Aspects of Quantum Resources: Coherence, Measurements, and Network Correlations

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Abstract

This thesis is devoted to different aspects in quantum information theory. We will put forward new results on the topics of coherence theory, correlations in quantum networks, measurement incompatibility, quantification of quantum resources, as well as a connection between channel compatibility and the quantum marginal problem.

First, we will introduce the notion of genuine correlated coherence, which is defined as the amount of coherence that remains after applying global incoherent unitaries, which are deemed to be free in a resource theoretic approach to coherence. This contributes to an ongoing discussion on the possible free operations in a resource theory of multipartite coherence and reveals a connection to genuine multilevel entanglement.

Then, we will derive monogamy relations that capture the trade-off in the coherence that can exist between multiple orthogonal subspaces. Such a trade-off puts limits on the distinguishability of quantum states under unitary time evolution when measurements are restricted to subspaces. Moreover, this will allow us to derive criteria detecting genuine multisubspace coherence of the density matrix, which has applications in, e.g., the characterization of quantum networks.

Next, we will turn our focus to correlations in quantum networks. We will show how the structure of the network limits the distribution of entanglement, focusing on the so-called triangle network. We derive several necessary criteria for states to be preparable in the triangle network, based on the independence of the sources, entanglement monogamy and constraints on the local ranks. Then, we will consider a different approach based on the coherence properties of covariance matrices that arise from performing measurements on a network state. We will use the theory of coherence to analyze the relevant properties of the covariance matrices. This allows us to witness probability distributions that are incompatible with the structure of the network.

Another large part of this thesis is concerned with the quantification of quantum resources. We will first show that incompatible measurements provide an advantage over all compatible measurements in certain instances of quantum state discrimination. This provides an operational characterization of measurement incompatibility and opens a possibility of its semi-device-independent verification. The result is based on properties of the so-called incompatibility robustness.

Subsequently, we will show that such a result is a rather generic feature of the generalized robustness. More precisely, we will show that in any convex resource theory of states, measurements, channels, and collections of those, the robustness with respect to the set of free elements quantifies the advantage of a resourceful element over all free ones, in a task that can be derived from the duality theory of conic optimization.

Finally, we will put forward a connection between the compatibility of channels and certain instances of the quantum marginal problem, which allows us to translate many structural results between the two fields.

Zusammenfassung

Diese Arbeit ist verschiedenen Aspekten der Quanteninformationstheorie gewidmet. Es werden neue Resultate auf den Gebieten der Kohärenztheorie, der Netzwerkkorrelationen, der Gemeinsamen Messbarkeit, und der Quantifizierung von quantenmechanischen Ressourcen präsentiert, sowie eine Zusammenhang zwischen inkompatiblen Kanälen und dem Marginalienproblem in der Quantenmechanik diskutiert.

Zuerst wird das Konzept der echt-korrelierten Kohärenz eingeführt. Diese entspricht dem kleinsten Anteil der globalen Kohärenz, der übrig bleibt, wenn man globale inkohärente unitäre Transformationen durchführt, die freie Operationen in der Ressourcentheorie der Kohärenz bilden. Dies trägt zu einer aktuellen Diskussion über mögliche freie Operationen in einer Ressourcentheorie von Vielteilchenkohärenz bei, und zeigt eine Verbindung zur echten Mehrniveauverschränkung auf.

Danach werden wir Monogamierelationen für Kohärenzen herleiten, die zwischen orthogonalen Unterräumen existieren kann. Diese Art von Monogamie limitiert die Unterscheidbarkeit von Zuständen unter unitärer Zeitentwicklung, wenn Messungen auf Unterräume beschränkt sind. Diese Monogamie wird es uns außerdem erlauben Kriterien für echte Unterraumkohärenz herzuleiten, die zum Beispiel Anwendung in der Charakterisierung von Quantennetzwerken finden werden.

