality given in [14] reads in our scenario

$$\frac{1}{M}\sum_{k=1}^{M}\text{tr}[(A_k\otimes\vec{c}_k\cdot\vec{\sigma}_k)|\psi_\lambda^+\rangle\langle\psi_\lambda^+|] \le C_M, \tag{5.0.0.20}$$

where $A_k = A_{+|k} - A_{-|k}$, $\vec{c}_k$ is a Bloch vector and the bound $C_M$ is

$$C_M = \max_{\vec{a}} \left( \lambda_{\max} \left( \frac{1}{M} \sum_{k=1}^{M} a_k \vec{c}_k \cdot \vec{\sigma}_k \right) \right), \tag{5.0.0.21}$$

$\lambda_{\max}(K)$ is the largest eigenvalue of a matrix $K$ and $\vec{a} = (a_1 \ldots a_M)$. Inserting $\vec{c}_k = \vec{y}_k$ and the transposes of the observables given by the Bloch vectors $\vec{x}_k$ for Alice we arrive at

$$\lambda \leq \frac{1}{M \sin(\frac{\pi}{2M})}. \tag{5.0.0.22}$$

Hence, a violation of the above inequality implies steerability and non-joint measurability of Alice's observables. As transposition does not affect joint measurability, we conclude that the equally distributed planar observables are jointly measurable if and only if the above condition holds.

In article VII we analyse various symmetric and non-symmetric scenarios together with a generalisation of the adaptive strategy to higher dimensional systems. We demonstrate the power of the strategy by providing optimal noise threshold for incompatibility in cases including symmetric sets and non-symmetric pairs of qubit observables and pairs of observables given by mutually unbiased bases in any finite-dimensional system.[1]

[1]**Author's contribution**: The author of this thesis contributed to the derivation of various joint measurement criteria.

# Conclusions

In this thesis we have used a quantum measurement theoretical approach towards classical models of quantum correlations. As the main result we have proved a deep connection between a type of measurement incompatibility, namely joint measurability, and a type of quantum correlations, namely quantum steering. As a consequence, we have translated various results between the two fields and built an alternative formalism for steering.

On top of the main result and its implications, we have also mapped entropic uncertainty relations to steering criteria and found ways to generate joint observables and, hence, local hidden state models for steering tests. These results are powerful in the sense that the former has managed to beat every known analytical steering criteria either in strength or in applicability, and the latter has managed to provide optimal noise thresholds for various incompatibility and steering scenarios.

The thesis has also broadened the typical steering setup by developing steering detection techniques for scenarios where one party is completely uncharacterised and the other party has only a dimension-bound on their system. Such techniques are based on mapping steering problems to entanglement and using dimension-bounded entanglement witnesses.

For future research it will be interesting to see if further connections between measurement theoretical concepts, such as non-disturbance and coexistence, and quantum information tasks, such as violations of Bell inequalities, macrorealism or contextuality, can be drawn. Such connections, if existing, may provide powerful tools for all of these fields.

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# Bibliography


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# Appendices

# Article I

- **Title**: Joint measurability of generalised measurements implies classicality
- **Authors**: Roope Uola, Tobias Moroder, Otfried Gühne
- **Journal reference**: Phys. Rev. Lett. **113**, 160403 (2014)
- **Abstract**: The fact that not all measurements can be carried out simultaneously is a peculiar feature of quantum mechanics and is responsible for many key phenomena in the theory, such as complementarity or uncertainty relations. For the special case of projective measurements, quantum behavior can be characterized by the commutator but for generalized measurements it is not easy to decide whether two measurements can still be understood in classical terms or whether the already show quantum features. We prove that a set of generalized measurements which does not satisfy the notion of joint measurability is nonclassical, as it can be used for the task of quantum steering. This shows that the notion of joint measurability is, among several definitions, the proper one to characterize quantum behavior. Moreover, the equivalence allows one to derive novel steering inequalities from known results on joint measurability and new criteria for joint measurability from known results on the steerability of states.
- **Author's contribution**: The author of this thesis contributed to proofs and examples.

# Joint Measurability of Generalized Measurements Implies Classicality

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The fact that not all measurements can be carried out simultaneously is a peculiar feature of quantum mechanics and is responsible for many key phenomena in the theory, such as complementarity or uncertainty relations. For the special case of projective measurements, quantum behavior can be characterized by the commutator but for generalized measurements it is not easy to decide whether two measurements can still be understood in classical terms or whether the already show quantum features. We prove that a set of generalized measurements which does not satisfy the notion of joint measurability is nonclassical, as it can be used for the task of quantum steering. This shows that the notion of joint measurability is, among several definitions, the proper one to characterize quantum behavior. Moreover, the equivalence allows one to derive novel steering inequalities from known results on joint measurability and new criteria for joint measurability from known results on the steerability of states.

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*Introduction.*—Quantum theory is formulated in the language of Hilbert spaces, where states correspond to vectors or density matrices, and measurements are described by Hermitian matrices, the so-called observables. As realized by M. Born and P. Jordan, two observables $A$ and $B$ do not necessarily commute, which means, in the first place, that the corresponding measurements cannot be carried out simultaneously in a direct way [1,2]. This intuition can be made precise by formulating uncertainty relations, where the commutator $[A,B] = AB - BA$ quantifies the degree of uncertainty about the values of $A$ and $B$ [2–4]. Consequently there is the widespread opinion that sets of noncommuting observables are central for many quantum effects, while commuting observables are considered to be classical.

It has turned out, however, that the notion of observables is far too narrow to describe all measurements procedures in quantum mechanics. This has led to the formulation of generalized measurements or positive operator valued measures (POVMs). Mathematically, a POVM consists of a collection of operators $E = \{E(i), i \in I\}$ which are positive, $E(i) \geq 0$, and sum up to the identity, $\sum_i E(i) = \mathbb{1}$. The POVM elements $E(i)$ describe the measurement outcomes and the probability of an outcome $i$ is given by $p(i) = \mathrm{tr}[\varrho E(i)]$. Physically, any POVM can be realized by first letting the physical system interact with an auxiliary system and then measuring an ordinary observable on the auxiliary system. Finally, any observable $A$ is also a POVM if one identifies the $E(i)$ with the projectors onto the eigenspaces of $A$, in which case the measurement is also called a projection valued measure (PVM).

Given the notion of generalized measurements the question arises, when two or more POVMs can be considered to be nonclassical. One possibility is to require the commutativity of all the POVM elements, but more refined notions are useful. Indeed, several notions such as "nondisturbance," "joint measurability," and "coexistence" have been introduced and their investigation is an active area of research [5–9].

In this Letter, we argue that the notion of joint measurability is the proper one to describe the classical behavior of two or more generalized measurements. To do so, we establish a connection between joint measurability and the task of quantum steering. Quantum steering refers to the scenario, where one party, usually called Alice, wishes to convince the other party, called Bob, that she can steer the state at Bob's side by making measurements on her side. This task was introduced by E. Schrödinger to demonstrate the puzzling effects of quantum correlations [10] and recently it has attracted increasing attention again [11–16].

More precisely, we show that a set of POVMs in the finite dimensional case is nonjointly measurable if and only if the set can be used for Alice to show the steerability of some quantum state. This allows one to derive new steering inequalities from results known for joint measurability, and we will also find new criteria for joint measurability from results on steering. Finally, we demonstrate that other possible extensions of commutativity to generalized measurements, such as coexistence, lead to nonclassical effects and we explore the relation of joint measurability to Bell inequality violations.

*Joint measurability.*—The notion of joint measurability is most conveniently introduced with an example. The Pauli spin matrices $\sigma_x$ and $\sigma_z$ are noncommuting and cannot be measured jointly. However, one can consider the smeared or unsharp measurements $S_x$ and $S_z$, defined by the POVM elements $S_x(\pm) = \frac{1}{2}(\mathbb{1} \pm (1/\sqrt{2})\sigma_x)$ and

 

$S_z(\pm) = \frac{1}{2}(\mathbb{1} \pm (1/\sqrt{2})\sigma_z)$. It was shown in Ref. [17] that these are jointly measurable: One can consider the joint observable

$$G(i,j) = \frac{1}{4}\left(\mathbb{1} + \frac{i}{\sqrt{2}}\sigma_x + \frac{j}{\sqrt{2}}\sigma_z\right), \qquad i,j \in \{-1,+1\}, \tag{1}$$

and since $S_x(\pm) = \sum_j G(\pm, j)$ and $S_z(\pm) = \sum_i G(i, \pm)$, one can jointly determine the probabilities of the generalized measurements $S_x$ and $S_z$ by measuring $G$.

More precisely, joint measurability of the set $\{E_k\}$ of POVMs can be formulated as the existence of a set of positive operators $\{G(\lambda)\}$ from which the original observables can be attained as

$$\sum_\lambda D_\lambda(x|k)G(\lambda) = E_k(x) \quad \text{for all } x,k, \tag{2}$$

with $\sum_\lambda G(\lambda) = \mathbb{1}$ and where $D_\lambda(x|k)$ are positive constants with $\sum_x D_\lambda(x|k) = 1$ [18]. In practice, this means that the probabilities of the results $E_k(x)$ can be determined by measuring the operators $G(\lambda)$ and classically postprocessing the data.

*Quantum steering.*—The essence of steering can also be described by an example. Let us assume that two parties, Alice and Bob, share a maximally entangled two-qubit state $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. If Alice measures the Pauli operators $\sigma_x$ or $\sigma_z$, the state on Bob's side will be an eigenstate $|x^\pm\rangle$ or $|z^\pm\rangle$ depending on Alice's measurement and result. Since all these states are pure, Bob cannot explain this by assuming that he has a fixed marginal state $\varrho_B$ which is only modified due to the additional knowledge from Alice's measurements. So Bob must conclude that Alice can steer the state in his lab by making measurements on her side. The question arises whether the same phenomenon occurs if Alice uses the smeared measurements $S_x$ and $S_z$ introduced above. This will be answered in full generality in the following.

First, we label Alice's and Bob's POVMs by $\{A_k\}$ and $\{B_l\}$ and the system's state by $\varrho_{AB}$. Clearly, the scenario is nonsteerable if the probabilities of possible events can be written in the form

$$\mathrm{tr}[\varrho_{AB} A_k(x) \otimes B_l(y)] = \sum_\lambda p(\lambda)p(x|k,\lambda)\mathrm{tr}[\varrho_\lambda B_l(y)] \tag{3}$$

because then Bob can assume that he has the collection of states $\varrho_\lambda$ with probabilities $p(\lambda)$ which is only modified by additional information from Alice's measurements quantified by conditional probability distributions $p(x|k,\lambda)$. We can write the left-hand side of this equation as

$$\mathrm{tr}(\mathrm{tr}_A\{[A_k(x) \otimes \mathbb{1}]\varrho_{AB}\}B_l(y)) =: \mathrm{tr}[\varrho_{x|k} B_l(y)] \tag{4}$$

and if Bob's measurements are tomographically complete it follows that $\varrho_{x|k} = \sum_\lambda p(\lambda)p(x|a,\lambda)\varrho_\lambda$. If, on the other hand, the quantities $\varrho_{x|k}$ admit this kind of a decomposition (also called a hidden state model) we conclude that the scenario is nonsteerable.

This can be reformulated as suggested in Refs. [12,13]: Steering is equivalent to the nonexistence of a set of positive operators $\{\sigma_\lambda\}$ such that

$$\sum_\lambda p(x|k,\lambda)\sigma_\lambda = \varrho_{x|k} \quad \text{for all } x,k, \tag{5}$$

with $\mathrm{tr}(\sum_\lambda \sigma_\lambda) = 1$ and where $\varrho_{x|k} = \mathrm{tr}_A\{[A_k(x) \otimes \mathbb{1}]\varrho_{AB}\}$ are Bob's not-normalized conditional states. The formal similarity between Eq. (2) and Eq. (5) is appealing and, as we will see now, no coincidence.

*Steering and joint measurements.*—Consider the case where Alice has observables $\{A_k\}$ which are jointly measurable. Using Eq. (2) we can write for any steering scenario the conditional states of Bob as

$$\varrho_{x|k} = \sum_\lambda D_\lambda(x|k)\mathrm{tr}_A\{[G(\lambda) \otimes \mathbb{1}]\varrho_{AB}\}, \tag{6}$$

which is a decomposition as in Eq. (5). Therefore, if Alice's observables are jointly measurable then the scenario is nonsteerable.

Conversely, if the measurements are nonjointly measurable, one can always find a state which can be used for steering: For the maximally entangled state $|\phi^+\rangle = 1/\sqrt{d}\sum_{i=1}^d |ii\rangle$ one can write Bob's conditional states as

$$\varrho_{x|k} = \mathrm{tr}_A[(A_k(x) \otimes \mathbb{1})|\phi^+\rangle\langle\phi^+|] = \frac{1}{d}[A_k(x)]^T. \tag{7}$$

If the scenario is not steerable then one can find a set of positive operators $\{\sigma_\lambda\}$ and a set of positive numbers $p(x|k,\lambda)$ such that

$$A_k(x) = d\sum_\lambda p(x|k,\lambda)\sigma_\lambda^T =: \sum_\lambda D_\lambda(x|k)G(\lambda), \tag{8}$$

where $G(\lambda) = d\sigma_\lambda^T$. This is just the joint measurability condition from Eq. (2). Note that by summing over $x$ in Eq. (8) we see that $G$ is properly normalized. We now state the main result of this article.

*Observation 1:* Generalized measurements are nonjointly measurable if and only if they can be used for quantum steering.

Let us note that the reasoning prior to Observation 1 was done for the maximally entangled state. Steering is, however, invariant under stochastic local operations and classical communication [19] on the characterized (Bob's) side. This means that any state which is obtained from the maximally entangled one by stochastic local operations and classical communication can be used to show steering for a set of nonjointly measurable observables. Therefore, any pure Schmidt rank $d$ state (possibly having an arbitrarily small amount of entanglement) reveals steering.

We exploit the connection by giving a generic incompatibility criteria for sharp observables, deriving a steering inequality based on the Fermat-Torricelli point, and pointing out two interesting notes on different formulations of simultaneous measurability.

*From steering to incompatibility.*—We show that there exists a threshold value of white noise [that is, adding the identity as in Eq. (11)] that one needs to add in order to get any set of PVMs jointly measurable. For this purpose we need the following connection between noisy states and noisy observables:

$$\mathrm{tr}_A[A_k(x) \otimes \mathbb{1}\varrho^\lambda_{AB}] = \mathrm{tr}_A[A^\lambda_k(x) \otimes \mathbb{1}\varrho_{AB}], \tag{9}$$

where

$$\varrho^\lambda_{AB} = \lambda\varrho_{AB} + \frac{1-\lambda}{d}\mathbb{1} \otimes \mathrm{tr}_A[\varrho_{AB}], \tag{10}$$

$$A^\lambda_k(x) = \lambda A_k(x) + \frac{1-\lambda}{d}\mathrm{tr}[A_k(x)]\mathbb{1}. \tag{11}$$

In order to obtain the threshold value we take the known result from Ref. [11] stating that the maximally entangled state is steerable with PVMs up to the amount $\lambda^* \coloneqq (H_d - 1)/(d-1)$ of white noise, where $H_d = \sum_{n=1}^d (1/n)$. Using Eq. (9) and Observation 1 one obtains that for any smearing parameter $\lambda \geq \lambda^*$ there must exist a set of PVMs which is noise resistant up to the amount $\lambda$ of white noise; i.e., one can add this amount of white noise to the PVMs without making them jointly measurable. On the other hand, the maximally entangled state reveals steering for nonjointly measurable observables, so all PVMs must be jointly measurable with the amount $\lambda^*$ of white noise. Thus, we arrive at the following result.

*Observation 2:* In a $d$-dimensional Hilbert space, any set of sharp observables is jointly measurable with the amount $\lambda^*$ of white noise. Moreover, for any amount of smearing above this limit there exists a set of PVMs which remains nonjointly measurable.

Note that this is formerly known to be sufficient for $d = 2$ [20]. The result leads to an interesting open question: Are there sets of POVMs which remain nonjointly measurable with the amount $\lambda^*$ of white noise? If this is the case then PVMs are not enough for concluding steerability of a state and if it is not the case then this directly leads to new local hidden variable models for POVMs.

*Fermat-Torricelli steering inequality.*—There are many results of joint measurability known in terms of white noise resistance [17,21,22]. As an example, consider that Alice has three dichotomic unbiased [i.e., $p(\pm|k) = \frac{1}{2}$] measurements while Bob's conditional (normalized) qubit states are characterized by the Bloch vector $\vec{x}_k$, $k = 1, 2, 3$. Using the joint measurability criterion of Ref. [23] we see steering iff

$$||\vec{x}_1 + \vec{x}_2 + \vec{x}_3 - \vec{x}_{FT}|| + ||\vec{x}_1 - \vec{x}_2 - \vec{x}_3 - \vec{x}_{FT}|| + ||\vec{x}_1 - \vec{x}_2 + \vec{x}_3 + \vec{x}_{FT}|| + ||\vec{x}_1 + \vec{x}_2 - \vec{x}_3 + \vec{x}_{FT}|| > 4, \tag{12}$$

where $\vec{x}_{FT}$ denotes the Fermat-Torricelli point of the vectors $\vec{x}_1 + \vec{x}_2 + \vec{x}_3$, $\vec{x}_1 - \vec{x}_2 - \vec{x}_3$, $-\vec{x}_1 + \vec{x}_2 - \vec{x}_3$, and $-\vec{x}_1 - \vec{x}_2 + \vec{x}_3$; i.e., it is the vector that minimizes the sum in Eq. (12).

*Coexistence leads to a nonclassical effect.*—Coexistence of POVMs $A_1$ and $A_2$ means the possibility of making a measurement $G$ of which statistics include the statistics of $A_1$ and $A_2$. To be more precise, $A_1$ and $A_2$ are coexistent if their POVM elements are contained in the range (i.e., all possible sums of POVM elements) of a third POVM $G$. Note that contrary to joint measurements, the statistics do not need to originate from a postprocessing scheme as in Eq. (2). To clarify the notion we present an example given in Ref. [5] which was originally used to show that coexistence is more general than joint measurability; for a similar example, see Ref. [8].

In $\mathbb{C}^3$ define $|\varphi\rangle = 1/\sqrt{3}(|1\rangle + |2\rangle + |3\rangle)$ and a POVM $G$ by the elements $\{\frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|2\rangle\langle 2|, \frac{1}{2}|3\rangle\langle 3|, \frac{1}{2}|\varphi\rangle\langle\varphi|, \frac{1}{2}(\mathbb{1} - |\varphi\rangle\langle\varphi|)\}$. One sees straightforwardly that the measurement statistics of a three-valued POVM $A_1$ defined as $A_1(i) = \frac{1}{2}(\mathbb{1} - |i\rangle\langle i|)$ and a two-valued POVM $A_2$ defined as $A_2(1) = \frac{1}{2}|\varphi\rangle\langle\varphi|$, $A_2(2) = \mathbb{1} - A_2(1)$ are contained in the measurement statistics of $G$; hence, they are coexistent. In Ref. [5] it was shown that these measurements are nevertheless nonjointly measurable due to the lack of a postprocessing relation. By Observation 1 we conclude the following.

*Observation 3:* As coexistence is more general than joint measurability it can reveal steering; i.e., it can lead to nonclassical effects in the distributed scenario.

*Disturbing measurements can be useless for steering.*—One way to define the classicality of two measurements, say $A_1$ and $A_2$, is to say that the measurement of $A_1$ does not disturb the measurement of $A_2$. This means that a measurement of $A_1$ updates the state in such a way that a subsequent measurement of $A_2$ has the same statistics for both the updated and the original state. It was shown in Ref. [9] that there exists pairs of observables that can be measured jointly even though they do not admit a nondisturbing sequential measurement. Using this together with Observation 1 we conclude that disturbing measurements do not necessarily lead to steering.

*Joint measurability and nonlocality.*—From the previous discussion we know that any nonjointly measurable set of POVMs can reveal its "quantumness" in a strictly nonclassical, nonlocal effect, more precisely, in the form of steering. Steering is, however, not the ultimate strongest form of nonlocality since one still needs a quantum description on one side. Thus, it is of course a natural question whether this connection can even be strengthened, so whether it also holds that any nonjointly measurable set of POVMs can show nonclassicality in a Bell-type scenario.

This is indeed the case for two dichotomic measurements as has been shown by Wolf *et al.* in Ref. [24]. It also holds for an arbitrary number of PVMs. In the following, we argue that it would be very surprising if this connection were to hold in general, since via a very simple example one encounters already large difficulties.

Consider the three dichotomic spin measurements of a qubit $A_k^\lambda(\pm) = (\mathbb{1} \pm \lambda\sigma_k)/2$ with $k \in \{x, y, z\}$. As already mentioned, the additional parameter $\lambda$ characterizes the noise on these measurements. For $\lambda = 1$ the measurements $A_k \coloneqq A_k^{\lambda=1}$ are noncommuting projectors, while for $\lambda \leq 1/\sqrt{3} \approx 0.5774$ the set of POVMs becomes jointly measurable. Suppose that joint measurability and nonlocality are as strongly connected as steering. This would mean that for any noisy, but nonjointly measurable set of these POVMs, i.e., for all $1/\sqrt{3} < \lambda$, it is possible to find a respective bipartite state $\varrho_{AB}$ and corresponding measurements for Bob $B_l(k)$, such that the obtained data $P(\pm, y|k, l) = \mathrm{tr}[\varrho_{AB} A_k^\lambda(\pm) \otimes B_l(y)]$ violate a Bell inequality.

In the search for such an appropriate state, first note that pure states $\varrho_{AB} = |\psi\rangle\langle\psi|$ are sufficient, since any mixed state can only violate a Bell inequality if at least one pure state from its range does so. Using the Schmidt decomposition together with the fact that $\dim(\mathcal{H}_A) = 2$ we can write the most general pure state as $|\psi\rangle = U_A \otimes U_B|\psi_s\rangle$ with $|\psi_s\rangle = s|00\rangle + \sqrt{1-s^2}|11\rangle$ where $1/\sqrt{2} \leq s \leq 1$. Since we optimize Bob's measurements we can additionally assume $U_B = \mathbb{1}$, meaning that Bob similarly holds a qubit. Next we also wish to transfer the noise of the measurements into the state, as given by Eq. (9). Thus, rather than looking for a pure state which violates a Bell inequality using the noisy measurements $A_k^\lambda$, we can equivalently search for a mixed state that violates a Bell inequality with perfect measurements $A_k$. To sum up, we would need to show that for any parameter $\lambda > 1/\sqrt{3}$, a state of the form

$$\varrho_{AB}(s; U_A) = \lambda U_A \otimes \mathbb{1}|\psi_s\rangle\langle\psi_s|U_A^\dagger \otimes \mathbb{1} + (1-\lambda)\mathbb{1}/2 \otimes \mathrm{tr}_A[|\psi_s\rangle\langle\psi_s|] \quad (13)$$

with appropriately chosen $1/\sqrt{2} \leq s \leq 1$ and $U_A$ violates a Bell inequality using the three perfect spin measurements on system $A$, and arbitrary measurements for system $B$.

Let us start with the maximally entangled state, $s = 1/\sqrt{2}$, for which it is known that it does not violate a Bell inequality using projective measurements if $\lambda \leq 0.6595$ [25]. Hence, for the given noisy nonjointly measurable set of POVMs within $1/\sqrt{3} < \lambda \leq 0.6595$, the data of the maximally entangled state, using also projective measurements for Bob, will not display any nonlocality. For nonmaximally entangled states the situation is much less analyzed, especially under the influence of nonwhite noise as in Eq. (13). The statement extends, however, to $1/\sqrt{3} < \lambda \leq 0.6009$ [25] for arbitrary, nonmaximally entangled states if one wants to reproduce the full correlations. Thus, the only Bell inequalities that remain are the ones with marginals.

A different way to prove that certain states do not violate a Bell inequality is to write them as a convex combination of states known to possess a local hidden variable model for the considered configuration

$$\varrho_{AB}(s; U_A) = \sum_i p_i \varrho_i^{\mathrm{LHV}}. \quad (14)$$

Generic states that we consider in this decomposition include (i) noisy Bell states with $\lambda \leq 0.6595$ and (ii) states with two symmetric extensions for system $A$ [26]. States of class (ii) are known to have a local hidden variable model for three generic measurements for system $A$ [27], such that we exploit the fact that Alice has only a restricted number of measurements. Such a search for symmetric extensions can be easily done with semidefinite programming [28]. Figure 1 shows, depending on the Schmidt coefficient $s$ (and for all $U_A$), the respective maximal values of $\lambda$ when such a decomposition is possible. As can be seen for $s \leq 0.835$, there is always a noise parameter $\lambda > 1/\sqrt{3}$ such that the given set of POVMs is nonjointly measurable, but the measured state will not violate a Bell inequality using an arbitrary number of projective measurements for Bob. Finally, if one additionally constrains Bob to perform only $n$ different dichotomic measurements then one can further add (iii) the class of states that have $n-1$ symmetric extensions for system $B$. As shown in Fig. 1 for $n \leq 6$, such a decomposition is possible for all values of $s$. Thus, there

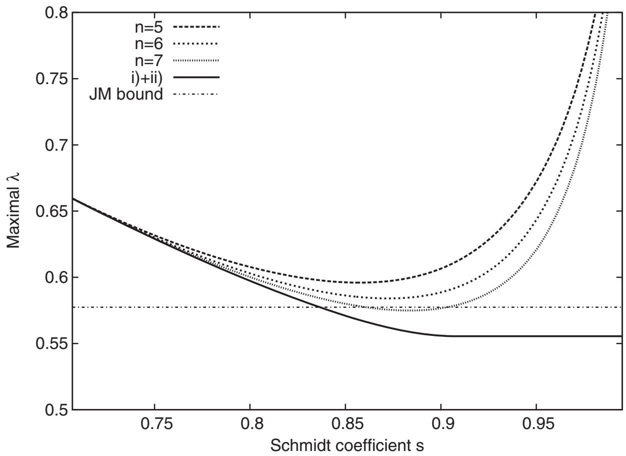


FIG. 1. Maximal values of $\lambda$ when a decomposition as given by Eq. (14) is possible for all $U_A$ depending on the Schmidt coefficient $s$. It shows that a pure state with $s \leq 0.835$ is never able to reveal Bell nonlocality for an arbitrary number of projective measurements, while for $n \leq 6$ projective measurements it is not possible for any state.

exists a parameter $\lambda > 1/\sqrt{3}$ such that the corresponding set of POVMs is nonjointly measurable but no quantum state will display nonlocality if Bob only carries out 6 dichotomic measurements.

These observations give strong hints that there are sets of POVMs which are nonjointly measurable, but which are nevertheless useless to certify nonlocality.

*Conclusions.*—We have shown that joint measurability and quantum steering are intrinsically connected: A collection of different measurements are nonjointly measurable if and only if they can reveal its "nonclassicality" as a violation of a steering inequality. This connects the abstract notion of joint measurability to an explicit nonlocality task, and thereby singles out nonjoint measurability as a special nonclassical property among other peculiar quantum features of measurements.

Since measurements are as relevant as quantum states, we believe that this connection will spur the resource theory of measurements, i.e., which kind of measurements are required for certain tasks. This investigation could provide some operational meaning to other quantum properties of measurements such as disturbance or noncoexistence in the distributed scenario.

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*Note added.*—After finishing this work we noticed that similar results were obtained in Ref. [29].

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# Article II

- **Title**: One-to-one mapping between steering and joint measurability problems
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- **Author's contribution**: The author of this thesis contributed to proofs and examples.

# One-to-One Mapping between Steering and Joint Measurability Problems

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Quantum steering refers to the possibility for Alice to remotely steer Bob's state by performing local measurements on her half of a bipartite system. Two necessary ingredients for steering are entanglement and incompatibility of Alice's measurements. In particular, it is known that for the case of pure states of maximal Schmidt rank the problem of steerability for Bob's assemblage is equivalent to the problem of joint measurability for Alice's observables. We show that such an equivalence holds in general; namely, the steerability of any assemblage can always be formulated as a joint measurability problem, and vice versa. We use this connection to introduce steering inequalities from joint measurability criteria and develop quantifiers for the incompatibility of measurements.

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*Introduction.*—Steering is a quantum effect by which one experimenter Alice can remotely prepare an ensemble of states for another experimenter Bob by performing a local measurement on her half of a bipartite system and communicating the results to Bob. Introduced by Schrödinger in 1935 [1], quantum steering is a form of quantum correlation intermediate between Bell nonlocality and entanglement. It has recently attracted increasing interest [2–7], both from a theoretical and experimental perspective, and it has been recognized as a resource for different tasks such as one-sided device-independent quantum key distribution [8,9] and subchannel discrimination [10]. In addition, the question of which quantum states can be used for steering can be addressed with efficient numerical techniques, contrary to the notion of entanglement or the question of which states violate a Bell inequality. In this way, the notion of steering has been used to find a counterexample to the Peres conjecture, a long-standing open problem in entanglement theory [11,12].

A successful implementation of a steering protocol involves different elements, e.g., entangled states and incompatible measurements, and therefore steering has been investigated under different perspectives. On the one hand, allowing for an optimization over all possible quantum states or, equivalently, considering the maximal entangled state, steering has been identified with the lack of joint measurability of Alice's local observables [13,14], similarly to the case of nonlocality [15]. On the other hand, if an optimization over all possible measurements for Alice has been considered, steering has been identified with a property of the state allowing for optimal subchannel discrimination when one is restricted to local measurements and one-way classical communication [10]. In addition, a very natural and interesting framework for steering is that of one-sided device-independent quantum information processing [16–18]. In the case of device-independent quantum information processing, both parties are untrusted; hence, no assumption is made on the system and the measurement apparatuses and the only resources are the observed (nonlocal) correlations. Similarly, in one-sided device-independent scenarios, where only one party (Bob) is trusted, it is natural to identify the resources for information processing tasks with the ensemble of states Bob obtains as a consequence of Alice's measurement (see also Ref. [19] for a discussion of this point).

Taking the above perspective, we are able to prove that any steerability problem can be translated into a joint measurability problem, and vice versa. This result connects the well-known theory of joint measurements [20,21] and uncertainty relations [22–25] to the relatively new research direction of steering. This is done by mapping any state ensemble for Bob in a corresponding steering-equivalent positive operator valued measure (POVM). This simple technique is shown to give an intuitive way of generalizing the known results [13,14]. Moreover, the power of the technique is demonstrated by mapping joint measurement uncertainty relations [22] into steering inequalities, and discussing the role of known steering monotones as monotones for incompatibility.

*Preliminary notions.*—Given a quantum state $\rho$, i.e., a positive operator with trace 1, an ensemble $\mathcal{E} = \{\rho_a\}$ for $\rho$ is a collection of positive operators such that $\sum_a \rho_a = \rho$. An assemblage $\mathcal{A} = \{\mathcal{E}_x\}_x$ is a collection of ensembles for the same state $\rho$, i.e., $\sum_a \rho_{a|x} = \rho$, for all $x$. Similarly, a measurement assemblage $\mathcal{M} = \{M_{a|x}\}_{a,x}$ is a collection of operators $M_{a|x} \geq 0$ such that $\sum_a M_{a|x} = \mathbb{1}$ for all $x$. Each subset $\{M_{a|x}\}_a$ is called a POVM, and it gives the outcome probabilities for a general quantum measurement via the formula $P(a|x) = tr[M_{a|x}\rho]$.

A measurement assemblage $\mathcal{M} = \{M_{a|x}\}_{a,x}$ is defined to be jointly measurable (JM) [26] if there exist numbers $p_M(a|x,\lambda)$ and positive operators $\{G_\lambda\}$ such that

 

$$M_{a|x} = \sum_{\lambda} p_M(a|x,\lambda) G_\lambda \quad (1)$$

with $\sum_\lambda G_\lambda = \mathbb{1}$, $p_M(a|x,\lambda) \geq 0$, and $\sum_a p_M(a|x,\lambda) = 1$. Physically, this means that all the measurements in the assemblage can be measured jointly by performing the measurement $\{G_\lambda\}$ and doing some postprocessing of the obtained probabilities.

In a steering scenario, a bipartite state $\rho_{AB}$ is shared by Alice and Bob. Alice performs measurements on her system with possible settings $x$ and possible outcomes $a$, that is, the measurement assemblage $\{A_{a|x}\}_{a,x}$. As a result of her measurement with the setting $x$, Bob obtains the reduced state $\varrho(a|x)$ with probability $P(a|x)$. Such a collection of reduced states and probabilities defines the state assemblage $\{\rho_{a|x}\}_{a,x}$, where

$$\rho_{a|x} = \mathrm{tr}_A[(A_{a|x} \otimes \mathbb{1})\rho_{AB}] \quad (2)$$

with $P(a|x) = \mathrm{tr}[(A_{a|x} \otimes \mathbb{1})\rho_{AB}] = \mathrm{tr}_B[\rho_{a|x}]$ and $\varrho(a|x) = \rho_{a|x}/P(a|x)$. In particular, elements of the assemblage satisfy

$$\rho_B = \sum_a \rho_{a|x} = \sum_{a'} \rho_{a'|x'}, \qquad \text{for all settings } x, x', \quad (3)$$

where $\rho_B = \mathrm{tr}_A[\rho_{AB}]$. This expresses the fact that Alice cannot signal to Bob by choosing her measurement $x$.

A state assemblage $\{\rho_{a|x}\}_{a,x}$ is called unsteerable if there exists a local hidden state (LHS) model, namely, numbers $p_\rho(a|x,\lambda) \geq 0$ and positive operators $\{\sigma_\lambda\}$ such that

$$\rho_{a|x} = \sum_\lambda p_\rho(a|x,\lambda)\sigma_\lambda \quad (4)$$

with $\mathrm{tr}[\sum_\lambda \sigma_\lambda] = 1$. A state assemblage is called steerable if it is not unsteerable. The physical interpretation is the following. If the assemblage has a LHS model, then Bob can interpret his conditional states $\rho_{a|x}$ as coming from the preexisting states $\sigma_\lambda$, where only the probabilities are changed due to the knowledge of Alice's measurement and result. Contrary, if no LHS model is possible, then Bob must believe that Alice can remotely steer the states in his lab by making measurements on her side.

*Steerability as a joint measurability problem.*—We now prove the main results of the Letter, namely, that the steerability properties of a state assemblage can always be translated in terms of the joint measurability properties of a measurement assemblage.

Let $\{\rho_{a|x}\}_{a,x}$ be a state assemblage and $\rho_B$ the corresponding total reduced state for Bob. We define $\Pi_B : \mathcal{H}_B \to \mathcal{K}_{\rho_B} \subset \mathcal{H}_B$ as the projection on the subspace $\mathcal{K}_{\rho_B}$:=range$(\rho_B)$; i.e., $\Pi_B \Pi_B^* = \mathbb{1}_{\mathcal{K}_{\rho_B}}$ and $\Pi_B^* \Pi_B$ is a Hermitian projector in $\mathcal{L}(\mathcal{H}_B)$.

Since $\rho_{a|x}$ are positive operators, Eq. (3) implies range$(\rho_{a|x}) \subset$ range$(\rho_B)$ for all $a$, $x$ [27]. Hence, we can define the restriction of our assemblage elements to the subspace $\mathcal{K}_{\rho_B}$ as $\tilde{\rho}_{a|x} = \Pi_B \rho_{a|x} \Pi_B^*$ and $\tilde{\rho}_B = \Pi_B \rho_B \Pi_B^*$, preserving the positivity of the operators. Such a restriction is needed in order to define $(\tilde{\rho}_B)^{-\frac{1}{2}}$ (see below). Then, we define Bob's steering-equivalent (SE) observables $B_{a|x} \in \mathcal{L}(\mathcal{K}_{\rho_B})$ as

$$B_{a|x} = (\tilde{\rho}_B)^{-\frac{1}{2}} \tilde{\rho}_{a|x} (\tilde{\rho}_B)^{-\frac{1}{2}}. \quad (5)$$

These operators are clearly positive and, by Eq. (3), $\sum_a B_{a|x} = \mathbb{1}_{\mathcal{K}_{\rho_B}}$; hence, $\{B_{a|x}\}_a$ forms a POVM. We can formulate the first equivalence.

**Theorem 1.** The state assemblage $\{\rho_{a|x}\}_{a,x}$ is unsteerable if and only if the measurement assemblage $\{B_{a|x}\}_{a,x}$ defined by Eq. (5) is jointly measurable.

*Proof.*—First, notice that it is sufficient to discuss the existence of a LHS model for $\{\tilde{\rho}_{a|x}\}_{a,x}$. From Eqs. (4) and (1), one can easily see that from a LHS for $\{\tilde{\rho}_{a|x}\}_{a,x}$ one can construct a joint observable for $\{B_{a|x}\}_{a,x}$ and vice versa. The corresponding LHS model and joint observable are obtainable via the relation

$$G_\lambda = (\tilde{\rho}_B)^{-\frac{1}{2}} \tilde{\sigma}_\lambda (\tilde{\rho}_B)^{-\frac{1}{2}}, \quad (6)$$

where $\tilde{\sigma}_\lambda$ denotes the elements of the LHS for $\tilde{\rho}_{a|x}$. □

The above theorem shows that every steerability problem can be recast as a joint measurability problem. The other direction is trivial, since every joint measurability problem corresponds, up to a multiplicative constant, to a steerability problem with $\rho_B = \mathbb{1}/d$. We can then state the main result.

**Theorem 2.** The steerability problem of any state assemblage $\{\rho_{a|x}\}_{a,x}$ can be translated into a joint measurability problem for a measurement assemblage $\{M_{a|x}\}_{a,x}$, and vice versa.

It is now interesting to discuss the interpretation of Bob's SE observables. Let $\rho = \sum_{i,j=1}^n \lambda_i \lambda_j |ii\rangle\langle jj|$ be a pure state on a finite-dimensional Hilbert $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\{|i\rangle_A\}_1^{d_A}$,$\{|i\rangle_B\}_1^{d_B}$ are the local bases associated with the above Schmidt decomposition of $\rho$, $n \leq \min\{d_A, d_B\}$, $\lambda_l > 0$, and $\mathrm{tr}[\rho] = \sum_i \lambda_i^2 = 1$.

The reduced states for Alice and Bob have in such basis an identical form, namely, $\rho_X = \sum_{i=1}^n \lambda_i^2 |i\rangle\langle i|_X$ with $X = A, B$; hence, their ranges $\mathcal{K}_{\rho_A}$, $\mathcal{K}_{\rho_B}$ are isomorphic through the obvious mapping $|i\rangle_A \leftrightarrow |i\rangle_B$. Using that, we can formally write

$$\rho_{a|x} = \mathrm{tr}_A[(A_{a|x} \otimes \mathbb{1})\rho] = \sum_{i,j=1}^n \lambda_i \lambda_j \langle j|A_{a|x}|i\rangle |i\rangle\langle j| = \rho_A^{1/2} A_{a|x}^t \rho_A^{1/2}, \quad (7)$$

recovering a similar relation as in Eq. (5). The only missing step is to invert the relation by projecting on $\mathcal{K}_{\rho_B}$ and writing the inverse $\rho_A^{-1/2}$. Hence, for any pure state, Theorem 1 gives us a clear interpretation of Bob's SE observables that generalizes the result given in Refs. [13,14], namely, that for a Schmidt rank $d$ state it is sufficient for Alice to use nonjointly measurable observables in order to demonstrate steering.

**Remark.** For a pure bipartite state, in order for Alice to demonstrate steering, her observable must be not jointly measurable even when restricted to the subspace where her reduced state $\rho_A$ does not vanish.

Notice that the above remark holds also for pure separable states; however, since the corresponding subspace $\mathcal{K}_{\rho_A}$ is one dimensional, joint measurability of Alice's observables is always trivially achieved.

For the case of mixed states, a straightforward generalization of the above argument, e.g., via convex combinations, is not possible. Hence, the physical interpretation of Bob's SE observable for mixed states remains an open problem.

*Steering inequalities.*—We use the above result to give new steering inequalities for an assemblage arising from two and three dichotomic measurements for Alice when Bob's system is a qubit. We begin with the assemblage arising from two dichotomic measurements.

Given the assemblage $\{\rho_{a|x}\}$, with $a = \pm$ and $x \in \{1, 2\}$, written in terms of Pauli matrices $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ as

$$\rho_{\pm|x} = t_x^{\pm}\mathbb{1} + \vec{s}_x^{\pm} \cdot \vec{\sigma} \tag{8}$$

with $\vec{s}_x^{\pm} = (s_{1x}^{\pm}, s_{2x}^{\pm}, s_{3x}^{\pm})$, the only nontrivial case corresponds to a reduced state $\rho_B = \sum_{a=\pm}\rho_{a|x}$ of rank 2; otherwise, the total state would be separable.