Anschließend werden wir unseren Fokus auf Korrelationen in Quantennetzwerken richten. Wir werden zeigen, wie die Struktur dieser Netzwerke die Erzeugung von Verschränkung limitiert, insbesondere im sogenannten Dreieck-Netzwerk. Wir werden notwendige Bedingungen herleiten die ein Zustand erfüllen muss, um im Dreieck-Netzwerk präparierbar zu sein. Diese Kriterien basieren auf der statistischen Unabhängigkeit der Quellen, der Monogamie von Verschränkung, und Bedingungen an die lokalen Ränge. Zudem werden wir einen anderen Ansatz verfolgen der auf den Eigenschaften von Kovarianzmatrizen beruht, die sich durch Messungen auf einem Netzwerkzustand ergeben. Wir werden zeigen, dass die Kohärenztheorie benutzt werden kann, um die relevanten Eigenschaften der Kovarianzmatrizen zu untersuchen. Dies wird es uns erlauben zu überprüften, ob eine Wahrscheinlichkeitsverteilung einem Netzwerk entstammen kann oder nicht.

Ein weiterer großer Teil dieser Arbeit beschäftigt sich mit der Quantifizierung quantenmechanischer Ressourcen. Genauer gesagt, werden wir zeigen, dass inkompatible Messungen einen Vorteil haben gegenüber allen kompatiblen Messungen in einem quantenmechanischen Zustandsunterscheidungsproblem. Dies resultiert in einer operationellen Charakterisierung inkompatibler Messungen und der Möglichkeit, diese teilweise gerätunabhängig zu verifizieren. Das Resultat beruht auf Eigenschaften der sogenannten Inkompatibilitätsrobustheit.

Anschließend werden wir zeigen, dass solche Resultate eine allgemeine Eigenschaft von Robustheitsmaßen sind. Genauer gesagt werden wir zeigen, dass in jeder kon-

vexen Ressourcentheorie von Zuständen, Messungen, Kanälen, oder Mengen von diesen, das Robustheitmaß den Vorteil quantifiziert, den eine Ressource gegenüber Nicht-Ressourcen in einem bestimmten Problem hat. Die Form des Problems kann man aus der Dualitätstheorie konischer Optimierungsprobleme ableiten.

Außerdem werden wir einen Zusammenhang zwischen inkompatiblen Kanälen und bestimmten Instanzen des Marginalienproblems in der Quantenmechanik diskutieren, welcher es uns erlauben wird Resultate, zwischen den beiden Gebieten auf das jeweils andere zu übersetzen.

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Introduction

Over the last decades, it has become evident that information processing based on the laws of quantum mechanics provides significant advantages over classical information processing. The idea of using quantum mechanical devices to perform computations was first put forward by Beninoff [1]. Later, it was suggested by Feynman [2] and Manin [3], that such a computing device had the potential to outperform any classical computer. Indeed, in quantum computing, Shor's algorithm [4] can be used to factor large numbers into their prime-factor decomposition in polynomial time, something that is believed to be impossible for any classical computer. Despite the fact that a general quantum computer is still out of reach, the advantages provided by quantum mechanics can already be observed in other areas. Quantum key distribution protocols [5, 6] allow to secretly establish a key between two distant parties that is provably secure. Due to the rapidly advancing experimental techniques, the originally purely theoretical concepts are nowadays routinely being demonstrated in the lab, e.g., to perform long-distance quantum key distribution via satellite [7, 8] or optical fibers [9]. Recent advances also lead to promising developments towards the first small scale quantum networks, which could in the future be used to scale up quantum key distribution to multiple parties [10–12]. Other examples can be found in the realm of quantum metrology, where it is known that using quantum systems, one can perform high-precision measurements that are classically impossible, i.e., one can get below the so-called shot-noise limit (see, e.g., Refs. [13, 14] and references therein).

At the same time we have made significant progress in deepening our understanding of the foundational aspects of quantum mechanics, and especially of the properties of quantum mechanics that make the application discussed above possible. One such property certainly is quantum entanglement, which was first termed by Schödinger *Verschränkung* [15]. It refers to the fact that certain states of distributed systems do not admit a local description but must rather be seen as a single entity. It is known that entanglement is necessary for certain quantum advantages in, e.g., teleportation [16], measurement-based quantum computation [17], cryptography [6], and metrology [18–20].

Entangled quantum states do not only lead to the quantum mechanical advantages discussed above, but also reveal some very fundamental properties of quantum mechanics. The fact that entangled states lead to observations that contradict our classical intuition was first criticized by Einstein, Podolsky, and Rosen in their seminal 1935 paper [21]. According to their view, the description of physical reality via the wave function was incomplete. However, in 1964, it was shown by Bell [22] that under certain natural assumptions a completion of quantum mechanics is not possible. Indeed, he showed that under the assumptions of *locality*, *realism*, and *free will*, the correlations in certain experimental scenarios are restricted, and that quantum mechanics allows for a violation of these restrictions, which are today known as Bell inequalities. Moreover, the first loophole free violations of Bell inequalities have been observed in recent experiments [23–25]. Hence, the observations predicted by quantum mechanics, and observed in the lab, cannot simply be completed by so-called local hidden variables. Although this seems to be a purely theoretical result, it is still conceptually important and therefore deserves a lot of attention. One reason certainly is that it provides a proof for the security of certain quantum key distribution protocols [6].

Another peculiarity in quantum mechanics is the measurement incompatibility. This refers to the fact that some measurements can be performed jointly, at the same time, while others cannot. A famous instance of such measurements are the position and momentum observables, for which the famous Heisenberg uncertainty relation $\Delta(x)\Delta(p) \leq \hbar/2$ holds [26]. Measurement incompatibility is conceptually important since it is necessary to reveal certain quantum mechanical correlations such as steering [27], which is a concept in between entanglement and Bell non-locality, and Bell non-locality itself.

This thesis is concerned with different aspects of quantum information theory. In particular, we will aim to gain a better understanding of the properties of quantum mechanics that allow for the advantages discussed above, namely quantum coherence, entanglement, the incompatibility of measurements and channels, and general quantum resources.

In the first chapter, we will introduce the basic mathematical and conceptual results that are important to understand the results of this thesis.

In the second chapter, we will introduce the notion of genuine correlated coherence, which is defined as the amount of coherence that remains after one applies global incoherent unitaries, which are deemed to be free in a resource theoretic approach to coherence. This contributes to an ongoing discussion on the possible free operations in a resource theory of multipartite coherence. Moreover, it reveals a connection to the concept of genuine multilevel entanglement.

In the third chapter, we will derive monogamy relations that capture the trade-off of the coherence that can exist between multiple orthogonal subspaces. On the one hand, we will find that these trade-offs put limits on how well quantum states can be distinguished under unitary evolution, when the measurements are restricted to act only on strict subspaces. Moreover, this will allow us to derive criteria detecting genuine multisubspace coherence of the density matrix, which has applications in, e.g., the characterization of quantum networks.

In the fourth chapter, we will turn our focus to correlations in quantum networks. We will show how the structure of the network limits the distribution of entanglement, focusing on the so-called triangle network. We derive several necessary criteria for states to be preparable in the triangle network, based on the statistical independence of the sources, the monogamy of entanglement and constraints on the local ranks.

In the fifth chapter, we will consider a different approach based on the properties of covariance matrices that arise from measurements on a network state. It was recently shown [28] that the topology of the network leads to a certain block structure of the covariance matrix, and we will show that the theory of coherence can be utilized to analyze this block structure. This will allow us to witness probability distributions that are incompatible with the structure of the network in scenarios that cannot be solved by numerical approaches due to the rapidly growing number of free parameters.

Another large part of this thesis is concerned with the quantification of quantum resources. To be more precise, we will show in chapter six that incompatible measurements provide an advantage over all compatible measurements in certain instances of quantum state discrimination. This provides an operational characterization of measurement incompatibility and opens up the possibility of its semi-device-independent verification. The result is based on the properties of the incompatibility robustness and the duality theory of semidefinite programming.

Subsequently, we will show in the remainder of chapter six, and chapter seven, that similar results, as the ones presented for incompatible measurements, are a generic feature of the generalized robustness quantifier. More precisely, we will show that in any convex resource theory of states, measurements, channels, and sets thereof, the robustness with respect to the set of free elements quantifies the advantage of a resourceful element over all free ones, in a task that can be derived from the duality theory of conic optimization.