Then, the SE observables for Bob can be written as

$$B_{+|x} = \frac{1}{2}[(1+\alpha_x)\mathbb{1} + \vec{r}_x \cdot \vec{\sigma}], \qquad B_{-|x} = \mathbb{1} - B_{+|x} \tag{9}$$

with $\alpha_x$ and $\vec{r}_x = (r_{1x}, r_{2x}, r_{3x})$ being functions of the assemblage $\{\rho_{a|x}\}$; the explicit forms of these functions are given in the the Supplemental Material [28]. For such observables Busch *et al.* [22] have defined the degree of incompatibility to be the amount of violation of the following inequality

$$\|\vec{r}_1 + \vec{r}_2\| + \|\vec{r}_1 - \vec{r}_2\| \le 2. \tag{10}$$

This inequality is a measurement uncertainty relation for joint measurements and as such it is a necessary condition for the joint measurability of two observables on a qubit (see also Ref. [21]). A violation of this inequality means that the SE observables of Bob are not jointly measurable and hence the setup is steerable. However, it has been shown that the degree of incompatibility does not capture all incompatible observables and a more fine-tuned version of this inequality, providing necessary and sufficient conditions, has been derived [29]:

$$(1 - F_1^2 - F_2^2)\left(1 - \frac{\alpha_1^2}{F_1^2} - \frac{\alpha_2^2}{F_2^2}\right) \le (\vec{r}_1 \cdot \vec{r}_2 - \alpha_1\alpha_2)^2 \tag{11}$$

with $F_i = \frac{1}{2}(\sqrt{(1+\alpha_i)^2 - \|\vec{r}_i\|^2} + \sqrt{(1-\alpha_i)^2 - \|\vec{r}_i\|^2})$, for $i = 1, 2$.

With the above definition, we can see the difference in the steerable assemblages detected by the steering inequality (10), which provides only a necessary condition, and inequality (11), which completely characterizes steerability. Consider an ensemble of two reduced states along the $z$ axis and symmetric with respect to the origin, i.e., $\rho_{\pm|1} = \frac{1}{2}(\mathbb{1} \pm \lambda\sigma_z)$. Given another ensemble $\rho_{\pm|2}$, by Eq. (3) only one of the two reduced states can be chosen freely, say $\rho_{+|2} = t_2^+ + \vec{s}_2^+ \cdot \vec{\sigma}$, with the conditions $t_2^+ \le 1/2$ and $\|\vec{s}_2^+\| \le t_2^+$. The steerability detected by Eqs. (10) and (11) is plotted in Fig. 1, for different values of the parameters $\lambda, r := \|\vec{s}_2^+\|$, and the angle $\theta$ between $\vec{s}_2^+$ and the $z$ axis.

Finally, for the case of three dichotomic measurements on Alice's side (and Bob holding a qubit) we get three steering equivalent observables of the form (9). For this case a joint measurement uncertainty relation and hence a steering inequality is given by [30]

$$\sum_{i=1}^{4} \|\vec{R}_i - \vec{R}_{\mathrm{FT}}\| \le 4, \tag{12}$$

where $\vec{R}_1 = \vec{r}_1 + \vec{r}_2 + \vec{r}_3$, $\vec{R}_i = 2\vec{r}_{i-1} - \vec{R}_1$ $(i = 2, 3, 4)$, and $\vec{R}_{\mathrm{FT}}$ is the Fermat-Torricelli point of the vectors $\vec{R}_i$,

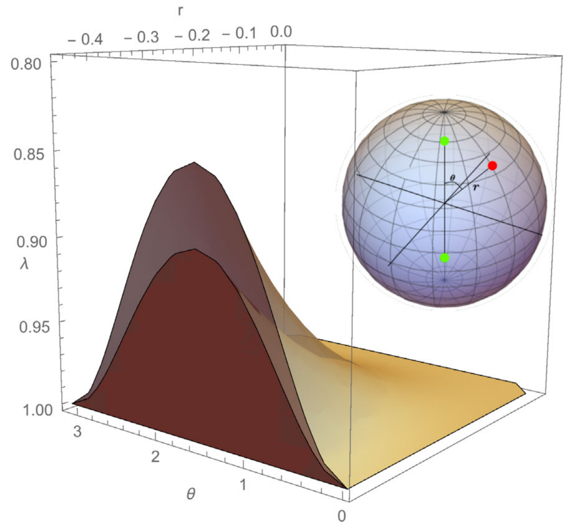


FIG. 1 (color online). Regions of the parameters $\lambda, r, \theta$ allowing for steering, detected by the inequality (10) (inner region) and inequality (11) (outer region), with $r = \|\vec{s}_2^+\|$ and $\theta$ the angle between $\vec{s}_2^+$ and the $z$ axis, and $t_2^+ = 0.45$ (fixed). Inset: representation in the Bloch sphere of the reduced states $\rho_{\pm|1}$ (green points) and $\rho_{+|2}$ (red point). The normalization factor $t_2^+ = \mathrm{tr}[\rho_{+|2}]$ is not represented.

i.e., the point that minimizes the left-hand side of Eq. (12). Analogously to the case of Eq. (10), Eq. (12) provides a necessary condition for the unsteerability of the state assemblage.

*Steering monotones.*—The previously known connection between joint measurability and steering [13,14] has inspired the definition of incompatibility monotones, i.e., measures of incompatibility that are nonincreasing under local channels, based on steering monotones [31] or associated with steering tasks [32].

Following the same spirit and in light of Theorem 2, we introduce an incompatibility monotone based on a recently proposed steering monotone, i.e., the steering robustness [10]. Given a measurement assemblage $\{M_{a|x}\}_{a,x}$ we define the incompatibility robustness ($\mathcal{IR}$) as the minimum $t$ such that there exists another measurement assemblage $\{N_{a|x}\}_{a,x}$ such that $\{(M_{a|x} + tN_{a|x})/(1+t)\}_{a,x}$ is jointly measurable. The idea is to quantify the robustness of the incompatibility properties of the measurement assemblage under the most general form of noise. It is easily proven that $\mathcal{IR}$ can be computed as a semidefinite program and that it is monotone under the action of a quantum channel (cf. the Supplemental Material [28]).

It is interesting to discuss the relation with previously proposed incompatibility monotones. In Ref. [31], the incompatibility weight (IW), a monotone based on the steerable weight (SW) of Ref. [3], was defined for a set of POVMs $\{M_{a|x}\}_x$ as the minimum positive number $\lambda$ such that the decomposition $M_{a|x} = \lambda O_{a|x} + (1-\lambda)N_{a|x}$ holds for the assemblage $\{N_{a|x}\}_{a,x}$ and the jointly measurable assemblage $\{O_{a|x}\}_{a,x}$. From the definition it is clear that the IW suffers from a similar problem as the SW, namely, that whenever the elements of the (state or measurement) assemblage are rank-1, such a weight is maximal. As a consequence, each pair of projective measurements, e.g., on a qubit, even along arbitrary close directions, is maximally incompatible according to the IW, and, similarly, the state assemblage arising from a bipartite pure state, even with arbitrary small entanglement, is maximally steerable according to the SW (see also the discussion in Ref. [10]).

Another monotone has been proposed by Heinosaari *et al.* [32], based on noise robustness of the incompatibility with respect to mixing with white biased noise. This definition can be obtained from $\mathcal{IR}$, with the substitution $N_{a|x} \mapsto \frac{\mathbb{1}}{d}$ (white noise) and, for the corresponding coefficient $\lambda := t/(1+t)$, the substitution $\lambda \mapsto (1+ab)\lambda$, in the case of dichotomic measurements, i.e., $a = \pm 1$. The notion of biasedness refers to the possibility of having a different disturbance for different outcomes.

As a consequence, $\mathcal{IR}$ is always a lower bound to the white noise tolerance. It is interesting to discuss such differences in a simple example. Consider a mixing of a measurement assemblage $\{M_{a|x}\}_{a,x}$ with white or general noise

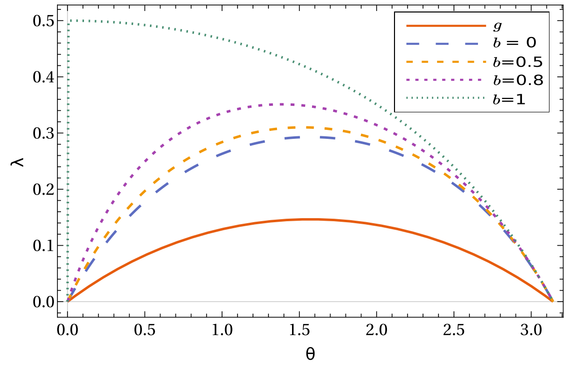


FIG. 2 (color online). Plot of noise robustness for white and general noise for two sharp qubit measurements separated by an angle $\theta$. The line denoted by $g$ corresponds to the parameter $\lambda_g$ of Eq. (13), whereas lines denoted by $b$ correspond to the parameter $\lambda_w$ of Eq. (14) for different level of bias, namely, $b = 0, 0.5, 0.8, 1$ (see main text). The plot shows that the white noise tolerance is always at least double that of the the general noise tolerance $\lambda_g$. Moreover, the introduction of biased noise, quantified by the parameter $b$, with $b = 0$ corresponding to unbiased white noise, only increases the noise tolerance.

$$\mathcal{M}_g = \{(1-\lambda_g)M_{a|x} + \lambda_g N_{a|x}\}_{a,x}, \tag{13}$$

$$\mathcal{M}_w = \left\{(1-\lambda_w)M_{a|x} + \lambda_w \frac{\mathbb{1}}{d}\right\}_{a,x}. \tag{14}$$

If we choose in a qubit case $M_{a|x} = \frac{1}{2}(\mathbb{1} + \vec{v}_{a|x}\cdot\vec{\sigma})$ and $N_{a|x} = \frac{1}{2}(\mathbb{1} - \vec{v}_{a|x}\cdot\vec{\sigma})$ we end up with the mixings $\mathcal{M}_g = \{\frac{1}{2}[\mathbb{1} + (1-2\lambda)\vec{v}_{a|x}\cdot\vec{\sigma}]\}_{a,x}$ and $\mathcal{M}_w = \{\frac{1}{2}[\mathbb{1} + (1-\lambda)\vec{v}_{a|x}\cdot\vec{\sigma}]\}_{a,x}$. It is then clear that in this case the noise robustness for general noise is always smaller than half the noise robustness with respect to white noise, namely,

$$\min\{\lambda_g|\mathcal{M}_g \text{is JM}\} \leq \frac{1}{2}\min\{\lambda_w|\mathcal{M}_w \text{ is JM}\}. \tag{15}$$

Explicit calculations (plotted in Fig. 2) show that the above choice for $N_{a|x}$ is not always the optimal one. The same noise robustness, for the case of orthogonal sharp measurements in dimension $d$, has been calculated in Ref. [33].

The case of biased white noise corresponds to the substitution in Eq. (14) $\lambda \mapsto \lambda(1+ab)$ for the case of binary measurements, i.e., $a = \pm 1$. For the simplest case, i.e., two sharp projective measurement on a qubit, the noise robustness for mixing with general noise or with white noise plus a bias is plotted in Fig. 2.

*Conclusions.*—We have proven that every steerability problem can be recast as a joint measurability problem, and vice versa. As opposed to previous results [13,14], our approach does not include any assumption on the state of the system, but it is applicable knowing solely Bob's state assemblage. This is arguably the most natural resource for steering, especially for one-sided device-independent quantum information protocols, where only Bob's side is characterized [19].

Our work connects the relatively new field of quantum steering with the much older topic of joint measurability. As we showed with concrete examples, this connection allows us to translate results from one field to the other. On the one hand, we were able to derive new steering inequalities for the two simplest steering scenarios based on joint measurability criteria for qubit observables. As opposed to previously defined steering inequalities based on semidefinite programming [3,10], our inequalities are not defined in terms of an optimization for a specific assemblage, but are valid in general. For example, Eq. (11) gives a complete analytical characterization of the simplest steering scenario for any state assemblage.

On the other hand, our result allowed us to introduce a new incompatibility monotone based on a steering monotone. This opens a connection to entanglement theory: similar quantities to the incompatibility monotone have been used to quantify entanglement [34–36]. So, for future work it would be very interesting to use ideas from entanglement theory to characterize the incompatibility of measurements.

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# Supplemental Material: A one-to-one mapping between steering and joint measurability problems

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### Explicit form of Bob's SE observables for a qubit and tight steering inequality

Given the assemblage $\{\rho_{a|x}\}$, with $a = \pm$ and $x \in \{1,2\}$, written in terms of Pauli matrices $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ as

$$\rho_{\pm|x} = t_x^{\pm} \mathbb{1} + \vec{s}_x^{\,\pm} \cdot \vec{\sigma}, \tag{1}$$

with $\vec{s}_x^{\,\pm} = (s_{1x}^{\pm}, s_{2x}^{\pm}, s_{3x}^{\pm})$, the only nontrivial case corresponds to a reduced state $\rho_B = \sum_{a=\pm} \rho_{a|x}$ of rank 2, otherwise the total state would be separable.

Since $\rho_B$ is full rank, we can directly compute first the square root $(\rho_B)^{\frac{1}{2}}$ and then its inverse $(\rho_B)^{-\frac{1}{2}}$ as a function of $\vec{s}_x^{\,\pm}$, either via a tedious direct calculation or with the aid of a symbolic mathematical computation program.

Then the SE observables for Bob can then be obtained from the equation

$$B_{\pm|x} = (\rho_B)^{-\frac{1}{2}}\ \rho_{\pm|x}\ (\rho_B)^{-\frac{1}{2}}. \tag{2}$$

as

$$B_{+|x} = \frac{1}{2}((1+\alpha_x)\mathbb{1} + \vec{r}_x \cdot \vec{\sigma}),\ B_{-|x} = \mathbb{1} - B_{+|x}, \tag{3}$$

with $\vec{r}_x = (r_{1x}, r_{2x}, r_{3x})$ and the substitutions

$$\alpha_x = -1 + (2t_x^+\beta_0^2 - 4s_{3x}^+\beta_0\beta_3 + 2t_x^+\beta_3^2)/\Gamma^2, \tag{4}$$

$$r_{1x} = (2s_{1x}^+\beta_1^2 - 4s_{2x}^+\beta_1\beta_2 - 2s_{1x}^+\beta_2^2)/\Gamma^2, \tag{5}$$

$$r_{2x} = 2(s_{2x}^+\beta_1^2 + 2s_{1x}^+\beta_1\beta_2 - s_{2x}^+\beta_2^2)/\Gamma^2, \tag{6}$$

$$r_{2x} = 2(s_{3x}^+\beta_0^2 + 2t_x^+\beta_0\beta_3 - s_{3x}^+\beta_3^2)/\Gamma^2, \tag{7}$$

$$\Gamma = (\beta_0^2 - |\vec{\beta}|^2), \tag{8}$$

$$\vec{\beta} = \frac{\lambda}{8\beta_0}(\vec{s}_1^{\,+} + \vec{s}_2^{\,+}), \tag{9}$$

$$\beta_0 = \frac{1}{2}\sqrt{1 - \sqrt{1-\lambda^2}}, \tag{10}$$

$$\lambda = |\vec{s}_x^{\,+} + \vec{s}_x^{\,-}|. \tag{11}$$

Notice that $\lambda$ can be computed both from $\vec{s}_1^{\,\pm}$ and $\vec{s}_2^{\,\pm}$, it corresponds to the norm of the Bloch vector associated with Bob's reduced state.

### Incompatibility robustness as a semidefinite program

The following construction is almost identical to the one presented in Ref. [1], we discuss it here for completeness. By definition

$$\mathcal{IR} = \min\Big\{ t \geq 0 \ \Big|\ \frac{M_{a|x} + tN_{a|x}}{1+t} := O_{a|x} \text{ are JM },\ \{N_{a|x}\}_{a,x} \text{ measurement assemblage} \Big\}. \tag{12}$$

We can then write

$$N_{a|x} = \frac{(1+t)O_{a|x} - M_{a|x}}{t} \geq 0, \tag{13}$$

where $\geq$ denotes a positive semidefiniteness condition. Eq. (13) is satisfied whenever

$$(1+t)O_{a|x} - M_{a|x} \geq 0, \tag{14}$$

which can be rewritten, using the joint measurability properties of $\{O_{a|x}\}_{a|x}$, i.e., $O_{a|x} = \sum_\lambda p_M(a|x,\lambda)G_\lambda$ for all $a, x$, as

$$(1+t)\sum_\lambda p_M(a|x,\lambda)G_\lambda \geq M_{a|x}\ \forall a, x \tag{15}$$

By incorporating the factor $1+t$ in the definition of $G_\lambda$, one can easily see that the value of $1+\mathcal{IR}$ can be obtained via the following SDP:

$$\begin{aligned} \text{minimize: } & \frac{1}{d}\sum_\lambda \text{tr}[G_\lambda] \\ \text{subject to: } & \sum_\lambda p_M(a|x,\lambda)G_\lambda \geq M_{a|x}\ \forall a, x, \\ & G_\lambda \geq 0. \\ & \sum_\lambda G_\lambda = \mathbb{1}\frac{1}{d}\left(\sum_\lambda \text{tr}[G_\lambda]\right), \end{aligned} \tag{16}$$

where the last equation encode the fact that $G$, up to the correct normalization, must be an observable. In addition, the postprocessing can be chosen, without loss of generality, as the deterministic strategy $p_M(a|x,\lambda) = \delta_{a,\lambda_x}$, where $\lambda := (\lambda_x)_x$ and $\lambda_x$ is the hidden variable associated with the setting $x$, taking as value the possible outcomes $a$.

It can be easily proven that the program is strictly feasible (e.g., take $G_\lambda = \mathbb{1}$) and bounded from below, i.e., the optimal value is always larger or equal one.

**Monotonocity of the incompatibility robustness under local channels**

To prove monotonocity of $\mathcal{IR}$ under the action of a quantum channel $\Lambda$ it is sufficient to prove that

$$\begin{aligned} &\left\{\frac{M_{a|x}+tN_{a|x}}{1+t}\right\}_{a,x} \text{ is JM} \\ &\Longrightarrow \left\{\Lambda\left(\frac{M_{a|x}+tN_{a|x}}{1+t}\right)\right\}_{a|x} \text{ is JM .} \end{aligned} \tag{17}$$

Let us denote again $O_{a|x} := (M_{a|x} + tN_{a|x})/(1 + t)$, with $\{O_{a|x}\}_{a,x}$ admitting a joint measurement, i.e., $O_{a|x} = \sum_\lambda p_M(a|x,\lambda)\ G_\lambda$. It is sufficient to check that $\{\Lambda(O_{a|x})\}_{a,x}$ again admits a joint measurement $\Lambda(O_{a|x}) = \sum_\lambda p_M(a|x,\lambda)\ \Lambda(G_\lambda)$. That $\Lambda(G_\lambda)$ is a POVM follows directly the properties of the channel $\Lambda$, since

$$\begin{aligned} &\Lambda(G_\lambda) \geq 0, \\ &\sum_\lambda \Lambda(G_\lambda) = \Lambda\left(\sum_\lambda G_\lambda\right) = \Lambda(\mathbb{1}) = \mathbb{1}. \end{aligned} \tag{18}$$

Notice that, since we are looking for the transformation of the observables, we use the channel in the Heisenberg picture, hence the fact that the map is trace preserving when acting on states (Schrödinger picture) corresponds to its adjoint (Heisenberg picture) being unital.

[1] M. Piani and J. Watrous, Phys. Rev. Lett. **114**, 060404 (2015).

# Article III

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- **Abstract**: The term Einstein-Podolsky-Rosen steering refers to a quantum correlation intermediate between entanglement and Bell nonlocality, which has been connected to another fundamental quantum property: measurement incompatibility. In the finite-dimensional case, efficient computational methods to quantify steerability have been developed. In the infinite-dimensional case, however, less theoretical tools are available. Here, we approach the problem of steerability in the continuous variable case via a notion of state-channel correspondence, which generalizes the well-known Choi-Jamiołkowski correspondence. Via our approach we are able to generalize the connection between steering and incompatibility to the continuous variable case and to connect the steerability of a state with the incompatibility breaking property of a quantum channel, with applications to noisy NOON states and amplitude damping channels. Moreover, we apply our methods to the Gaussian steering setting, proving, among other things, that canonical quadratures are sufficient for steering Gaussian states.
- **Author's contribution**: The author of this thesis contributed to the proof of the connection between steering and joint measurements for Schmidt rank-$d$ states in the infinite-dimensional case (which was independently proven by the first author) and to the search for applications of the main result.

# Continuous-variable steering and incompatibility via state-channel duality

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The term Einstein-Podolsky-Rosen steering refers to a quantum correlation intermediate between entanglement and Bell nonlocality, which has been connected to another fundamental quantum property: measurement incompatibility. In the finite-dimensiona case, efficien computational methods to quantify steerability have been developed. In the infinite-dimensiona case, however, less theoretical tools are available. Here, we approach the problem of steerability in the continuous variable case via a notion of state-channel correspondence, which generalizes the well-known Choi-Jamiołkowski correspondence. Via our approach we are able to generalize the connection between steering and incompatibility to the continuous variable case and to connect the steerability of a state with the incompatibility breaking property of a quantum channel, with applications to noisy NOON states and amplitude damping channels. Moreover, we apply our methods to the Gaussian steering setting, proving, among other things, that canonical quadratures are sufficien for steering Gaussian states.

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## I. INTRODUCTION

The phenomenon of Einstein-Podolsky-Rosen (EPR) steering combines two central features of quantum theory: entanglement and incompatibility, namely, the impossibility of determine precisely and simultaneously certain properties of a physical system, e.g., position and momentum. In practice, steering is a quantum effect by which one experimenter, Alice, can remotely prepare (i.e., steer) an ensemble of states for another experimenter, Bob, by performing local measurement on her half of a bipartite system shared by them, and communicating the results to Bob [1].

Due to the fact that steering is a form of quantum correlation intermediate between entanglement and Bell nonlocality [2], it has been proven useful to solve foundational problems [3–7] and important for applications in quantum information processing such as one-sided-device-independent (1SDI) quantum information [8–10].

In the finite-dimensiona case, several methods are available to attack the steering problem. In particular, efficien methods based on semidefinit programming [11] are able to detect and quantify steerability of a given state and set of measurements [3,12–14]. Notwithstanding the existence of several methods (see, e.g., Refs. [1,15–19] and the review [14]), such a systematic approach is missing in the continuous variable case.

In this paper, we will develop a general tool for discussing steering in the continuous variable case, which is based on an extension of the Choi-Jamiołkowski state-channel duality [20–22]. The Choi-Jamiołkowski correspondence associates a state to each channel, but not all states can be mapped to a channel in this way. We will extend this idea by showing that one can associate to each bipartite state a channel, such that the steerability property of a state is equivalent to the property of the corresponding channel being incompatibility breaking [23], when all possible measurements are allowed for steering. This result, in turn, extends to the continuous variable case the result on equivalence between steering and joint-measurability [24–26].

In addition to these conceptual results, we fin that the channel picture reduces seemingly different steering problems to a single one. For instance, we show that steerability of noisy NOON-states (cf. Ref. [19]) corresponds to the decoherence of incompatibility under an amplitude damping channel (cf. [27,28]), and how to use steering to investigate its Markovianity properties. Using incompatibility techniques we investigate both analytically and numerically the noise tolerance of these states with two quadrature measurements. Finally, we apply our methods in the continuous variable Gaussian settings, showing that steerability by a pair of canonical quadrature measurements already ensures steerability by all Gaussian measurements, and connecting this to Gaussian incompatibility breaking channels [29]. We also show in passing how the method yields an independent proof of the known Gaussian steering criterion [1].

The paper is organized as follows: We begin by introducing preliminary notions in Sec. II, including the general formalism for measurements, joint measurability, steering, the formal connection between hidden state models and positive-operator valued measures (POVMs), and quantificatio of steering and incompatibility. Section III contains our main results on the role of state-channel duality in the connection between steering and incompatibility. In Sec. IV we present all the above mentioned applications, except for the Gaussian case, which is treated separately in Sec. V. Technical proofs of four Lemmas are given in Appendices A–D, and Appendix E contains the derivation of the Gaussian LHS, which is not essential for understanding the main results.

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## II. PRELIMINARY NOTIONS

### A. Measurements as POVMs

A POVM with a discrete outcome set $\Lambda$ is a collection $\{G_\lambda\}_{\lambda\in\Lambda}$ of positive semidefinit operators such that $\sum_{\lambda\in\Lambda} G_\lambda = \mathbb{1}$. Such operators represent the probability of the outcome $\lambda$ for a measurement on a state $\rho$ via the rule $\text{Prob}(\lambda) = \text{tr}[\rho M_\lambda]$. This notion is not sufficien for this paper, since we also consider Gaussian measurements. A POVM with a *continuous outcome set* is one for which $\Lambda = \mathbb{R}^n$, i.e., the Euclidean space. This space comes with the usual integration measure $d\lambda$, and a POVM $\{G_\lambda\}_{\lambda\in\Lambda}$ consist of elements $G_\lambda$ that may be "infinitesimal" so, in general, only the integrals $\int_X G_\lambda d\lambda$ with $X \subset \Lambda$ defin proper operators. This definitio clarifie the name positive-operator valued measure [30], i.e., a map from measurable sets to positive operators $X \mapsto \int_X G_\lambda d\lambda$ with normalization $\int_\Lambda G_\lambda d\lambda = \mathbb{1}$ and countable additivity on disjoint sets. To illustrate this well-known technical issue with a typical example relevant for the main text, consider the position operator $Q = \int_{\mathbb{R}} q|q\rangle\langle q|dq$. The corresponding POVM has elements $|q\rangle\langle q|$, which are *not* proper operators as they map wave functions $\psi$ into improper states $\psi(q)|q\rangle$. The symbols $|q\rangle\langle q|$ only make up operators when integrated into $\int_{[a,b]} |q\rangle\langle q|dq$, which projects $\psi$ into the wave function coinciding with $\psi(q)$ for $a \leqslant q \leqslant b$ and vanishing elsewhere.

### B. Joint measurability

A collection of POVMs, indexed by measurement settings $x$, will be denoted as $\mathcal{M} = \{M_{a|x}\}_{a,x}$ and called a *measurement assemblage*. In the discrete case, it is said to be *jointly measurable* [30] if there is a POVM $\{G_\lambda\}_\lambda$ such that each $M_{a|x}$ can be obtained from $G_\lambda$ via classical postprocessing, i.e., $M_{a|x} = \sum_\lambda D(a|x,\lambda)G_\lambda$ for all $x,a$, where $D(a|x,\lambda) \geqslant 0$ and $\sum_a D(a|x,\lambda) = 1$. For the continuous case, with $\mathcal{A}_x$ the set of outcomes for the POVM $M_x$, one has joint measurability if

$$M_{X|x} := \int_X M_{a|x} da = \int_\Lambda D(X|x,\lambda)G_\lambda d\lambda, \tag{1}$$

where the postprocessing $D(\cdot|x,\cdot) = \mathcal{A}_x \times \Lambda \to [0,1]$ is generally known as a *weak Markov kernel* [31]. An assemblage not jointly measurable is called *incompatible*.

### C. Quantum steering

Another main ingredient for our discussions is bipartite quantum steering. Alice can prepare an ensemble of states for Bob by performing a local measurement ($x$) on her half of the bipartite state $\rho$ and communicating the result ($a$) to Bob. This is related to the measurement assemblage $\{A_{a|x}\}_{a,x}$ via $\varrho(a|x) := \text{tr}_A[(A_{a|x} \otimes \mathbb{1})\rho]/P(a|x)$, where $P(a|x) := \text{tr}[(A_{a|x} \otimes \mathbb{1})\rho]$ is the probability of the outcome $a$ for the setting $x$, and $\varrho(a|x)$ is the reduced state obtained by Bob in this case. We call the collection $\{\rho_{a|x}\}_{a,x}$, with $\rho_{a|x} := \text{tr}_A[(A_{a|x} \otimes \mathbb{1})\rho]$, a *state assemblage*. It satisfie the nonsignalling rule $\rho_B = \sum_a \rho_{a|x}$ for all $x$, with $\rho_B := \text{tr}_A[\rho]$ the reduced state for Bob. An assemblage $\{\rho_{a|x}\}_{a,x}$ is called *unsteerable* if it admits a local hidden state (LHS) model [1], i.e., a collection of positive operators $\{\sigma_\lambda\}_\lambda$ with $\text{tr}[\sum_\lambda \sigma_\lambda] = 1$ and $\rho_{a|x} = \sum_\lambda D(a|x,\lambda)\,\sigma_\lambda$ for all $a,x$, where $D(a|x,\lambda) \geqslant 0$ and $\sum_a D(a|x,\lambda) = 1$. If a LHS model exists, Bob can interpret each $\rho_{a|x}$ as coming from some preexisting states $\sigma_\lambda$, where only the classical probabilities are updated due to the information obtained by Alice from her measurement. In the continuous case the assemblage consists of operators $\sigma_x(X) := \int_X \sigma_{a|x} da$, where $X \subset \mathcal{A}_x$, and the unsteerable case with LHS $\{\sigma_\lambda\}_\lambda$ is define by

$$\int_X \sigma_{a|x} da = \int_\Lambda D(X|x,\lambda)\sigma_\lambda d\lambda, \tag{2}$$

where $D(\cdot|x,\cdot)$ is a weak Markov kernel for each $x$. In the steerable case we also say that the state $\rho$ *is steerable by* the the measurement assemblage $\{A_{a|x}\}_{a,x}$.

Our main results (Theorems 1 and 2 below) can be applied to reduce seemingly different steering problems to a single one. To formulate this precisely, we need a few extra notions. First, we say that states $\rho_1$ and $\rho_2$ are *steering-equivalent* if they are steerable by the exact same measurement assemblages $\{A_{a|x}\}_{a,x}$. For a weaker version, suppose instead that there is a quantum channel $\Lambda$ (with Heisenberg picture $\Lambda^*$), such that $\rho_1$ is steerable by an assemblage $\{A_{a|x}\}_{a,x}$ exactly when $\rho_2$ is steerable by $\{\Lambda^*(A_{a|x})\}_{a,x}$. Generalizing the notion in Ref. [26], we then call $B_{a|x} := \Lambda^*(A_{a|x})$ the *steering-equivalent observables* (for $A_{a|x}$). A related (state-independent) notion is that of an *incompatibility breaking channel* (IBC) [23], namely, a channel $\Lambda$ such that $\{\Lambda^*(A_{a|x})\}_{x,a}$ is jointly measurable for any measurement assemblage $\{A_{a|x}\}_{x,a}$. For instance, entanglement breaking channels [32] belong to this class. It is known [23] that when such a channel is applied to one side of a maximally entangled state, the resulting state is not steerable by any measurement assemblage. Corollary 1(d) extends this to arbitrary states in the broader context of state-channel duality (see below).

### D. Hidden state models and measurements in terms of POVMs

We now review the fact that hidden state models and general quantum observables can both be described by POVMs. Since we are interested in the infinite-dimensiona case with POVMs having continuous outcome sets, some technical considerations are unavoidable, and we discuss them briefl . These technicalities are not essential for understanding the main text, but they are needed to make the proofs mathematically sound.

The connection between hidden state models and POVMs is fairly obvious when $d < \infty$ and $\Lambda$ is discrete. Suppose now we have a general family $\{\sigma_\lambda\}_{\lambda\in\Lambda}$ of positive operators on Bob's side of a bipartite setting. Here $\Lambda$ is the set of hidden variables, either discrete or continuous as above. *The crucial difference to POVMs is that each $\sigma_\lambda$ is a proper trace class operator, i.e., not "infinitesimal" even in the continuous case.* The function $\lambda \mapsto \sigma_\lambda$ must satisfy the technical condition of measurability in the trace class norm, to allow the (Bochner) integrals $\int f(\lambda)\sigma_\lambda d\lambda$ to exist with finit trace for every measurable scalar function $f$ on $\Lambda$. We also assume the normalization $\sum_\lambda \sigma_\lambda = \sigma$ (discrete case) and $\int \sigma_\lambda d\lambda = \sigma$ (continuous case), where $\sigma$ is again a fi ed density operator. Then there exists a unique POVM $G$ with outcomes in $\Lambda$, satisfying

$$\sigma^{\frac{1}{2}} G_\lambda \sigma^{\frac{1}{2}} = \sigma_\lambda. \tag{3}$$

This is clear in the finite-dimensiona case with finit outcome set $\Lambda$—we just multiply with $\sigma^{-\frac{1}{2}}$, which preserves positivity, and normalization translates into $\sum_\lambda G_\lambda = \mathbb{1}$. For $d = \infty$ we need a technical density argument analogous to that used in the proof of Lemma 1 below (see Appendix A). In the case of continuous outcome set, Eq. (3) is again understood via the corresponding integrals.

Suppose then that we start with a POVM $\{G_\lambda\}_{\lambda\in\Lambda}$; the question is how to get the states $\sigma_\lambda$. If $\Lambda$ is discrete, this is trivial: we defin $\sigma_\lambda := \sigma^{\frac{1}{2}} G_\lambda \sigma^{\frac{1}{2}}$. However, the case of continuous outcome set $\Lambda$ introduces a subtlety: we have to show that the possibly infinitesima POVM elements $G_\lambda$ yield trace class operators $\sigma_\lambda$. In general, this is nontrivial, and follows from the Radon-Nikodym property of the trace class (cf. p. 79 of Ref. [33]). In the relevant case of a position operator (and more generally a Gaussian POVM), this is easier to prove: $\sigma^{\frac{1}{2}}|q\rangle\langle q|\sigma^{\frac{1}{2}}$ maps $\psi$ into $\langle q|\sigma^{\frac{1}{2}}\psi\rangle\sigma^{\frac{1}{2}}|q\rangle$, which is indeed a proper wave function since $\sigma^{\frac{1}{2}}|q\rangle = \sum_n \sqrt{s_n}\langle n|q\rangle|n\rangle$ has finit norm $\sum_n s_n|\langle n|q\rangle|^2 < \infty$ for all $q$ due to $\sum_n s_n < \infty$, assuming the basis functions are continuous (which is the case for the number basis considered in the main text). Here, $\sigma = \sum_n s_n|n\rangle\langle n|$ is the eigendecomposition of $\sigma$.

### E. Robustness quantification

Both incompatibility and steering can be quantifie by the amount of classical noise required to destroy these quantum properties. There are different ways of setting up a precise definitio for this idea; here we only introduce the quantifier which turn out to be naturally compatible with our state-channel duality.

We recall from Ref. [34] that *Consistent Steering Robustness* (CSR) of a state assemblage is given by

$$\mathrm{CSR}(\{\sigma_{a|x}\}) = \inf\Bigg\{t \geqslant 0 | \{\pi_{a|x}\}\ \sigma\text{-consistent}, \times \left\{\frac{\sigma_{a|x} + t\pi_{a|x}}{1+t}\right\} \text{ unsteerable}\Bigg\}, \tag{4}$$

where $\sigma$-consistence means $\sum_a \sigma_{a|x} = \sum_a \tau_{a|x}$ for all $x$. Similarly, the *Incompatibility Robustness* (IR) [26] of a measurement assemblage is given by

$$\mathrm{IR}(\{M_{a|x}\}) = \inf\left\{t \geqslant 0 | \frac{M_{a|x} + tN_{a|x}}{1+t} \text{ jointly measurable}\right\}. \tag{5}$$

We stress that these definitions although typically interpreted as SDPs in the finite-dimensiona case, can also be stated in infinit dimensions with possibly continuous outcomes for the measurements. We note that in such a case they can only be formulated as SDPs by firs restricting to a subspace and discretizing the outcomes, as in our numerical example in Sec. IV B.

### III. MAIN RESULT: STATE-CHANNEL CORRESPONDENCE AND STEERING

Our key idea for attacking steering problems is a state-channel duality valid in infinit dimensions. It goes beyond the familiar Choi-Jamiołkowski (CJ) isomorphism, which maps channels $\mathsf{T}: \mathcal{L}(\mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_A)$ into states $\rho = (\mathsf{T}\otimes \mathrm{Id})(|\Omega_0\rangle\langle\Omega_0|)$ on $\mathcal{H}_A \otimes \mathcal{H}_B$, where $|\Omega_0\rangle = \frac{1}{\sqrt{d}}\sum_n |nn\rangle$ is the maximally entangled state on $\mathcal{H}_B \otimes \mathcal{H}_B$ and $\dim\mathcal{H}_B = d < \infty$. The CJ isomorphism is a one-to-one map between channels and states $\rho$ with completely mixed $\mathcal{H}_B$ marginals, i.e., $\sigma = \mathrm{tr}_A[\rho] = \mathbb{1}/d$. It has been used in the definitio of channel steering [35] and the verificatio of the quantumness of a channel [36]. Our extension is as follows.

*Lemma 1.* There is a 1-to-1 correspondence between bipartite states $\rho$ sharing a full-rank marginal $\sigma = \mathrm{tr}_A[\rho]$, and quantum channels $\mathsf{T}$ from Bob to Alice, such that

$$\rho = (\mathsf{T}\otimes \mathrm{Id})(|\Omega_\sigma\rangle\langle\Omega_\sigma|), \tag{6}$$

where $|\Omega_\sigma\rangle := \sum_{n=1}^d \sqrt{s_n}|nn\rangle \in \mathcal{H}_B \otimes \mathcal{H}_B$ is define as the purificatio of $\sigma = \sum_n s_n|n\rangle\langle n|$.

We postpone the detailed proof to Appendix A. However, since one aim of the paper is to pay due attention to the technicalities related to the infinite-dimensiona case, we briefl sketch the relevant points here in the main text: Given a channel $\mathsf{T}$, $\rho$ is clearly a valid state with $\mathrm{tr}_A[\rho] = \sigma$. Vice versa, given $\rho$ with marginal $\sigma$, the idea is to fin a channel $\mathsf{T}$, such that

$$\sigma^{\frac{1}{2}}\mathsf{T}^*(A)\sigma^{\frac{1}{2}} = \mathrm{tr}_A[\rho(A\otimes \mathbb{1})]^{\mathsf{T}}, \tag{7}$$

where the transpose is taken with respect to the basis $\{|n\rangle\}$. Equation (7) can then be seen to be equivalent to Eq. (6) by direct computation. To fin $\mathsf{T}$, one can invert $\sigma^{\frac{1}{2}}$ and solve for $\mathsf{T}^*(A)$ provided that $d < \infty$. For $d = \infty$, one cannot directly invert $\sigma^{\frac{1}{2}}$, since it will be an unbounded operator. However, one can still construct the Kraus operators $\{M_k\}_k$ for the channel $\mathsf{T}^*$ from the Kraus operators $R_k$ of $\sigma^{\frac{1}{2}}\mathsf{T}^*(\cdot)\sigma^{\frac{1}{2}}$, obtained via Eq. (7). This is achieved by extending $R_k\sigma^{-\frac{1}{2}}$ to a bounded operator on $\mathcal{H}_B$; see Appendix A.

Using Lemma 1, we can prove the equivalence between steering of a state assemblage and incompatibility of a measurement assemblage [26] in full generality and from a quantitative perspective [34,37].

*Theorem 1.* The state assemblage $\{\sigma_x(X)\}_{X,x}$ define by $\rho$ and $\{A_x\}_x$ is steerable $\Leftrightarrow$ the measurement assemblage $\{\mathsf{T}^*(A_x)\}_x$ is incompatible. Here, $\mathsf{T} \leftrightarrow \rho$ via Lemma 1, with $\sigma = \mathrm{tr}_A[\rho] = \sigma_x(\mathcal{A}_x)$. This correspondence is quantitative in that the incompatibility robustness (IR) of $\{\mathsf{T}^*(A_x)\}_x$ coincides with the consistent steering robustness (CSR) of $\{\sigma_x(X)\}_{X,x}$.

*Proof.* Using Lemma 1 with any f xed state $\sigma$, we have the correspondences

$$\begin{aligned} \{\mathsf{T}^*(A_{a|x})\} &\mapsto \{\rho_{a|x}\}, \\ \mathsf{T} &\mapsto \rho = (\mathsf{T}\otimes \mathrm{Id})(|\Omega_\sigma\rangle\langle\Omega_\sigma|), \end{aligned} \tag{8}$$

between the measurement assemblage $A_{a|x}$ transformed, via the Heisenberg-picture channel $\mathsf{T}^*$ and the steering assemblage obtained via measurements $A_{a|x}$ on the state $\rho$. Note that the measurements $\{A_{a|x}\}$ stay fi ed. Now, $\{\mathsf{T}^*(A_{a|x})\}$ is jointly measurable if and only if

$$\mathsf{T}^*(A_{a|x}) = \sum_\lambda D(a|x,\lambda)G_\lambda. \tag{9}$$

By multiplying this with $\sigma^{\frac{1}{2}}$ on both sides, we obtain

$$\rho_{a|x}^{\mathsf{T}} = \sum_\lambda D(a|x,\lambda)\sigma_\lambda, \tag{10}$$

where the hidden states $\sigma_\lambda$ correspond to $G_\lambda$ via Eq. (3), and $\rho^{\mathsf{T}}_{a|x} := \sigma^{\frac{1}{2}}\mathsf{T}^*(A_{a|x})\sigma^{\frac{1}{2}} = \mathrm{tr}_A[\rho(A_{a|x}\otimes\mathbb{1})]^{\mathsf{T}}$ is the assemblage. As we have established above, all the correspondences are one-to-one, and hence steerability of the setting $(\rho,\{\mathsf{A}_{a|x}\})$ is equivalent to the incompatibility of $\{\mathsf{T}^*(A_{a|x})\}$.