Finally, in the eighth chapter, we will put forward a connection between the compatibility of channels and certain instances of the quantum marginal problem, which allows us to translate many structural results between the two fields. For instance, we will discuss a semidefinite programming hierarchy to compute the robustness of quantum memories and solve the quantum marginal problem for pairs of states under depolarizing noise and pairs of two-qubit Bell-diagonal states. Moreover, the connection will allow us to solve the self-compatibility of qubit-to-qubit channels and derive new conditions for channel incompatibility based on entropic criteria.

1 Preliminaries

In this chapter, we will first introduce the basic mathematical framework of quantum mechanics and the important conceptual results, both of which are needed in order to understand the results of this thesis.

1.1 States and effects

The mathematical formalism of quantum mechanics is aimed at predicting the statistics of possible observations when a system is subjected to a measurement. As such, the fundamental elements of quantum mechanics are *states*, which contain all the information that is necessary to predict the statistics of every possible measurement, and *effects*, which correspond to the possible outcomes of a measurement.[1] Quantum mechanics is formulated in terms of linear operators on Hilbert spaces. A Hilbert space $\mathcal{H}$ is a complete vector space, equipped with an inner product.
Quantum states are represented by positive (semidefinite) operators on the Hilbert space $\mathcal{H}$ with unit trace, so-called *density matrices*.[2] We call the set of all possible density matrices the *state space* and it is formally defined as

$$\mathcal{S}(\mathcal{H}) = \{\varrho | \varrho \geq 0, \mathrm{tr}[\varrho] = 1\}. \tag{1.1}$$

This set is convex, since for $\varrho_1, \varrho_2 \in \mathcal{S}(\mathcal{H})$ their convex mixture $\lambda\varrho_1 + (1-\lambda)\varrho_2 \in \mathcal{S}(\mathcal{H})$ for $\lambda \in [0,1]$. This corresponds to our intuition that if we are given a state ϱ_1 with probability λ and ϱ_2 with probability $1-\lambda$, the resulting state should also be a valid quantum state. Any state ϱ can be decomposed using the spectral decomposition

$$\varrho = \sum_j p_j P_j, \tag{1.2}$$

where $\{p_j\}$ is a probability distribution and P_j are mutually orthogonal rank-1 projectors, i.e., $P_i P_j = \delta_{ij}$ and $P_i^2 = P_i$. The rank-1 projectors themselves are also states and they are of the form $|\psi\rangle\langle\psi|$, such states are called *pure* states and they correspond to

[1] We will give a brief overview on the fundamental concepts that are needed throughout this thesis, for a more detailed discussion we refer to the excellent book by Heinosaari and Ziman [29].

[2] An operator is called *positive semidefinite*, if all its eigenvalues are non-negative. For convenience we will simply say that an operator is *positive*, if it is positive semidefinite and *strictly positive* if all its eigenvalues are strictly positive.

the case where we have the maximum amount of information about the state of the system. In the most common formulation of quantum mechanics the object $|\psi\rangle \in \mathcal{H}$ is also called a pure state, where $\langle\psi|\psi\rangle = 1$. Note, that all pure states $|\psi'\rangle = e^{i\varphi}|\psi\rangle$ lead to the same density matrix $|\psi\rangle\langle\psi|$, and thus, describe the same state of the system. In the following we will also work with vectors as pure states whenever it is convenient.

There are two convenient ways to quantify the amount of knowledge that is provided by a density matrix ϱ. One is called the *purity* and it is defined by $\mathcal{P}(\varrho) = \mathrm{tr}[\varrho^2]$. One finds that this quantity reaches its maximum value of $\mathcal{P}(\varrho) = 1$ if and only of ϱ is pure, and its minimum value $\mathcal{P}(\varrho) = \frac{1}{d}$ if and only if ϱ is the maximally mixed state $\frac{\mathbb{1}}{d}$. The purity also enjoys other useful properties, i.e., it is convex and independent of the choice of basis. Another quantifier is called the von Neumann entropy, which is defined by $S(\varrho) = -\mathrm{tr}[\varrho\log(\varrho)] = -\sum_i p_i \log(p_i)$, where the last equality follows from the properties of the matrix logarithm and p_i are the eigenvalues of ϱ. The von Neumann entropy quantifies our uncertainty about the state of the system. It reaches its minimum value, i.e., the state of minimum uncertainty, $S(\varrho) = 0$ if and only if ϱ is pure, and its maximum value, i.e., the state of maximum uncertainty, $S(\varrho) = \log(d)$ for the maximally mixed state $\varrho = \frac{\mathbb{1}}{d}$. Similar to the purity, the von Neumann entropy is concave and independent of the choice of basis.

The Bloch sphere. If the dimension of the Hilbert space is two, we will refer to the system as *qubit* and for such systems the set of density matrices has a simple geometrical representation, which is called the *Bloch sphere*. To represent any state we first choose a complete basis of the space of hermitian 2×2 matrices. Such a basis is given by the *Pauli matrices* together with the identity matrix

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{1.3}$$

Then, any qubit density matrix can be decomposed in the following way

$$\varrho = \frac{1}{2}\left[\mathbb{1} + \sum_{i\in\{x,y,z\}} r_i\sigma_i\right], \tag{1.4}$$

where $\vec{r} = (r_i)_i$ is called the Bloch vector of the state ϱ. From the positive semidefiniteness of ϱ it follows that $|\vec{r}| \leq 1$, and the purity is given by $\mathcal{P}[\varrho] = \frac{1}{2}[1 + |\vec{r}|^2]$. Hence, the possible vectors that correspond to quantum states correspond to a unit ball, with the maximally mixed state in the origin and the pure states on the surface. For higher dimensional systems a similar construction exists, but it is not a ball anymore [30].

Until now we have only discussed how to describe the state of a system, and not how to obtain predictions about possible observations. To each observation we wish to predict the probability of its occurrence. Hence, we need some mapping from the state space to probabilities. Such a map is provided by $\varrho \mapsto \mathrm{tr}[\varrho E]$, where the operator E is called *effect*. From the condition that $\mathrm{tr}[\varrho E]$ is a probability for any possible input state the operator inequality $0 \leq E \leq \mathbb{1}$ directly follows. Any E that fulfills this relation is an effect, and the map it called *Born rule*. A collection $\{E_i\}$ of effects, for which $\sum_i E_i = \mathbb{1}$ is called a positive operator valued measure (POVM) and it represents the most general measurement in quantum mechanics. For any POVM $\{E_i\}$ the probabilities for the different outcomes are given by the Born rule

$$p_i = \mathrm{tr}[\varrho E_i]. \tag{1.5}$$

In case the E_i are projectors the matrix $A = \sum_i \lambda_i E_i$, where $\lambda_i \in \mathbb{R}$, is called an *observable*, and its expectation value is given by $\langle A \rangle = \mathrm{tr}[\varrho A]$.

To conclude this section we note that the state space can be seen as the intersection of the cone of semidefinite operators and the unit trace hyperplane. To be more precise, a subset C of a vector space V is called a *cone*, if for any $x \in C$ the element $\lambda x \in C$ for all $\lambda \geq 0$, and, clearly, the set of positive operators is a cone in the space of Hermitian operators. For any cone C one can define its *dual cone* to be the set $C^* = \{y | \langle x, y \rangle \geq 0 \, \forall x \in C\}$. For the positive semidefinite cone one finds that $C = C^*$, which is called *self duality*. Therefore we can visualize states and effects as shown in Fig. 1.1b, where the effects are contained in a *double cone* within the positive semidefinite cone.