To prove the equivalence of the quantifiers we follow a similar reasoning as the one in Ref. [37]: We need to prove that for each noise term $N_{a|x}$ of the IR problem, i.e., a term making the measurement assemblage jointly measurable for a given $t$, we can fin a noise term $\pi_{a|x}$ of the CSR problem, i.e., a term making the state assemblage unsteerable for the same $t$, and vice versa. We use again the relation

$$\pi^{\mathsf{T}}_{a|x} = \sigma^{\frac{1}{2}}N_{a|x}\sigma^{\frac{1}{2}} \tag{11}$$

to obtain a a one-to-one mapping between $\sigma$-consistent assemblages and arbitrary POVMs. In the finite-dimensiona case, we can argue as follows: Given a $\sigma$-consistent assemblage $\{\pi_{a|x}\}_{a,x}$, $\{N_{a|x}\}_{a,x}$ define as in Eq. (11) is a valid measurement assemblage. Vice versa, given $\{N_{a|x}\}_{a,x}$ a valid measurement assemblage, we can construct the $\sigma$-consistent assemblage $\{\pi_{a|x}\}_{a,x}$ as

$$\pi^{\mathsf{T}}_{a|x} = \mathrm{tr}_A[N_{a|x}\otimes\mathbb{1}|\Omega_\sigma\rangle\langle\Omega_\sigma|] = \sigma^{\frac{1}{2}}N_{a|x}\sigma^{\frac{1}{2}}, \tag{12}$$

where $|\Omega_\sigma\rangle := \sum_n\sqrt{s_n}|nn\rangle$ is the purificatio of $\sigma := \sum_n s_n|n\rangle\langle n|$. Hence $\mathrm{CSR}(\{\sigma_{a|x}\}) = \mathrm{IR}(\{\mathsf{T}^*(A_{a|x})\})$. When the Hilbert space is infinite-dimensional with possibly continuous outcomes for the POVMs, we again need the same argument as in Sec. II D, since $N_{a|x}$ may not be a proper operator, while we need $\pi_{a|x}$ to actually be in the trace class. This establishes the correspondence Eq. (11) between POVMs and $\sigma$-consistent assemblages in the same way as we obtained Eq. (3). Then the equality $\mathrm{CSR}(\{\sigma_{a|x}\}) = \mathrm{IR}(\{\mathsf{T}^*(A_{a|x})\})$ clearly follows, and so we can extend the equivalence of quantifier to the infinite-dimensiona case. ■

We remark that the above reasoning also provides the connection with the steering equivalent observables define in the introduction. Given a state assemblage $\{\rho_{a|x}\}_{a,x}$, with a full rank reduced state $\sigma := \sum_a\rho_{a|x}$, its steering equivalent (SE) observables [26] are given by

$$B_{a|x} := \sigma^{-1/2}\rho_{a|x}\sigma^{-1/2} = \mathsf{T}^*_\rho(A_{a|x}). \tag{13}$$

We stress that we have here used Theorem 1 above to make a connection between the notion in Ref. [26] and the one given in the introduction in terms of channels. In particular, this extends the former notion to the infinite-dimensiona case.

Furthermore, it is easy to show that if we have only access to the assemblage $\{\rho_{a|x}\}_{a,x}$, and not to the bipartite state $\rho$, we can always interpret $B_{a|x}$ as the observables giving the assemblage when measured on the purificatio $|\Omega_\sigma\rangle := \sum_n\sqrt{s_n}|nn\rangle$ of $\sigma := \sum_n s_n|n\rangle\langle n|$. Namely,

$$\sigma^{1/2}B_{a|x}\sigma^{1/2} = \mathrm{tr}_A[B_{a|x}\otimes\mathbb{1}|\Omega_\sigma\rangle\langle\Omega_\sigma|]^{\mathsf{T}}. \tag{14}$$

From Theorem 1 we know that $\{\rho_{a|x}\}_{a,x}$ is unsteerable $\Leftrightarrow$ $\{B_{a|x}\}_{a,x}$ is jointly measurable. If we compare that with the definitio of the channel $\mathsf{T}_\rho$, we fin that $\mathsf{T}^*_{|\Omega_\sigma\rangle\langle\Omega_\sigma|}(B_{a|x}) = B_{a|x}$. Hence, the observables $B_{a|x} = \mathsf{T}^*(A_{a|x})$, when measured on $|\Omega_\sigma\rangle$, reproduce the state assemblage $\{\sigma_{a|x}\}_{a,x}$. We record this conclusion, along with some other direct implications of Theorem 1, into the following Corollary, which generalizes several existing results.

*Corollary 1.* (a) Two states $\rho_1,\rho_2$ are steering-equivalent if the corresponding channels of Lemma 1 have $\mathsf{T}^*_1(\cdot) = U\mathsf{T}^*_2(\cdot)U^*$, where $U$ is unitary. (b) A pure state $|\Psi\rangle$ of full Schmidt rank is steerable by assemblage $\{A_{X|x}\}_{X,x}$ iff the latter is incompatible. (c) A state $\rho$ is steerable by measurements $\{A_{X|x}\}$ iff the purificatio $|\Omega_\sigma\rangle$ of Lemma 1 is steerable by the steering-equivalent measurements $\{\mathsf{T}^*(A_{X|x})\}$. (d) A state $\rho$ is unsteerable iff the channel $\mathsf{T}^*$ is incompatibility breaking.

*Proof.* Part (a) follows directly from Theorem 1 and the fact that incompatibility is preserved in unitary operations. We demonstrate the use of (a) with NOON-states below. Part (b) is the infinite-dimensiona version of the result in Refs. [24,25] and can be obtained by definin a Hilbert-Schmidt operator $R$ with $\langle n|R|m\rangle = \langle nm|\Psi\rangle$, where the basis on Bob's side is chosen as in Lemma 1. Since $R$ and $R^*$ have full rank, $U = R\sigma^{-\frac{1}{2}}$ is unitary and $|\Psi\rangle = (U\otimes\mathbb{1})|\Omega_\sigma\rangle$, so that $\mathsf{T}^*(A) = U^*AU$ and hence preserves incompatibility. Part (c) was proved above, while (d) is a direct consequence of Theorem 1 on the theory of incompatibility breaking channels. ■

We stress the difference with respect to Ref. [23], where the incompatibility breaking property of a given quantum channel was related to the unsteerability property of specifi bipartite states derived from it. Here we have devised a way (via the above state-channel duality) to do the converse: for any given state $\rho$, we can fin a quantum channel $\mathsf{T}$ that is incompatibility breaking exactly when the state is steerable. This allows us to treat *any* given steering problem as an IBC problem, which might open up new possibilities for investigating steering. In the following section, we illustrate this with different applications.

## IV. APPLICATIONS

### A. Separable and pure states

Consider firs separable states $\rho = \sum_i p_i\rho^i_A\otimes\rho^i_B$, which are of course not steerable. We easily fin the channel of Lemma 1 as $\mathsf{T}^*(A) = \sum_i \mathrm{tr}[\rho^i_A A]F_i$, where $F_i = p_i\sigma^{-\frac{1}{2}}(\rho^i_B)^{\mathsf{T}}\sigma^{-\frac{1}{2}}$ satisfie $0\leqslant F_i\leqslant\mathbb{1}$ and $\sum_i F_i = \mathbb{1}$, that is, $\mathsf{T}$ is entanglement breaking [32].

At the other extreme, pure states of full Schmidt rank correspond to unitary channels by Corollary 1(b). As an infinite-dimensiona example, the channel for the two-mode coherent state $|z\rangle$ with $z = re^{i\theta}$ is the phase shift $\mathsf{T}^*(A) = e^{i\theta a^\dagger a}Ae^{-i\theta a^\dagger a}$ if we identify the photon number bases of Alice and Bob. Importantly, the problem of nonunique regularization of maximally entangled states in $d=\infty$ is circumvented by our method.

### B. Noisy NOON states

Consider the "NOON state,"

$$|N00N\rangle = \frac{1}{\sqrt{2}}(|0N\rangle - e^{iN\alpha}|N0\rangle),$$

shared by Alice and Bob [38], with $\{|n\rangle\}$ photon number basis of 1-mode electromagnetic field Via random photon loss, the state becomes

$$\rho_\eta = \eta|N00N\rangle\langle N00N| + (1-\eta)|00\rangle\langle 00|,$$

which is unsteerable for $\eta = 0$ and steerable for $\eta = 1$. Hence, there is a threshold $\eta_c$ (depending on the allowed measurements) such that $\rho_\eta$ is steerable iff $\eta > \eta_c$ (cf. [19] and the references therein, for previous results on the problem). Using Lemma 1 we fin the channel of $\rho_\eta$ as

$$\begin{aligned}\mathsf{T}^*(A) &= \sigma^{-\frac{1}{2}}\,\mathrm{tr}_A[\rho(A\otimes\mathbb{1})]^\mathsf{T}\sigma^{-\frac{1}{2}} \\ &= \begin{pmatrix} r^2A_{NN} + (1-r^2)A_{00} & -rA_{N0}e^{-iN\alpha} \\ -rA_{0N}e^{+iN\alpha} & A_{00}\end{pmatrix} \\ &= U^*\Lambda_r^*(A)U, \end{aligned} \tag{15}$$

where $r = \sqrt{\eta/(2-\eta)}$, $\sigma = \mathrm{tr}_A(\rho) = (1-\eta/2)|0\rangle\langle 0| + \eta/2|N\rangle\langle N|$,

$$\begin{aligned}\Lambda_r^*(A) &= \sum_{i=0}^{1} K_{i,r}^* A K_{i,r} \\ &= \begin{pmatrix} A_{00} & rA_{0N} \\ rA_{N0} & r^2A_{NN} + (1-r^2)A_{00}\end{pmatrix},\end{aligned} \tag{16}$$

is the *amplitude damping channel* [39] with Kraus operators

$$K_{0,r} = \begin{pmatrix} 1 & 0 \\ 0 & r\end{pmatrix},\ K_{1,r} = \begin{pmatrix} 0 & \sqrt{1-r^2} \\ 0 & 0\end{pmatrix}, \tag{17}$$

and

$$U := |0\rangle\langle N| - e^{iN\alpha}|N\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ -e^{iN\alpha} & 0\end{pmatrix} \tag{18}$$

is a unitary matrix. By Corollary 1(a), the unitary is irrelevant for steering, and we will ignore it in what follows.

The problem, then, reduces to the question of how $\Lambda_r$ breaks incompatibility. We introduce the following necessary criterion for this:

*Lemma 2.* Let $\{A_x\}_{x=1}^n$ be any finit assemblage of qubit measurements (with arbitrary outcome sets $\mathcal{A}_x$). Then the "damped measurements" $\Lambda_r^*(A_x)$ are jointly measurable if

$$\sum_{x=1}^{n} \det\frac{\Lambda_r^*(A_{X_x|x})}{\langle 0|A_{X_x|x}|0\rangle} \geqslant n-1 \text{ for each } X_x \subset \mathcal{A}_x. \tag{19}$$

A proof of this result is given in Appendix B.

Next, we proceed to introduce the relevant measurements: we focus on the case of Alice attempting to steer Bob using rotated quadratures $Q_\theta = (e^{i\theta}a^\dagger + e^{-i\theta}a)/\sqrt{2}$. They act in the infinite-dimensiona Hilbert space, with spectral projections (PVM) $Q_{q|\theta} = e^{i\theta a^\dagger a}|q\rangle\langle q|e^{-i\theta a^\dagger a}$. As our state lives in $\mathrm{span}\{|0\rangle,|N\rangle\}$, only the $2\times 2$ matrix $(\tilde{Q}_{q|\theta})_{nm} = \langle n|Q_{q|\theta}|m\rangle = e^{i\theta(n-m)}\langle n|q\rangle\langle q|m\rangle$ with $n,m = 0,N$ contributes. Explicitly, this matrix reads

$$\tilde{Q}_{q|\theta} = \begin{pmatrix} 1 & e^{-iN\theta}h(q) \\ e^{iN\theta}h(q) & h(q)^2\end{pmatrix}\frac{e^{-q^2}}{\sqrt{\pi}},\quad q\in\mathbb{R}, \tag{20}$$

where $h(x) := \frac{H_N(x)}{\sqrt{2^N N!}}$ with $H_N(x)$ a Hermite polynomial. Note that indeed $\int_\mathbb{R}\tilde{Q}_{q|\theta}dq = \mathbb{1}$ and $\tilde{Q}_{q|\theta}\geqslant 0$, so this is a valid qubit POVM with continuous outcomes.

We assume that Alice only has one pair, i.e., an assemblage $\{\tilde{Q}_{q|0},\tilde{Q}_{q|\theta}\}_q$ for f xed $\theta$, and this pair is incompatible (despite the truncation), if $\theta \neq 0,\pi$. Indeed, since these are rank-1 POVMs, they can only be compatible if $\tilde{Q}_{q|\theta} = \int D(q|q')\tilde{Q}_{q|0}$ for some classical postprocessing $D(q|q')$ [40], implying $e^{i\theta}\in\mathbb{R}$, a contradiction. Hence, the pure NOON state (no damping, $r = 1$) is steerable with these measurements.

The next step is then to compute the steering-equivalent (SE) observables by applying the channel $\Lambda_r$. With $0 < r < 1$, the SE observables become

$$\mathsf{T}_r^*(\tilde{Q}_{q|\theta}) = \begin{pmatrix} 1 & re^{-iN\theta}h(q) \\ re^{iN\theta}h(q) & r^2h(q)^2 + 1 - r^2\end{pmatrix}\frac{e^{-q^2}}{\sqrt{\pi}}. \tag{21}$$

The determinant of this kernel matrix is $(1-r^2)\frac{e^{-2q^2}}{\pi}$ so that the joint measurability criterion of Lemma 2 reduces to $r^2\leqslant 1/2$. From this we conclude that

$$r_c \geqslant 1/\sqrt{2},$$

corresponding to $\eta_c\geqslant 2/3$ (independently of $\theta$ and $N$). The value $\eta_c\approx 2/3$ has previously been obtained numerically [19] for $N = 1$; up to our knowledge, ours is the firs fully analytical proof of a lower bound on $\eta_c$.

We also remark that when $\eta\leqslant 2/3$, Eq. (B3) used in the proof of Lemma 2 gives an explicit joint observable and hence a local hidden state model preventing steering of $\rho_\eta$ by the two quadrature measurements.

Independently of Ref. [19], we show that our method can provide also upper bounds on $r_c$ for $N = 1$. We do this by binarizing the POVMs, and recalling that incompatibility of binarizations is sufficien for that of the original POVMs, as coarse-graining is an instance of post-processing. Choosing the split at $q = 0$ (i.e., Alice only records if $q > 0$ or not) gives the POVM with elements $\frac{1}{2}(\mathbb{1}\pm\mathbf{n}\cdot\sigma)$, where $\mathbf{n} = (2r\sqrt{2}/\pi)(\cos\theta,\sin\theta,0)$. Using an exact criterion [41] we conclude that the binarizations are incompatible for $r^2\geqslant\pi(1-\sin\theta)/(2\cos^2\theta)$. Notice that the bound depends on $\theta$; with $\theta = \pi/2$ (orthogonal quadratures) we get $r_c\leqslant\sqrt{\pi}/2$, or $\eta_c\leqslant 2\pi/(4+\pi)\approx 0.88$.

Since the split at $q = 0$ is the most incompatible binarization of quadratures [42], fine coarse-grainings are needed to get better bounds. By dividing the real line in $N_{\mathrm{int}} = 2,4,6,8,10,12,14$ parts, we obtain bounds via SDP methods, cf. Fig. 1 for pairs with varying $\theta$, and Table I for larger values of $N_{\mathrm{int}}$ with $\theta = \pi/2$. In particular, for $N_{\mathrm{int}} = 20$ and $\theta = \pi/2$, we obtain the value $\eta_c\leqslant 0.671$, which is rather close to the lower bound $\eta_c\geqslant 2/3$.

We obtained these numerical results by implementing the SDP of the incompatibility robustness (IR) [see Eq. (5)], searching for the values of $\eta_c$ for which IR > 0. We used the coarse-graining where $\mathbb{R}$ is divided into the intervals $(-\infty,-c]$, $[-c,-c+c/N_{\mathrm{int}}]$, ..., $[-c/N_{\mathrm{int}},0]$, ..., $[0,c/N_{\mathrm{int}}]$, ...,

TABLE I. Minimal $\eta$ such that the obtained $N_{\mathrm{int}}$-valued observables become incompatible.

| $N_{\mathrm{int}}$ | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|---|---|---|
| $\eta$ | 0.742 | 0.698 | 0.684 | 0.678 | 0.675 | 0.674 | 0.673 | 0.672 | 0.671 |

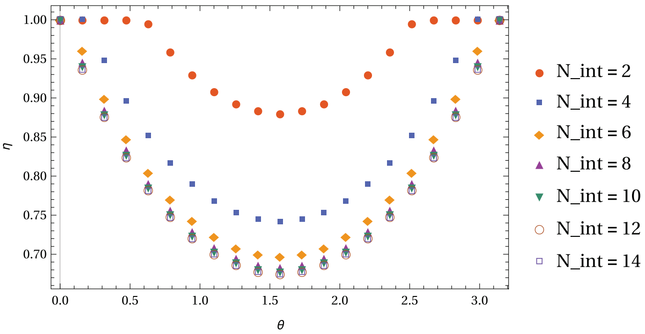


FIG. 1. Critical noise bound for steering through the 1001-state by a coarse-grained pair of quadrature measurements, as a function of the separation angle $\theta$, with coarse-grainings of different number of intervals $N_{\text{int}}$. The case $N_{\text{int}} = 2$ can be reproduced analytically. Below $\eta = 2/3$ the setting is unsteerable by joint measurability criterion Eq. (19).

$[c,\infty)$, where $c \approx 1.4$. The corresponding qubit observables were obtained by integrating over the intervals $I_k$, i.e., $Q_{I_k|\theta} = \int_{I_k} \tilde{Q}_{q|\theta} dq$; such integrals can be explicitly written in terms of error functions.

One can try the same approach in the different subspaces with a higher number of photons. For instance, we investigated the case of 0 or 6 photons, which turned out to be more sensitive to noise, e.g., for the case of $N_{\text{int}} = 16$ one can reach $\eta_{\min} = 0.89$. If one further increases the number of intervals, the computation becomes too slow and practically impossible.

### C. A dynamical example with non-Markovian noise

We now illustrate how the above steering problem for the NOON state arises from a different context, and how our techniques provide a solution in that case as well.

Consider a setup where physical noise arises on Alice's side due to coupling to a zero-temperature heat bath. Starting from the 1001 state, the photon dissipates into the bath on Alice's side via a channel $\mathcal{E}_t$ given by the amplitude damping master equation [43] $d\mathcal{E}_t(\rho_0)/dt = \gamma(t)[\sigma_-\mathcal{E}_t(\rho_0)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-,\mathcal{E}_t(\rho_0)\}]$, where $\sigma_+ = |1\rangle\langle 0|$, $\sigma_- = |0\rangle\langle 1|$, and $\gamma(t) = -2\text{Re}\frac{d}{dt}\log G(t)$ with $G(t)$ depending on the bath spectral density. The state at time $t$ is $\rho_t = (\mathcal{E}_t \otimes \text{Id})(|1001\rangle\langle 1001|)$ so by Eq. (6), its channel $\mathsf{T} = \mathsf{T}_t$ equals $\mathcal{E}_t$ up to a unitary. Using the form of $\mathcal{E}_t$ [27], we fin $\mathsf{T}_t^*(A) = U^*\Lambda_{r(t)}^*(A)U$, as in Eq. (15), where now $r(t) = |G(t)|$, and $U$ is an irrelevant unitary. Interestingly, in this scenario our state-channel duality connects the steerability problem with the non-Markovian properties of the bath (cf., e.g., Ref. [44]), previously associated with temporal correlations [45] and decoherence of incompatibility [27].

The result of the preceding subsection can now be directly applied to characterize steering in the heat bath scenario: for any time $t$, the state $\rho_t$ is steerable by $\{Q_{q|0}, Q_{q|\pi/2}\}$ iff $r(t) \geqslant r_c$. For the typical Lorentzian spectral density, $r(t) = e^{-\lambda t/2}|\cosh(w\lambda t/2) + \sinh(w\lambda t/2)/w|$ where $\lambda$ is the linewidth, and $w = \sqrt{1 - 2u/\lambda}$ with $u$ the coupling strength [27]. We can then evaluate $r(t) \geqslant r_c$ with the numerical value $r_c \approx 1/\sqrt{2}$, to get the region of points $(u,t)$ where the state is steerable; cf. Fig. 2 and its caption.

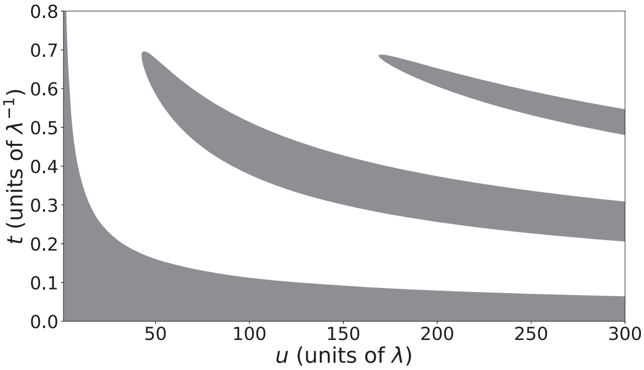


FIG. 2. Steerability region for the dynamical setting (shaded area). The parameter $u$ is the coupling strength (in units of the spectral linewidth $\lambda$ of the bath) and $t$ is time (in units of $\lambda^{-1}$). The two revival regions reflec the non-Markovian character of the evolution in the strong coupling regime, which allows steerability to re-emerge at later times.

## V. GAUSSIAN CASE

In this section, we establish the correspondence between steering of Gaussian states and incompatibility of Gaussian measurements, via a Gaussian version of our general state-channel duality. To do this we f rst need to establish the required formalism and introduce the notation.

Starting with the basics, an optical system with $N$ modes is a continuous variable (CV) quantum system (see, e.g., Ref. [46]) with the infinite-dimensiona Hilbert space $\mathcal{H}^{\otimes N} = \otimes_{j=1}^{N} L^2(\mathbb{R}) \simeq L^2(\mathbb{R}^N)$. The associated phase space is $\mathbb{R}^{2N}$, with canonical coordinates $\mathbf{x} = (q_1,p_1,\ldots,q_N,p_N)^T$ in a f xed symplectic basis. The corresponding standard quadrature operators are denoted by $Q_j$ and $P_j$; they satisfy $[Q_i,P_j] = i\delta_{ij}\mathbb{1}$, $[Q_i,Q_j] = [P_i,P_j] = 0$ and we set $\mathbf{R} = (Q_1,P_1,\ldots,Q_N,P_N)^T$, so that $[R_i,R_j] = i\Omega_{ij}\mathbb{1}$ with $\Omega = \oplus_{j=1}^{N}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We further denote

$$Q_{\mathbf{x}} = \mathbf{x}^T\mathbf{R};$$

these operators are called (generalized) *quadratures*. For a pair of quadratures $(Q_{\mathbf{x}},Q_{\mathbf{y}})$ the commutator is given by $[Q_{\mathbf{x}},P_{\mathbf{y}}] = i\mathbf{x}^T\Omega\mathbf{y}\mathbb{1}$, and any pair for which $\mathbf{x}^T\Omega\mathbf{y} = 1$ is called *canonical*.

The *Weyl operators* $W(\mathbf{x}) = e^{-iQ_{\mathbf{x}}}$ satisfy the canonical commutation relation (CCR),

$$W(\mathbf{x})W(\mathbf{y}) = e^{-i\mathbf{x}^T\Omega\mathbf{y}}W(\mathbf{y})W(\mathbf{x}), \tag{22}$$

and we defin displacement operators $D_{\mathbf{c}} := W(\Omega^T\mathbf{c})$ so that $D_{\mathbf{c}}^*W(\mathbf{x})D_{\mathbf{c}} = e^{-i\mathbf{c}^T\mathbf{x}}W(\mathbf{x})$. A matrix $\mathbf{S}$ is *symplectic* if $\mathbf{S}^T\Omega\mathbf{S} = \Omega$; then by Stone-von Neumann theorem there is a unitary $U_{\mathbf{S}}$ with $U_{\mathbf{S}}^*W(\mathbf{x})U_{\mathbf{S}} = W(\mathbf{S}\mathbf{x})$.

### A. Gaussian states, measurements, channels, and postprocessings

In the following, we firs review the characteristic function formalism for Gaussian quantum objects [46,47]; see also Refs. [29,48]. We then use this to prove the Gaussian version of the state-channel correspondence, after which we proceed to establish the connection between steering and incompatibility.

The characteristic function formalism treats Gaussian state, channels, measurements, and postprocessings in the same footing, is a transparent quantum analog of the corresponding classical objects by way of a rigorous correspondence theory [49], does not require the use of ancillas, circumvents the technical problem of the POVM elements not always being proper operators (see the discussion above), and is especially convenient to use with concatenation, making explicit the idea that a Gaussian channel applied to a Gaussian state (Schrodinger picture) or measurement (Heisenberg picture) produces a new Gaussian state and measurement, respectively. We note that this approach differs from the alternative (equivalent) one introduced by Giedke and Cirac [50], on which Wiseman *et al.* based their derivation of the Gaussian steering criterion [1].

A state on a CV system is *Gaussian* if its characteristic function $\hat{\rho}(\mathbf{x}) := \mathrm{tr}[\rho W(\mathbf{x})]$ is a Gaussian function:

$$\hat{\rho}(\mathbf{x}) = e^{-\frac{1}{4}\mathbf{x}^T\mathbf{V}_\rho\mathbf{x} - i\mathbf{r}^T\mathbf{x}}, \tag{23}$$

where $\mathbf{V}_\rho$ is the *covariance matrix* (CM) $[V_\rho]_{ij} = \mathrm{tr}[\rho\{R_i - r_i, R_j - r_j\}]$ with displacement vector $r_j = \mathrm{tr}[\rho R_j]$. The CM satisfie the *uncertainty relation*

$$\mathbf{V}_\rho + i\Omega \geqslant 0. \tag{24}$$

Crucially, *every* real and symmetric matrix $\mathbf{V}$ satisfying Eq. (24) is a CM of some Gaussian state $\rho$.

A measurement (POVM) $M_\mathbf{a}$ with outcomes $\mathbf{a} \in \mathbb{R}^d$ is *Gaussian* if its outcome distribution for any Gaussian state is a Gaussian (i.e., normal) distribution. This is the case when the operator-valued characteristic function $\hat{M}(\mathbf{p}) := \int e^{i\mathbf{p}^T\mathbf{a}} M_\mathbf{a}\, d\mathbf{a}$ is of the form

$$\hat{M}(\mathbf{p}) = W(\mathbf{K}\mathbf{p}) e^{-\frac{1}{4}\mathbf{p}^T\mathbf{L}\mathbf{p} - i\mathbf{m}^T\mathbf{p}}, \tag{25}$$

where $\mathbf{K}$ is an $N \times d$-matrix and $\mathbf{L}$ is an $d \times d$-matrix satisfying the positivity condition

$$\mathbf{C}_{\mathbf{K},\mathbf{L}} := \mathbf{L} - i\mathbf{K}^T\Omega\mathbf{K} \geqslant 0, \tag{26}$$

and $\mathbf{m}$ is a displacement vector. Importantly, *every* triple $(\mathbf{K},\mathbf{L},\mathbf{m})$ satisfying Eq. (26) define a Gaussian measurement.

In the case $d = 1$ we have $\mathbf{K} = \mathbf{x}$, a column vector, while $\mathbf{L} = 2\xi^2$ and $\mathbf{m} = m$ are just numbers. Since shifts in outcomes are irrelevant for steering, we consider $m = 0$ so that the corresponding POVM $M_{a|\mathbf{x},\xi}$ has characteristic function

$$\hat{M}_{a|\mathbf{x},\xi}(p) = e^{-ipQ_\mathbf{x}} e^{-\frac{1}{2}p^2\xi^2}.$$

With $\xi^2 = 0$, we simply obtain the PVM with characteristic function $\hat{M}(p) = e^{-ipQ_\mathbf{x}}$, that is, the unitary group generated by the quadrature operator $Q_\mathbf{x}$. Consistently with the notation in previous section, we use $Q_{a|\mathbf{x}}$ to denote the corresponding PVM elements. Hence, Gaussian PVMs with $d = 1$ are just quadrature measurements. In general, the product form of the characteristic function implies that $M_{a|\mathbf{x},\xi}$ has the convolution form [48]:

$$M_a = M_{a|\mathbf{x},\xi} := \frac{1}{\xi\sqrt{2\pi}} \int e^{-\frac{1}{2}(a-a')^2/\xi^2} Q_{a'|\mathbf{x}} da'.$$

Hence, any Gaussian POVM $M_a$ with $a \in \mathbb{R}$ is, up to a shift, a "noisy" quadrature. Interestingly, noise exceeding the uncertainty limit renders quadratures jointly measurable:

*Lemma 3.* The noisy versions $M_{\mathbf{x},\xi}$ and $M_{\mathbf{y},\xi'}$ of two quadratures $Q_\mathbf{x}$, $Q_\mathbf{y}$ are jointly measurable if and only if

$$\xi\xi' \geqslant \|[Q_\mathbf{x}, P_\mathbf{y}]\|/2,$$

in which case they have a Gaussian joint measurement.

This result generalizes a known joint measurability criterion for position and momentum [51–53]; see Appendix C for a proof. The crucial point here is the existence of joint Gaussian measurement, which follows from the nontrivial averaging argument of Ref. [53].

A quantum channel between two CV systems with respective degrees of freedom $N$ and $N'$ is *Gaussian*, if it maps Gaussian states into Gaussian states. In the Heisenberg picture, this entails

$$\Lambda^*[W(\mathbf{x})] = W(\mathbf{M}\mathbf{x}) e^{-\frac{1}{4}\mathbf{x}^T\mathbf{N}\mathbf{x} - i\mathbf{c}^T\mathbf{x}}, \tag{27}$$

where $\mathbf{M}$ is a real $2N \times 2N'$-matrix, and $\mathbf{N}$ is a real $2N' \times 2N'$-matrix. Due to complete positivity, they satisfy

$$\mathbf{C}_{\mathbf{M},\mathbf{N}} + i\Omega \geqslant \mathbf{0}, \tag{28}$$

where (interestingly) $\mathbf{C}_{\mathbf{M},\mathbf{N}}$ is as in Eq. (26). Again, *every* triple $(\mathbf{M},\mathbf{N},\mathbf{c})$ with Eq. (28) define a Gaussian channel via Eq. (27). Unitary channels $B \mapsto U^*BU$ have $\mathbf{N} = \mathbf{0}$ and $\mathbf{M} = \mathbf{S}$ symplectic, i.e., $U = D_\mathbf{c}U_\mathbf{S}$. Using Eqs. (23) and (27) we get the general transformation rule for states in terms of CMs and displacement vectors:

$$\mathbf{V} \mapsto \mathbf{M}^T\mathbf{V}\mathbf{M} + \mathbf{N}, \quad \mathbf{r} \mapsto \mathbf{M}^T\mathbf{r} + \mathbf{c}. \tag{29}$$

Similarly, a Gaussian channel with matrices $(\mathbf{M},\mathbf{N},\mathbf{c})$, followed by a Gaussian measurement with matrices $(\mathbf{K},\mathbf{L},\mathbf{m})$ is clearly a Gaussian measurement as well, and we can easily derive the associated matrices by combining Eqs. (25) and (27); there the result is

$$(\mathbf{K},\mathbf{L},\mathbf{m}) \mapsto (\mathbf{M}\mathbf{K}, \mathbf{L} + \mathbf{K}^T\mathbf{N}\mathbf{K}, \mathbf{m} + \mathbf{K}^T\mathbf{c}). \tag{30}$$

Using Eq. (30), we observe that (for $\mathbf{c} = 0$) the channel transforms a quadrature PVM $Q_\mathbf{x}$ into the noisy POVM $M_{\mathbf{M}\mathbf{x},\xi}$ where now $\xi^2 = \mathbf{x}^T\mathbf{N}\mathbf{x}/2$.

Finally, a Gaussian post-processing (*classical* channel) is one which transforms every Gaussian probability distribution into another one. These are determined by triples $(\mathbf{M},\mathbf{N},\mathbf{c})$ as in the above quantum case, *except that only* $\mathbf{N} \geqslant \mathbf{0}$ *is required* as complete positivity does not appear in the classical case. One can show that the matrices are associated with linear coordinate transformations, convolutions, and translations, respectively [29]. Note that linear transformations include the deterministic post-processings, which simply project on a lower-dimensional subspace. A Gaussian measurement $(\mathbf{K},\mathbf{L},\mathbf{m})$, followed by a Gaussian postprocessing $(\mathbf{M},\mathbf{N},\mathbf{c})$, is again a Gaussian measurement, with parameters obtained by the transformation rule

$$(\mathbf{K},\mathbf{L},\mathbf{m}) \mapsto (\mathbf{K}\mathbf{M}, \mathbf{N} + \mathbf{M}^T\mathbf{L}\mathbf{M}, \mathbf{c} + \mathbf{M}^T\mathbf{m}). \tag{31}$$

### B. State-channel correspondence and Gaussian steering

We are now ready to prove our main results on Gaussian steering. We start with the Gaussian version of the state-channel duality:

*Lemma 4.* There is a 1-to-1 correspondence between bipartite Gaussian states $\rho$ sharing a marginal $\sigma = \mathrm{tr}_A[\rho]$ with CM $\mathbf{V}_\sigma$ of full symplectic rank and displacement $\mathbf{r}_\sigma$, and Gaussian channels $\mathsf{T}$ from Bob to Alice, such that Eq. (6) holds with $|\Omega\rangle$ having CM and displacement

$$\mathbf{V}_\Omega = \begin{pmatrix} \mathbf{V}_\sigma & \mathbf{S}^T\mathbf{ZS} \\ \mathbf{S}^T\mathbf{ZS} & \mathbf{V}_\sigma \end{pmatrix}, \quad \mathbf{r}_\Omega = \mathbf{r}_\sigma \oplus \mathbf{r}_\sigma.$$

Here $\mathbf{S}$ is a symplectic matrix diagonalising $\mathbf{V}_\sigma$, and $\mathbf{Z} = \oplus_{i=1}^N \sqrt{\nu_i^2 - 1}\,\sigma_z$, with $\nu_i$ the symplectic eigenvalues of $\mathbf{V}_\sigma$. The correspondence between the parameters $(\mathbf{V},\mathbf{r})$ and $(\mathbf{M},\mathbf{N},\mathbf{c})$ of $\rho$ and $\mathsf{T}$, respectively, is explicitly given by

$$\begin{cases} \mathbf{V} = \begin{pmatrix} \mathbf{V_A} & \Gamma^T \\ \Gamma & \mathbf{V}_\sigma \end{pmatrix}, \\ \mathbf{r} = \mathbf{r}_A \oplus \mathbf{r}_B \end{cases} \leftrightarrow \begin{cases} \mathbf{M} = (\mathbf{S^T ZS})^{-1}\Gamma \\ \mathbf{N} = \mathbf{V}_A - \mathbf{M}^T\mathbf{V}_\sigma\mathbf{M}, \\ \mathbf{c} = \mathbf{r}_A - \mathbf{M}^T\mathbf{r}_\sigma \end{cases}$$

where the positivity conditions are equivalent: $\mathbf{V} + i\Omega \geqslant 0$ iff $\mathbf{C_{M,N}} + i\Omega \geqslant 0$.

The proof of this Lemma is given in Appendix D. Interestingly, the equivalence of the inequalities is obtained via Schur complements, which have recently found applications in the investigation of quantum correlations [54]. Using Lemmas 3 and 4, we finall prove

*Theorem 2.* Let $\rho$ be a bipartite Gaussian state with CM $\mathbf{V}$ and displacement $\mathbf{r}$, and $(\mathbf{M},\mathbf{N},\mathbf{c})$ the matrices of the channel $\mathsf{T}$ given by Lemma 4. The following are equivalent:

(i) $\rho$ is steerable by the set of Gaussian measurements.

(ii) $\rho$ is steerable by some canonical pair of quadratures.

(iii) $\mathbf{V} + i(\mathbf{0} \oplus \Omega)$ is not positive semidefinite

(iv) $(\mathbf{M},\mathbf{N},\mathbf{c})$ do not defin a valid Gaussian observable.

*Proof.* We firs note that (ii) trivially implies (i). Next, we repeat the calculation (D6) in the proof of Lemma 4 (see Appendix D) without $\Omega_A$, which establishes that $\mathbf{C_{M,N}}$ is the Schur complement of $\mathbf{V}_\sigma + i\Omega_B$ in $\mathbf{V}_\rho + i(\mathbf{0} \oplus \Omega_B)$. This shows that (iii) and (iv) are equivalent. Furthermore, using [29, Prop. 2] we conclude that $\mathsf{T}$ maps the set of all Gaussian measurements into a set having a joint (Gaussian) measurement, if $\mathbf{C_{M,N}} \geqslant 0$. Hence (i) implies (iv).

We are left with the proof of the main result, stating that (iv) implies (ii). Assuming (iv) let $\mathbf{x},\mathbf{y}$ be vectors such that $(\mathbf{y}^T - i\mathbf{x}^T)\mathbf{C_{M,N}}(\mathbf{y} + i\mathbf{x}) < 0$. Then by complete positivity $(\mathbf{y}^T - i\mathbf{x}^T)(\mathbf{C_{M,N}} + i\Omega)(\mathbf{y} + i\mathbf{x}) \geqslant 0$, which implies $r := \mathbf{x}^T\Omega\mathbf{y} > 0$ and

$$(\mathbf{Mx})^T\Omega\mathbf{My} > \tfrac{1}{2}(\mathbf{x}^T\mathbf{Nx} + \mathbf{y}^T\mathbf{Ny}). \tag{32}$$

Clearly, we may replace $\mathbf{x}$ and $\mathbf{y}$ with $r^{-\frac{1}{2}}\mathbf{x}$ and $r^{-\frac{1}{2}}\mathbf{y}$ and (32) still holds. Then the pair $Q_\mathbf{x} = \mathbf{x}^T\mathbf{R}$ and $P_\mathbf{y} = \mathbf{y}^T\mathbf{R}$ of quadratures is canonical since $\mathbf{x}^T\Omega\mathbf{y} = 1$. It is easy to check using the transformation rule Eq. (30) that the channel $\mathsf{T}$, having parameters $(\mathbf{M},\mathbf{N},\mathbf{c})$, transforms the associated PVMs into the POVMs $M_{\mathbf{Mx},\xi}$ and $M_{\mathbf{My},\xi'}$ (up to irrelevant shifts in outcomes), where $\xi^2 = \mathbf{x}^T\mathbf{Nx}/2$ and $\xi'^2 = \mathbf{y}^T\mathbf{Ny}/2$. By Eq. (32) we have $2\xi\xi' \leqslant \xi^2 + \xi'^2 < (\mathbf{Mx})^T\Omega\mathbf{My}$ so from Lemma 3 we conclude that the POVMs are not jointly measurable. This means we have found a canonical pair $(Q_\mathbf{x}, Q_\mathbf{y})$ of quadratures such that $\mathsf{T}(Q_\mathbf{x})$ and $\mathsf{T}(Q_\mathbf{y})$ are not jointly measurable, so according to Theorem 1, the state $\rho$ is steerable by this pair. Hence, (ii) holds. The proof is complete. ■

We remark that the equivalence between (i) and (iii) was originally proven in Ref. [1]. Here we use Lemma 3 to show that quadratures are enough [(ii)]; this comes closest to the original notion of steering of an EPR-state via position and momentum as discussed by Schrödinger [55]. Note that the above proof shows explicitly how one can construct quadrature pairs for which steering is possible when the conditions of the theorem hold.

Furthermore, an interpretation emerges from (iv): the Gaussian POVM determined by the *channel parameters* $(\mathbf{M},\mathbf{N},\mathbf{c})$ is exactly the joint observable for the assemblage $\{\mathsf{T}^*(A_\mathbf{a}) : A_\mathbf{a}$ Gaussian$\}$ that rules out steering in (i) by Th. 1. To explain this in detail, we follow the argument in Ref. [29, Prop. 2] mentioned in the above proof: we f rst note that an arbitrary Gaussian measurement $(\mathbf{K},\mathbf{L},\mathbf{m})$ on Alice's side is transformed by the channel $(\mathbf{M},\mathbf{N},\mathbf{c})$ into one with parameters $(\mathbf{K}',\mathbf{L}',\mathbf{m}') = (\mathbf{MK},\mathbf{L} + \mathbf{K}^T\mathbf{NK},\mathbf{m} + \mathbf{K}^T\mathbf{c})$ by Eq. (30). To show that such POVMs are all jointly measurable, we only need to reinterpret the channel parameters $(\mathbf{M},\mathbf{N},\mathbf{c})$ as the joint measurement $G_\lambda$. Indeed, with $(\mathbf{K},\mathbf{L},\mathbf{m})$ taken as post-processing parameters, Eq. (31) becomes identical to Eq. (30), showing how $(\mathbf{K}',\mathbf{L}',\mathbf{m}')$ is postprocessed from $G_\lambda$. We stress that the nontrivial part is in the positivity requirements, which are *not* in general identical. Indeed, the reinterpretation is possible only when (iv) does not hold, i.e., $\mathbf{C_{M,N}} \geqslant \mathbf{0}$, which is not true for general channels.

For the sake of completeness, and to further demonstrate that the existing formulation of Gaussian steering [1] follows from our theory, we also show how one can easily derive the Gaussian LHS model given in Ref. [1] from the above results. Since this is not essential for understanding our main results, the derivation is given in Appendix E.

Finally, in addition to its impact on Gaussian steering, Theorem 2 yields

*Corollary 2.* A Gaussian channel which maps each canonical quadrature pair into a jointly measurable pair, is *Gaussian incompatibility breaking* in the sense of Ref. [29].

This considerably strengthens the theory in Ref. [29], by showing that *canonical* pairs are sufficient and that a *Gaussian* joint observable always exists for the jointly measurable Gaussian POVMs. The latter is a nontrivial and a fairly fundamental result which requires Lemma 3.

## VI. CONCLUSIONS

Steering is a genuine quantum phenomenon, with important applications both in quantum information processing and foundations of quantum mechanics. Notwithstanding the growing interest in it in the past few years [14], limited results and tools are available in the continuous variable case. We introduced a state-channel correspondence that allows us to discuss the steering problem in a completely general context. In particular, we extend many of the results previously known only in the finite-dimensiona case, such as the mathematical equivalence of steering and joint-measurability problems [26] and the equivalence of steering and joint-measurability for the case of full Schmidt rank states [24,25]. Moreover, via state-channel duality we are able to connect steerability properties of noisy NOON states with Markovianity properties of the corresponding channel and to provide an analytical lower bound to the steerability noise threshold for any $N$. Finally, we apply our methods to the Gaussian setting, introducing a channel characterization of steerability and proving that canonical quadratures are enough for steering. An interesting future direction would be to extensively investigate the capability of the state-channel duality to provide steering, joint measurability, and incompatibility breaking criteria in the continuous variable case for states, observables, and channels, respectively.

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### APPENDIX A: PROOF OF THE GENERAL STATE-CHANNEL DUALITY (LEMMA 1)

Let $|\Omega_\sigma\rangle$ and $\sigma$ be as in Lemma 1. Given any quantum channel $\mathsf{T}$, the state

$$\rho = (\mathsf{T}\otimes \mathrm{Id})(|\Omega_\sigma\rangle\langle\Omega_\sigma|) \tag{A1}$$

clearly has the property $\mathrm{tr}_A[\rho]=\sigma$, so we have managed to produce more general states than ones obtained by the Choi-Jamiołkowski correspondence. We now need to prove that the new correspondence is one-to-one *onto the set of states with* $\mathrm{tr}_A[\rho]=\sigma$. Given such as state, we firs compute

$$\begin{aligned}\mathrm{tr}[\rho(A\otimes B)] &= \langle\Omega_\sigma|\mathsf{T}^*(A)\otimes B|\Omega_\sigma\rangle\\ &= \sum_{nm}\sqrt{s_n s_m}\langle nn|\mathsf{T}^*(A)\otimes B|mm\rangle\\ &= \sum_{nm}\sqrt{s_n s_m}\langle n|\mathsf{T}^*(A)|m\rangle\langle n|B|m\rangle\\ &= \sum_{nm}\langle n|\sqrt{\sigma}\mathsf{T}^*(A)\sqrt{\sigma}|m\rangle\langle n|B|m\rangle\\ &= \mathrm{tr}[\sqrt{\sigma}\mathsf{T}^*(A)\sqrt{\sigma}B^\mathsf{T}],\end{aligned} \tag{A2}$$

where $B^\mathsf{T}$ is the transpose of $B$ in the f xed basis. Hence,

$$\sigma^{\frac{1}{2}}\mathsf{T}^*(A)\sigma^{\frac{1}{2}} = \mathrm{tr}_A[\rho(A\otimes\mathbb{1})]^\mathsf{T}. \tag{A3}$$

From this we see immediately that distinct channels correspond to distinct states, since the matrix elements of the state are clearly uniquely determined by those of the channel: $\langle nm|\rho|n'm'\rangle = \mathrm{tr}[\sqrt{\sigma}\mathsf{T}^*(|n'\rangle\langle n|)\sqrt{\sigma}(|m'\rangle\langle m|)^\mathsf{T}] = \sqrt{s_m}\sqrt{s_{m'}}\langle m'|\mathsf{T}^*(|n'\rangle\langle n|)|m\rangle$, where we have now also fi ed a basis $\{|n\rangle\}$ on Alice's side.

What remains to be shown is that for *any* state $\rho$ with $\mathrm{tr}_A[\rho]=\sigma$ there exists a channel $\mathsf{T}$ such that Eq. (A1) [or, equivalently, Eq. (A3)] holds. If $d<\infty$ we can invert $\sigma^{-\frac{1}{2}}$ in Eq. (A3) to solve for $\mathsf{T}^*(A)$; however, we still need to show that this define a channel, i.e., a CPTP map. We therefore proceed by writing the state $\rho$ as

$$\rho = \sum_k |\psi_k\rangle\langle\psi_k| = \sum_{\substack{k,n,m\\ n',m'}}\langle nm|\psi_k\rangle\langle\psi_k|n'm'\rangle\,|nm\rangle\langle n'm'|, \tag{A4}$$

so that, for all bounded operators $A$ and $B$ (for Alice and Bob, respectively), we get

$$\mathrm{tr}[\rho(A\otimes B)] = \sum_k \mathrm{tr}[R_k^* A R_k B^\mathsf{T}], \tag{A5}$$

where $R:\mathcal{H}_B\to\mathcal{H}_A$ is the Hilbert-Schmidt operator define by $\langle n|R_k m\rangle = \langle nm|\psi_k\rangle$. Hence, $\mathrm{tr}_A[\rho(A\otimes\mathbb{1})]^\mathsf{T} = \sum_k R_k^* A R_k$. In particular, $\sigma=\sigma^\mathsf{T}=\sum_k R_k^* R_k$.

Next, we need a little of functional analysis, so as to allow the proof to go through also for $d=\infty$, in which case the inverse of any full rank state is unbounded and requires some care. Let $\mathcal{R}$ be the dense range of $\sigma$, containing all the basis vectors. Then, $\mathcal{R} = \mathrm{ran}\,\sigma^{\frac{1}{2}}$, $\sigma^{\frac{1}{2}}$ is injective, and for any $|\psi\rangle\in\mathcal{R}$ we have

$$\|R_k\sigma^{-\frac{1}{2}}\psi\|^2 \leqslant \sum_k\langle\sigma^{-\frac{1}{2}}\psi|R_k^* R_k\sigma^{-\frac{1}{2}}\psi\rangle = \|\psi\|^2,$$

which implies that each $R_k\sigma^{-\frac{1}{2}}$ extends to a bounded operator $M_k:\mathcal{H}_B\to\mathcal{H}_A$, for which $M_k\sigma^{\frac{1}{2}} = R_k$.

Since $\sum_k M_k^* M_k = \mathbb{1}$, the operators $M_k$ set up a Kraus decomposition of a channel: we defin

$$\mathsf{T}(T) := \sum_k M_k T M_k^* \tag{A6}$$

for all (trace class) operators $T$. This is by construction completely positive, and it is trace-preserving since $\sum_k M_k^* M_k = \mathbb{1}$. In the infinite-dimensiona case the series converges, e.g., in the weak topology. Plugging this channel in Eq. (A3) immediately gives

$$\begin{aligned}\sigma^{\frac{1}{2}}\mathsf{T}^*(A)\sigma^{\frac{1}{2}} &= \sum_k\left(M_k\sigma^{\frac{1}{2}}\right)^* A M_k\sigma^{\frac{1}{2}} = \sum_k R_k^* A R_k\\ &= \mathrm{tr}_A[\rho(A\otimes\mathbb{1})]^\mathsf{T},\end{aligned} \tag{A7}$$

so that Eq. (A3), and hence also Eq. (A1) holds, that is, the channel gives back the original state $\rho$. This proves that the correspondence is one-to-one, and completes the proof.

### APPENDIX B: A JOINT MEASURABILITY CRITERION FOR QUBIT POVMS WITH ARBITRARY OUTCOMES (LEMMA 2)

In contrast, to most existing criteria, this one applies to qubit POVMs with continuous outcome sets. In the main text, it was shown to be useful for findin noise bounds for quadrature measurements restricted to two-dimensional photon number eigenspaces.

More generally, we prove that an assemblage of $n$ qubit observables $\{B_{b|i}\}_{i=1}^n$ is jointly measurable if

$$\Delta(b_1,\dots,b_n) := \sum_i r_i(b_i) - n + 1 \geqslant 0, \tag{B1}$$

where $r_i(b) := \det M_i(b)$, $M_i(b) := B_{b|i}/p_i(b)$, and $p_i(b) = \langle 0|B_{b|i}|0\rangle$. Indeed,

$$M_i(b) = \begin{pmatrix} 1 & \overline{f_i(b)}\\ f_i(b) & r_i(b)+|f_i(b)|^2\end{pmatrix} \tag{B2}$$

for some complex valued functions $f_i$. The normalization forces $\int f_i(b)p_i(x)dx = 0$ and $\int(|f_i(b)|^2 + r_i(b))p_i(b)db = 1$.

We defin $G_{b_1,\dots,b_n}$ via

$$\frac{G_{b_1,\dots,b_n}}{\prod_{i=1}^n p_i(b_i)} = \begin{pmatrix} 1 & \sum_i \overline{f_i(b_i)} \\ \sum_i f_i(b_i) & |\sum_i f_i(b_i)|^2 + \Delta(b_1,\dots,b_n) \end{pmatrix}. \quad (B3)$$

Using the constraints we see that it is normalized, and that

$$B_{b|i} = M_i(b)p_i(b) = \int \delta_{b,b_i} G_{b_1,\dots,b_n} db_1 \cdots db_n. \quad (B4)$$

The critical constraint is $G_{b_1,\dots,b_n} \geqslant 0$ now follows from

$$\det G_{b_1,\dots,b_n} = \Delta(b_1,\dots,b_n) \prod_{i=1}^n p_i(b_i)^2 \geqslant 0, \quad (B5)$$

which is ensured by the assumption. This means that the $B_i$ have a joint observable with deterministic response functions, so they are jointly measurable.

By taking $B_{a|i} = \Lambda_r^*(A_{a|i})$, where $\Lambda_r$ is the amplitude damping channel define in the main text, the assumption Eq. (B1) becomes Eq. (19) of the main text, once we notice that $\langle 0|\Lambda_r^*(A_{a|i})|0\rangle = \langle 0|A_{a|i}|0\rangle$; see Eq. (16). This completes the proof.

## APPENDIX C: PROOF OF THE JOINT MEASURABILITY CRITERION FOR CONVOLUTED QUADRATURES (LEMMA 3)

This lemma was critical for the characterization of Gaussian steering. To prove it we let $r = \mathbf{x}^T\Omega\mathbf{y}$, so that $[Q_\mathbf{x}, Q_\mathbf{y}] = ir\mathbb{1}$. If $r = 0$, then $Q_\mathbf{x}$ and $Q_\mathbf{y}$ commute and the claim is trivial since they stay jointly measurable after convolution. We suppose $r > 0$, and look at the scaled quadrature $Q_\mathbf{y}/r = \mathbf{y}^T\mathbf{R}/r = Q_{\mathbf{y}/r}$. By using the connection $Q_\mathbf{y} = \int a\, Q_{a|\mathbf{y}} da$ between the operator $Q_\mathbf{y}$ and the corresponding PVM $Q_{a|\mathbf{y}}$, we see that $Q_{a|\mathbf{y}/r} = rQ_{ra|\mathbf{y}}$. A direct computation then shows that scaling of the noisy POVM gives $M_{a|\mathbf{y}/r,\xi'/r} = rM_{ra|\mathbf{y},\xi'}$. Since scaling is a postprocessing and hence does not affect joint measurability, the original pair $(M_{\mathbf{x},\xi}, M_{\mathbf{y},\xi'})$ is jointly measurable if and only if $(M_{\mathbf{x},\xi}, M_{\mathbf{y}/r,\xi'/r})$ is. But the corresponding quadrature pair $(Q_\mathbf{x}, Q_{\mathbf{y}/r})$ is canonical, as

$$[Q_\mathbf{x}, Q_{\mathbf{y}/r}] = [\mathbf{x}^T\mathbf{R}, \mathbf{y}^T\mathbf{R}/r] = i(\mathbf{x}^T\Omega\mathbf{y}/r)\mathbb{1} = i\mathbb{1},$$

and hence unitarily equivalent to the pair $(Q_0, Q_{\pi/2})$ via a symplectic transformation, where $Q_\theta = (e^{i\theta}a^\dagger + e^{-i\theta}a)/\sqrt{2}$ are the rotated quadratures of a single-mode system. The same unitary then transforms the convoluted pair $(M_{\mathbf{x},\xi}, M_{\mathbf{y}/r,\xi'/r})$ into the pair $(M_{0,\xi}, M_{\pi/2,\xi'/r})$, where

$$M_{a|\theta,\xi} := \frac{1}{\sqrt{2\pi}\xi} \int e^{-\frac{1}{2}(a-a')^2/\xi^2} Q_{a'|\theta},$$

and hence it suffice to show that the joint measurability of $(M_{0,\xi}, M_{\pi/2,\xi'/r})$ is equivalent to the inequality $\xi(\xi'/r) \geqslant 1/2$, and that the joint observable, when exists, can be chosen Gaussian.

To prove this, we use known results on joint measurability of "unsharp" position and momentum [51,52], which is exactly what our convoluted quadratures are. In particular, if $(M_{0,\xi}, M_{\pi/2,\xi'/r})$ are jointly measurable, they must have a joint observable of the Weyl-covariant form $G_{a_1,a_2} = W(a_1,a_2)\rho_0 W(a_1,a_2)^*/(2\pi)$, where $\rho_0$ is a state with

$$\mathrm{tr}[\rho_0 Q_{a_1|0}] = \frac{e^{-\frac{1}{2}a_1^2/\xi^2}}{\sqrt{2\pi}\xi}, \quad \mathrm{tr}[\rho_0 Q_{a_2|\pi/2}] = \frac{e^{-\frac{1}{2}a_2^2/(\xi'/r)^2}}{\sqrt{2\pi}(\xi'/r)}. \quad (C1)$$

This implies that $\xi$ and $\xi'/r$ are the standard deviations of $Q_0$ and $Q_{\pi/2}$ in the state $\rho_0$, hence satisfying $\xi(\xi'/r) \geqslant 1/2$ by the Heisenberg uncertainty principle. Conversely, if the inequality holds, we can defin $\rho_0 = |\psi_0\rangle\langle\psi_0|$ in the coordinate representation as $\psi_0(a) = (2c/\pi)^{\frac{1}{4}} e^{-(c+iw)a^2}$ with $\xi^2 = 1/(4c)$ and $\xi'^2/r^2 = (c^2+d^2)/d$; then a direct computation shows that the corresponding $G_{a_1,a_2}$ is a joint observable for $M_{0,\xi}$ and $M_{\pi/2,\xi'/r}$. This observable is Gaussian since $\rho_0$ is a Gaussian state [48].

Finally, since all the above unitary equivalences were done via symplectic transformations, the original POVMs have a Gaussian joint observable as well. This completes the proof.

## APPENDIX D: PROOF OF THE GAUSSIAN STATE-CHANNEL DUALITY (LEMMA 4)

The difference to the general case (considered above) is that to preserve Gaussianity, we need to do the diagonalization of the reference state $\sigma$ "symplectically" (see, e.g., Ref. [56]): Let $\mathbf{V}_\sigma$ be the CM of $\sigma$ and $\mathbf{r}_\sigma$ the displacement. By Williamson's theorem [57] there is a symplectic matrix $\mathbf{S}$ such that $\mathbf{V}_\sigma = \mathbf{S}^T\mathbf{D}\mathbf{S}$ with $\mathbf{D} = \oplus_{k=1}^N \nu_k \mathbb{1}_2$, where $\nu_k$ are the symplectic eigenvalues of $\mathbf{V}_\sigma$, and we assume $\nu_i > 1$ (full symplectic rank). This is not restrictive as any $\nu_i = 1$ corresponds to a vacuum mode, which we may factor out from the system. Then $U = D_{\mathbf{r}_\sigma} U_\mathbf{S}$ diagonalizes $\sigma$ in the photon number basis $|\mathbf{n}\rangle = |n_1,\dots,n_N\rangle$:

$$U^*\sigma U = \sum_\mathbf{n} p_\mathbf{n} |\mathbf{n}\rangle\langle\mathbf{n}|, \quad p_\mathbf{n} = \prod_{k=1}^N \frac{2}{1+\nu_k}\left(\frac{\nu_k - 1}{\nu_k + 1}\right)^{n_k}. \quad (D1)$$

Moreover, the purificatio $\sum_\mathbf{n} \sqrt{p_\mathbf{n}}|\mathbf{n}\rangle \otimes |\mathbf{n}\rangle$ has the CM

$$\begin{pmatrix} \mathbf{D} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{D} \end{pmatrix} \quad \text{with } \mathbf{Z} = \bigoplus_{i=1}^N \sqrt{\nu_i^2 - 1}\,\sigma_z. \quad (D2)$$

The eigenbasis $\{U|\mathbf{n}\rangle\}$ of $\sigma$ is the one we use to construct the steering channels following the general scheme (see Lemma 1). Hence, we form the purificatio

$$\Omega_\sigma = \sum_\mathbf{n} \sqrt{p_\mathbf{n}}\, U|\mathbf{n}\rangle \otimes U|\mathbf{n}\rangle, \quad (D3)$$

which by Eq. (29) has displacement vector $\mathbf{r}_\sigma \oplus \mathbf{r}_\sigma$ and CM,

$$\mathbf{V}_{\Omega_\sigma} = (\mathbf{S}^T \oplus \mathbf{S}^T)\begin{pmatrix} \mathbf{D} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{D} \end{pmatrix}(\mathbf{S} \oplus \mathbf{S}) = \begin{pmatrix} \mathbf{V}_\sigma & \mathbf{S}^T\mathbf{Z}\mathbf{S} \\ \mathbf{S}^T\mathbf{Z}\mathbf{S} & \mathbf{V}_\sigma \end{pmatrix}, \quad (D4)$$

as stated in the Lemma. Again by Eq. (29), the application of a Gaussian channel $\Lambda$ with matrices $(\mathbf{M},\mathbf{N},\mathbf{c})$ yields the state

$\rho := (\Lambda \otimes \mathrm{Id})(|\Omega_\sigma\rangle\langle\Omega_\sigma|)$ with CM,

$$\mathbf{V} = (\mathbf{M}^T \oplus \mathbf{I})\begin{pmatrix} \mathbf{V}_\sigma & \mathbf{S}^T\mathbf{Z}\mathbf{S} \\ \mathbf{S}^T\mathbf{Z}\mathbf{S} & \mathbf{V}_\sigma \end{pmatrix}(\mathbf{M} \oplus \mathbf{I}) + \mathbf{N} \oplus \mathbf{0}$$
$$= \begin{pmatrix} \mathbf{M}^\mathbf{T}\mathbf{V}_\sigma\mathbf{M} + \mathbf{N} & \mathbf{M}^T\mathbf{S}^T\mathbf{Z}\mathbf{S} \\ \mathbf{S}^T\mathbf{Z}\mathbf{S}\mathbf{M} & \mathbf{V}_\sigma \end{pmatrix}, \quad \text{(D5)}$$

and displacement $\mathbf{r} = (\mathbf{M}^T\mathbf{r}_\sigma + \mathbf{c}) \oplus \mathbf{r}_\sigma$. Now $\mathbf{V}_\rho + i\Omega \geqslant 0$ if and only if $\mathbf{C} \geqslant 0$ where $\mathbf{C}$ is the Schur complement of the block $\mathbf{V}_\sigma + i\Omega_B$ in $\mathbf{V}_\rho + i\Omega$. But

$$\mathbf{C} = \mathbf{M}^T\mathbf{V}_\sigma\mathbf{M} + \mathbf{N} + i\Omega_A$$
$$- \mathbf{M}^T\mathbf{S}^T\mathbf{Z}\mathbf{S}(\mathbf{V}_\sigma + i\Omega_B)^{-1}\mathbf{S}^T\mathbf{Z}\mathbf{S}\mathbf{M}$$
$$= \mathbf{N} + i\Omega_A + \mathbf{M}^T\mathbf{S}^T(\mathbf{D} - \mathbf{Z}(\mathbf{D} + i\Omega_B)^{-1}\mathbf{Z})\mathbf{S}\mathbf{M}$$
$$= \mathbf{C}_{\mathbf{M},\mathbf{N}} + i\Omega_A, \quad \text{(D6)}$$

where we have used $\mathbf{D} - \mathbf{Z}(\mathbf{D} + i\Omega)^{-1}\mathbf{Z} = \Omega$, which is straightforward to verify. This shows that $\mathbf{C}_{\mathbf{M},\mathbf{N}} + i\Omega_A \geqslant 0$ is *equivalent* to $\mathbf{V}_\rho$ being a valid CM. Now for any given Gaussian state $\rho$ with CM and displacement vector

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_\mathbf{A} & \Gamma^T \\ \Gamma & \mathbf{V}_\sigma \end{pmatrix}, \quad \mathbf{r} = \mathbf{r}_A \oplus \mathbf{r}_\sigma, \quad \text{(D7)}$$

we can defin

$$(\mathbf{M},\mathbf{N},\mathbf{c}) = ((\mathbf{S}^T\mathbf{Z}\mathbf{S})^{-1}\Gamma,\ \mathbf{V}_A - \mathbf{M}^T\mathbf{V}_\sigma\mathbf{M},\ \mathbf{r}_A - \mathbf{M}^T\mathbf{r}_\sigma), \quad \text{(D8)}$$

which then satisfie Eq. (D5), so that $\mathbf{C}_{\mathbf{M},\mathbf{N}} + i\Omega_A \geqslant 0$ by the above equivalence, showing that $(\mathbf{M},\mathbf{N},\mathbf{c})$ determines a Gaussian channel $\Lambda$ with $\rho = (\Lambda \otimes \mathrm{Id})(|\Omega_\sigma\rangle\langle\Omega_\sigma|)$. This completes the proof.

## APPENDIX E: THE DERIVATION OF THE LHS FROM THE JOINT GAUSSIAN MEASUREMENT

Here we show that our joint Gaussian POVM (discussed in the main text) is consistent with the LHS of Ref. [1]. According to the general discussion in Sec. II D, joint POVM $G_\lambda$ and the LHS $\sigma_\lambda$ are related by $\sigma_\lambda = \sigma^{\frac{1}{2}} G_\lambda \sigma^{\frac{1}{2}} = \mathrm{tr}_A[G_\lambda \otimes \mathbb{1}|\Omega_\sigma\rangle\langle\Omega_\sigma|]$. Now $\sigma_\lambda$ has finit trace, and $\tilde{\sigma}_\lambda := \sigma_\lambda/\mathrm{tr}[\sigma_\lambda]$ is an actual state; we show that it is Gaussian by computing the characteristic function $\widehat{\tilde{\sigma}_\lambda}(\mathbf{x}) := \mathrm{tr}[W(\mathbf{x})\tilde{\sigma}_\lambda] = f_\mathbf{x}(\lambda)/f_\mathbf{0}(\lambda)$, where $f_\mathbf{x}(\lambda) := \mathrm{tr}[W(\mathbf{x})\sigma_\lambda] = \mathrm{tr}[G_\lambda \otimes W(\mathbf{x})|\Omega_\sigma\rangle\langle\Omega_\sigma|]$. For simplicity we assume $\mathbf{c} = 0$. Due to Eq. (25), the function $f_\mathbf{x}$ is determined via its Fourier transform, in terms of the channel parameters $(\mathbf{M},\mathbf{N},\mathbf{c})$. For simplicity, we will assume $\mathbf{c} = 0$, and compute

$$\widehat{f_\mathbf{x}}(\mathbf{p}) = \int e^{i\mathbf{p}^T\lambda}\,\mathrm{tr}[G_\lambda \otimes W(\mathbf{x})|\Omega_\sigma\rangle\langle\Omega_\sigma|]\,d\lambda$$
$$= \mathrm{tr}[\hat{G}(\mathbf{p}) \otimes W(\mathbf{x})|\Omega_\sigma\rangle\langle\Omega_\sigma|]$$
$$= \mathrm{tr}[W(\mathbf{M}\mathbf{p}) \otimes W(\mathbf{x})|\Omega_\sigma\rangle\langle\Omega_\sigma|]e^{-\frac{1}{4}\mathbf{p}^T\mathbf{N}\mathbf{p}}. \quad \text{(E1)}$$

Now by definition the f rst factor in the last expression is the characteristic function of the state $\Omega_\sigma$, evaluated at $\mathbf{M}\mathbf{p} \oplus \mathbf{x}$; hence, by Eqs. (23) and (D5) we get

$$\widehat{f_\mathbf{x}}(\mathbf{p}) = e^{-\frac{1}{4}((\mathbf{M}\mathbf{p})^T \oplus \mathbf{x}^T)\mathbf{V}_{\Omega_\sigma}(\mathbf{M}\mathbf{p}\oplus\mathbf{x})}e^{-\frac{1}{4}\mathbf{p}^T\mathbf{N}\mathbf{p}}$$
$$= e^{-\frac{1}{4}(\mathbf{p}^T\oplus\mathbf{x}^T)\mathbf{V}(\mathbf{p}\oplus\mathbf{x})} = e^{-\frac{1}{4}(\mathbf{p}^T\mathbf{V}_A\mathbf{p} + 2\mathbf{p}^T\Gamma^T\mathbf{x} + \mathbf{x}^T\mathbf{V}_\sigma\mathbf{x})}$$
$$= e^{-\frac{1}{4}(\mathbf{p}-\mu_\mathbf{x})^T\mathbf{V}_A(\mathbf{p}-\mu_\mathbf{x})}\ e^{-\frac{1}{4}\mathbf{x}^T(\mathbf{V}_\sigma - \Gamma\mathbf{V}_A^{-1}\Gamma^T)\mathbf{x}}, \quad \text{(E2)}$$

where $\mu_\mathbf{x} = -\mathbf{V}_A^{-1}\Gamma^T\mathbf{x}$, and we have used the notation Eq. (D7). Taking the inverse Fourier transform, we obtain

$$f_\mathbf{x}(\lambda) = Ce^{-\lambda^T\mathbf{V}_A^{-1}\lambda - i\lambda^T\mu_\mathbf{x}}\ e^{-\frac{1}{4}\mathbf{x}^T(\mathbf{V}_\sigma - \Gamma\mathbf{V}_A^{-1}\Gamma^T)\mathbf{x}}, \quad \text{(E3)}$$

where $C$ depends only on $\mathbf{V}_A$. Hence, $\widehat{\tilde{\sigma}_\lambda}(\mathbf{x}) = f_\mathbf{x}(\lambda)/f_\mathbf{0}(\lambda) = e^{-\frac{1}{4}\mathbf{x}^T(\mathbf{V}_\sigma - \Gamma\mathbf{V}_A^{-1}\Gamma^T)\mathbf{x} + i(\Gamma\mathbf{V}_A^{-1}\lambda)^T\mathbf{x}}$, so by Eq. (23), $\tilde{\sigma}_\lambda$ is Gaussian with CM and displacement

$$\mathbf{V}_\lambda = \mathbf{V}_\sigma - \Gamma\mathbf{V}_A^{-1}\Gamma^T, \qquad \mathbf{r}_\lambda = -\Gamma\mathbf{V}_A^{-1}\lambda. \quad \text{(E4)}$$

Furthermore, each $\tilde{\sigma}_\lambda$ occurs in the LHS decomposition with Gaussian probability $p_\lambda = \mathrm{tr}[\sigma_\lambda] = f_\mathbf{0}(\lambda) \propto e^{-\lambda^T\mathbf{V}_A^{-1}\lambda}$. By changing the hidden variable $\lambda$ to $\mathbf{r}_\lambda$ we recover exactly the LHS of Ref. [1].


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# Article IV

- **Title**: Unified picture for spatial, temporal, and channel steering
- **Authors**: Roope Uola, Fabiano Lever, Otfried Gühne, Juha-Pekka Pellonpää
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- **Abstract**: Quantum steering describes how local actions on a quantum system can affect another, spacelike separated, quantum state. Lately, quantum steering has been formulated also for timelike scenarios and for quantum channels. We approach all the three scenarios as one using tools from Stinespring dilations of quantum channels. By applying our technique we link all three steering problems one-to-one with the incompatibility of quantum measurements, a result formerly known only for spatial steering. We exploit this connection by showing how measurement uncertainty relations can be used as tight steering inequalities for all three scenarios. Moreover, we show that certain notions of temporal and spatial steering are fully equivalent and prove a hierarchy between temporal steering and macrorealistic hidden variable models.
- **Author's contribution**: The author of this thesis contributed to proofs and examples.

# Unified picture for spatial, temporal, and channel steering

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Quantum steering describes how local actions on a quantum system can affect another, spacelike separated, quantum state. Lately, quantum steering has been formulated also for timelike scenarios and for quantum channels. We approach all the three scenarios as one using tools from Stinespring dilations of quantum channels. By applying our technique we link all three steering problems one-to-one with the incompatibility of quantum measurements, a result formerly known only for spatial steering. We exploit this connection by showing how measurement uncertainty relations can be used as tight steering inequalities for all three scenarios. Moreover, we show that certain notions of temporal and spatial steering are fully equivalent and prove a hierarchy between temporal steering and macrorealistic hidden variable models.

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## I. INTRODUCTION

Quantum steering refers to the possibility of one party, typically called Alice, to affect the quantum state of a spatially separated party, typically called Bob, by making only local measurements on her system and classically communicating the measurement outcome and setting to Bob. Quantum steering formalizes spooky action at a distance [1], and as such it is an entanglement verificatio method intermediate to trust-based entanglement witnesses and no trust-requiring device-independent scenarios, e.g., Bell inequalities. Steering provides a natural framework for semi-device-independent quantum information protocols [2–4] and a guideline for theoretical and experimental work on both entanglement theory and nonlocality [5–10]. Moreover, steering is known to be closely connected to incompatibility of quantum measurements [11,12]. To be more precise, it has been shown that steering and joint measurability problems are in one-to-one correspondence [13] and that unsteerability of quantum states can be checked through incompatibility breaking properties of quantum channels [14].

Extending the spatial case, steering has recently found its temporal counterpart [15] (see Fig. 1). The idea of temporal steering is to ask whether steeringlike phenomena can happen on a single quantum system, where Alice measures a single particle firs and then hands it to Bob. One could argue that some sort of steering effect is easy to reach in such scenarios, because Alice's measurement choice can in principle affect Bob's state, i.e., Alice can signal to Bob. However, signaling can be excluded by using well-chosen input states. The remaining scenarios have found connections to, for example, non-Markovianity [16]. In this work we want to characterize quantum measurements so we do not restrict ourselves to specifi input states. Instead, we take an approach where nonsignaling is a feature of the measurement instruments. Physically, these are the scenarios where the original system is firs interacting with a probe system in some predefine manner, and then different measurements on the probe system are carried out. We show that all (nontrivial) temporal scenarios can be mapped into our formulation.

State steering has a natural extension to the level of quantum channels through the well-known state-channel isomorphism [17]. This extension is called channel steering, and it investigates the possibility of Alice to affect Bob's end of a broadcast channel from Charlie to Alice and Bob. Technically, channel steering can be seen as a (semi-device-independent) method of verifying the coherence of a channel extension [17].

By now channel steering has been introduced as a theoretical construction, but in this article we show how a certain modifi cation of it provides a powerful framework for all three steering scenarios. Namely, we connect all three steering scenarios one-to-one with the incompatibility of quantum measurements, provide universally applicable steering inequalities through measurement uncertainty relations, show an equivalence between spatial and temporal steering, and prove a hierarchy between temporal steering and macrorealistic hidden variable models.

## II. SPATIAL STEERING

Steering scenarios can be seen as processes where an untrusted party (Alice) sends a trusted party (Bob) a state assemblage $\{\rho_{a|x}\}_{a,x}$, where $x$ labels the measurements and $a$ the respective outcomes, satisfying the nonsignaling condition $\sum_a \rho_{a|x} = \sum_a \rho_{a|x'}$ for all $x,x'$. The nonsignaling property is crucial in our scenarios for reasons to become clear in the following sections. The steerability of a state assemblage is decided by checking the existence of a so-called local hidden state model (see below).

In *spatial* steering the state assemblage originates from spacelike separated local measurements on one party and is hence naturally nonsignaling. Formally, consider a bipartite system described by a quantum state $\rho_{AB}$. When Alice performs measurements described by positive operator valued measures (POVMs) $\{A_{a|x}\}_{a,x}$ (i.e., $A_{a|x} \geqslant 0$ and $\sum_a A_{a|x} = \mathbb{1}$

 

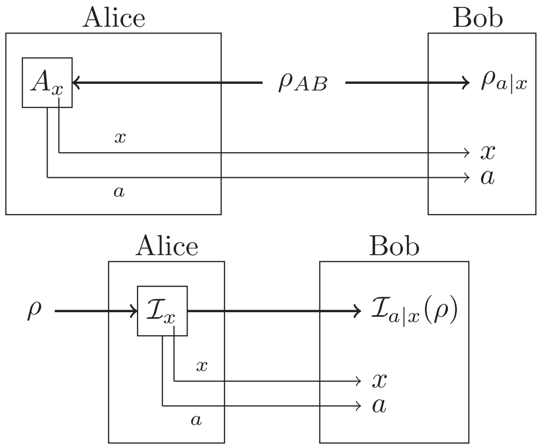


FIG. 1. Spatial steering (top): Alice and Bob share a bipartite state $\rho_{AB}$, Alice measures $A_x$ and classically communicates the measurement setting ($x$) and result ($a$) to Bob. The (non-normalized) postmeasurement state assemblage Bob receives is given as $\rho_{a|x} = \text{tr}_A[(A_{a|x} \otimes \mathbb{1})\rho_{AB}]$. Temporal steering (bottom): Alice applies an instrument $\mathcal{I}_x$ on a single-system state $\rho$ and classically communicates the measurement setting ($x$) and result ($a$) to Bob, together with the (non-normalized) output state $\mathcal{I}_{a|x}(\rho)$.

for all $a,x$) on her system, Bob is left with a non-normalized state assemblage

$$\rho_{a|x} := \text{tr}_A[(A_{a|x} \otimes \mathbb{1})\rho_{AB}]. \tag{1}$$

Here, $\mathbb{1}$ is the identity operator on Bob's system. The setup is called (spatially) unsteerable if Bob can recover his state assemblage from a local state ensemble (or state preparator) $\{p(\lambda),\sigma_\lambda\}_\lambda$ together with additional information about Alice's choice of measurement $x$ and obtained outcome $a$ by means of classical postprocessing, i.e., if for every $a,x$

$$\rho_{a|x} = \sum_\lambda p(\lambda)p(a|x,\lambda)\sigma_\lambda \tag{2}$$

and steerable otherwise. Here $p(\lambda) \geqslant 0$ is the probability that Bob's state $\sigma_\lambda$ occurs and $p(a|x,\lambda) \geqslant 0$ are conditional probabilities so that $\sum_a p(a|x,\lambda) = 1$ for each $x$, $\lambda$. The right-hand side of Eq. (2) is called, when existing, a local hidden state (LHS) model for the assemblage $\{\rho_{a|x}\}_{a,x}$.

## III. TEMPORAL STEERING

For temporal steering one needs the concept of quantum instruments. Quantum instruments are collections of completely positive maps which sum up to a completely positive trace preserving (cptp) map, i.e., to a quantum channel. Physically, instruments describe the state transformation caused by a measurement, and one can think of them as a generalization of the projection postulate to the case of POVMs. For a POVM $\{A_a\}_a$ the most typical instrument is the von Neumann–Lüders instrument $\mathcal{I}_a^L(\rho) = \sqrt{A_a}\rho\sqrt{A_a}$, and all possible instruments compatible with $\{A_a\}_a$ are the ones which code the measurement outcome probabilities into the postmeasurement state, i.e., $\text{tr}[\mathcal{I}_a(\rho)] = \text{tr}[A_a\rho]$ for all $\rho$. It can be shown [18] that any instrument implementing $\{A_a\}_a$ can be described by the quantum channels $\{\Lambda_a\}_a$ from Alice to Bob applied to the von Neumann–Lüders instrument via $\mathcal{I}_a(\rho) = \Lambda_a[\mathcal{I}_a^L(\rho)]$.

In temporal steering one is interested in state assemblages $\{\rho_{a|x}^{\text{temp}}\}_{a,x}$ which are given by the actions of a set of quantum instruments $\{\mathcal{I}_{a|x}\}_{a,x}$ on a single system state $\rho_A$. The steerability of this assemblage is decided by checking the existence of a LHS model, i.e., the scenario is temporally nonsteerable if

$$\rho_{a|x}^{\text{temp}} := \mathcal{I}_{a|x}(\rho_A) = \sum_\lambda p(\lambda)p(a|x,\lambda)\sigma_\lambda \tag{3}$$

and steerable otherwise. In temporal steering the nonsignaling condition is not a built-in feature. However, as some input states lead to steering trivially, it makes sense to talk about temporal steering only in the case of nonsignaling assemblages. Finally, note that sometimes temporal state assemblages are define through an instrument and an additional time evolution. As a concatenation of an instrument and a channel is an instrument, we do not write the channel explicitly to our state assemblages.

## IV. MAIN TECHNIQUE

As our main technique we use the Stinespring dilation of quantum channels. In textbook quantum mechanics any quantum channel $\Lambda$ on a finite-dimensiona system is given through the representation $\Lambda(\rho) = \text{tr}_E[U(\rho_0 \otimes \rho)U^\dagger]$, where $U$ is a unitary operator on the total space of the system and an environment $E$, and $\rho_0$ is a quantum state of the environment [19]. This type of representation is, however, not the only way to dilate a channel. It appears that a slightly modifie version of Stinespring dilation is better tailored for our purposes. Namely, instead of using a unitary operator on the system and its environment, we defin an isometry $V : \mathcal{H} \to \mathcal{A} \otimes \mathcal{K}$, where $\mathcal{H}$ and $\mathcal{K}$ are the Hilbert spaces of the input and output systems and $\mathcal{A}$ is the Hilbert space of a dummy system. For a channel given in the Kraus form $\Lambda(\rho) = \sum_{k=1}^r K_k\rho K_k^\dagger$, the isometry $V$ can be constructed as $V|\psi\rangle = \sum_{k=1}^r |\varphi_k\rangle \otimes K_k|\psi\rangle$ for all $|\psi\rangle$, with $\{|\varphi_k\rangle\}_{k=1}^r$ being an orthonormal basis of the dummy system. With this isometry the dilation simply reads $\Lambda(\rho) = \text{tr}_\mathcal{A}[V\rho V^\dagger]$. Note that this dilation does not have a specifi initial state on the environment and, hence, in order to make a clear distinction between the textbook unitary dilation and our isometric dilation we talk about a dummy system instead of an environment.

We are specificall interested in sets of instruments $\{\mathcal{I}_{a|x}\}_{a,x}$ which do not allow signaling, i.e., which have the same total channel $\Lambda := \sum_a \mathcal{I}_{a|x} = \sum_a \mathcal{I}_{a|x'}$ for every $x,x'$. Nonsignaling instruments are related to observables on the dummy space of their total channel $\Lambda$ [20,21]. Namely, the actions of nonsignaling instruments $\{\mathcal{I}_{a|x}\}_{a,x}$ can be written as actions of a set of POVMs $\{\tilde{A}_{a|x}\}_{a,x}$ on the dummy system, i.e.,

$$\mathcal{I}_{a|x}(\rho) = \text{tr}_\mathcal{A}[(\tilde{A}_{a|x} \otimes \mathbb{1})V\rho V^\dagger]. \tag{4}$$

Note that in general the dummy POVMs $\{\tilde{A}_{a|x}\}_{a,x}$ do not coincide with the POVMs $\{A_{a|x}\}_{a,x}$ one measures on the actual system. Note, moreover, that the isometry $V$ is constructed from $\Lambda$ and, due to nonsignaling, does not depend on $x$. Hence, nonsignaling instruments can be implemented using a predefine interaction with a probe system, as described in the Introduction.