1.2 Multipartite systems and their subsystems

Composite systems that consist of multiple subsystems are described by the tensor product of Hilbert spaces. In the simplest case of a bipartite system the two subsystems are traditionally called Alice (A) and Bob (B). Fixing a basis $\{ |a_i\rangle \}$ for $\mathcal{H}_A$ and $\{ |b_j\rangle \}$ for $\mathcal{H}_B$, the joint Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ has a basis $\{ |a_i\rangle \otimes |b_j\rangle \}$ and we will adopt

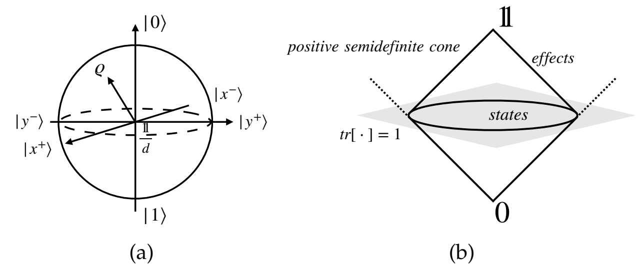


Figure 1.1: (a) Any state of a single qubit can be represented on the Bloch sphere. The pure states lie on its surface and the maximally mixed state in its origin. (b) The state space is the intersection of the positive semidefinite cone with the unit trace hyperplane, and the effects are contained in a double cone that is contained in the positive semidefinite cone.

the short-hand notation $|a_i b_j\rangle$. Hence, any pure state of the combined system can be written as

$$|\psi_{AB}\rangle = \sum_{ij} \lambda_{ij} |a_i b_j\rangle . \tag{1.6}$$

This construction generalized straightforwardly to density matrices by convex combination.

Sometimes one is faced with the problem that one can only access a subsystem of a much larger system and one aims to describe the state of the subsystem while remaining ignorant about the state of the whole system. Starting with the global state ϱ_{AB}, a local description is obtained by the means of the *partial trace*. The partial trace can be seen as the unique operation that fulfills $\mathrm{tr}_{AB}[(X_A \otimes \mathbb{1})Y_{AB}] = \mathrm{tr}_A[X_A \, \mathrm{tr}_B[Y_{AB}]]$ for all operators X, Y. An equivalent formulation is that the description of the local state of Alice can be obtained by

$$\varrho_A = \mathrm{tr}_B[\varrho_{AB}] = \sum_j \langle b_j| \varrho_{AB} |b_j\rangle , \tag{1.7}$$

where $\{ |b_j\rangle \}_j$ is a complete orthonormal basis on Bobs system. For a pure bipartite state $|\psi_{AB}\rangle$ it might happen, that, although the global state is pure, its reduced states are maximally mixed. An example for such a state is the state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle], \tag{1.8}$$

and one can easily verify that $\varrho_A = \varrho_B = \frac{\mathbb{1}}{d}$. This raises the question if any mixed state can be interpreted as a smaller part of a larger system that itself is in a pure state. This leads to the notion of *purification*. Given a mixed state $\varrho_A \in \mathcal{S}(\mathcal{H}_A)$, there always exists a Hilbert space $\mathcal{H}_B$ of dimension at least $\mathrm{rk}(\varrho_A)$ and a pure state $|\Omega\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ such that $\varrho_A = \mathrm{tr}_B[\,|\Omega\rangle\langle\Omega|]$. Such a state can be constructed in the following way. Consider the eigendecomposition $\varrho_A = \sum_n t_n |n\rangle\langle n|$ of ϱ_A, then the state $|\Omega\rangle = \sum_{n=1}^{rk(\varrho_A)} \sqrt{t_n} |nn\rangle$ is a purification of ϱ_A. In particular, such a purification is not unique, since one can apply arbitrary local isometries on the purifying system, that leave the marginals of the original system invariant, i.e., $(V \otimes \mathbb{1}) |\Omega\rangle$ is also a purification whenever V is an isometry with $V^\dagger V = \mathbb{1}$. Indeed all possible purifications of ϱ_A are equivalent under local isometries on the purifying system.