In what follows, we will mainly concentrate on minimal dummy systems, i.e., minimal Stinespring dilations. The minimality means that $r$ is the smallest possible dimension, in which case the Kraus operators of the total channel are linearly independent. In this case (for a given total channel) the correspondence between the dummy POVMs and the instruments they defin is one-to-one [20,21]. Namely, we have that

$$\mathcal{I}_{a|x}(\rho) = \sum_{k,l=1}^{r} \langle\varphi_l|\tilde{A}_{a|x}|\varphi_k\rangle K_k \rho K_l^\dagger, \quad (5)$$

from where the matrix elements $\langle\varphi_l|\tilde{A}_{a|x}|\varphi_k\rangle$ of the dummy POVMs can be computed.

## V. MINIMAL DILATION FOR A STATE ASSEMBLAGE

A crucial concept for our study is joint measurability. A set of POVMs $\{A_{a|x}\}_{a,x}$ is called jointly measurable if there exists a common POVM $\{G_\lambda\}_\lambda$ from which the original POVMs can be postprocessed, i.e., if for every $a,x$

$$A_{a|x} = \sum_\lambda p(a|x,\lambda) G_\lambda \quad (6)$$

and incompatible otherwise. Here $p(\cdot|x,\lambda)$ is a probability distribution for every $x,\lambda$.

Because of the one-to-one connection between dummy POVMs and instruments, we see that a set $\{\tilde{A}_{a|x}\}_{a,x}$ of dummy POVMs is jointly measurable if and only if the instruments $\{\mathcal{I}_{a|x}\}_{a,x}$ (define through the minimal dilation) have a common refinement i.e., for every $a,x$ one has

$$\mathcal{I}_{a|x} = \sum_\lambda p(a|x,\lambda)\mathcal{I}_\lambda. \quad (7)$$

To relate this connection to spatial and temporal steering, note that any state ensemble $\{p(\lambda),\rho_\lambda\}_\lambda$, where $\sum_\lambda p(\lambda) = 1$, is an output of a state preparator, i.e., an instrument with a trivial input space $\mathbb{C}$. Even though using a one-dimensional Hilbert space might sound unconventional, it appears to be useful for our purposes, as any nonsignaling state assemblage $\{\rho_{a|x}\}_{a,x}$ corresponds to a nonsignaling set $\{\mathcal{I}_{a|x}\}_{a,x}$ of state preparators through the minimal Stinespring dilation as [see Eq. (4)]

$$\rho_{a|x} = \mathcal{I}_{a|x}(|1\rangle\langle 1|) = \mathrm{tr}_\mathcal{A}[(\tilde{A}_{a|x}\otimes \mathbb{1})|\psi\rangle\langle\psi|], \quad (8)$$

where $|1\rangle$ is a complex number with norm 1 and $|\psi\rangle := V|1\rangle$ is a unit vector on the compound system. As the dummy POVMs $\{\tilde{A}_{a|x}\}_{a,x}$ are unique for a given minimal dilation, and as the state preparator corresponding to a LHS model has the same total channel as state preparators associated to the assemblage, we arrive at our firs Observation [see also Eq. (7)].

*Observation 1.* Any nonsignaling state assemblage $\{\rho_{a|x}\}_{a,x}$ is unsteerable if and only if the associated observables $\{\tilde{A}_{a|x}\}_{a,x}$ on the minimal dilation of the corresponding state preparator are jointly measurable.

In order to make Observation 1 more concrete, consider a state assemblage given by a set of state preparators $\{\mathcal{I}_{a|x}\}_{a,x}$ through Eq. (8). The state of the total system $V|1\rangle\langle 1|V^\dagger$ is clearly a purificatio of $\rho_B := \sum_a \rho_{a|x}$. One possible choice of this purificatio is the canonical one $|\psi\rangle = (\mathbb{I}\otimes\rho_B^{1/2})|\psi^+\rangle$, where $|\psi^+\rangle = \sum_i |ii\rangle$ is a non-normalized singlet state written in the eigenbasis of $\rho_B$. For this choice Eq. (8) reads

$$\rho_{a|x} = \mathrm{tr}_\mathcal{A}[(\tilde{A}_{a|x}\otimes\mathbb{1})|\psi\rangle\langle\psi|] = \rho_B^{1/2}\tilde{A}_{a|x}^T\rho_B^{1/2}, \quad (9)$$

where the transpose is taken in the eigenbasis of $\rho_B$. Hence, the dummy POVMs whose joint measurability solves spatial and temporal steerability are given as $\tilde{A}_{a|x} = \rho_B^{-1/2}\rho_{a|x}^T\rho_B^{-1/2}$.

Noting that joint measurability is invariant under transposition, we can reproduce the known result [13] for spatial steering stating that a state assemblage $\{\rho_{a|x}\}_{a,x}$ is unsteerable if and only if the so-called Bob's steering equivalent observables define as $B_{a|x} := \rho_B^{-1/2}\rho_{a|x}\rho_B^{-1/2}$ are jointly measurable.

It is worth mentioning that Observation 1 can also be used to reproduce a known example of the connection between temporal steering and joint measurability for scenarios using Lüders instruments and a maximally mixed input state [22]. The result of the article states that a set of observables is nonjointly measurable if and only if it can be used for temporal steering. Whereas this claim works perfectly for the maximally mixed input state, it is worth noting that, for example, a typical joint measurement scenario with orthogonal noisy qubit observables $A^\eta_{\pm1|x} := \frac{1}{2}(\mathbb{1}\pm\eta\vec{x}\cdot\vec{\sigma})$, where $0<\eta\leqslant 1$ is the noise parameter, leads to signaling assemblages with any other input state than the maximally mixed one. Hence, even jointly measurable observables, i.e., $\eta\leqslant\frac{1}{\sqrt{3}}$ [23], can lead to temporal steering in the state-dependent framework, providing a counterexample for the general claim in [22].

For scenarios including the maximally mixed input state and Lüders instruments, one sees that our approach gives the transposed versions of Alice's measurements as dummy POVMs. Hence, one sees that the claims made in [22] for the specifi input state and instruments can be reproduced using our method.

## VI. CHANNEL STEERING

In channel steering [17] one is interested in an assemblage of instruments $\{\mathcal{I}_{a|x}\}_{a,x}$ instead of states. This assemblage originates from a process where Charlie sends quantum states to Bob through a quantum channel $\Lambda^{C\to B}$ which possibly entangles some of the states to an environment (Alice) (see Fig. 2). The task is to decide if the entanglement between Alice and Bob is strong enough to allow Alice to steer Bob's outputs of the channel. Mathematically this means that one takes a channel extension $\Lambda^{C\to A\otimes B}$ of the channel $\Lambda^{C\to B}$ and define an instrument assemblage through

$$\mathcal{I}_{a|x}(\rho) = \mathrm{tr}_A[(A_{a|x}\otimes\mathbb{1})\Lambda^{C\to A\otimes B}(\rho)]. \quad (10)$$

Note that here the assemblage is nonsignaling by definition The unsteerability of this instrument assemblage is define as the existence of a common instrument $\mathcal{I}_\lambda$ and postprocessings $p(a|x,\lambda)$ such that

$$\mathcal{I}_{a|x} = \sum_\lambda p(a|x,\lambda)\mathcal{I}_\lambda. \quad (11)$$

Noticing that Eq. (11) and Eq. (7) are identical and using a minimal dummy system instead of a generic extension in Eq. (10) we arrive to the following Observation:

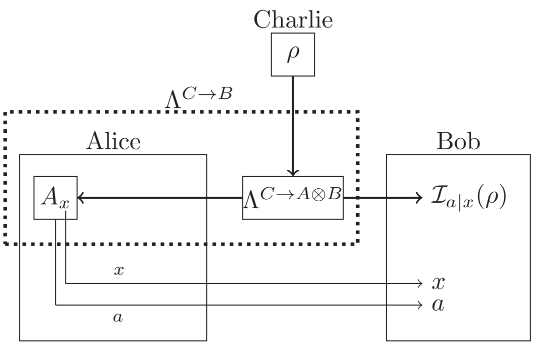


FIG. 2. Channel steering: The setup is similar to the spatial steering scenario, but in the channel case the shared state is prepared by Charlie via the broadcast channel $\Lambda^{C\to A\otimes B}$. The operations enclosed in the dotted line are then viewed by Bob as instruments which have the total channel $\Lambda^{C\to B}$. The main difference to spatial steering is that here Bob's task is to build a local (instrument) model for all possible input states, see Eq. (11).

*Observation 2.* An instrument assemblage $\{\mathcal{I}_{a|x}\}_{a,x}$ define through a minimal dilation $\Lambda^{C\to A\otimes B}$ is unsteerable if and only if the associated dummy POVMs $\{\tilde{A}_{a|x}\}_{a,x}$ are jointly measurable.

There exists a former result [24] reporting a one-to-one connection between joint measurability of measurements $\{A_{a|x}\}_{a,x}$ on any dilation (or extension) of the total channel and the nonsteerability of the instrument assemblage they define This result, however, can be proven false. Namely, while it is true that compatible measurements will not lead to channel steering no matter which dilation (or extension) is used, the other direction is not true in general. Take, for example, any instrument assemblage $\{\mathcal{I}_{a|x}\}_{a,x}$ define through linearly dependent Kraus operators $K_1 = \frac{1}{\sqrt{2}}U$, $K_2 = \frac{1}{\sqrt{2}}U$ of some unitary channel $\Lambda_U(\rho) = U\rho U^\dagger$. The instrument assemblage is given by

$$\mathcal{I}_{a|x}(\rho) = \frac{1}{2}\sum_{k,l=1}^{2}\langle\varphi_l|\tilde{A}_{a|x}|\varphi_k\rangle U\rho U^\dagger. \quad (12)$$

Hence definin $p(a|x,\lambda) = \frac{1}{2}\sum_{k,l}\langle\varphi_l|\tilde{A}_{a|x}|\varphi_k\rangle$ (which is a probability distribution as $\{\tilde{A}_{a|x}\}_{a,x}$ is a POVM), $\Lambda_\lambda = \Lambda_U$, and the hidden variable space to be trivial, one sees that the setup is unsteerable for compatible as well as for incompatible sets $\{\tilde{A}_{a|x}\}_{a,x}$ of POVMs.

To relate our result to the above example, note that as the *minimal* dilation of the channel $\Lambda_U$ is one dimensional, observables in this space are always jointly measurable and hence the instrument assemblage is nonsteerable.

## VII. STEERING INEQUALITIES FROM INCOMPATIBILITY

As various joint measurement uncertainty relations have been analytically characterized [23,25–31], our Observation 1 and Observation 2 open up a possibility to use them as steering inequalities for all three scenarios. As an example, take the simplest case of two two-valued qubit observables $A_{\pm|x} = \frac{1}{2}(\mathbb{1} \pm \vec{a}_x\cdot\vec{\sigma})$, $x = 1,2$. These observables are jointly measurable [23] if and only if $\|\vec{a}_1+\vec{a}_2\| + \|\vec{a}_1-\vec{a}_2\| \leqslant 2$. This inequality is universally applicable to all three steering scenarios and gives an "if and only if" condition for each of them. Inserting the measurements $\{A_{\pm|x}\}_{x=1,2}$ as the dummy POVMs to, for example, Eq. (5) gives instruments for which channel steering can directly be decided. We are ready to state our next Observation.

*Observation 3.* Joint measurement uncertainty relations can be used as steering inequalities for spatial, temporal, and channel steering.

## VIII. EQUIVALENCE BETWEEN TEMPORAL AND SPATIAL STEERING

Applying the Stinespring dilation to a set of nonsignaling instruments $\{\mathcal{I}_{a|x}\}_{a,x}$ shows that the temporal steering scenario they defin can be mapped into the spatial steering scenario, see Eq. (4). The question remains which spatial scenarios can be reached by these instruments as the mapping is in general not injective.

To answer this, take a nonsignaling state assemblage $\{\rho_{a|x}\}_{a,x}$ with a $d$-dimensional support. Notice that this state assemblage can be prepared through spatial steering using a purificatio of the total state $\rho_B := \sum_a \rho_{a|x}$ [32–34] [see also Eq. (9)]. Hence, we need an isometry $V$ which has such purificatio in its range. One possible choice is the set of Kraus operators $K_k = |k\rangle\langle k|$, where $\{|k\rangle\}_{k=1}^d$ is the eigenbasis of $\rho_B$. Taking the input state $|\psi\rangle := \sum_{i=1}^d \sqrt{\lambda_i}|i\rangle$, where the numbers $\lambda_i > 0$ are the eigenvalues of the state $\rho_B$, and the observables $\tilde{A}_{a|x} := \rho_B^{-1/2}\rho_{a|x}^T\rho_B^{-1/2}$, where the transpose is taken in the eigenbasis of $\rho_B$, we get through the minimal dilation of the channel $\Lambda(\rho) := \sum_k K_k\rho K_k^\dagger$ the desired state assemblage

$$\mathcal{I}_{a|x}(|\psi\rangle\langle\psi|) = \sum_{k,l=1}^{d}\langle l|\tilde{A}_{a|x}|k\rangle K_k|\psi\rangle\langle\psi|K_l^\dagger = \rho_{a|x}. \quad (13)$$

With this, we are ready to state the next Observation:

*Observation 4.* Temporal and spatial steering are fully equivalent problems in that temporal steering can be embedded into the spatial scenario and the two can produce exactly the same assemblages. Moreover, any nonsignaling state assemblage on a $d$-level system can be reproduced with nonsignaling instruments acting on a $d$-level system.

The above Observation has two crucial consequences. First, for nontrivial instances of temporal steering the restriction to nonsignaling instruments is actually not a restriction at all. Second, Observation 4 allows one to prove a hierarchy between temporal steering and macrorealistic hidden variable models (see below).

## IX. TEMPORAL STEERING AND MACROREALISM

We now proceed to show that steering has an analogous role in the temporal scenario to that of the spatial case. Namely, whereas spatially nonsteerable correlations are a proper subset of local correlations, we show that temporally nonsteerable correlations are a proper subset of macrorealistic correlations.

To do so, recall that the probabilities in a sequential measurement scenario (consisting here of two different time steps) are said to have a macrorealistic hidden variable model if they can be written in the form $\text{tr}[\mathcal{I}_{a|x}(\rho)B_{b|y}] = \sum_\lambda p(\lambda)p(a|x,\lambda)p(b|y,\lambda)$, where $p(\cdot), p(\cdot|x,\lambda)$ and $p(\cdot|y,\lambda)$ are probability distributions for all $x, y$ and $\lambda$ [35]. Provided that one uses nonsignaling instruments, the left-hand side of the above equation can be written in the distributed scenario simply as $\text{tr}[(\tilde{A}_{a|x} \otimes B_{b|y})V\rho V^\dagger]$. As the nonsignaling condition is automatically satisfie for a given total channel, our question boils down to findin an isometry $V$ and a state $\rho$ such that the state $V\rho V^\dagger$ is steerable but local. As an example, consider the Kraus operators $K_0 = |0\rangle\langle 0| + |1\rangle\langle 1|$ and $K_1 = |0\rangle\langle 2| + |1\rangle\langle 3|$. Now the state $\rho := \lambda|\psi\rangle\langle\psi| + (1-\lambda)\frac{1}{4}\mathbb{1}_4$, where $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |3\rangle)$, maps to the isotropic state $V\rho V^\dagger = \lambda|\psi^+\rangle\langle\psi^+| + (1-\lambda)\frac{1}{4}\mathbb{1}_4$. Isotropic states are steerable but local for projective measurements with $\frac{1}{2} < \lambda \leqslant \frac{1}{K_G(3)}$, where $K_G(3)$ is a Grothendieck constant and $\frac{1}{K_G(3)} \geqslant 0.6829$ [8,36,37].

However, considering only projective measurements does not cover all possible instruments compatible with the total channel. In order to cover this more general scenario, we recall that all possible instrument assemblages $\{\mathcal{I}_{a|x}\}_{a,x}$ compatible with a channel $\Lambda$ are given by the minimal Stinespring dilation:

$$\{\mathcal{I}_{a|x}(\cdot)\}_{a,x} = \{\text{tr}_\mathcal{A}[(\tilde{A}_{a|x} \otimes \mathbb{1})V(\cdot)V^\dagger] | \{\tilde{A}_{a|x}\}_a \text{ is a POVM}\}. \tag{14}$$

To provide the desired example, we use a known steerable qutrit-qutrit state which is local for POVMs [38] as our target state $V\rho V^\dagger$. The state reads

$$\tilde{\rho} := \frac{1}{9}[a|\varphi^-\rangle\langle\varphi^-| + (3-a)\frac{1}{2}\mathbb{1} \otimes |2\rangle\langle 2| + 2a|2\rangle\langle 2| \otimes \frac{1}{2}\mathbb{1} + (6-2a)|22\rangle\langle 22|], \tag{15}$$

where $|\varphi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$, and $0 < a \leqslant \frac{3}{2}$. To reach this state we can use a channel acting on $\mathbb{C}^7$ define through the Kraus operators

$$K_0 = |1\rangle\langle 0| + |2\rangle\langle 2|, \tag{16}$$

$$K_1 = -|0\rangle\langle 1| + |2\rangle\langle 3|, \tag{17}$$

$$K_2 = |0\rangle\langle 4| + |1\rangle\langle 5| + |2\rangle\langle 6|. \tag{18}$$

Now the state

$$\rho := \frac{1}{9}[a|\psi\rangle\langle\psi| + (3-a)\frac{1}{2}(|2\rangle\langle 2| + |3\rangle\langle 3|) + a(|4\rangle\langle 4| + |5\rangle\langle 5|) + (6-2a)|6\rangle\langle 6|], \tag{19}$$

where $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, maps to the state $\tilde{\rho}$ on the minimal dilation space, hence completing the example.

As the (nontight) inclusion of temporally nonsteerable correlations to the set of macrorealistic correlations follows from their definitions we can formulate:

*Observation 5.* The set of temporally nonsteerable correlations is a proper subset of macrorealistic correlations.

The above Observation shows that there exists instances of temporal steering where a certain steerable channel-state pair can never lead to nonmacrorealistic behavior, no matter what (nonsignaling) measurements (compatible with the channel) are performed on the f rst party.

## X. CONCLUSIONS

In this work, we have approached spatial, temporal, and channel steering through a modifie version of the well-known Stinespring dilation. We have demonstrated the power of our approach by showing that incompatibility of quantum measurements is one-to-one connected to quantum steering in all three scenarios. In addition, we have shown how measurement uncertainty relations can be used as universal steering inequalities through this connection.

In contrast to the formerly known connections between spatial steering and joint measurability [11–14], the current approach is not limited to incompatibility. Using the Stinespring approach, we have mapped temporal steering into a framework where nonsignaling is a built-in state-independent feature. Moreover, we have shown an equivalence between temporal and spatial steering and shown that temporally unsteerable correlations are a proper subset of nonmacrorealistic correlations. For future works it would be interesting to investigate other possible connections between temporal and spatial correlations, e.g., investigate if our approach can be used to translate such concepts as entanglement in a meaningful way to the temporal scenario, and to see if our approach can be related to the recent works [39,40] comparing spatial and temporal scenarios.

*Note added.* Recently, we became aware of the work of Ref. [41], which independently proved a hierarchy between temporal steering and macrorealism.

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[1] The fact that steering is the essence of the EPR argument, was, to our knowledge, f rst noted by E. Schrödinger in a letter to A. Einstein (13/07/1935): "... All the others told me that there is no incredible magic in the sense that the system in America gives q = 6 if I perform in the European system nothing or a certain action (you see, we put emphasis on spatial separation), while it gives q = 5 if I perform another action; but I only repeated myself: It does not have to be so bad in order to be silly. I can, by maltreating the European system, steer the American system deliberately into a state where either q is sharp, or into a state which is certainly not of this class, for example where p is sharp. This is also magic!" see also, K. v. Meyenn, *Eine Entdeckung von ganz außerordentlicher Tragweite* (Springer, Berlin, 2011), p. 551.

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# Article V

- **Title**: Steering criteria from general entropic uncertainty relations
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- **Author's contribution**: The author of this thesis contributed to miscellaneous calculations concerning the proofs and examples.

# Steering criteria from general entropic uncertainty relations

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The effect of steering describes a possible action at a distance via measurements but characterizing the quantum states that can be used for this task remains difficult. We provide a method to derive sufficient criteria for steering from entropic uncertainty relations using generalized entropies. We demonstrate that the resulting criteria outperform existing criteria in several scenarios; moreover, they allow to detect weakly steerable states.

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*Introduction.—* Steering is a term coined by Schrödinger in 1935 in order to capture the essence of the Einstein-Podolsky-Rosen argument [1]. It describes Alice's ability to affect Bob's quantum state through her choice of a measurement basis, without allowing for instantaneous signaling. In the modern view, steering is based on a quantum correlation between entanglement and the violation of Bell inequalities, meaning that not every entangled state can be used for steering and not every steerable state violates a Bell inequality [2].

In the last years the theory of steering has evolved quickly. It has been shown that the concept of steering is closely related to fundamental problems and open questions in quantum physics. For instance, steering has been used to find counterexamples to so-called Peres conjecture, which was an open problem in entanglement theory for more than fifteen years [3–5]. In addition, steering was shown to be equivalent to the notion of joint measurability of generalized measurements [6–10] and results from one problem can be transfered to the other. Finally, steering has been shown to be useful for tasks in quantum information processing, such as one-sided device-independent quantum key distribution [11] and subchannel discrimination [12].

Despite of all these results, the simple question whether or not a given bipartite quantum state is useful for steering is not easy to answer. If the conditional states of Bob are known, the problem can be solved via semidefinite programming [13–15], but this approach requires knowledge of Alice's measurements and is restricted to small dimensions. Other steering criteria exist [2, 16–20], but general concepts for the derivation of them are missing. This is in contrast to entanglement theory, where concepts such as the theory of positive, but not completely positive maps provide a guiding line for developing separability criteria [21].

In this paper we identify entropic uncertainty relations as a fundamental tool to develop steering criteria. Uncertainty relations in terms of entropies have already become important in many areas of quantum information theory [22, 23]. We show that various entropic uncertainty relations can be transformed into a steering criterion. As examples, we consider generalized entropies such as the so-called Tsallis entropy and demonstrate that the resulting criteria outperform known criteria in many cases. Our approach is motivated by previous works on entanglement criteria from entropic uncertainty relations [24] and generalizes recent entropic criteria for steering [25, 26], which were, however, restricted to the special case of the Shannon entropy.

*Steering and entropies.—* In steering scenarios, one assumes that Alice and Bob share a quantum state $\varrho_{AB}$. Then, Alice makes measurements on her system and claims that with these measurements she can steer the state inside Bob's laboratory. Bob, of course, is not convinced of Alice's abilities. In a more formal manner, we can assume that Alice performs a measurement $A$ with outcome $i$ on her part of the system, while Bob performs a measurement $B$ with outcome $j$ on his part. From that, they can obtain the joint probability distribution of the outcomes. If for all possible measurements $A$ and $B$ one can express the joint probabilities in the form

$$p(i, j|A, B) = \sum_\lambda p(\lambda)p(i|A, \lambda)p_q(j|B, \lambda), \qquad (1)$$

then the system is called unsteerable. Here, $p(i|A, \lambda)$ is a general probability distribution, while $p_q(j|B, \lambda) = \mathrm{Tr}_B[B(j)\sigma_\lambda]$ is a probability distribution originating from a quantum state $\sigma_\lambda$ being the same for all measurements $B$ on Bob's side. Furthermore, $B(j)$ denotes a measurement operator such that $\sum_j B(j) = \mathbb{1}$, and $\sum_\lambda p(\lambda) = 1$, where $\lambda$ is a label for the hidden quantum state $\sigma_\lambda$. A model as in Eq. (1) is called a local hidden state (LHS) model, and if it exists, Bob can explain all the results through a set of local states $\{\sigma_\lambda\}$ which is not altered by Alice's measurements. But if it is not possible to find states $\sigma_\lambda$ that make this probability distribution feasible, Bob has to assume that Alice can steer the state.

Let us now explain some basic facts about entropy. For a general probability distribution $\mathcal{P} = (p_1, \ldots, p_N)$, the Shannon entropy is defined as [27]

$$S(\mathcal{P}) = -\sum_i p_i \ln(p_i). \qquad (2)$$

Entropic uncertainty relations can easily be explained with an example. Consider the Pauli measurements $\sigma_x$

and $\sigma_z$ on a single qubit. For any quantum state these measurements give rise to a probability distribution of the outcomes $\pm 1$ and the corresponding entropy $S(\sigma_k)$. The fact that $\sigma_x$ and $\sigma_z$ do not share a common eigenstate can be expressed by [22]

$$S(\sigma_x) + S(\sigma_z) \geq \ln(2), \tag{3}$$

where the lower bound does not depend on the state.

For our approach, we also need the relative entropy, also known as Kullback-Leibler divergence [27], between two probability distributions $\mathcal{P}$ and $\mathcal{Q}$,

$$D(\mathcal{P}||\mathcal{Q}) = \sum_i p_i \ln\left(\frac{p_i}{q_i}\right). \tag{4}$$

Two properties are essential: First, the relative entropy is additive for independent distributions, that is if $\mathcal{P}_1, \mathcal{P}_2$ are independent distributions, with the joint probability distribution $\mathcal{P}(x,y) = \mathcal{P}_1(x)\mathcal{P}_2(y)$ and the same for $\mathcal{Q}_1, \mathcal{Q}_2$ then one has that

$$D(\mathcal{P}||\mathcal{Q}) = D(\mathcal{P}_1||\mathcal{Q}_1) + D(\mathcal{P}_2||\mathcal{Q}_2). \tag{5}$$

Second, the relative entropy is jointly convex. This means that for two pairs of distributions $\mathcal{P}_1, \mathcal{Q}_1$ and $\mathcal{P}_2, \mathcal{Q}_2$ one has

$$\begin{aligned} &D[\lambda\mathcal{P}_1 + (1-\lambda)\mathcal{P}_2||\lambda\mathcal{Q}_1 + (1-\lambda)\mathcal{Q}_2] \\ &\quad\leq \lambda D(\mathcal{P}_1||\mathcal{Q}_1) + (1-\lambda)D(\mathcal{P}_2||\mathcal{Q}_2). \end{aligned} \tag{6}$$

*The main idea.*— The starting point of our method is the relative entropy between two distributions, namely

$$F(A,B) = -D(A \otimes B||A \otimes \mathbb{I}). \tag{7}$$

Here, $A \otimes B$ denotes the joint probability distribution $p(i,j|A,B)$, which we denote by $p_{ij}$ for convenience, $A$ is the marginal distribution $p(i|A)$, which we denote by $p_i$, and $\mathbb{I}$ is a uniform distribution with $q_j = 1/N$ for all $j$. As the relative entropy is jointly convex, $F(A,B)$ is concave in the probability distribution $A \otimes B$. We can directly calculate that

$$F(A,B) = -\sum_{ij} p_{ij} \ln\left(\frac{p_{ij}}{p_i/N}\right) = S(B|A) - \ln(N), \tag{8}$$

where $S(B|A) = S(A,B) - S(A)$ is the conditional entropy. On the other hand, considering a product distribution $p(i|A,\lambda)p_q(j|B,\lambda)$ with a fixed $\lambda$ and the property from Eq. (5), we have

$$\begin{aligned} F(A,B) &= -D[p(i|A,\lambda)||p(i|A,\lambda)] - D[p_q(j|B,\lambda)||\mathbb{I}] \\ &= S(B|\lambda) - \ln(N). \end{aligned} \tag{9}$$

Consequently, for a product distribution and a set of measurements $A_k \otimes B_k$, we have

$$\sum_k S(B_k|A_k) = \sum_k S(B_k|\lambda). \tag{10}$$

The right-hand side of this equation depends on probability distributions taken from the quantum state $\sigma_\lambda$. Such distributions typically obey an entropic uncertainty relation,

$$\sum_k S(B_k|\lambda) \geq C_B. \tag{11}$$

So, for product distributions we have

$$\sum_k S(B_k|A_k) \geq C_B. \tag{12}$$

Finally, since $F$ is concave, the same bound holds for convex combinations of product distributions $p(i|A,\lambda)p_q(j|B,\lambda)$ from Eq. (1), meaning that any non-steerable quantum system obeys this relation. In this way entropic uncertainty relations can be used to derive steering criteria. The intuition behind these criteria is based on the interpretation of Shannon conditional entropy. In Eq. (12), one can see that the knowledge that Alice has about Bob's outcomes is bounded. If this inequality is violated, then the system is steerable, meaning that Alice can do better predictions than those allowed by an entropic uncertainty relation.

So far, this criterion is the same as the one in Ref. [26], but our proof highlights the three central ingredients:

First, we needed an additivity relation for independent distributions in Eq. (5), second we needed the state independent entropic uncertainty relation in Eq. (11), and finally we needed the joint convexity of the relative entropy in Eq. (6). These properties are not at all specific for the Shannon entropy, so our strategy works also for generalized entropies.

*Steering criteria for generalized entropies.*— As a possible generalized entropy, we consider the so-called Tsallis entropy [28, 29] which depends on a parameter $q > 1$. It is given by

$$S_q(\mathcal{P}) = -\sum_i p_i^q \ln_q(p_i), \tag{13}$$

where the $q$-logarithm is defined as $\ln_q(x) = (x^{1-q} - 1)/(1-q)$. Note that in the limit $q \to 1$ this entropy converges to the Shannon entropy. The generalized relative entropy can be defined as [30]

$$D_q(\mathcal{P}||\mathcal{Q}) = -\sum_i p_i \ln_q\left(\frac{q_i}{p_i}\right), \tag{14}$$

it is jointly convex and obeys the following relation for product distributions:

$$\begin{aligned} D_q(\mathcal{P}||\mathcal{Q}) &= D_q(\mathcal{P}_1||\mathcal{Q}_1) + D_q(\mathcal{P}_2||\mathcal{Q}_2) \\ &\quad + (q-1)D_q(\mathcal{P}_1||\mathcal{Q}_1)D_q(\mathcal{P}_2||\mathcal{Q}_2). \end{aligned}$$

The additional term is due to non-additivity of the generalized entropy.

Now we can apply the machinery derived above and consider the quantity $F(A,B) = -D_q(A \otimes B||A \otimes \mathbb{I})$. It follows by direct calculation that if the measurements $B_k$ obey the entropic uncertainty relation

$$\sum_k S_q(B_k) \geq C_B^{(q)} \quad (15)$$

then one has the steering criterion

$$\sum_k \Big[S_q(B_k|A_k) + (1-q)C(A_k, B_k)\Big] \geq C_B^{(q)}, \quad (16)$$

and violation of it implies steerability of the state. Here $S_q(B|A) = S_q(A,B) - S_q(A)$ is the conditional entropy [31] and the additional term is given by

$$C(A,B) = \sum_i p_i^q [\ln_q(p_i)]^2 - \sum_{i,j} p_{ij}^q \ln_q(p_i)\ln_q(p_{ij}). \quad (17)$$

From Eq. (16) it is easy to see that if we consider $q \to 1$, we arrive at Eq. (12). Note that we can also rewrite Eq. (16) in terms of probabilities as

$$\frac{1}{q-1}\Big[\sum_k \big(1 - \sum_{ij} \frac{(p_{ij}^{(k)})^q}{(p_i^{(k)})^{q-1}}\big)\Big] \geq C_B^{(q)}. \quad (18)$$

Here, $p_{ij}^{(k)}$ is the probability of Alice and Bob for outcome $(i,j)$ when measuring $A_k \otimes B_k$, and $p_i^{(k)}$ are the marginal outcome probabilities of Alice's measurement $A_k$. This form of the criterion is very easy to evaluate.

*Application I: Isotropic states.*— To test the strength of our steering criteria we consider $d$-dimensional isotropic states [32]

$$\varrho_{\text{iso}} = \alpha|\phi_d^+\rangle\langle\phi_d^+| + \frac{1-\alpha}{d^2}\mathbb{1}, \quad (19)$$

where $|\phi^+\rangle = (1/\sqrt{d})\sum_{i=0}^{d-1}|i\rangle|i\rangle$ is a maximally entangled state. These states are known to be entangled for $\alpha > 1/(d+1)$ and separable otherwise. As observables, we consider a set of mutually unbiased bases (MUBs) in dimension $d$. One can directly check that the marginal probabilities for this class of states are $p_i = 1/d$ for all $i$ and the joint probabilities are $p_{ii} = [1+(d-1)\alpha]/d^2$ (occurring $d$ times), and $p_{ij} = (1-\alpha)/d^2$ [for $i \neq j$ and occurring $d(d-1)$ times]. These probabilities are the same for all measurements. Inserting them in Eq. (18), the condition for non-steerability reads

$$\frac{m}{q-1}\{1 - \frac{1}{d^q}[(1+(d-1)\alpha)^q + (d-1)(1-\alpha)^q]\} \geq C_B^{(q)}, \quad (20)$$

which depends on the parameter $q$ and the number of MUBs $m$. For certain values of $q$ and $m$, the bounds of the entropic uncertainty relations $C_B^{(q)}$ are known (see Appendix A). For other cases they can be approximated numerically.

Let us discuss the strength of this criterion. First, numerical investigations suggest that the criterion is strongest for $q = 2$. For this value of $q$ the violation of Eq. (20) occurs for $\alpha > 1/\sqrt{m}$. Considering a complete set of MUBs ($m = d+1$) (this exists for $d$ being a power of a prime) the violation happens for $\alpha > 1/\sqrt{d+1}$.

For qubits ($d = 2$) isotropic states are equivalent to Werner states [33]. Then, with a complete set of MUBs the violation occurs for $\alpha > 1/\sqrt{3} \approx 0.577$, which is known to be the optimal threshold [34]. More generally, in Ref. [35], a steering inequality for MUBs and isotropic states has been presented which is violated for $\alpha > (d^{3/2}-1)/(d^2-1)$. It is straightforward to show that our inequality is stronger. Recently, the same problem has been investigated using semi-definite programming [36]. For $3 \leq d \leq 5$ a better threshold than ours was obtained, but it is worth to mention that our criteria directly use probability distributions from few measurements, without the need of performing full tomography on Bob's conditional state. In addition, numerical approaches are naturally limited to small dimensions.

In Fig. 1, we compare our criterion with the ones mentioned above. We concentrate in the values of $q \to 1$ and $q = 2$, since the former is related to the usual entropic steering criteria and the latter is the optimal value of $q$ for the detection of steerable states.

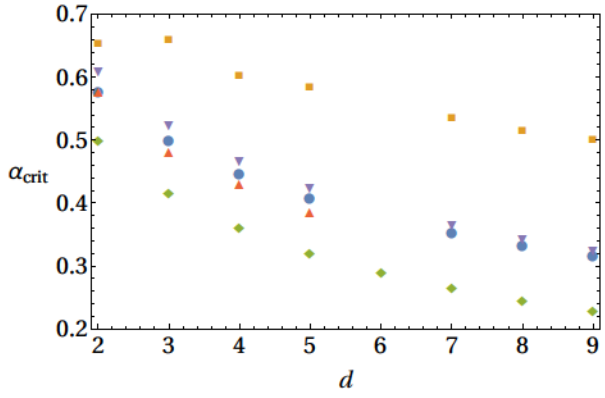


FIG. 1. (Color online) The critical value of white noise $\alpha$ for different dimensions $d$, considering a complete set of MUBs. In this plot, yellow squares correspond to our criterion in Eq. (20) for $q \to 1$ and the blue circles to $q = 2$. The purple reversed triangles correspond to the results for the inequality presented in Ref. [34] and the red triangles in Ref. [36], where $\alpha_{\text{crit}}$ was calculated via SDP. Below the green diamonds the existence of an LHS model for projective measurements is known [2].

*Connection to existing entanglement criteria.*— At this point, it is interesting to compare our approach with entanglement criteria derived from entropic uncertainty relations [24]. The mathematical formulation goes as follows. Let $A_1$ and $A_2$ ($B_1$ and $B_2$) be observables on Alice's (Bob's) laboratory. Assume that Bob's observables obey an entropic uncertainty relation $S(B_1) + S(B_2) \geq C_B$, where $S(B_i)$ is a generalized entropy, such as the Shannon or Tsallis entropy. Then it can be shown that for separable states

$$S(A_1 \otimes B_1) + S(A_2 \otimes B_2) \geq C_B \tag{21}$$

holds. Here, $S(A_k \otimes B_k)$ is the entropy of the probability distribution of the outcomes of the *global* observable $A_k \otimes B_k$. Note that this implies that for a degenerate $A_k \otimes B_k$ the probability distribution differs from the local ones. For instance, measuring $\sigma_z \otimes \sigma_z$ gives four possible local probabilities $p_{++}, p_{+-}, p_{-+}, p_{--}$, but for the evaluation of $S(A_k \otimes B_k)$ one combines them according to $q_+ = p_{++} + p_{--}$ and $q_- = p_{+-} + p_{-+}$, as these correspond to the global outcomes.

Some connections to our derivation of steering inequalities are interesting. First, if one reconsiders the proof in Ref. [24] one realizes that Eq. (21) is indeed a steering criterion and not a criterion for entanglement. That is, all probability distributions of the form in Eq. (1) fulfill it. Second, also in Ref. [24] it was observed that the criterion is strongest for values $2 \leq q \leq 3$. Finally, if one asks for a direct comparison between Eq. (21) and Eqs. (16,12) one finds that Eq. (21) is of the same strength for special scenarios (e.g. Bell-diagonal two-qubit states and Pauli measurements), while it seems weaker in the general case (see below).

*Application II: General two-qubit states.—* Let us now consider the application of our methods to general two-qubit states. Any two-qubit state can, after application of local unitaries, be written as

$$\varrho_{AB} = \frac{1}{4}[\mathbb{1}\otimes\mathbb{1} + (\vec{a}\vec{\sigma})\otimes\mathbb{1} + \mathbb{1}\otimes(\vec{b}\vec{\sigma}) + \sum_{i=1}^{3} c_i\sigma_i\otimes\sigma_i] \tag{22}$$

where $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ are vectors with norm less than one, $\vec{\sigma}$ is a vector composed of the Pauli matrices and $(\vec{a}\vec{\sigma}) = \sum_i a_i\sigma_i$. Let us assume that Alice performs projective measurements with effects $P_k^A = [\mathbb{1} + \mu_k(\vec{u}_k\vec{\sigma})]/2$ and Bob with the effects $P_k^B = [\mathbb{1} + \nu_k(\vec{v}_k\vec{\sigma})]/2$ with $\mu_k, \nu_k = \pm 1$ and $\{\vec{u}, \vec{v}\} \in \mathbb{R}^3$. Then, Eq. (18) can be written as

$$\sum_k \Big[1 - \sum_{\mu_k,\nu_k} \frac{[1 + \mu_k(\vec{a}\vec{u}_k) + \nu_k(\vec{b}\vec{v}_k) + \mu_k\nu_k T_k]^q}{2^{q+1}[1 + \mu_k(\vec{a}\hat{u}_k)]^{q-1}}\Big] \geq (q-1)C_B^{(q)}, \tag{23}$$

where $T_k = \sum_{i=1}^3 c_i u_{ik} v_{ik}$. The optimization over measurements of this criterion for general two-qubit states is involving. We will focus on the simple case of Pauli measurements, meaning that $\vec{u}_k = \vec{v}_k = \{(1,0,0)^T, (0,1,0)^T, (0,0,1)^T\}$ and $q = 2$. Then we have that

$$\sum_{i=1}^{3} \left[\frac{1 - a_i^2 - b_i^2 - c_i^2 + 2a_ib_ic_i}{2(1 - a_i^2)}\right] \geq 1. \tag{24}$$

If this inequality is violated, then the system is steerable.

Now, we can compare our criteria with other proposals for the detection of steerable states using three measurements, see Appendix B for detailed calculations. The criteria from Eq. (21) prove steerability if $\sum_{i=1}^3 c_i^2 > 1$, and from the linear criteria [2, 37] steerability follows if $(\sum_{i=1}^3 c_i^2)^{1/2} > 1$, which is equivalent. Not surprisingly, Eq. (24) is stronger, since it uses more information about the state. This statement can be made hard by analyzing $10^6$ two-qubit states randomly generated from a process based on Hilbert-Schmidt ensemble [38]. 94.34% of the states do not violate any of the criteria, 3.81% are steerable according to all criteria, 1.85% violate only criterion (24), and no state violates only the linear criteria.

A special case of two-qubit states are Bell diagonal states, which can be obtained if we set $\vec{a} = \vec{b} = 0$ in Eq. (22). For this class of states it is easy to see that the three criteria are equivalent. Note, however, that a necessary and sufficient condition for steerability of this class for projective measurements has recently been found [18].

*Application III: One-way steerable states.—* As an example of weakly steerable states that can be detected with our methods we consider one-way steerable states, i.e., states that are steerable from Alice to Bob and not the other way around. We consider the state

$$\varrho_{AB} = \beta|\psi(\theta)\rangle\langle\psi(\theta)| + (1-\beta)\frac{\mathbb{1}}{2}\otimes\varrho_B^\theta, \tag{25}$$

where $|\psi(\theta)\rangle = \cos(\theta)|00\rangle + \sin(\theta)|11\rangle$ and $\varrho_B^\theta = \mathrm{Tr}_A[|\psi(\theta)\rangle\langle\psi(\theta)|]$. It has been shown that for $\theta \in [0, \pi/4]$ and $\cos^2(2\theta) \geq (2\beta - 1)((2-\beta)\beta^3)$ this state is not steerable from Bob to Alice considering an infinite number of projective measurements [19], while Alice can steer Bob for $\beta > 1/2$.

Considering three measurement settings, this state is one way-steerable for $1/\sqrt{3} < \beta \leq \beta_{\max}$ with $\beta_{\max} = [1 + 2\sin^2(2\theta)]^{-1/2}$ [39]. For our entropic steering criteria we consider three Pauli measurements and $q = 2$ and we find that this state is one-way steerable for

$$\frac{1}{2\cos(2\theta)}\sqrt{3 - \sqrt{1 + 8\sin^2(2\theta)}} < \beta \leq \beta_{\max}. \tag{26}$$

For any $\theta$ this gives a non-empty interval of $\beta$ for which our criterion detects these weakly steerable states. An attempt of optimizing over the set of measurements will be addressed in a future work.

*Conclusions.—* In this work we have proposed a straightforward technique for the construction of strong steering criteria from entropic uncertainty relations. These criteria are easy to implement using a finite set of measurement settings only, and do not need the use of semi-definite programming and full tomography on Bob's conditional states.

For future work, several directions seem promising. First, besides the usual entropic uncertainty relations, such as entropic uncertainty relations in the presence of quantum memory [40] or relative entropy formulations of the uncertainty principle [41] are promising starting points for other criteria. Second, one can try to make quantitative statements on steerability from steering criteria. Recently, some attempts in this direction have been pursued [42]. Finally, it would be highly desirable to embed our approach in a general theory of multiparticle steering.

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## APPENDIX

### A: Known entropic uncertainty relations

In this Appendix we will present different entropic uncertainty relations that were used in this work and known from literature. For the Shannon entropy ($q \to 1$) and a complete set of MUBs, entropic uncertainty relations were analytically derived in Ref. [43] and are given by

$$C_B = \begin{cases} (d+1)\log\left(\frac{d+1}{2}\right), & d\,\text{odd} \\ \frac{d}{2}\log\left(\frac{d}{2}\right) + \left(\frac{d}{2}+1\right)\log\left(\frac{d}{2}+1\right), & d\,\text{even.} \end{cases} \qquad (27)$$

For the Tsallis entropy and $m$ MUBs it has been shown in Ref. [44] that, for $q \in (0;2]$, the bounds are given by

$$C_B^{(q)} = m \ln_q \left(\frac{md}{d+m-1}\right). \qquad (28)$$

If we consider the case $q \to 1$, this bound is not optimal for even dimensions, so in this case it is more appropriate to consider the bounds given in Eq. (27).

### B: Calculations for two-qubit states

First, consider the steering criterion in Eq. (21), developed in Ref. [24]. For three Pauli measurements and the Tsallis entropy, we have the following relation

$$\sum_{k=1}^{3} S_q(A_k \otimes B_k) \geq C_B^{(q)}, \qquad (29)$$

where $A_k = (\vec{u}_k\vec{\sigma})$ and $B_k = (\vec{v}_k\vec{\sigma})$. In terms of probabilities this criterion can be rewritten as

$$\frac{1}{q-1}\sum_{k=1}^{3}\Big\{1 - \Big[p_{\vec{u}_k,\vec{v}_k}(+1,+1) + p_{\vec{u}_k,\vec{v}_k}(-1,-1)\Big]^q - \Big[p_{\vec{u}_k,\vec{v}_k}(+1,-1) + p_{\vec{u}_k,\vec{v}_k}(-1,+1)\Big]^q\Big\} \geq C_B^{(q)}. \qquad (30)$$

Inserting the probabilities for general two-qubit systems, we have that

$$\frac{1}{q-1}\sum_{k=1}^{3}\Big\{1 - 2^{-q}\Big[(1+T_k)^q + (1-T_k)^q\Big]\Big\} \geq C_B^{(q)}. \qquad (31)$$

If we fix the measurements and the value of $q$ in the same way as in Eq. (24), this criterion gives $\sum_{i=1}^{3} c_i^2 \leq 1$. Then, if this inequality is violated, the system is steerable.


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# Article VI

- **Title**: Steering maps and their application to dimension-bounded steering
- **Authors**: Tobias Moroder, Oleg Gittsovich, Marcus Huber, Roope Uola, Otfried Gühne
- **Journal reference**: Phys. Rev. Lett. **116**, 090403 (2016)
- **Abstract**: The existence of quantum correlations that allow one party to steer the quantum state of another party is a counterintuitive quantum effect that was described at the beginning of the past century. Steering occurs if entanglement can be proven even though the description of the measurements on one party is not known, while the other side is characterized. We introduce the concept of steering maps, which allow us to unlock sophisticated techniques that were developed in regular entanglement detection and to use them for certifying steerability. As an application, we show that this allows us to go beyond even the canonical steering scenario; it enables a generalized dimension-bounded steering where one only assumes the Hilbert space dimension on the characterized side, with no description of the measurements. Surprisingly, this does not weaken the detection strength of very symmetric scenarios that have recently been carried out in experiments.
- **Author's contribution**: The author of this thesis contributed to the search of examples through the connection between steering and joint measurements.

# Steering Maps and Their Application to Dimension-Bounded Steering

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The existence of quantum correlations that allow one party to steer the quantum state of another party is a counterintuitive quantum effect that was described at the beginning of the past century. Steering occurs if entanglement can be proven even though the description of the measurements on one party is not known, while the other side is characterized. We introduce the concept of steering maps, which allow us to unlock sophisticated techniques that were developed in regular entanglement detection and to use them for certifying steerability. As an application, we show that this allows us to go beyond even the canonical steering scenario; it enables a generalized dimension-bounded steering where one only assumes the Hilbert space dimension on the characterized side, with no description of the measurements. Surprisingly, this does not weaken the detection strength of very symmetric scenarios that have recently been carried out in experiments.

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*Introduction.*—While the term *steering* was coined in the early days of quantum mechanics [1], its precise treatment only started alongside modern developments in quantum information theory [2,3]. The possibility of steering the ensemble in a two-party shared state in quantum mechanics requires that the two subsystems are entangled. To show steering, however, entanglement is not sufficient, since there are some entangled states that are nonsteerable. In fact, steering can be seen as a kind of entanglement verification where one relaxes all assumptions about the devices used by one of the parties, thus sacrificing the ability to detect all entangled states.

This fundamental fact is also what has motivated some recent interest in certifying the steerability of quantum states: Any successful steering test constitutes an entanglement test that is completely device independent for one of the parties, and can thus be exploited to design more secure quantum protocols in situations where one of the parties may be untrusted. Apart from this, it has been observed recently that steering is fundamentally asymmetric [4] and that it is closely connected to joint measurability [5,6]. Furthermore, steering is known to give an advantage for tasks like subset channel discrimination [7]. Naturally, this also spurred interest in devising strong steering criteria [2,8–12], i.e., to investigate their violation [13] or to develop and to use it quantitatively [14–16]. It has been shown that bound entangled quantum states also exhibit steering [17]. Steering has been successfully shown experimentally in several recent experiments [18–20], all of which demonstrate that steering, taking into account various loopholes, is already reachable with today's technology.

In this Letter, we operationally connect steering with regular entanglement verification: We develop a framework that maps the steering certification problem to a regular entanglement detection problem. More explicitly, we construct a matrix from the measurement data that exhibits entanglement if the state is steerable. These steering maps, as we call them, allow us to harness the sophisticated techniques developed in entanglement theory and to go beyond the current state of the art in steering. Contrary to intuition, this does not complicate the construction of steering criteria at all. In fact, at no additional expense, we can use the resulting entanglement tests to derive nonlinear or other improved steering tests that are not straightforward to derive with the standard semidefinite programming (SDP) approach. As an example of the vast possibilities of this framework, we introduce a new concept, which we call dimension-bounded steering, and show that it is accessible with our techniques. In this scenario, one also removes all assumptions of the usually trusted side, with the exception that all measurements operate in the same Hilbert space of dimension $d$. In that, this dimension-bounded steering lies between nonlocality and regular steering. Nonetheless, we also show that the robustness to experimental noise of dimension-bounded steering can be comparable or even equal to regular steering certification. This implies that recent loophole-free steering experiments could have also shown loophole-free dimension-bounded steering.

The Letter is organized as follows. First, we define steering and set the notation. We continue by demonstrating our approach in a dichotomic setting and then discuss our main technique, the steering maps. With this, we show that deciding steerability of an ensemble is equivalent to a separability problem. We then discuss how our approach can be used to derive criteria for the dimension-bounded case. We end with an explicit example of this criterion for recent experiments and a discussion of its strength.

*Steering.*—In the steering scenario, two parties (Alice and Bob) share a quantum state $\rho$. Alice can choose between $n$ different measurements, each having $m$ possible results. Her choice is denoted by $x = 1, \ldots, n$ for the setting while the results are labeled by $a = 1, \ldots, m$. For Bob, we assume that he performs full tomography on his reduced state depending on Alice's measurement and result; thus, he is able to reconstruct the conditional states $\rho_{a|x}$, and the data of this experiment is summarized by the ensemble $\mathcal{E} = \{\rho_{a|x}\}_{a,x}$ of unnormalized density operators, where Alice's probability is $P(a|x) = \mathrm{tr}(\rho_{a|x})$.

Originally, the question of steering asks whether Alice can convince Bob that she can steer the state at Bob's side via her measurements. This means that Bob cannot explain the reduced states $\rho_{a|x}$ as coming from some probability distribution $p(\lambda)$ of states $\rho_\lambda$, where Alice's measurements just give additional information about the probability. As shown in Ref. [16], this can be reformulated as follows: An ensemble $\mathcal{E}$ is nonsteerable if and only if there exist unnormalized density operators $\omega_{i_1,\ldots,i_n}$ with $i_k = 1, \ldots, m$ for each $k = 1, \ldots, n$ such that

$$\rho_{a|x} = \sum_{i_1,\ldots,i_n} \delta_{i_x,a}\omega_{i_1,\ldots,i_n}, \tag{1}$$

and steerable otherwise. This is the definition from which we start our considerations.

*A dichotomic warm-up.*—Let us first discuss the idea via the simplest scenario where Alice has two dichotomic measurements, i.e., $n = m = 2$, in which case we use labels $a = \pm$ to provide easier distinguishable formulas. In this scenario, the ensemble $\mathcal{E} = \{\rho_{+|1}, \rho_{-|1}, \rho_{+|2}, \rho_{-|2}\}$ is called nonsteerable if and only if there exists positive semidefinite operators $\omega_{ij}$ with $i, j = \pm$ such that

$$\rho_{+|1} = \omega_{++} + \omega_{+-}, \qquad \rho_{+|2} = \omega_{++} + \omega_{-+},$$
$$\rho_{-|1} = \omega_{-+} + \omega_{--}, \qquad \rho_{-|2} = \omega_{+-} + \omega_{--} \tag{2}$$

holds. Note that these linear equations are not linearly independent; therefore, $\mathcal{E}$ does not completely determine the unknowns $\omega_{ij}$. Choosing for instance an arbitrary $\omega_{++}$, the choices

$$\omega_{++}, \qquad \omega_{+-} = \rho_{+|1} - \omega_{++},$$
$$\omega_{-+} = \rho_{+|2} - \omega_{++}, \qquad \omega_{--} = \rho_\Delta + \omega_{++}, \tag{3}$$

with $\rho_\Delta = \rho - \rho_{+|1} - \rho_{+|2}$, satisfy the linear constraints, where $\rho$ denotes the reduced density matrix of Bob.

Recall that steering constitutes one-sided device-independent entanglement verification, because a nonsteerable ensemble can always be reproduced by measurements on a separable state $\sigma_{AB}$. This works by using

$$\sigma_{AB} = \sum_{ij} |i, j\rangle_A\langle i, j| \otimes \omega_{ij}, \tag{4}$$

where $|\pm, \pm\rangle_A$ label computational basis states and measurements $M_{\pm|1} = |\pm\rangle\langle\pm| \otimes \mathbb{1}$, $M_{\pm|2} = \mathbb{1} \otimes |\pm\rangle\langle\pm|$.

One might guess there is not much difference whether we explicitly search for appropriate $\omega_{ij}$ satisfying Eq. (2) or for the separable state $\sigma_{AB}$ in Eq. (4). However, looking for a separable state is a task with which we are currently well familiar, due to extensive research on separability criteria during the past two decades [21,22]. There are two things to take into account, though. First, obviously, the state $\sigma_{AB}$ is not completely known to us. Also, $\sigma_{AB}$ is not just a separable state, because Alice's states are very special; they are called classical-quantum states [23] or are described to have a zero "quantum discord" [24,25]. Thus, if one naïvely applies a separability criterion, one loses this required extra structure and the criterion will not be very strong. In the following we show how to circumvent these drawbacks.

*Steering maps.*—In the following, we reformulate the original SDP in an equivalent manner by using the duality of semidefinite programs [26]. This will later allow us to treat dimension-bounded steering. First, to remove the discord zero structure, we replace the basis states $|i, j\rangle\langle i, j|$ by other positive semidefinite operators $Z_{ij}$ of our choice, so that we get a generic separable structure

$$\Sigma_{AB} = \sum_{ij} Z_{ij} \otimes \omega_{ij}. \tag{5}$$

To get a unit trace for $\Sigma_{AB}$ and to remove the problem that not all $\omega_{ij}$ are known, one enforces certain linear relations on $Z_{ij}$. Using for instance the solution of Eq. (3), in Eq. (5) one obtains

$$\Sigma_{AB} = Z_{+-} \otimes \rho_{+|1} + Z_{-+} \otimes \rho_{+|2} + Z_{--} \otimes \rho_\Delta$$
$$+ (Z_{++} - Z_{+-} - Z_{-+} + Z_{--}) \otimes \omega_{++},$$

from which one sees that $\Sigma_{AB}$ is completely determined if the last term vanishes, i.e., $Z_{++} = Z_{+-} + Z_{-+} - Z_{--}$. With this identity, the normalization of $\mathrm{tr}(\Sigma_{AB}) = 1$ is then equal to $\mathrm{tr}(Z_{+-})\mathrm{tr}(\rho_{+|1}) + \mathrm{tr}(Z_{-+})\mathrm{tr}(\rho_{+|2}) + \mathrm{tr}(Z_{--})\mathrm{tr}(\rho_\Delta) = 1$. This is exactly what we were looking for, and we get the following sufficient criterion for steerability: For any nonsteerable ensemble $\mathcal{E}$ and any choice of positive semidefinite operators $Z_{ij}$, which satisfy the two just-mentioned extra relations, the operator

$$\Sigma_{AB} = Z_{+-} \otimes \rho_{+|1} + Z_{-+} \otimes \rho_{+|2} + Z_{--} \otimes \rho_\Delta \tag{6}$$

is a separable quantum state.

If for a given set of $Z_{ij}$ the state $\Sigma_{AB}$ is not separable, i.e., it is entangled or no quantum state at all, then operators $\omega_{ij}$ with the properties from Eqs. (2), (3) do not exist and the underlying ensemble is steerable. In order to check this, we can employ any separability criterion, e.g., partial transposition [27], positive maps [28], entanglement witness [28,29], computable cross norm or realignment [30,31], or covariance matrices [32], to name only a few. The whole power of this is unlocked by the mapping $|i,j\rangle\langle i,j| \mapsto Z_{ij}$, which from now on we refer to as the *steering map*.

In the most general steering case, we know that a nonsteerable ensemble can always be obtained by measuring the separable state $\sigma_{AB}=\sum_{i_1,\ldots,i_n}|i_1,\ldots,i_n\rangle_A\langle i_1,\ldots,i_n|\otimes\omega_{i_1,\ldots,i_n}$, with appropriate measurements that only act nontrivially on the respective subsystem for Alice. Each computational basis state is now mapped to a new positive semidefinite operator $Z_{i_1,\ldots,i_n}$ to obtain

$$\Sigma_{AB} = \sum Z_{i_1,\ldots,i_n} \otimes \omega_{i_1,\ldots,i_n}. \tag{7}$$

This operator is uniquely determined by the given ensemble $\mathcal{E}$ if and only if the chosen operators $Z_{i_1,\ldots,i_n}$ satisfy

$$Z_{i_1 i_2,\ldots,i_n} = Z_{i_1 j_2,\ldots,j_n} + Z_{j_1 i_2 j_3,\ldots,j_n} + \cdots + Z_{j_1 j_2,\ldots,i_n} - (n-1) Z_{j_1 j_2,\ldots,j_n} \tag{8}$$

for all possible choices of $i_1, \ldots, i_n$ and $j_1, \ldots j_n$. With this we are ready to state our first main result, which says that the developed criterion via steering maps is also sufficient. The proof is given in the Supplemental Material [33].

*Proposition 1*.—For any nonsteerable ensemble $\mathcal{E}$ and any set of positive semidefinite operators $\mathcal{Z} = \{Z_{i_1,\ldots,i_n}\}_{i_1,\ldots,i_n}$ fulfilling (8) the operator given by Eq. (7) has a separable structure.For any steerable ensemble $\mathcal{E}$ there exists a set of operators $\mathcal{Z}$ which uniquely determines $\Sigma_{AB}$ and satisfies $\mathrm{tr}(\Sigma_{AB}) = 1$, but where nonseparability of $\Sigma_{AB}$ is detected by the swap entanglement witness. Here, the swap entanglement witness is the flip operator $V = \sum_{ij}|ij\rangle\langle ji|$ where $\mathrm{tr}(\rho V) < 0$ signals entanglement.

Let us remark that the steering map criterion is strictly stronger than a single steering inequality, which is similarly characterized by $\mathcal{Z}$, but where one only checks the swap entanglement witness. Moreover, the proposition also applies to steering scenarios where Bob measures a few observables rather than a tomographic complete set; in this case, nonseparability of $\Sigma_{AB}$ must be verified via this partial information only. Note that since steering is closely related to joint measurability, Proposition 1 can directly be employed for this task also; we are using a result from this field [34] to deduce a collection of $\mathcal{Z}$ for the case $n = 2, m = d$, cf. Supplemental Material [33].

*Dimension-bounded steering*.—Next let us turn to the dimension-bounded steering case. Contrary to the standard steering setup, where it is essential that the measured observables on the characterized side are fully known, these criteria require only that Bob's measurements act on a fixed finite-dimensional Hilbert space.

To be precise, we assume that Bob can choose between $n_B$ different settings $y$, each yielding one of $m_B$ possible outcomes $b$. Each measurement is described by a positive operator valued measure (POVM), i.e., a set of operators $\{M_{b|y}\}_b$ which satisfies positivity $M_{b|y} \geq 0$ and normalization $\sum_b M_{b|y} = \mathbb{1}$. As the sole restriction, we have to assume that they all act on the same Hilbert space with at most dimension $d_B$. Thus, if Bob observes different distributions, $P(b|y,i)$, possibly conditioned onto a separate event $i$ like a measurement result by Alice, then there must exist a collection of different density operators $\{\rho_i\}_i$ and a single set of appropriate POVMs, both on a $d_B$-dimensional Hilbert space, which reproduce the data, $P(b|y,i) = \mathrm{tr}(M_{b|y}\rho_i)$ [35]. To complete the description of the problem, we assume that $n_A$, $m_A$ are the subsystem-labeled specifications for Alice, who is the fully uncharacterized side, and we refer to it as a $d_B$-dimension-bounded steering scenario with parameters $n_A, m_A, n_B, m_B$.

In order to derive steering criteria for this scenario, we employ a fixed steering map to transform the problem into a standard separability question according to Proposition 1. Afterwards, we use the entanglement detection techniques of Ref. [37], which require only a dimension constraint.

The criteria that we derive work best if Bob has dichotomic measurements $n_B = 2$. Before we give the general framework we would like to explain the ideas behind it. As shown in the previous section, we know that any steerable ensemble $\mathcal{E}$ can be detected by an appropriate collection $\mathcal{Z}$ such that $\Sigma_{AB} = \sum_{i_1,\ldots,i_n} Z_{i_1,\ldots,i_n} \otimes \omega^{\mathrm{spec}}_{i_1,\ldots,i_n}$ is not a separable state. Here, $\omega^{\mathrm{spec}}_{i_1,\ldots,i_n}$ should express that the $\omega_{i_1,\ldots,i_n}$, when using a $\mathcal{Z}$ satisfying Eq. (8), is given by a special solution of the linear relations given by Eq. (1), e.g., as in Eq. (6). To show that $\Sigma_{AB}$ is not separable, we can employ the computable cross-norm or realignment criterion [30,31]. This criterion states that the correlation matrix $[C(\rho_{AB})]_{kl} = \mathrm{tr}(G^A_k \otimes G^B_l \rho_{AB})$ of any separable state $\rho^{\mathrm{sep}}_{AB}$ satisfies $\|C(\rho^{\mathrm{sep}}_{AB})\|_1 \leq 1$. The norm that appears here is the trace norm $\|C\|_1 = \sum_i s_i(C)$ given by the sum of the singular values $s_i(C)$, while the sets $\{G_i\}_i$ are orthonormal Hermitian operators (not necessarily forming a basis) for the respective local side. Thus, whenever $\|C(\Sigma_{AB})\|_1 > 1$, the data $\mathcal{E}$ shows steering. Note that since $\|\cdot\|_1$ is unitarily equivalent, only the corresponding spanned local operator spaces matter.

However, one cannot directly evaluate this for the dimension-bounded scenario, because Bob can neither reconstruct $\rho_{a|x}$ nor compute values $\mathrm{tr}(G^B_k\rho_{a|x})$ because he lacks the precise description of his measurements $M_{b|y}$. Still, we can build a matrix that looks similar to the correlation matrix and for which the dichotomic choice of Bob's measurements becomes important. For each dichotomic measurement, consider the operators given by the difference of the two POVM elements $B_y = M_{+|y} - M_{-|y}$ for $y = 1, \ldots, n_B$ and $B_0 = \mathbb{1}$. Then, define the matrix $[D(\Sigma_{AB})]_{ky}$ with entries

$$\mathrm{tr}(G_k^A \otimes B_y \Sigma_{AB}) = \sum_{i_1,\ldots,i_n} \mathrm{tr}(G_k^A Z_{i_1,\ldots,i_n}) \mathrm{tr}(B_y \omega^{\mathrm{spec}}_{i_1,\ldots,i_n}). \quad (9)$$

For convenience we assume that we only pick $n_B + 1$ different operators $G_k^A$, such that $D$ is a square matrix with a determinant. We call this matrix the *data matrix* $D$ to further express that $D$ is determined by the observed data $P(a, b|x, y)$ once having selected $\mathcal{Z}$ and $\{G_k^A\}_k$.

From the data matrix $D$ we obtain a correlation matrix $C = DT$ if $T$ describes a linear transformation that maps $\{B_y\}_y$ into an orthonormal set $\{G_l^B = \sum_y T_{yl} B_l\}_l$. Though we have only the limited information about $n_B$ being dichotomic measurements on a $d_B$-dimensional Hilbert space, this transformation $T$ satisfies [37]

$$|\det(T)| \geq d_B^{-[(n_B+1)/2]}. \quad (10)$$

To be precise, this only holds if $\{B_y\}_y$ is linearly independent, but that can be inferred directly from a data matrix with $|\det(D)| \neq 0$. Through this, one can then lower bound the trace norm of $C$ by

$$\begin{aligned} \|C\|_1 &= \sum s_i(C) \geq (n_B + 1)|\det(C)|^{[1/(n_B+1)]} \\ &= (n_B + 1)(|\det(D)\|\det(T)|)^{[1/(n_B+1)]} \\ &\geq \frac{n_B + 1}{\sqrt{d_B}} |\det(D)|^{[1/(n_B+1)]}, \end{aligned} \quad (11)$$

using the inequality of the arithmetic and geometric means in the first step, the determinant rule, and finally Eq. (10). If this lower bound is strictly above 1, we certify that $\Sigma_{AB}$ is not separable and thus steerability of the underlying state. This is effectively the second condition of the following proposition; the other statement employs a slightly better bounding technique.

*Proposition 2.*—Consider a $d_B$-dimension-bounded steering scenario with parameters $n_A, m_A, n_B$, and $m_B = 2$. From the observed data, build up the data matrix

$$D_{ky} = \sum_{i_1,\ldots,i_n} \mathrm{tr}(G_k^A Z_{i_1,\ldots,i_n}) \mathrm{tr}(B_y \omega^{\mathrm{spec}}_{i_1,\ldots,i_n}) \quad (12)$$

using $B_0 = \mathbb{1}$ and $B_y = M_{+|y} - M_{-|y}$ for $y = 1, \ldots, n_B$, any set of steering operators $\mathcal{Z}$ with $n_A, m_A$, and any choice of $n_B + 1$ orthonormal operators $G_k^A$.Let $d_A$ be the dimension of the chosen $\mathcal{Z}$. If the observed data are nonsteerable, then the determinant of $D$ satisfies

$$|\det(D)| \leq \frac{1}{\sqrt{d_A}} \left( \frac{\sqrt{d_A d_B} - 1}{n_B \sqrt{d_A}} \right)^{n_B} \quad (13)$$

if $n_B > \sqrt{d_A d_B} - 1$ and $\mathbb{1} \in \mathrm{span}(\{G_i^A\})$. If this is not the case, nonsteerable data give

$$|\det(D)| \leq \left( \frac{\sqrt{d_B}}{n_B + 1} \right)^{n_B+1}. \quad (14)$$

*Application to experiments.*—We now give an explicit example of Proposition 2, in order to demonstrate its application and compare its strength. We pick the scenario that has been implemented in the loophole-free steering experiment performed in Vienna [19]. We follow the procedure outlined in our Letter to arrive at the data matrix (for details see the Supplemental Material [33])

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \langle B_1 \rangle & \langle B_2 \rangle & \langle B_3 \rangle \\ \langle A_1 \rangle/\sqrt{3} & \langle A_1 B_1 \rangle/\sqrt{3} & \langle A_1 B_2 \rangle/\sqrt{3} & \langle A_1 B_3 \rangle/\sqrt{3} \\ \langle A_2 \rangle/\sqrt{3} & \langle A_2 B_1 \rangle/\sqrt{3} & \langle A_2 B_2 \rangle/\sqrt{3} & \langle A_2 B_3 \rangle/\sqrt{3} \\ \langle A_3 \rangle/\sqrt{3} & \langle A_3 B_1 \rangle/\sqrt{3} & \langle A_3 B_2 \rangle/\sqrt{3} & \langle A_3 B_3 \rangle/\sqrt{3} \end{bmatrix}.$$

Because $n_B = 3 > \sqrt{d_A d_B} - 1 = 1$, and since the full operator basis for $A$ includes the identity, we can use the bound given by Eq. (13). Thus, if

$$|\det(D)| > \frac{1}{108}, \quad (15)$$

then the observed data show steering under the sole assumption that Bob's measurements act onto a qubit.

If one evaluates this criterion for a noisy maximally entangled state $p|\psi^-\rangle\langle\psi^-| + (1 - p)\mathbb{1}/4$, measuring along the three spin directions $\sigma_1, \sigma_2, \sigma_3$, one verifies steering if $p > 1/\sqrt{3}$. This is surprising, because the visibility to show standard steering, i.e., requiring the knowledge that Bob perfectly measures $\sigma_1, \sigma_2, \sigma_3$, is exactly the same. Thus, we learn that for this symmetric case, the only crucial knowledge of the measurements is that they act onto a qubit, and no further characterization is needed. In the Supplemental Material we discuss this scenario under experimentally realistic conditions showing that current technology indeed allows (or has already allowed) a loophole-free dimension-bounded steering experiment [33].

*Conclusion.*—We have introduced a framework that allows us to map the steering problem to a standard separability problem. This opened the possibility of exploiting the sophisticated tools available in entanglement detection, and thereby creating strong steering criteria. We showed dimension-bounded steering to be one particularly promising further application. Considering that many quantum protocols also require a certain level of trust, we believe that this dimension-bounded scenario is of high relevance for scenarios where at least one of the parties has some degree of confidence in his or her local device. We have shown that this "nearly" device-independent scenario is a lot stronger than the undoubtedly harder to achieve fully device-independent scenario. This scenario will help to make quantum key distribution more robust [38,39] and will assist in unifying frameworks of resource theories that exist for nonlocality [40] and steering [41] in order to approach a resource theory of partially device-independent entanglement certification.

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# Supplemental material: Steering maps and their application to dimension-bounded steering


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### Proof of Eq. (8) in the main text

Let us summarize the statement in the following proposition:

**Proposition 1.** *The set $\mathcal{Z} = \{Z_{i_1\dots i_n}\}_{i_1\dots i_n}$ uniquely determines $\Sigma_{AB}$ if and only if Eq. (8) in the main text holds for any choices of $i_1,\dots,i_n$ and $j_1,\dots j_n$.*

Before we prove this proposition let us note a technical lemma, which will be useful in the following. It describes the most general solution of $\omega_{i_1\dots i_n}$ which satisfy the relations demanded for a local hidden state model.

**Lemma 1.** *Any collection of hidden states $\omega_{i_1\dots i_n}$ which satisfies the set of linear equations given by Eq. (1) in the main text for $\mathcal{E}$ can be written as $\omega = \omega^{\text{spec}} + \omega^{\text{homo}}$. A special solution $\omega^{\text{spec}}$ is given by $\omega^{\text{spec}}_{i_1\dots i_n} = 0$ for all indices $i_1,\dots,i_n$ except*

$$\omega^{\text{spec}}_{am\dots m} = \rho_{a|1}, \omega^{\text{spec}}_{mam\dots m} = \rho_{a|2}, \dots \omega^{\text{spec}}_{m\dots ma} = \rho_{a|n}, \quad (1)$$

*for $a < m$ and*

$$\omega^{\text{spec}}_{m\dots m} = \sum_x \rho_{m|x} - (n-1)\rho. \quad (2)$$

*The general solution of the corresponding homogeneous system is given by*

$$\omega^{\text{homo}}_{i_1\dots i_n} = \sum_{\mathbf{k}} v^{(\mathbf{k})}_{i_1\dots i_n} X_{\mathbf{k}} \quad (3)$$

*using arbitrary Hermitian operators $X_{\mathbf{k}}$. Here $\mathbf{k} = k_1\dots k_n$ is an $n$-length index similar to the subscripts of $\omega$, where only the distinct possibilities with at least two $k_i < m$ are considered. For a fixed $\mathbf{k}$ the vector $v^{(\mathbf{k})}$ is given by*

$$\begin{aligned} v^{(\mathbf{k})}_{i_1\dots i_n} = \quad & \delta_{i_1\dots i_n, k_1\dots k_n} - \delta_{i_1\dots i_n, k_1 m\dots m} - \dots \\ & -\delta_{i_1\dots i_n, m\dots m k_n} + (n-1)\delta_{i_1\dots i_n, m\dots m}. \end{aligned} \quad (4)$$

*Proof.* Note that Eq. (1) in the main text is a standard set of linear equations, except that we have Hermitian operators rather than scalar variables. Therefore all the basic linear algebra results apply.

In total we have $m^n$ unknowns but only $n(m-1)+1$ linear independent relations recalling once more that $\sum_a \rho_{a|x} = \rho$ is independent of the setting. Hence the general solution can be written as a combination of a special solution and the general solution of the homogeneous system $\sum \delta_{i_x,a}\omega_{i_1\dots i_n} = 0$.

That $\omega^{\text{spec}}$ as given in the Lemma is a special solution can be checked straightforwardly. For the general solution of the homogeneous system $\omega^{\text{homo}}$ note that via the Ansatz of Eq. (3) this breaks down to the relation

$$\sum_{i_1,\dots,i_n} \delta_{i_x,a} v^{(\mathbf{k})}_{i_1\dots i_n} = 0. \quad (5)$$

The dimension of this linear subspace is $m^n - [n(m-1)+1]$, which is precisely the number of the considered $\mathbf{k}$'s. Now first note that the given $\{v^{(\mathbf{k})}\}_{\mathbf{k}}$ are linearly independent, since vector $v^{(\mathbf{k})}$ is the only vector which has a non-zero entry at the position $i_1\dots i_n = k_1\dots k_n$. Thus we are left to show that they indeed solve Eq. (5). For the $x=1$ and $a<m$ this follows for instance by

$$\sum_{i_2\dots i_n} v^{(\mathbf{k})}_{a i_2\dots i_n} = \underbrace{+1}_{a k_2\dots k_n} \underbrace{-1}_{am\dots m} = 0 \quad (6)$$

if $k_1 = a$, otherwise it holds trivially. The same arguments holds if one picks a different index $i_x$. At last we still need to check the relation corresponding to reduced state, which is given by

$$\sum_{i_1\dots i_n} v^{(\mathbf{k})}_{i_1\dots i_n} = \underbrace{+1}_{k_1 k_2\dots k_n} \underbrace{-n}_{\{k_1 m\dots m,\dots,m\dots k_n\}} \underbrace{n-1}_{m\dots m} = 0. \quad (7)$$

which finishes the proof. □

*Proof of Prop. 1.* Using the general solution $\omega^{\text{sol}}$ as given the Lemma 1 in the operator $\Sigma_{AB}$ one sees that

$$\Sigma_{AB} = \sum_{i_1\dots i_n} Z_{i_1\dots i_n} \otimes \omega^{\text{spec}}_{i_1\dots i_n} + \sum_{\mathbf{k}} \left( \sum_{i_1\dots i_n} v^{(\mathbf{k})}_{i_1\dots i_n} Z_{i_1\dots i_n} \right) \otimes X_{\mathbf{k}} \quad (8)$$

is uniquely determined by the given ensemble $\mathcal{E}$ if and only if

$$\sum_{i_1\dots i_n} v^{(\mathbf{k})}_{i_1\dots i_n} Z_{i_1\dots i_n} = 0 \quad (9)$$

holds for all possibilities $\mathbf{k}$. Using the explicit form of the vectors $v^{(\mathbf{k})}$ as given in Eq. (4) these constraints can be re-written as

$$Z_{k_1\dots k_n} = Z_{k_1 m\dots m} + Z_{m k_2\dots m} + \dots + Z_{m\dots k_n} - (n-1) Z_{m\dots m} \quad (10)$$

for all admissible $k_1 \dots k_n$ with at least two $k_i < m$. However, this condition also holds also for each $k_1 \dots k_n$ without this restriction, because then the vectors $v^{(\mathbf{k})}$ in Eq. (4) vanish. Thus we have proven Eq. (8) in the main text for all $i_1 \dots i_n$, but only for the special index set $j_1 \dots j_n = m \dots m$. Still, these conditions already imply the general (more symmetric looking) relation, using an arbitrary $j_1 \dots j_n$. This can be inferred more easily directly from the problem formulation by relabeling the individual outcomes of the conditional states. □

**Proof of Prop. 1 in the main text**

We prove this in two parts; the first only considers the statement without the extra condition $\text{tr}(\Sigma_{AB}) = 1$, but which is discussed in the second part then.

As mentioned in the main text, the proof rests on the duality properties of semidefinite programs. In fact, the first part of the proof can be considered as a special interpretation of the dual program of the original semidefinite program. Since the dual might be of independent interest, we compactly summarizes it in Remark 1.

*Proof, Part* 1. The idea of the proof is to employ the duality statements given by respective semidefinite programs. Recall that the problems $\inf_{x\in\mathbb{R}^n}\{c^T x | F_0 + \sum_i x_i F_i \geq 0\}$ and $\sup_{Z\geq 0}\{-\text{tr}(Z F_0) | \text{tr}(Z F_i) = c_i \forall i\}$, called primal and dual semidefinite programs, are connected by a couple of important relations. The most relevant is strong duality, which states that both optimal values are equal. This holds for instance under the Slater regularity condition that either problem has a strictly feasible point, *i.e.*, either an $x$ such that $F_0 + \sum_i x_i F_i > 0$ or a $Z > 0$ satisfying $\text{tr}(Z F_i) = c_i$ [32]. The proof goes along the following lines: We parse the original steering problem into the form of the primal semidefinite program, then we invoke its dual, show strong duality such that we can ensure that it gives the same solution, and finally we interpret this dual program as a the swap witness on $\Sigma_{AB}$.

To start let us write the original problem into the form of a primal semidefinite program, which is given by

$$\begin{aligned} &\inf \quad 0 \\ &\text{s.t.} \quad \omega^{\text{spec}}_{i_1\dots i_n} + \sum_{\mathbf{k}} v^{(\mathbf{k})}_{i_1\dots i_n} X_{\mathbf{k}} \geq 0 \;\; \forall i_1 \dots i_n. \end{aligned} \quad (11)$$

This can be transformed to the standard form if one uses, i) a Hermitian operator basis $\{S_r\}$ to transform the matrix-valued variables $X_{\mathbf{k}}$ into $X_{\mathbf{k}} = \sum_r x_{\mathbf{k},r} S_r$ to scalar-valued variables $x_{\mathbf{k},r}$, and ii) that several positivity constraints are equivalent to a single positivity constraint of a corresponding block matrix. We emphasize that Eq. (11) is a special primal problem called feasibility problem, since we effectively do not optimize anything. By convention, if the constraint cannot be fulfilled then the infimum is $+\infty$.

Working out the dual gives

$$\begin{aligned} &\sup \quad -\sum_{i_1\dots i_n} \text{tr}(Z_{i_1\dots i_n} \omega^{\text{spec}}_{i_1\dots i_n}) \\ &\text{s.t.} \quad Z_{i_1\dots i_n} \geq 0 \;\; \forall i_1 \dots i_n, \\ &\qquad \sum_i v^{(\mathbf{k})}_{i_1\dots i_n} Z_{i_1\dots i_n} = 0 \;\; \forall \mathbf{k}. \end{aligned} \quad (12)$$

If one has used the standard form for the previous problem, one simply reverses here the points i) and ii); the block-structure can be removed directly, while the linear relations in the last line of Eq. (12) appear since one has respective linear relations for all Hermitian operator basis elements.

This dual has a strictly feasible point $Z_{i_1\dots i_n} = \mathbb{1} > 0$, noting $\sum_i v^{(\mathbf{k})}_{i_1\dots i_n} = 0$ was already proven in Lemma 1. Therefore we have strong duality, and consequently the statement that, whenever the primal problem is infeasible ($\mathcal{E}$ steerable) then there exists a sequence of appropriate $Z_{i_1\dots i_n}$ such that $C = \sum_i \text{tr}(Z_{i_1\dots i_n} \omega^{\text{spec}}_{i_1\dots i_n})$ will tend to $-\infty$, saying that Eq. (12) is unbounded. We summarize this more direct dual SDP in Remark 1.

Now let us interpret this as the detection statement of the proposition. That we labeled the dual variables by $Z_{i_1\dots i_n}$ as also used in $\Sigma_{AB}$ is no coincidence. Effectively the solutions $Z_{i_1\dots i_n}$ of the dual program will be the ones used in the operator $\Sigma_{AB}$ that shows steering. Note that the variables of the dual program already satisfy positivity $Z_{i_1\dots i_n} \geq 0$ and the linear relations in Eq. (12) uniquely determine $\Sigma_{AB} = \sum_i Z_{i_1\dots i_n} \otimes \omega^{\text{spec}}_{i_1\dots i_n}$, as already shown in the proof of Prop. 1. Finally, note here the formal operator connection between $\Sigma_{AB}$ and the objective function $C$. Using the swap operator $V$, *i.e.*, $\text{tr}(V A \otimes B) = \text{tr}(AB)$, one directly sees that the swap operator evaluated on $\Sigma_{AB}$ gives the objective value $\mathrm{tr}(V\Sigma_{AB}) = C$. Since the swap operator $V$ is an entanglement witness a negative $\mathrm{tr}(V\Sigma_{AB}) = C < 0$ signals that the optimal $\Sigma_{AB}$ has not a separable structure. This finishes the first part of the proof. □

**Remark 1.** *The dual problem to the feasibility problem for the collection of positive semidefinite operators satisfying the relations given by Eq. (1) reads as*

$$\begin{aligned} \sup \quad & -\sum_{i_1\dots i_n} \mathrm{tr}(Z_{i_1\dots i_n}\omega_{i_1\dots i_n}) \\ \text{s.t.} \quad & Z_{i_1\dots i_n} \geq 0 \quad \forall i_1\dots i_n, \\ & Z_{i_1 i_2\dots i_n} = Z_{i_1 j_2\dots j_n} + Z_{j_1 i_2 j_3\dots j_n} + \dots + Z_{j_1 j_2\dots i_n} \\ & \qquad -(n-1)Z_{j_1 j_2\dots j_n} \quad \forall i_1,\dots j_n. \end{aligned} \tag{13}$$

*Via the linear equations for $Z_{i_1\dots i_n}$ and by Eq. (1) one can evaluate the objective $C = \sum_{i_1\dots i_n} \mathrm{tr}(Z_{i_1\dots i_n}\omega_{i_1\dots i_n})$. For instance, if one picks fixed indices $j_1,\dots,j_n$ one arrives at*

$$\begin{aligned} C = & \sum_{i_1\dots i_n} \mathrm{tr}(Z_{i_1 j_2\dots j_n}\omega_{i_1\dots i_n}) + \dots + \sum_{i_1\dots i_n} \mathrm{tr}(Z_{j_1\dots i_n}\omega_{i_1\dots i_n}) \\ & -(n-1)\sum_{i_1\dots i_n} \mathrm{tr}(Z_{j_1 j_2\dots j_n}\omega_{i_1\dots i_n}) \\ = & \sum_{i_1} \mathrm{tr}\left[Z_{i_1 j_2\dots j_n}(\sum_{i_2\dots i_n}\omega_{i_1\dots i_n})\right] + \dots \\ & + \sum_{i_n} \mathrm{tr}[Z_{j_1\dots i_n}(\sum_{i_2\dots i_n}\omega_{i_1\dots i_n})] \\ & -(n-1)\,\mathrm{tr}[Z_{j_1 j_2\dots j_n}(\sum_{i_1\dots i_n}\omega_{i_1\dots i_n})] \\ = & \sum_{i_1} \mathrm{tr}(Z_{i_1 j_2\dots j_n}\rho_{i_1|1}) + \dots + \sum_{i_n} \mathrm{tr}(Z_{j_1\dots i_n}\rho_{i_n|n}) \\ & -(n-1)\,\mathrm{tr}(Z_{j_1 j_2\dots j_n}\rho). \end{aligned}$$

*Note that any other choice gives the same value; this is expressed by $C = \sum_{i_1\dots i_n} \mathrm{tr}(Z_{i_1\dots i_n}\omega^{\mathrm{spec}}_{i_1\dots i_n})$.*

*Proof, Part* 2. It is left to show that we can also find a solution $\mathcal{Z}$ which satisfies $\mathrm{tr}(\Sigma_{AB}) = 1$, since such a condition does not appear in Eq. (12). Note that since the value of an objective function of any steerable ensemble will tend to $-\infty$, there are for sure parameters $\mathcal{Z}$ such that $C < 0$. Suppose that for these $Z_{i_1\dots i_n}$, the operator $\Sigma_{AB}$ is not normalized. If $\mathrm{tr}(\Sigma_{AB}) > 0$, then one can directly used a rescaled version $Z_{i_1\dots i_n}/\mathrm{tr}(\Sigma_{AB})$, now also satisfying the trace condition, but still detecting the state. Note that this trick fails if $\mathrm{tr}(\Sigma_{AB}) \leq 0$, either due to a division by zero, or due to $Z_{i_1\dots i_n}$ being not positive semidefinite anymore. Thus we are left to prove that $\mathrm{tr}(\Sigma_{AB}) > 0$.

To verify $\mathrm{tr}(\Sigma_{AB}) \geq 0$ we employ that $C \geq 0$ holds for any non-steerable ensemble. From the given ensemble $\mathcal{E}$ such a non-steerable ensemble is for instance $\tilde{\mathcal{E}} = \{\tilde{\rho}_{a|x} = \mathrm{tr}(\rho_{a|x})\mathbb{1}/d\}$, having a special solution $\tilde{\omega}_{i_1\dots i_n} = \mathrm{tr}(\omega^{\mathrm{spec}}_{i_1\dots i_n})\mathbb{1}/d$ as can be checked by Eqs. (1, 2). Thus evaluating the objective function of this non-steerable ensemble and the chosen selection $\mathcal{Z}$ one finds

$$\begin{aligned} & \sum_{i_1\dots i_n} \mathrm{tr}(Z_{i_1\dots i_n}\tilde{\omega}^{\mathrm{spec}}_{i_1\dots i_n}) \\ & \quad = \frac{1}{d}\sum_{i_1\dots i_n} \mathrm{tr}(Z_{i_1\dots i_n})\,\mathrm{tr}(\omega^{\mathrm{spec}}_{i_1\dots i_n}) = \frac{1}{d}\,\mathrm{tr}(\Sigma_{AB}) \geq 0. \end{aligned}$$

Finally, we show that from $\mathcal{Z}$ with $C < 0$ and $\mathrm{tr}(\Sigma_{AB}) = 0$ it is always possible to find a different solution $\bar{\mathcal{Z}}$ with $\bar{C} < 0$ but $\mathrm{tr}(\Sigma_{AB}) > 0$ such that we can employ the rescaling trick again. Note first that the only negative part in the $C$ must be due to $\mathrm{tr}(Z_{m\dots m}\omega^{\mathrm{spec}}_{m\dots m}) < 0$, since all other terms involve only positive semidefinite operators. Now pick any $\omega^{\mathrm{spec}}_{i_1\dots i_n}$ with $\mathrm{tr}(\omega^{\mathrm{spec}}_{i_1\dots i_n}) > 0$, and assume this is $\omega^{\mathrm{spec}}_{am\dots m}$ with $a < m$. Then define the new set of operator

$$\begin{aligned} \bar{Z}_{am\dots m} &= Z_{am\cdots} + \epsilon\mathbb{1}, \\ \bar{Z}_{mam\dots m} &= Z_{mam\dots m}, \dots, \bar{Z}_{m\dots m} = Z_{m\dots m} \end{aligned} \tag{14}$$

which by Eq. (10) are enough to fully determine the set $\bar{\mathcal{Z}}$. This set still contains only positive semidefinite operators because the only operators that change are $\bar{Z}_{ai_2\dots i_n} = Z_{ai_2\dots i_n} + \epsilon\mathbb{1}$. For this new solution $\bar{\mathcal{Z}}$ we get $\mathrm{tr}(\bar{\Sigma}_{AB}) = \epsilon\,\mathrm{tr}(\omega^{\mathrm{spec}}_{am\dots m})$ and $\bar{C} = C + \epsilon\,\mathrm{tr}(\omega^{\mathrm{spec}}_{am\cdots})$, thus choosing $\epsilon$ small enough one obtains the given statement. This completes the proof. □

**Proof of Prop. 2 in the main text**

The ideas and bounding techniques are the same as in Ref. [35], which derived similar determinant constraints for the dimension-bounded entanglement verification; here we only need to apply them to a single side.

*Proof.* Inequality (14) in the main text is just a rearrangement of Eq. (11) in the main text. We remark once more that the bound of $T$ as given by Eq. (10) in the main text holds only if $\{B_y\}_y$ is linearly independent, which follows from the observation $|\det(D)| \neq 0$.

The first and stronger condition in Eq. (13) in the main text follows using the extra information of $C$ that if both sets $\{G^A_k\}_k$, $\{G^B_l\}_l$ have the identity in its linear span, then the largest singular value satisfies $\sigma_0(C) \geq q = \mathrm{tr}(\mathbb{1}/\sqrt{d_A} \otimes \mathbb{1}/\sqrt{d_B}\Sigma_{AB}) = 1/\sqrt{d_A d_B}$. This follows from the fact that the ordered singular values of $C$ are lower bounded by the ordered singular values of any submatrix $C^{\mathrm{sub}}$ of $C$. While $\{G^B_l\}_l$ satisfies this extra condition automatically since $B_0 = \mathbb{1}$, we need this requirement for the choice of $\{G^A_k\}_k$.

Via this extra condition we can achieve a better bound using the inequality of arithmetic and geometric means only to $n_B$ singular values and then checking whether the minimal value of $\sigma_0(C)$ can be reached, more precisely one obtains

$$\min_{\sigma_0(C)\geq q} \|C\|_1 \geq \min_{\sigma_0(C)\geq q} \left[\sigma_0(C) + \left(\frac{|\det(C)|}{\sigma_0(C)}\right)^{\frac{1}{n_B}}\right]$$
$$= \begin{cases} (n_B+1)|\det(C)|^{\frac{1}{n_B+1}} & \text{if } |\det(C)|^{\frac{1}{n_B+1}} \geq q \\ q + n_B\left(\frac{|\det(C)|}{q}\right)^{\frac{1}{n_B}} & \text{else} \end{cases}, \quad (15)$$

depending on the determinant of $C$. Note that both bounds are monotonically increasing functions. By the determinant rule $|\det(C)| = |\det(D)||\det(T)|$ and the bound of Eq. 10 in the main text, the possible values are constrained to satisfy

$$|\det(C)| \geq |\det(D)| d_B^{-\frac{n_B+1}{2}}. \quad (16)$$

Thus, depending on the value of $|\det(D)|$ the second bound in Eq. (15) can be used or not. If $|\det(D)|^{1/(n_B+1)} \geq 1/\sqrt{d_A}$ the determinant of $C$ will always satisfy the constraint in Eq. (15) and one obtains

$$\min_{\sigma_0(C)\geq q} \|C\|_1 \geq \frac{n_B+1}{\sqrt{d_B}} |\det(D)|^{\frac{1}{n_B+1}}. \quad (17)$$

Otherwise one can split the possible region and minimize separately, yielding

$$\min_{\sigma_0(C)\geq q} \|C\|_1 \geq \quad (18)$$
$$\min\left\{\frac{1}{\sqrt{d_A d_B}} + n_B\left(\sqrt{d_A} d_B^{-\frac{n_B}{2}} |\det(D)|\right)^{\frac{1}{n_B}}, \frac{n_B+1}{\sqrt{d_A d_B}}\right\}.$$

At last, if $n_B > \sqrt{d_A d_B} - 1$ note that the bound given by Eq. (17) and the second argument in minimum of Eq. (18) are strictly larger than 1. Thus only the first argument of Eq. (18) must be checked, which is the stated condition. This completes the proof. □

### Steering scenario for $n = 2$ and $m = d$

In this section we exemplify the construction of respective $\mathcal{Z} = \{Z_{ij}\}_{ij}$ for the case of two settings but arbitrary number of outcomes. The idea and construction rely on Fourier connected mutually unbiased bases [33]. Thus we need a couple of definitions first.

Consider a Hilbert space $\mathbb{C}^d$ and suppose that one has a basis $\{|\phi_k\rangle\}_{k\in\mathbb{Z}_d}$ with $\mathbb{Z}_d = \{0, \ldots, d-1\}$, which we also use to label the outcomes. Then one obtains another basis, which is mutually unbiased, by the Fourier transform

$$|\psi_k\rangle = \mathcal{F}|\phi_k\rangle = \frac{1}{\sqrt{d}} \sum_{l\in\mathbb{Z}_d} q^{kl} |\phi_l\rangle \quad (19)$$

with $q = e^{2\pi i/d}$.

These two bases even admit further structure which becomes convenient in the following. Consider two representations $U, V$ of the cyclic group $\mathbb{Z}_d$ on $\mathcal{H}$ defined by its action onto the first basis, $U_x|\psi_k\rangle = |\psi_{k+x}\rangle$ and $V_y|\psi_k\rangle = q^{yk}|\psi_k\rangle$ for all $x, y, k$. These two representations further satisfy $U_x V_y = q^{-xy} V_y U_x$ and the Fourier transform is the intertwining map, $U_x\mathcal{F} = \mathcal{F}V_x^\dagger$ and $V_y\mathcal{F} = \mathcal{F}U_y$. Via this one can identify the action on both basis states that we summarize as

$$U_x|\phi_k\rangle = |\phi_{k+x}\rangle, \qquad U_x|\psi_k\rangle = q^{-xk}|\psi_k\rangle, \quad (20)$$
$$V_y|\phi_k\rangle = q^{yk}|\phi_k\rangle, \qquad V_y|\psi_k\rangle = |\psi_{k+y}\rangle \quad (21)$$

for all $x, y \in \mathbb{Z}_d$. Then the following set of operators will be our characterization of the steering inequality. The structure can be guessed once one knows the so-called mother observable for the respective joint measurability problem [33], from whose result one further knows that the current form is optimal.

**Proposition 2.** *Consider the set of operators* $\mathcal{Z} = \{Z_{kl} = U_k V_l Z_{00} V_l^\dagger U_k^\dagger\}$ *with*

$$Z_{00} = \mu_1|\chi_-\rangle\langle\chi_-| + \mu_2(\mathbb{1} - |\chi_+\rangle\langle\chi_+| - |\chi_-\rangle\langle\chi_-|), \quad (22)$$

*pure states* $|\chi_\pm\rangle \propto |\phi_0\rangle \pm |\psi_0\rangle$ *and parameters*

$$\mu_1 = \frac{2}{\sqrt{d}(\sqrt{d}-1)(\sqrt{d}+2)}, \quad (23)$$
$$\mu_2 = \frac{1+\sqrt{d}}{\sqrt{d}(\sqrt{d}-1)(\sqrt{d}+2)}. \quad (24)$$

*Then this set of operators can be used in the steering map, since all operators are positive semidefinite and uniquely determines the operator* $\Sigma_{AB}$ *and satisfies* $\mathrm{tr}(\Sigma_{AB}) = 1$.

*Proof.* Using the form of $Z_{00}$ as given by Eq. (22) one sees that $Z_{00}$ is positive semidefinite, since both $\mu_i$ are strictly positive and $|\chi_-\rangle$ and $|\chi_-\rangle$ are orthogonal, moreover it has unit trace. Since all other $Z_{kl}$ are obtained by a unitary transformation each $Z_{kl}$ is positive semidefinite and satisfies $\mathrm{tr}(Z_{kl}) = 1$, which directly shows that $\Sigma_{AB}$ has unit trace. Thus we are left to show that $Z_{kl}$ uniquely determines $\Sigma_{AB}$, for which we have to show

$$Z_{kl} = Z_{kt} + Z_{sl} - Z_{st} \quad (25)$$

for all $k, l, s, t \in \mathbb{Z}_d$ according to Prop. 1. In order to show this we expand the states $|\chi_\pm\rangle$ in $Z_{00}$ which results into the structure

$$Z_{00} = c_1(|\phi_0\rangle\langle\phi_0| + |\psi_0\rangle\langle\psi_0|) + c_2\mathbb{1} \quad (26)$$

with appropriate coefficients $c_1, c_2$. Note that at this point the very specific choices of $\mu_1$ and $\mu_2$ become important; they are chosen such that cross terms of $|\phi_0\rangle\langle\psi_0|$ or $|\psi_0\rangle\langle\phi_0|$ vanish. Applying now the rules given by

Eqs. (20, 21) one gets

$$Z_{kl} = c_1(|\phi_k\rangle\langle\phi_k| + |\psi_l\rangle\langle\psi_l|) + c_2\mathbb{1} \tag{27}$$

from which the necessary relation given by Eq. (25) can be verified. □

In order to obtain a steering criterion one can use the given operators $Z_{kl}$ of the proposition to build up $\Sigma_{AB}$, which is uniquely determined by the given ensemble $\mathcal{E}$ in the $n = 2$ and $m = d$ steering case. Whenever this operator $\Sigma_{AB}$ is then not a separable state the underlying distribution is steerable.

### Dimension-bounded steering in a loophole free experiment of Ref. [19]

First let us reiterate how to arrive at the data matrix necessary for employing the dimension bounded steering criterion. Alice and Bob have three different dichotomic measurements, $n_A = n_B = 3$ and $m_A = m_B = 2$, and we assume that Bob's measurement act onto a qubit $d_B = 2$. The settings will be labeled by $x, y \in \{1, 2, 3\}$ and the outcomes by $a, b \in \{\pm 1\}$.

According to Prop.2, let us first pick operators $Z_{ijk}$ with $i, j, k \in \{\pm 1\}$ that characterize a steering map with parameters $n_A = 3$ and $m_A = 2$. Here we choose $Z_{ijk} = \left[\mathbb{1} + \left(i\sigma_1 + j\sigma_2 + k\sigma_3\right)/\sqrt{3}\right]/2$, which can be interpreted as pure states, whose Bloch vectors point towards the 8 different corners of the cube. It can be checked that these choices satisfy all relations given by Eq.(8) of the main text, so that, by construction, the operator $\Sigma_{AB}$ is uniquely determined by the ensemble $\mathcal{E}$ and furthermore normalized. This operator is given by

$$\Sigma_{AB} = \frac{1}{2}\left[\mathbb{1}\otimes\rho + \frac{1}{\sqrt{3}}\sum_{s=1}^{3}\sigma_s\otimes(\rho_{+|s} - \rho_{-|s})\right]. \tag{28}$$

In order to get to the data matrix $D$ we still need to fix the operator set $\{G_k^A\}_k$, for which the properly normalized identity and Pauli-operators, $\{\mathbb{1}, \sigma_1, \sigma_2, \sigma_3\}/\sqrt{2}$, are convenient choices since they only act non-trivially on certain terms in Eq. (28). Since only the subspace of $\{G_k^A\}_k$ matters in the criteria of Prop.2, any other basis choice will perform equally well. As the final step we rewrite the abstract values $\mathrm{tr}(B_y\rho_{a|x})$, with $B_0 = \mathbb{1}$ and $B_y = M_{+|y} - M_{-|y}$, in terms of the directly observable quantities $P(a, b|x, y)$. Looking at

$$\begin{aligned}
&\mathrm{tr}[B_y(\rho_{+|x} - \rho_{-|x})] \\
&\quad= \mathrm{tr}[(M_{+|y} - M_{-|y})\rho_{+|x}] - \mathrm{tr}[(M_{+|y} - M_{-|y})\rho_{-|x}] \\
&\quad= P(+,+|x,y) - P(+,-|x,y) - \\
&\qquad [P(-,+|x,y) - P(-,-|x,y)] \equiv \langle A_x B_y\rangle,
\end{aligned}$$

one sees that correlations $\langle A_x B_y\rangle$ and respective marginals $\langle A_x\rangle, \langle B_y\rangle$, which similarly appear in Bell inequalities, give an appropriate formulation. Hence, to sum up one gets the data matrix $D$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \langle B_1\rangle & \langle B_2\rangle & \langle B_3\rangle \\ \langle A_1\rangle/\sqrt{3} & \langle A_1B_1\rangle/\sqrt{3} & \langle A_1B_2\rangle/\sqrt{3} & \langle A_1B_3\rangle/\sqrt{3} \\ \langle A_2\rangle/\sqrt{3} & \langle A_2B_1\rangle/\sqrt{3} & \langle A_2B_2\rangle/\sqrt{3} & \langle A_2B_3\rangle/\sqrt{3} \\ \langle A_3\rangle/\sqrt{3} & \langle A_3B_1\rangle/\sqrt{3} & \langle A_3B_2\rangle/\sqrt{3} & \langle A_3B_3\rangle/\sqrt{3} \end{bmatrix}.$$

Next let us explain how the developed criterion can be employed for the real setup used in Vienna [19]. The main difference is that in the actual experiment one additionally observes an inconclusive outcome "inc" due to no click or even double click events. On Bob's side, the side which is at least partially trusted, this event can safely be discarded [19] assuming that this event is independent of the measurement choice such that it can be viewed as a kind of filter telling whether the final result will be conclusive or not. Only if this filter succeeds one looks at the corresponding state. For those measurements (acting on the conditional state) the measurements are assumed to act on a qubit, respective single photon in two polarization modes. However for Alice, the uncharacterized side, this is not possible. In order to incorporate the inconclusive event for Alice we consider the case that each inconclusive outcome "inc" is randomly assigned to either of the +1 or −1 outcome. This is also the standard for Bell experiments. Then one is left with the dimension-bounded steering scenario considered in the main section.

To finally give an example of the strength of our developed criterion we employ the following model to simulate real data: For the quantum state we assume a noisy maximally entangled singlet which has passed through a lossy channel for Alice, more precisely the state given by

$$\begin{aligned}
\rho_{AB} = &p\left[\lambda|\psi^-\rangle\langle\psi^-| + (1-\lambda)\mathbb{1}/4\right] \\
&+ (1-p)|\Omega\rangle\langle\Omega|\otimes\mathbb{1}/2.
\end{aligned} \tag{29}$$

Here $p$ denotes the transmission probability, $|\Omega\rangle$ is the vacuum state and $\lambda$ a parameter characterizing the quality of the Werner state. In the true experiment there will be also loss on Bob's side, but as mentioned before, we look at the conditional state. Next we imagine that Alice and Bob perform projective measurements in the $\sigma_1, \sigma_2, \sigma_3$ basis, while the additional "inc" event for Alice is given by the projection onto the vacuum state. Then the observed data, if Alice and Bob are using the same settings $x, y$, are given by

$$P(+,-|x,y) = P(-,+|x,y) = \frac{1}{4}p(1+\lambda\delta_{x,y}), \tag{30}$$

$$P(+,+|x,y) = P(-,-|x,y) = \frac{1}{4}p(1-\lambda\delta_{x,y}), \tag{31}$$

$$P(inc,+|x,y) = P(inc,-|x,y) = \frac{1}{2}(1-p). \tag{32}$$

If one reassign each "inc" one obtains

$$P(+,-|x,y) = P(-,+|x,y) = \frac{1}{4}(1 + p\lambda\delta_{x,y}), \qquad (33)$$

$$P(+,+|x,y) = P(-,-|x,y) = \frac{1}{4}(1 - p\lambda\delta_{x,y}), \qquad (34)$$

and thus

$$\langle A_x B_y \rangle = -\delta_{x,y} p\lambda, \;\; \langle A_x \rangle = \langle B_y \rangle = 0. \qquad (35)$$

Putting these observations into the data matrix from the main text one obtains

$$D = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & -\frac{p\lambda}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & -\frac{p\lambda}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & -\frac{p\lambda}{\sqrt{6}} \end{bmatrix}, \qquad (36)$$

which shows steering according to Eq. (16) in the main text if $p\lambda > 1/\sqrt{3} \approx 0.577$. Let us point out that this is also the condition if we would know that the performed measurements are perfect projective measurements in the eigenbasis of $\sigma_1, \sigma_2, \sigma_3$. Thus, we see that we have here a scenario where this further characterization is totally redundant and only the knowledge that one measures a qubit is essential.

Assuming the visibility and detection efficiency parameters from Ref. [19], one would obtain the values $\{0.74, 0.73, 0.73\}$ for the respective $p\lambda$, which are all well above the threshold. Assuming that all other correlations and marginals vanish, this would strongly show steering also in the case where one has only the very limited knowledge that the conclusive outcomes were qubit measurements. However, note, that these other observations are essential for the inequality, otherwise one could not gain the required extra knowledge of the uncharacterized qubit measurements. Unfortunately, these experimental data are not available anymore for the experiment of Ref. [19].

# Article VII

- **Title**: Adaptive strategy for joint measurements
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- **Abstract**: We develop a technique to find simultaneous measurements for noisy quantum observables in finite-dimensional Hilbert spaces. We use the method to derive lower bounds for the noise needed to make incompatible measurements jointly measurable. Using our strategy together with recent developments in the field of one-sided quantum information processing we show that the attained lower bounds are tight for various symmetric sets of quantum measurements. We use this characterisation to prove the existence of so called 4-Specker sets, i.e. sets of four incompatible observables with compatible subsets in the qubit case.
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# Adaptive strategy for joint measurements

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We develop a technique to fin simultaneous measurements for noisy quantum observables in finite-dimensiona Hilbert spaces. We use the method to derive lower bounds for the noise needed to make incompatible measurements jointly measurable. Using our strategy together with recent developments in the fiel of one-sided quantum information processing we show that the attained lower bounds are tight for various symmetric sets of quantum measurements. We use this characterisation to prove the existence of so called 4-Specker sets, i.e. sets of four incompatible observables with compatible subsets in the qubit case.

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## I. INTRODUCTION

Incompatibility of quantum devices is one of the fundamental features of quantum theory with a wide range of consequences [1]. In particular, incompatibility of quantum measurements is known to be a very powerful tool in many branches of quantum information theory including entanglement detection [2], uncertainty relations [3,4], and tasks demanding a Bell violation, like quantum cryptography [5,6]. Recent developments [7–9] are suggesting that incompatibility can be seen as a resource for quantum information processing. The resource theoretical aspect has spurred a development of monotones quantifying quantum incompatibility or nonjoint measurability [10–13] as a general quantum resource. These monotones are based on adding noise to a set of incompatible measurements and concluding numerically the noise threshold for the measurements to become jointly measurable. Besides the numerical methods a few measurement setups have been analyzed analytically, including two- [14,15] and three-qubit measurements [16,17], two mutually unbiased measurements [18,19], and a measurement setup including a Clifford algebra generalization of the Pauli matrices [20]. There are also some general methods to derive either necessary [21] or sufficien [22] conditions for a set of measurements to be incompatible.

In this article we present a method to derive analytical bounds for the noise resistance of incompatible measurements on a $d$-level quantum system. Our method is based on an adaptive algorithm which starts with a set of well-chosen Hilbert space vectors and results as a set of noisy compatible measurements. The algorithm gives lower bounds for the noise needed to make a set of $M$ quantum measurements compatible. We demonstrate the power of our method by rederiving some of the known [14,18] joint measurement uncertainty relations, but then also applying the technique to various new symmetric measurement scenarios obtaining lower bounds for the noise robustness of the involved measurements. Moreover, we translate known quantum steering [23,24] techniques into inequalities which, when violated, witness incompatibility. These inequalities are shown to coincide with our lower bounds proving the optimality of our results. We conclude by constructing a so called 4-Specker sets, i.e. sets of four incompatible observables with compatible subsets of incompatible quantum measurements in the qubit scenario.

The paper is organized as follows: in Sec. II we recall the mathematical definitio of joint measurability and we explain the general idea of the method, in Sec. III we exploit the strategy for qubit measurements, in Sec. IV we generalize our strategy for $d$-level quantum systems and apply it to two mutually unbiased bases (MUBs), and in Sec. V we prove the optimality of our results using known steering techniques. We conclude by proving the existence of a 4-Specker set in the qubit case in Sec. VI and by stating our conclusions in Sec. VII.

## II. QUANTUM INCOMPATIBILITY AND THE ADAPTIVE STRATEGY

Quantum incompatibility means the impossibility of measuring two or more quantum observables simultaneously. For the case of projective measurements incompatibility is characterized by the noncommutativity of the measurements, but for general observables (i.e., positive operator-valued measures or POVMs for short) this is no longer the case. Indeed, there exist noncommuting observables which allow a simultaneous measurement. Generally, incompatibility is formulated as the nonexistence of a joint measurement: a set $\{\mathsf{A}_k\}_{k=1}^{M}$ of observables is compatible if there exists a joint observable $\mathsf{G}$ from which one recovers the observables as marginals, i.e.,

$$\mathsf{A}_k(x_k) = \sum_{x_i, i\neq k} \mathsf{G}(x_1, \ldots, x_m). \tag{1}$$

A set of observables which is not compatible is called incompatible. In what follows, we develop an intuitive strategy for implementing joint measurements for several quantum measurement scenarios.

Suppose we have measurement devices for two observables $\mathsf{A}_1$ and $\mathsf{A}_2$, but that these observables are incompatible. We can thus only hope to simultaneously implement their *approximate* versions. But how do we perform a joint measurement even of their approximate versions if the only available devices are the ones for $\mathsf{A}_1$ and $\mathsf{A}_2$?

A rough method as presented in Fig. 1 is to toss a coin and measure either $\mathsf{A}_1$ or $\mathsf{A}_2$, depending on the result of the coin toss. Since a joint measurement should give an outcome for both observables, one can then for example draw randomly

 

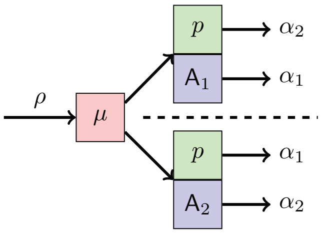


FIG. 1. Nonadaptive strategy to obtain a joint observable for approximations of two observables. A system in a state $\rho$ enters the measurement device. With probability $\mu$ the observable $\mathsf{A}_1$ is measured and for observable $\mathsf{A}_2$ a random outcome is drawn. With probability $1-\mu$ the observable $\mathsf{A}_2$ is measured and a value for the observable $\mathsf{A}_1$ is drawn randomly.

an outcome for the other observable. A POVM describing this procedure is

$$\mathsf{G}(\alpha_1,\alpha_2) = \mu p(\alpha_2)\mathsf{A}_1(\alpha_1) + (1-\mu)p(\alpha_1)\mathsf{A}_2(\alpha_2), \qquad (2)$$

where $p$ is a probability distribution and $\mu$ is the probability that the coin toss leads to the choice of the f rst observable. The noisy versions of $\mathsf{A}_1$ and $\mathsf{A}_2$ are $\mu\mathsf{A}_1 + (1-\mu)p\mathbb{1}$ and $(1-\mu)\mathsf{A}_2 + \mu p\mathbb{1}$, respectively.

There is a way to improve the previous procedure if we know something about the relation of $\mathsf{A}_1$ and $\mathsf{A}_2$. Namely, we can take an advantage of the obtained measurement outcome and adapt the random choice accordingly. This means that we replace the probability distribution $p$ by conditional probability distributions, hence leading to a joint observable

$$\begin{aligned}\mathsf{G}(\alpha_1,\alpha_2) = {} & \mu p(\alpha_2 \mid \mathsf{A}_1 = \alpha_1)\mathsf{A}_1(\alpha_1) \\ & + (1-\mu)p(\alpha_1 \mid \mathsf{A}_2 = \alpha_2)\mathsf{A}_2(\alpha_2). \qquad (3)\end{aligned}$$

This has an obvious generalization to any finit number of observables. The strategy is illustrated in Fig. 2.

There is still an important modificatio of the previous procedure that adds a new dimension of f exibility (see Fig. 3). Suppose we want to perform a joint measurement of noisy versions of $\mathsf{A}_1,\dots,\mathsf{A}_M$, but we have measurement devices for some other (incompatible) observables $\mathsf{B}_1,\dots,\mathsf{B}_N$ in our possession. Again, we make a random choice of which observable $\mathsf{B}_k$ we measure. Based on the obtained outcome, we create $M$ outcomes $\alpha_1,\dots,\alpha_M$ that are interpreted as the outcomes of noisy versions of $\mathsf{A}_1,\dots,\mathsf{A}_M$. If an observable $\mathsf{B}_k$ gives an outcome $\beta_k$, then the resulting set of outcomes

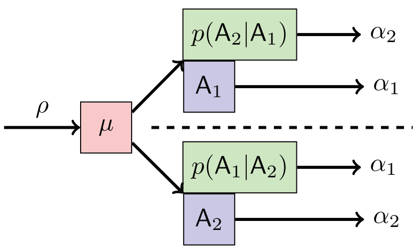


FIG. 2. Adaptive strategy to obtain a joint observable for approximations of two observables. In comparison to the nonadaptive strategy (Fig. 1), in the adaptive strategy the outcome of the nonmeasured observable is conditioned on the outcome of the measured observable.

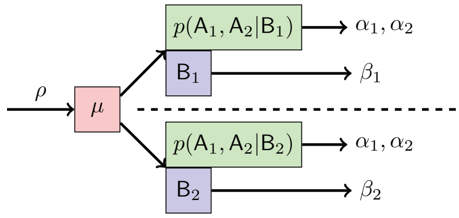


FIG. 3. Adaptive strategy with auxiliary observables. In the general case the adaptive strategy uses some other observables $\mathsf{B}_1,\mathsf{B}_2$ to obtain approximations of the original observables $\mathsf{A}_1,\mathsf{A}_2$.

$\alpha_1,\dots,\alpha_M$ is obtained with some conditional probability $p(\alpha_1,\dots,\alpha_M \mid \mathsf{B}_k = \beta_k)$. We also note that the observables $\mathsf{B}_k$ do not have to be chosen with equal probability, but in general we can throw an $N$-sided biased die giving the outcome $k$ (i.e., telling to measure $\mathsf{B}_k$) with probability $\mu_k$. In this general case the obtained POVM is thus

$$\begin{aligned}&\mathsf{G}(\alpha_1,\dots,\alpha_M) \\ &\quad = \mu_1 \sum_\beta p(\alpha_1,\dots,\alpha_M \mid \mathsf{B}_1 = \beta_1)\mathsf{B}_1(\beta_1) + \cdots \\ &\qquad + \mu_N \sum_\beta p(\alpha_1,\dots,\alpha_M \mid \mathsf{B}_N = \beta_N)\mathsf{B}_N(\beta_N). \qquad (4)\end{aligned}$$

There is no guarantee that the marginals of this joint observable are good approximations of $\mathsf{A}_1,\dots,\mathsf{A}_M$. This obviously depends on the choices of $\mathsf{B}_1,\dots,\mathsf{B}_N$ and the conditional postprocessing. In what follows we show that our technique captures many of the known examples of joint measurability and, moreover, we present various new joint measurement scenarios.

## III. ADAPTIVE STRATEGY FOR QUBIT OBSERVABLES

The general procedure described in the last sections has two critical choices: test observables $\mathsf{B}_1,\dots,\mathsf{B}_N$ and conditional postprocessing functions $p(\alpha_1,\dots,\alpha_M \mid \mathsf{B}_k = \beta)$. We now specify the latter in the case of unbiased qubit observables.

First, we start from unbiased binary qubit observables. For each unit vector $\mathbf{a} \in \mathbb{R}^3$ and $0 \leqslant \lambda \leqslant 1$, we denote by $\mathsf{S}^{\lambda\mathbf{a}}$ the binary qubit observable

$$\mathsf{S}^{\lambda\mathbf{a}}(\pm 1) = \tfrac{1}{2}(\mathbb{1} \pm \lambda\mathbf{a}\cdot\boldsymbol{\sigma}).$$

For values $0 < \lambda < 1$ we consider $\mathsf{S}^{\lambda\mathbf{a}}$ as a noisy version of the sharp qubit observable $\mathsf{S}^{\mathbf{a}}$ [14,15].

Let $\mathbf{a}_1,\dots,\mathbf{a}_M \in \mathbb{R}^3$ be a finit set of unit vectors. We are seeking a joint observable for noisy versions of $\mathsf{S}^{\mathbf{a}_1},\dots,\mathsf{S}^{\mathbf{a}_M}$; i.e., we want to construct an observable $\mathsf{G}$ such that

$$\sum_{\alpha_2,\dots,\alpha_M=\pm1} \mathsf{G}(\alpha_1,\dots,\alpha_M) = \mathsf{S}^{\lambda\mathbf{a}_1}(\alpha_1),$$

and similarly for the other marginals. Following the general guideline of the adaptive strategy, we proceed as follows.

(1) We f x a set of unit vectors $\mathbf{b}_1,\dots,\mathbf{b}_N \in \mathbb{R}^3$ such that $\mathbf{a}_\ell\cdot\mathbf{b}_k \neq 0$ for all $\ell = \{1,\dots,M\}$ and $k = \{1,\dots,N\}$.

(2) We choose randomly $k \in \{1,\dots,N\}$.

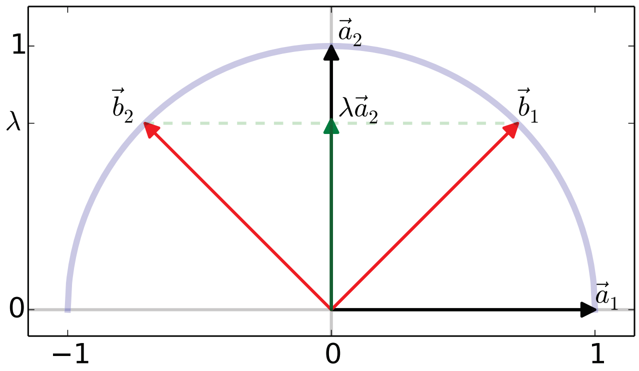


FIG. 4. Two orthogonal observables in plane.

(3) We perform a measurement of $\mathsf{S}^{\mathbf{b}_k}$ in the input state $\varrho$, hence obtaining an outcome $\beta_k = \pm 1$ with the probability $\mathrm{tr}[\varrho \mathsf{S}^{\mathbf{b}_k}(\pm 1)]$.

(4) For each $\ell = \{1, \ldots, M\}$, we decide that the outcome $\alpha_\ell$ is $\beta_k$ if $\mathbf{a}_\ell \cdot \mathbf{b}_k > 0$ and $-\beta_k$ if $\mathbf{a}_\ell \cdot \mathbf{b}_k < 0$.

(5) As a result, we get a list $(\alpha_1, \ldots, \alpha_M)$ of outcomes.

(6) It is possible that some combination $(\alpha_1, \ldots, \alpha_M)$ does not result in the process at all. In this case we set $\mathsf{G}(\alpha_1, \ldots, \alpha_M) = 0$.

The marginals of $\mathsf{G}$ are binary qubit observables, but they are not guaranteed to be unbiased noisy versions of the original observables $\mathsf{S}^{\mathbf{a}_1}, \ldots, \mathsf{S}^{\mathbf{a}_M}$. However, we next see that the marginals are unbiased noisy versions of the original observables in many symmetric situations if the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_N$ are chosen properly.

### A. Planar directions

We begin our discussion with a well-known example of two orthogonal qubit observables $\mathsf{S}^{\mathbf{x}}$ and $\mathsf{S}^{\mathbf{y}}$ [14] (see Fig. 4). We want to build our joint measurement by findin two measurement directions and then randomly performing one of the measurements. For this purpose we employ projective measurements in the directions which are equal superpositions of the Bloch vectors $\mathbf{x}, \mathbf{y}$ of $\mathsf{S}^{\mathbf{x}}(+)$ and $\mathsf{S}^{\mathbf{y}}(+)$. If an outcome of a measurement in the direction $\mathbf{b}_1 = \frac{1}{\sqrt{2}}(\mathbf{x}+\mathbf{y})$ is positive (negative), then the measurement outcomes of $\mathsf{S}^{\mathbf{x}}$ and $\mathsf{S}^{\mathbf{y}}$ are decided to be positive (negative) as $\mathbf{x} \cdot \mathbf{b}_1 > 0$ and $\mathbf{y} \cdot \mathbf{b}_1 > 0$. In a similar way, if a measurement in the direction $\mathbf{b}_2 = \frac{1}{\sqrt{2}}(\mathbf{x}-\mathbf{y})$ is performed and a positive (negative) outcome is obtained, then the measurement outcomes of $\mathsf{S}^{\mathbf{x}}$ and $\mathsf{S}^{\mathbf{y}}$ are decided to be $+$ and $-$ ($-$ and $+$) as $\mathbf{x} \cdot \mathbf{b}_2 > 0$ and $\mathbf{y} \cdot \mathbf{b}_2 < 0$. In this procedure the actually measured observable is a POVM $\mathsf{G}$ given by the operators

$$\mathsf{G}(+,+) = \tfrac{1}{2}\mathsf{S}^{\mathbf{b}_1}(+), \quad \mathsf{G}(-,-) = \tfrac{1}{2}\mathsf{S}^{\mathbf{b}_1}(-),$$
$$\mathsf{G}(+,-) = \tfrac{1}{2}\mathsf{S}^{\mathbf{b}_2}(+), \quad \mathsf{G}(-,+) = \tfrac{1}{2}\mathsf{S}^{\mathbf{b}_2}(-).$$

The marginals (postprocessings) of $\mathsf{G}$ are given as

$$\mathsf{G}(+,+) + \mathsf{G}(+,-) = \lambda \mathsf{S}^{\mathbf{x}}(+) + (1-\lambda)\tfrac{1}{2}\mathbb{1}, \tag{5}$$

$$\mathsf{G}(+,+) + \mathsf{G}(-,+) = \lambda \mathsf{S}^{\mathbf{y}}(+) + (1-\lambda)\tfrac{1}{2}\mathbb{1}, \tag{6}$$

with $\lambda = \frac{1}{\sqrt{2}} \approx 0.7071$. The parameter $\lambda$ is usually called the noise parameter. It tells how much white noise is added to the observable. Surprisingly, the obtained value of $\lambda$ is known to be necessary and sufficien for the joint measurability of the observables $\mathsf{S}^{\mathbf{x}}$ and $\mathsf{S}^{\mathbf{y}}$ [14].

The following proposition generalizes the previous example for symmetrical arrangement of qubit observables in a plane.

*Proposition 1*. Let $M \geqslant 2$ be an integer and

$$\mathbf{a}_k = \cos\theta_k \mathbf{x} + \sin\theta_k \mathbf{y}, \quad \theta_k = (k-1)\pi/M$$

for $k = 1, \ldots, M$. The observables $\mathsf{S}^{\lambda\mathbf{a}_1}, \ldots, \mathsf{S}^{\lambda\mathbf{a}_M}$ are jointly measurable if

$$\lambda \leqslant \frac{1}{M\sin(\frac{\pi}{2M})}.$$

*Proof*. Suppose that $M$ is odd. We choose $\mathbf{b}_k = \mathbf{a}_k$ for $k = 1, \ldots, M$ and follow the previously described procedure. We have

$$\begin{aligned}\mathbf{a}_k \cdot \mathbf{a}_\ell &= \cos\theta_k \cos\theta_\ell + \sin\theta_k \sin\theta_\ell \\ &= \cos(\theta_k - \theta_\ell) \\ &= \cos\frac{(k-\ell)\pi}{M}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbf{a}_k \cdot \mathbf{a}_\ell > 0 \quad &\text{if } |k-\ell| < M/2, \\ \mathbf{a}_k \cdot \mathbf{a}_\ell < 0 \quad &\text{if } |k-\ell| > M/2.\end{aligned}$$

The f rst marginal of $\mathsf{G}$ is

$$\sum_{\alpha_2,\ldots,\alpha_M=\pm1} \mathsf{G}(\alpha_1, \ldots, \alpha_M) = \mathsf{S}^{\lambda\mathbf{a}_1}(\alpha_1),$$

where

$$\lambda = \frac{1}{M}\left(1 + 2\sum_{k=1}^{(M-1)/2} \cos\left(\frac{k\pi}{M}\right)\right).$$

By Lagrange's trigonometric identity we have

$$\sum_{k=1}^{(M-1)/2} \cos\left(\frac{k\pi}{M}\right) = -\frac{1}{2} + \frac{1}{2\sin(\frac{\pi}{2M})}.$$

Suppose then that $M$ is even. We choose

$$\begin{aligned}\mathbf{b}_k &= \cos\left(\theta_k + \frac{\pi}{2M}\right)\mathbf{x} + \sin\left(\theta_k + \frac{\pi}{2M}\right)\mathbf{y}, \\ \theta_k &= \frac{(k-1)\pi}{M},\end{aligned}$$

for $k = 1, \ldots, M$. The firs marginal of $\mathsf{G}$ is

$$\sum_{\alpha_2,\ldots,\alpha_M=\pm1} \mathsf{G}(\alpha_1, \ldots, \alpha_M) = \mathsf{S}^{\lambda\mathbf{a}_1}(\alpha_1),$$

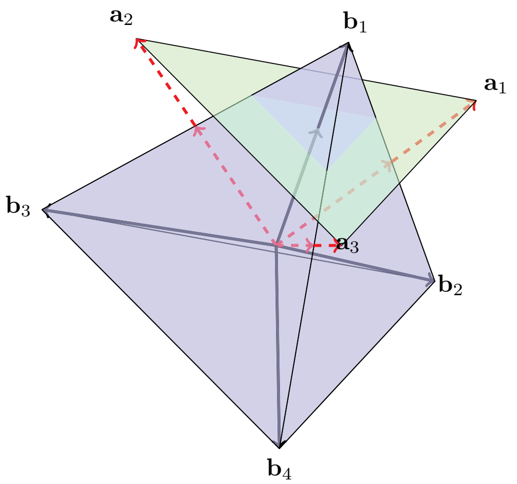


FIG. 5. Three observables. The vectors $\mathbf{a}_i, i = 1,2,3$, are the desired directions and the vectors $\mathbf{b}_j, j = 1,2,3,4$, are the guessing directions.

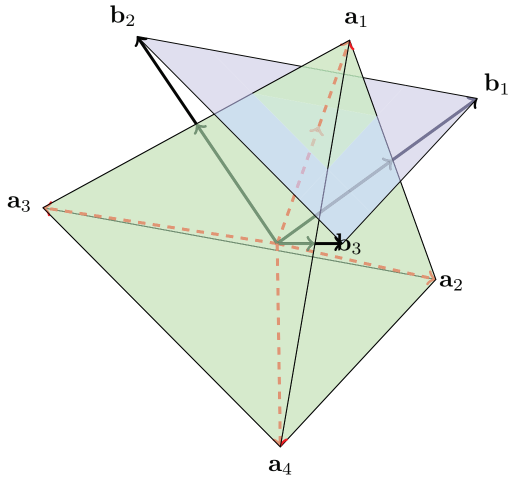


FIG. 6. Four observables. The vectors $\mathbf{a}_i, i = 1,2,3,4$, are the desired directions and the vectors $\mathbf{b}_j, j = 1,2,3$, are the guessing directions.

where

$$\lambda = \frac{2}{M}\sum_{k=1}^{M/2}\cos\left(\frac{(2k-1)\pi}{2M}\right) = \frac{1}{M\sin(\frac{\pi}{2M})}.$$

■

### B. Nonplanar directions

In this section we apply our method for qubit observables whose Bloch vectors are not in the same plane but are situated in a symmetric way. We explain the method for the case of three and four observables and we note that the noise parameter for these settings is the same. Moreover, we give the results concerning two more complicated cases, and the proofs for these results are given in the Appendix.

#### 1. *Three observables*

Let us first choose $\mathbf{a}_1 = \mathbf{x}$, $\mathbf{a}_2 = \mathbf{y}$, and $\mathbf{a}_3 = \mathbf{z}$. Adding also opposite directions the vectors would form an octahedron. However, since two vectors $\mathbf{a}$ and $-\mathbf{a}$ determine the same binary observable up to the permutation of outcomes, we keep only the positive directions.

For our adaptive strategy we choose $\mathbf{b}_1 = \frac{1}{\sqrt{3}}(\mathbf{x}+\mathbf{y}+\mathbf{z})$, $\mathbf{b}_2 = \frac{1}{\sqrt{3}}(-\mathbf{x}+\mathbf{y}+\mathbf{z})$, $\mathbf{b}_3 = \frac{1}{\sqrt{3}}(\mathbf{x}-\mathbf{y}+\mathbf{z})$, and $\mathbf{b}_4 = \frac{1}{\sqrt{3}}(-\mathbf{x}-\mathbf{y}+\mathbf{z})$; see Fig. 5. If in this case we measure for example in the direction of $\mathbf{b}_1$ and get a positive (negative) outcome our adaptive strategy assigns the value $+$ $(-)$ to all of the observables $\mathsf{A}_1$, $\mathsf{A}_2$, and $\mathsf{A}_3$. The nonzero elements of the constructed joint observable $\mathsf{G}$ are thus (see Fig. 5)

$$\mathsf{G}(+,+,+) = \tfrac{1}{4}\mathsf{S}^{\mathbf{b}_1}(+), \quad \mathsf{G}(-,-,-) = \tfrac{1}{4}\mathsf{S}^{\mathbf{b}_1}(-),$$
$$\mathsf{G}(+,+,-) = \tfrac{1}{4}\mathsf{S}^{\mathbf{b}_4}(-), \quad \mathsf{G}(-,-,+) = \tfrac{1}{4}\mathsf{S}^{\mathbf{b}_4}(+),$$
$$\mathsf{G}(+,-,+) = \tfrac{1}{4}\mathsf{S}^{\mathbf{b}_3}(+), \quad \mathsf{G}(-,+,-) = \tfrac{1}{4}\mathsf{S}^{\mathbf{b}_3}(-),$$
$$\mathsf{G}(+,-,-) = \tfrac{1}{4}\mathsf{S}^{\mathbf{b}_2}(-), \quad \mathsf{G}(-,+,+) = \tfrac{1}{4}\mathsf{S}^{\mathbf{b}_2}(+).$$

The marginals of $\mathsf{G}$ are $\mathsf{S}^{\lambda\mathbf{a}_1}$, $\mathsf{S}^{\lambda\mathbf{a}_2}$, and $\mathsf{S}^{\lambda\mathbf{a}_3}$ with the noise parameter $\lambda = 1/\sqrt{3} \approx 0.5774$, and we have thus reproduced the result first proved in [14]. Let us note that this noise parameter is known to be the boundary point, meaning that for any $\lambda > 1/\sqrt{3}$ the three observables are incompatible [16].

#### 2. *Four observables*

We then choose $\mathbf{a}_1 = \frac{1}{\sqrt{3}}(\mathbf{x}+\mathbf{y}+\mathbf{z})$, $\mathbf{a}_2 = \frac{1}{\sqrt{3}}(\mathbf{x}-\mathbf{y}-\mathbf{z})$, $\mathbf{a}_3 = \frac{1}{\sqrt{3}}(-\mathbf{x}+\mathbf{y}-\mathbf{z})$, and $\mathbf{a}_4 = \frac{1}{\sqrt{3}}(-\mathbf{x}-\mathbf{y}+\mathbf{z})$, so the vectors go to the vertices of a tetrahedron. For our adaptive strategy we choose $\mathbf{b}_1 = \mathbf{x}$, $\mathbf{b}_2 = \mathbf{y}$, and $\mathbf{b}_3 = \mathbf{z}$; see Fig. 6. Now the nonzero elements of the joint observable $\mathsf{G}$ are

$$\mathsf{G}(+,+,-,-) = \tfrac{1}{3}\mathsf{S}^{\mathbf{b}_1}(+), \quad \mathsf{G}(-,-,+,+) = \tfrac{1}{3}\mathsf{S}^{\mathbf{b}_1}(-),$$
$$\mathsf{G}(+,-,+,-) = \tfrac{1}{3}\mathsf{S}^{\mathbf{b}_2}(+), \quad \mathsf{G}(-,+,-,+) = \tfrac{1}{3}\mathsf{S}^{\mathbf{b}_2}(-),$$
$$\mathsf{G}(+,-,-,+) = \tfrac{1}{3}\mathsf{S}^{\mathbf{b}_3}(+), \quad \mathsf{G}(-,+,+,-) = \tfrac{1}{3}\mathsf{S}^{\mathbf{b}_3}(-).$$

The marginals of $\mathsf{G}$ are $\mathsf{S}^{\lambda\mathbf{a}_j}$ with $\lambda = \frac{1}{\sqrt{3}}$. This value of the noise parameter coincides with the one for three observables (see previous section).

#### 3. *Six and ten observables*

An icosahedron has 12 vertices and these directions determine six binary observables, while a dodecahedron has 20 vertices and these directions determine ten binary observables. Using the adaptive strategy we obtain the following values for the noise parameters (see the Appendix): $\lambda_6 = \frac{1+\sqrt{5}}{6} \approx 0.5393$ and $\lambda_{10} = \frac{3+\sqrt{5}}{10} \approx 0.5236$. As we later see, these are the least noise values making the binary observables jointly measurable.

### C. Adaptive strategy for the nonsymmetric case

The adaptive strategy gives an optimal joint observable also for the case of two arbitrary unbiased qubit observables. For this purpose, let $\mathbf{a}_1$ and $\mathbf{a}_2$ be two Bloch vectors. We choose $\mathbf{b}_1 = \frac{1}{\|\mathbf{a}_1+\mathbf{a}_2\|}(\mathbf{a}_1+\mathbf{a}_2)$ and $\mathbf{b}_2 = \frac{1}{\|\mathbf{a}_1-\mathbf{a}_2\|}(\mathbf{a}_1-\mathbf{a}_2)$ to be the equal superpositions of these vectors. The adaptive strategy gives a joint observable G define by

$$\begin{aligned}
\mathsf{G}(+,+) &= \mu \mathsf{S}^{\mathbf{b}_1}(+),\\
\mathsf{G}(+,-) &= (1-\mu)\mathsf{S}^{\mathbf{b}_2}(+),\\
\mathsf{G}(-,+) &= (1-\mu)\mathsf{S}^{\mathbf{b}_2}(-),\\
\mathsf{G}(-,-) &= \mu \mathsf{S}^{\mathbf{b}_1}(-),
\end{aligned}$$

where the constant $\mu$ is the probability of making the measurement in the direction $\mathbf{b}_1$ and $1-\mu$ is the probability of making the measurement in the direction $\mathbf{b}_2$.

In order to get the correct marginals for G, i.e., $\mathsf{S}^{\lambda\mathbf{a}_1}$ and $\mathsf{S}^{\lambda\mathbf{a}_2}$, one needs to have

$$\mu = \frac{\|\mathbf{a}_1+\mathbf{a}_2\|}{\|\mathbf{a}_1+\mathbf{a}_2\|+\|\mathbf{a}_1-\mathbf{a}_2\|},$$

and this gives

$$\lambda = \frac{2}{\|\mathbf{a}_1+\mathbf{a}_2\|+\|\mathbf{a}_1-\mathbf{a}_2\|},$$

which coincides with the optimal value originally presented in [14].

## IV. GENERALIZED ADAPTIVE STRATEGY FOR MUBS

In order to use our strategy for measurements with more than two outcomes in a Hilbert space whose dimension is larger than or equal to two, one needs to introduce two minor modifications First, in step 3 of the strategy a measurement of a two-valued observable $\mathsf{S}^{\mathbf{b}_i}(\pm)$ is performed. In the general case such a measurement is not enough to distinguish between the outcomes of the desired observables. For example in the case where one of the desired observables (say A) is three valued, a two-valued observable could at best distinguish one of the outcomes of A and leave the other two open. This lack of distinguishability can be circumvented. Namely, instead of flippin a coin between two-valued observables, we build up the joint observable from the fi ed vectors and not from the two-valued observables assigned to them.

Second, in the qubit case our strategy requires f xed Bloch vectors. The values of the joint observable are decided by checking the overlaps between these vectors and the Bloch vectors of the desired observables. In order to generalize our strategy we note that the inner products of the Bloch vectors (say $\mathbf{a}$ and $\mathbf{b}$) and the inner products of the corresponding Hilbert space vectors (say $\psi_{\mathbf{a}}$ and $\psi_{\mathbf{b}}$) are related by the formula

$$|\langle\psi_{\mathbf{a}}|\psi_{\mathbf{b}}\rangle|^2 = \tfrac{1}{2}(1+\mathbf{a}\cdot\mathbf{b}). \tag{7}$$

Hence we can fi Hilbert space vectors instead of Bloch vectors and decide the values of the joint observable by maximizing the inner products between the f xed rank-1 operators and the effects of the desired observables. More generally, one could fi higher rank effects and maximize the overlaps between these effects and the effects of the observables.

The generalization withdraws step 3 from our strategy. Step 3 can be seen as a requirement for the normalization of the joint observable and, hence, we replace the third step by the condition that the f xed Hilbert space vectors as rank-1 operators must sum up to the identity.

To illustrate the generalized adaptive strategy we consider the case of two MUBs in a $d$-dimensional Hilbert space. Let $\{\varphi_j\}_{j=1}^n$ and $\{\psi_k\}_{k=1}^n$ be two mutually unbiased bases of $\mathbb{C}^n$, i.e., $|\langle\varphi_j\,|\,\psi_k\rangle| = 1/\sqrt{d}$, and let $\mathsf{A}_1(j) = |\varphi_j\rangle\langle\varphi_j|$ and $\mathsf{A}_2(k) = |\psi_k\rangle\langle\psi_k|$, $j,k = 1,\ldots,d$, be the corresponding observables.

We defin unit vectors $b_{j,k}$ like in the previous section as equal superpositions of the desired directions $\varphi_j$ and $\psi_k$:

$$b_{j,k} = N(\varphi_j + e^{i\theta_{j,k}}\psi_k), \quad e^{i\theta_{j,k}} = \sqrt{d}\langle\psi_k|\varphi_j\rangle,$$

where $N$ is a normalization factor. It is easy to see that the overlaps $|\langle\varphi_m|b_{j,k}\rangle|^2$ and $|\langle\psi_n|b_{j,k}\rangle|^2$ are both maximal when $m = j$ and $n = k$. Hence we build up a joint measurement candidate by definin

$$\mathsf{G}(j,k) = \frac{N^2}{d}|b_{j,k}\rangle\langle b_{j,k}|.$$

It is now straightforward to check that G sums up to identity and that it gives as marginals the smeared versions of $\mathsf{A}_1$ and $\mathsf{A}_2$ with the amount

$$\lambda = \frac{1}{2}\left(1+\frac{1}{1+\sqrt{d}}\right)$$

of white noise. This value was earlier shown to be the optimal noise parameter in the case of Fourier-connected MUBs [18,19].

## V. PROVING OPTIMALITY OF ADAPTIVE JOINT MEASUREMENTS USING STEERING

In this latter part of our investigation we prove that the joint measurements considered in the previous sections are actually optimal. In other words, we show that their marginals possess the least possible amount of white noise, meaning that for any larger value of the noise parameter $\lambda$ the respective observables are incompatible. For proving the optimality, we employ some known steering techniques. In this section we recall the basic setup of steering and explain why it helps to prove the optimality of the presented joint measurements.

Consider a bipartite scenario (Alice and Bob) sharing a quantum state $\rho_{AB}$. In a quantum steering task Alice tries to convince Bob that the shared state is entangled by making only local measurements $\mathsf{A}_k$, $k = 1,\ldots,n$, on her system and sending Bob the respective outcomes $x = 1,\ldots,m$ by classical communication.

When Alice measures an observable $\mathsf{A}_k$ and gets an outcome $x$ Bob's conditional (non-normalized) state reads

$$\sigma_{x|k} = \mathrm{tr}_A[(\mathsf{A}_k(x)\otimes\mathbb{1})\rho_{AB}]. \tag{8}$$

We firs notice that for any separable state $\sum_i p_i\rho_A^i\otimes\rho_B^i$ the conditional states always read

$$\sigma_{x|k} = \sum_i \mathrm{tr}\big[\mathsf{A}_k(x)\rho_A^i\big]p_i\rho_B^i. \tag{9}$$

This kind of an ensemble could also be created by classically postprocessing a local set of (non-normalized) states $\{p_i\rho_B^i\}_i$ on Bob's side.

If, however, Bob runs over all local ensembles of non-normalized states $\{\sigma_i\}_i$ together with all possible postprocessings and find out that he cannot reproduce his conditional states, he concludes that the shared state must be entangled. To be more precise, a bipartite setup (i.e., $\rho_{AB}$ and Alice's measurements $\{\mathsf{A}_k\}_{k=1}^n$) is nonsteerable if there exists an ensemble of positive semidefinit operators $\{\sigma_\eta\}_\eta$ together with classical postprocessings $p(x|k,\eta)$ such that

$$\sigma_{x|k} = \sum_\eta p(x|k,\eta)\sigma_\eta. \tag{10}$$

If this is not the case, the setup is called steerable. For a steerable (nonsteerable) setup it is said that Alice can (cannot) steer Bob.

One notices that the definition of steerability (10) and joint measurability (1) look very similar. It was proven in [8,9] that this similarity originates from a one-to-one correspondence between these concepts: compatible measurements never allow Alice to steer Bob and steering is always possible with incompatible observables provided that the Schmidt rank of the shared pure state is $d$. In order to emphasize the link between steering and joint measurements it is convenient to use the maximally entangled state $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_i |ii\rangle$ in a steering scenario. For the maximally entangled state the conditional states read

$$\sigma_{x|k} = \frac{1}{d}\mathsf{A}_k(x)^T, \tag{11}$$

where $T$ denotes the transpose. In what follows, we use this link to show that adaptive joint measurements are optimal in the sense that they possess the least possible amount of white noise.

### A. Necessary condition for qubit observables

In this part we use known steering results to obtain the necessary condition for joint measurability of qubit observables. The steering inequality introduced in [23] reads

$$\frac{1}{n}\sum_{k=1}^n \mathrm{tr}[(\mathsf{A}_k \otimes \mathbf{c}_k\cdot\boldsymbol{\sigma}_k)\rho_{AB}] \leqslant C_n, \tag{12}$$

where $\mathsf{A}_k = \mathsf{A}_k(+) - \mathsf{A}_k(-)$, $\mathbf{c}_k$ is a Bloch vector, and $\rho_{AB}$ is a state of the composite system. The bound $C_n$ is the maximum value for the expression

$$\frac{1}{n}\sum_{k=1}^n \sum_{x_k=\pm1} x_k \mathrm{tr}[\sigma_{x|k}\mathbf{c}_k\cdot\boldsymbol{\sigma}_k], \tag{13}$$

provided that the operators $\sigma_{x|k}$ are of the form (10). Here we have labeled the outcomes of the measurement $\mathsf{A}_k$ by $x_k$. The bound $C_n$ is obtained as

$$C_n = \max_{x_k=\pm1}\left(\lambda_{\max}\left(\frac{1}{n}\sum_{k=1}^n x_k\mathbf{c}_k\cdot\boldsymbol{\sigma}_k\right)\right), \tag{14}$$

where $\lambda_{\max}(K)$ is the largest eigenvalue of a matrix $K$.

For example, in the case of planar observables (Proposition 1) one gets

$$C_M^{\mathrm{planar}} = \frac{1}{M\sin\left(\frac{\pi}{2M}\right)}. \tag{15}$$

This is the maximum value for $C_M^{\mathrm{planar}}$ such that the scenario is not steerable. This means, like we discussed in the previous section, that a violation of inequality (12) is only possible if Alice's observables are incompatible.

To see that in the planar case our adaptive strategy gives the optimal joint observable, let $|\psi\rangle\langle\psi|$ denote the maximally entangled state and notice that

$$\mathrm{tr}_A[\mathsf{A}_k(x)\otimes\mathbb{1}|\psi^\lambda\rangle\langle\psi^\lambda|] = \mathrm{tr}_A\left[\mathsf{A}_k^\lambda(x)\otimes\mathbb{1}|\psi\rangle\langle\psi|\right], \tag{16}$$

where $|\psi^\lambda\rangle\langle\psi^\lambda| = \lambda|\psi\rangle\langle\psi| + \frac{1-\lambda}{4}\mathbb{1}\otimes\mathbb{1}$ and $\mathsf{A}_k^\lambda(x) = \lambda\mathsf{A}_k(x) + \frac{1-\lambda}{2}\mathrm{tr}[\mathsf{A}_k(x)]\mathbb{I}$.

Inserting the noisy maximally entangled state $|\psi^\lambda\rangle\langle\psi^\lambda|$, the Bloch vectors of the planar observables from Proposition 1 for Bob, and the transposed planar observables for Alice[1] into the left-hand side of Eq. (12), one arrives at the condition $\lambda \leqslant C_M^{\mathrm{planar}}$. In other words, if $\lambda$ is larger than this threshold one violates the steering inequality and consequently Alice's observables are nonjointly measurable (transposition does not affect joint measurability) with the amount $\lambda$ of white noise [see Eq. (16)]. This means that joint measurability of Alice's observables implies that $\lambda \leqslant C_M^{\mathrm{planar}}$. Hence, we arrive at the following result.

*Proposition 2*. The planar observables introduced in Proposition 1 are jointly measurable if and only if

$$\lambda \leqslant \frac{1}{M\sin\left(\frac{\pi}{2M}\right)}.$$

Calculating the values $C_M$ for the cases of Platonic solids and repeating the same procedure as above one find that the joint measurements given by the adaptive strategy are optimal also in these cases. We summarize the results of this section in the following statement.

*Proposition 3*. The sufficien joint measurability conditions given in Secs. III A and III B are also necessary.

### B. Necessary condition for MUBs

In this part we prove that the joint measurability conditions given in Sec. IV are optimal. For this purpose, suppose that Alice performs two different $d$-valued projective measurements given by two sets of MUBs $\{\varphi_j\}_{j=1}^d$ and $\{\psi_k\}_{k=1}^d$ as follows:

$$\mathsf{A}_1(j) = |\varphi_j\rangle\langle\varphi_j|, \quad \mathsf{A}_2(k) = |\psi_k\rangle\langle\psi_k|. \tag{17}$$

The ensemble of states that Bob observes is denoted by $\{\sigma_{x|i}\}_{x\in\mathbb{Z}_d}$, $i = 1,2$.

For this scenario the underlying state is nonsteerable if and only if there exists a collection $\{\omega_{jk}\}_{jk}$ of positive semidefinit operators $\omega_{jk} \geqslant 0$ such that

$$\sigma_{j|1} = \sum_k \omega_{jk}, \quad \sigma_{k|2} = \sum_j \omega_{jk}, \tag{18}$$

[1] We choose transposed observables because using the properties of the maximally entangled state gives then a simple condition for steering:

$$\mathrm{tr}\left[\mathsf{A}_k^T\otimes\mathsf{A}_k|\psi^\lambda\rangle\langle\psi^\lambda|\right] = \lambda.$$

Note that as this is the case for any choice of observables $\mathsf{A}_k$, one can use Eq. (14) to calculate necessary joint measurability conditions for arbitrary qubit observables.

hold for all possible $j,k$. Note that here the notation $\omega_{jk}$ combines the postprocessings and the positive operators in Eq. (10). To certify that a decomposition of this form is impossible one needs to violate a steering inequality. For our purpose a useful inequality is characterized by a collection of operators $\{Z_{jk}\}_{jk}$ which are all positive semidefinit $Z_{jk} \geqslant 0$ and furthermore satisfy the linear constraints

$$Z_{jk} = Z_{jt} + Z_{sk} - Z_{st} \tag{19}$$

for all possible $j,k,s,t$. If the observed ensembles $\{\sigma_{j|1}\}_{j\in\mathbb{Z}_d}$ and $\{\sigma_{k|2}\}_{k\in\mathbb{Z}_d}$ do have a decomposition of the above form, then it holds that

$$\begin{aligned}\sum_{jk}\mathrm{tr}[Z_{jk}\omega_{jk}] &= \sum_{jk}\mathrm{tr}[(Z_{jt}+Z_{sk}-Z_{st})\omega_{jk}]\\ &= \sum_j \mathrm{tr}\left[Z_{jt}\left(\sum_k \omega_{jk}\right)\right]\\ &\quad + \sum_k \mathrm{tr}\left[Z_{sk}\left(\sum_j \omega_{jk}\right)\right]\\ &\quad - \mathrm{tr}\left[Z_{st}\left(\sum_{j,k}\omega_{jk}\right)\right] \qquad (20)\\ &= \sum_j \mathrm{tr}[Z_{jt}\sigma_{j|1}] + \sum_k \mathrm{tr}[Z_{sk}\sigma_{k|2}]\\ &\quad - \mathrm{tr}[Z_{st}\rho] \geqslant 0, \qquad (21)\end{aligned}$$

for all $s,t$. Note that non-negativity holds because all $Z_{jk}$ and $\omega_{jk}$ are positive semidefinite In addition we have used the fact that $\rho = \sum_j \sigma_{j|1} = \sum_k \sigma_{k|2}$. Hence if one violates this inequality one proves that such a positive semidefinit decomposition is impossible.

### C. Employed inequality

In the following we use the following steering inequality. For the two given mutually unbiased basis sets $\{|\varphi_j\rangle\}_{j\in\mathbb{Z}_d}$, $\{|\psi_k\rangle\}_{k\in\mathbb{Z}_d}$, we defin

$$Z_{jk} = a(|\varphi_j\rangle\langle\varphi_j| + |\psi_k\rangle\langle\psi_k|) + b\mathbb{1}, \tag{22}$$

with

$$a = -\frac{1}{(\sqrt{d}-1)(\sqrt{d}+2)}, \quad b = \frac{\sqrt{d}+1}{\sqrt{d}(\sqrt{d}-1)(\sqrt{d}+2)}. \tag{23}$$

In this form one directly verifie that all linear constraints given by Eq. (19) are satisfied In order to show that these operators are also positive semidefinit we express them in a different way.

Since the sets are mutually unbiased one has $\langle\varphi_j\,|\,\psi_k\rangle = e^{i\theta_{jk}}/\sqrt{d}$. If one incorporates this phase into the superposition states

$$|\chi^{\pm}_{jk}\rangle = \frac{1}{N_\pm}(|\varphi_j\rangle \pm e^{-i\theta_{jk}}|\psi_k\rangle), \tag{24}$$

one achieves that the resulting states become orthogonal, $\langle\chi^+_{jk}\,|\,\chi^-_{jk}\rangle = 0$. Furthermore, note that the normalization is then also independent of $j$ and $k$.

Via these definition one can rewrite the operators given by Eq. (22) as

$$Z_{jk} = c\,|\chi^-_{jk}\rangle\langle\chi^-_{jk}| + b(\mathbb{1} - |\chi^-_{jk}\rangle\langle\chi^-_{jk}| - |\chi^+_{jk}\rangle\langle\chi^+_{jk}|), \tag{25}$$

with

$$c = \frac{2}{\sqrt{d}(\sqrt{d}-1)(\sqrt{d}+2)}. \tag{26}$$

Since both terms in Eq. (25) consist of a positive constant multiplied with a positive semidefinit operator, this shows that all $Z_{jk}$ are positive semidefinite

Note that in order to derive the coefficients one can start with $Z_{jk}$ as given by Eq. (25) but with open $b,c$. In order to achieve a form like Eq. (22) the coefficient in front of the terms $|\varphi_j\rangle\langle\psi_k|$ and $|\psi_j\rangle\langle\varphi_k|$ must vanish, which holds if

$$\frac{c-b}{N_-^2} + \frac{b}{N_+^2} = 0. \tag{27}$$

The solution which additionally satisfie $\mathrm{tr}[Z_{jk}] = 1$ is the given solution.

### D. The implication

In this part we apply the steering inequality to derive the statement that the two observables $\mathsf{A}_1^\lambda$ and $\mathsf{A}_2^\lambda$ are not jointly measurable if $\lambda_{\max} \leqslant \lambda$.

We measure the maximally entangled state $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_i |ii\rangle$ with the observables $\mathsf{A}_1^{\lambda\,T}$ and $\mathsf{A}_2^{\lambda\,T}$.[2] By the properties of the maximally entangled state, the conditional states are given by

$$\sigma_{j|1} = \frac{1}{d}\mathsf{A}_1^\lambda(j), \quad \sigma_{k|2} = \frac{1}{d}\mathsf{A}_2^\lambda(k). \tag{28}$$

If we now apply the steering inequality from Eq. (21) with the operators $Z_{jk}$ given by Eq. (22) one obtains

$$\begin{aligned} d\sum_{jk}\mathrm{tr}[Z_{jk}\omega_{jk}] &= \sum_j \mathrm{tr}\big[Z_{jt}\mathsf{A}_1^\lambda(j)\big]\\ &\quad + \sum_k \mathrm{tr}\big[Z_{sk}\mathsf{A}_2^\lambda(k)\big] - \mathrm{tr}[Z_{st}\mathbb{1}]\\ &= 2d\left\{\lambda\left[a\left(1+\frac{1}{d}\right)+b\right] + \frac{1-\lambda}{d}\right\} - 1\\ &= 2\{1-\lambda[1-a(d+1)-db]\} - 1\\ &= 2\left\{1-\lambda\left(1-\frac{1}{\sqrt{d}+2}\right)\right\} - 1 \geqslant 0.\end{aligned}$$

The inequality only holds if

$$\lambda \leqslant \frac{1}{2}\left(1+\frac{1}{1+\sqrt{d}}\right) = \lambda_{\max}, \tag{29}$$

and this proves the statement.

[2] If we violate a steering inequality it means that $\mathsf{A}_1^{\lambda\,T}$,$\mathsf{A}_2^{\lambda\,T}$ are not jointly measurable; this implies further that also the transposed measurements are not jointly measurable.

## VI. 4-SPECKER SET OF QUBIT OBSERVABLES

As a consequence of our previous results we can now show that there exists a so-called 4-Specker set [17] consisting of qubit observables. This means that there exists a set of four incompatible qubit observables which are not jointly measurable but any triplet of these observables forms a jointly measurable set. The precise set of these four observables is the following.

*Proposition 4*. Consider four qubit observables $\mathsf{S}^{\lambda \mathbf{a}_k}$ define by equally distributed Bloch vectors on the upper side of the $xy$ plane; i.e., the angle between the vector $k$ and $k+1$ is $\pi/4$. There exists a smearing parameter $\lambda$ such that the observables $\mathsf{S}^{\lambda \mathbf{a}_k}$ form a 4-Specker set.

*Proof 2*. Because of the symmetry of the situation, every subset of three observables $\{\mathsf{S}^{\lambda \mathbf{a}_k}\}_{k \neq i}$ for some $i$ is jointly measurable if and only if (see Eq. (16) in [16])

$$\lambda \leqslant [\cos(\pi/4) + 2\sin(\pi/8)]^{-1} \approx 0.679. \tag{30}$$

By Proposition 2, however, the set of four observables is incompatible if and only if $\lambda > \frac{1}{4\sin(\pi/8)} \approx 0.65$. Hence, choosing any $\lambda$ which is between these thresholds gives a 4-Specker set $\{\mathsf{S}^{\lambda \mathbf{a}_k}\}_{k=1}^{4}$ of qubit observables. ■

It has been earlier shown that qubit observables can form a 3-Specker set [17,25], and that a 4-Specker set exists in dimension four [20].

## VII. CONCLUSION

Our method opens up possibilities for future research on both quantum incompatibility and steering. First, we have tested our method on basic symmetric and nonsymmetric measurement setups and shown its power in these scenarios by reproducing some known and various new noise thresholds. The open question is how far the method can be fetched, i.e., how to fin  optimal vectors $\mathbf{b}_i$ for more complicated measurement settings. Second, it is an open problem to decide which pairs of quantum observables are the most noise resistant ones in finit  dimension. Our method gives lower bounds on the noise robustness of sets of measurements. Third, our technique might have interesting applications in quantum steering as every joint observable works as a local hidden state model for steering attempts with a restricted set of measurements [13]. These questions are left for now for future works.

## ACKNOWLEDGMENT

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## APPENDIX A: SIX OBSERVABLES

Denote $\chi = \frac{1+\sqrt{5}}{2}$ and $n = \frac{5+\sqrt{5}}{2}$. Let

$$\mathbf{a}_1 = \frac{1}{\sqrt{n}}(\mathbf{y} + \chi\mathbf{z}), \quad \mathbf{a}_2 = \frac{1}{\sqrt{n}}(-\mathbf{y} + \chi\mathbf{z}),$$

$$\mathbf{a}_3 = \frac{1}{\sqrt{n}}(\mathbf{x} + \chi\mathbf{y}), \quad \mathbf{a}_4 = \frac{1}{\sqrt{n}}(-\mathbf{x} + \chi\mathbf{y}),$$

$$\mathbf{a}_5 = \frac{1}{\sqrt{n}}(\chi\mathbf{x} + \mathbf{z}), \quad \mathbf{a}_6 = \frac{1}{\sqrt{n}}(-\chi\mathbf{x} + \mathbf{z}).$$

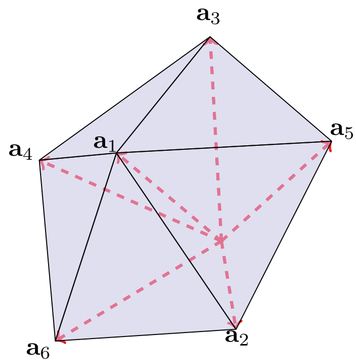


FIG. 7. Six observables (icosahedron). The desired directions $\mathbf{a}_i$ are the same as the guessing directions $\mathbf{b}_j$.

We choose $\mathbf{b}_i = \mathbf{a}_i$ for $i = 1, \ldots, 6$ and obtain the elements of the joint observable $\mathsf{G}$ according to Fig. 7 as follows.

The nonzero elements of the constructed joint observable $\mathsf{G}$ are

$$\mathsf{G}(+,+,+,+,+,+) = \tfrac{1}{6}\mathsf{S}^{\mathbf{b}_1}(+),$$
$$\mathsf{G}(-,-,-,-,-,-) = \tfrac{1}{6}\mathsf{S}^{\mathbf{b}_1}(-),$$
$$\mathsf{G}(+,+,-,-,+,+) = \tfrac{1}{6}\mathsf{S}^{\mathbf{b}_2}(+),$$
$$\mathsf{G}(-,-,+,+,-,-) = \tfrac{1}{6}\mathsf{S}^{\mathbf{b}_2}(-),$$
$$\mathsf{G}(+,-,+,+,+,-) = \tfrac{1}{6}\mathsf{S}^{\mathbf{b}_3}(+),$$
$$\mathsf{G}(-,+,-,-,-,+) = \tfrac{1}{6}\mathsf{S}^{\mathbf{b}_3}(-),$$
$$\mathsf{G}(+,-,+,+,-,+) = \tfrac{1}{6}\mathsf{S}^{\mathbf{b}_4}(+),$$
$$\mathsf{G}(-,+,-,-,+,-) = \tfrac{1}{6}\mathsf{S}^{\mathbf{b}_4}(-),$$
$$\mathsf{G}(+,+,+,-,+,-) = \tfrac{1}{6}\mathsf{S}^{\mathbf{b}_5}(+),$$

TABLE I. Outcomes of the joint observable $\mathsf{G}$ for ten observables.

| $k$ | Outcome of $\mathsf{S}^{\mathbf{b}_k}$ | Outcome of $\mathsf{G}$ |
|---|---|---|
| 1 | + | (+,+,+,+,+,+,+,+,+,−) |
|  | − | (−,−,−,−,−,−,−,−,−,+) |
| 2 | + | (+,+,−,−,+,+,−,−,+,−) |
|  | − | (−,−,+,+,−,−,+,+,−,+) |
| 3 | + | (+,−,+,−,+,+,+,+,−,+) |
|  | − | (−,+,−,+,−,−,−,−,+,−) |
| 4 | + | (+,−,−,+,−,−,+,+,+,−) |
|  | − | (−,+,+,−,+,+,−,−,−,+) |
| 5 | + | (+,+,+,−,+,+,+,+,+,+) |
|  | − | (−,−,−,+,−,−,−,−,−,−) |
| 6 | + | (+,+,+,−,+,+,−,−,+,+) |
|  | − | (−,−,−,+,−,−,+,+,−,−) |
| 7 | + | (+,−,+,+,+,−,+,+,+,−) |
|  | − | (−,+,−,−,−,+,−,−,−,+) |
| 8 | + | (+,−,+,+,+,−,+,+,−,+) |
|  | − | (−,+,−,−,−,+,−,−,+,−) |
| 9 | + | (+,+,−,+,+,+,+,−,+,−) |
|  | − | (−,−,+,−,−,−,−,+,−,+) |
| 10 | + | (−,−,+,−,+,+,−,+,−,+) |
|  | − | (+,+,−,+,−,−,+,−,+,−) |

$$\mathsf{G}(-,-,-,+,-,+) = \tfrac{1}{6}\mathsf{S}^{\mathbf{b}_5}(-),$$
$$\mathsf{G}(+,+,-,+,-,+) = \tfrac{1}{6}\mathsf{S}^{\mathbf{b}_6}(+),$$
$$\mathsf{G}(-,-,+,-,+,-) = \tfrac{1}{6}\mathsf{S}^{\mathbf{b}_6}(-).$$

Marginals of $\mathsf{G}$ are $\mathsf{S}^{\lambda\mathbf{a}_i}$, $i \in 1,\dots,6$, with $\lambda = \frac{1+\sqrt{5}}{6}$.

## APPENDIX B: TEN OBSERVABLES

We denote $\chi = \frac{1+\sqrt{5}}{2}$. Let

$$\mathbf{a}_1 = \frac{1}{\sqrt{3}}(\mathbf{x}+\mathbf{y}+\mathbf{z}), \quad \mathbf{a}_2 = \frac{1}{\sqrt{3}}(\mathbf{x}-\mathbf{y}+\mathbf{z}),$$
$$\mathbf{a}_3 = \frac{1}{\sqrt{3}}(-\mathbf{x}+\mathbf{y}+\mathbf{z}), \quad \mathbf{a}_4 = \frac{1}{\sqrt{3}}(\mathbf{x}+\mathbf{y}-\mathbf{z}),$$
$$\mathbf{a}_5 = \frac{1}{\sqrt{3}}(\chi^{-1}\mathbf{y}+\chi\mathbf{z}), \quad \mathbf{a}_6 = \frac{1}{\sqrt{3}}(-\chi^{-1}\mathbf{y}+\chi\mathbf{z}),$$
$$\mathbf{a}_7 = \frac{1}{\sqrt{3}}(\chi^{-1}\mathbf{x}+\chi\mathbf{y}), \quad \mathbf{a}_8 = \frac{1}{\sqrt{3}}(-\chi^{-1}\mathbf{x}+\chi\mathbf{y}),$$
$$\mathbf{a}_9 = \frac{1}{\sqrt{3}}(\chi\mathbf{x}+\chi^{-1}\mathbf{z}), \quad \mathbf{a}_{10} = \frac{1}{\sqrt{3}}(-\chi\mathbf{x}+\chi^{-1}\mathbf{z}).$$

We choose $\mathbf{b}_i = \mathbf{a}_i$ for $i = 1,\dots,10$ and obtain the outcomes of the joint observable $\mathsf{G}$ according to Table I.

The nonzero elements of the constructed joint observable $\mathsf{G}$ are the ones given by Table I divided by the number of guessing observables. Marginals of $\mathsf{G}$ are $\mathsf{S}^{\lambda\mathbf{a}_i}$, $i \in 1,\dots,10$, with $\lambda = \frac{1+\chi}{5}$.


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