1.3 Quantum dynamics and the measurement process

After the state preparation and before the measurement process, a non-trivial time evolution might occur that changes the state of the system, e.g., by an interaction of the system with the environment or an ambient field. To model such a process we assume that the system was initially in the state ϱ, and the environment in a pure state

$|\varphi\rangle$. The effective dynamics that describes the time evolution of the system is then obtained by

$$\varrho \mapsto \sigma = \mathrm{tr}_E[U\varrho \otimes |\varphi\rangle\langle\varphi|\, U^\dagger]. \tag{1.9}$$

Any time evolution that is of this form is described by a *completely positive trace preserving* (cptp) map that we call a *channel*. A linear map Λ is called *positive* if it maps positive operators on positive operators, and *completely positive* if the map $\mathbb{1}_d \otimes \Lambda$ maps positive operators on positive operators for any d. On the other hand, it is guaranteed by the *Stinespring dilation theorem*, that any channel admits a representation as in Eq. (1.9).[3]

Another prominent way to represent quantum channels is by the so-called *Kraus decomposition*, which states that any cptp map can be written as

$$\Lambda(\varrho) = \sum_i K_i \varrho K_I^\dagger, \quad \text{where } \sum_i K_i^\dagger K_i = \mathbb{1}. \tag{1.10}$$

The operators $\{K_i\}$ are called *Kraus operators* and the number of K_i is called the *Kraus rank*. Unitary operations are special cases of quantum channels with Kraus rank one.

Quantum channels on a single system can be naturally related in a one-to-one way to quantum states on a bipartite system. This is known as the *Choi-Jamiołkowski isomorphism* [31]. Given a cptp map Λ, its *Choi state* is obtained by acting on one half of the $|\Phi^+\rangle$ state, i.e.,

$$\Lambda \mapsto J_{AB}^{\Lambda} = (\mathbb{1} \otimes \Lambda)\, |\Phi^+\rangle\langle\Phi^+|\,. \tag{1.11}$$

The Choi-Jamiołkowski isomorphism enjoys many useful properties some of which we will mention here, namely: (i) $J_{AB}^{\Lambda} \geq 0$ if and only if Λ is completely positive, (ii) Λ is trace-preserving if and only if $\mathrm{tr}_B[J_{AB}^{\Lambda}] = \frac{\mathbb{1}}{d}$, (iii) Λ is unital if and only if $\mathrm{tr}_A[J_{AB}^{\Lambda}] = \frac{\mathbb{1}}{d}$, and (iv) the inverse of the isomorphism relates each Choi state to a cptp map by $\Lambda_{J_{AB}}[X] = \mathrm{tr}_A[(X \otimes \mathbb{1})\varrho_{AB}]$.

Quantum Instruments. Consider again the scenario from the beginning of this paragraph, but now the ancillary system E is not thrown away, but subjected to a measurement and the outcome of this measurement is recorded. Any such scenario leads to what is called an *instrument*. An instrument is a set $\{\mathcal{I}_x\}_x$ of completely positive trace non-increasing maps, with x the possible outcomes, which sum to a quantum channel, i.e., $\sum_x \mathcal{I}_x = \Lambda$. Applying an instrument to an input state ϱ produces the non-normalized states $\mathcal{I}_x[\varrho]$, depending on the outcome x. The probability of obtaining outcome x is given by $p(x) = \mathrm{tr}[\mathcal{I}_x[\varrho]]$, and the state of the system is $\varrho_x = \mathcal{I}_x[\varrho] / \mathrm{tr}[\mathcal{I}_x[\varrho]]$. Any such instrument can be implemented by a unitary interaction with an ancilla

[3]We note that this also implies the possibility of purifying any mixed state, as any state can be seen as the output of a channel with trivial input.

followed by a projective measurement on the ancilla, this is the statement of Ozawas theorem [32]. This allows us to describe the measurement process of a POVM in more detail. So far we have only formally defined a POVM without actually describing how it can be implemented or how the state of the system changes after the measurement. The most straightforward way of implementing any POVM $\{E_x\}$ is via the so-called *Lüders instrument* that is defined by $\mathcal{I}_x[\varrho] = E_x^{1/2}\varrho E_x^{1/2}$. One can simply verify that this correctly recovers the outcome probabilities by computing $p(x) = \mathrm{tr}[\mathcal{I}_x[\varrho]] = \mathrm{tr}\left[E_x^{1/2}\varrho E_x^{1/2}\right] = \mathrm{tr}[E_x\varrho]$. The state of the system after the measurement is then

$$\varrho_x = \frac{E_x^{1/2}\varrho E_x^{1/2}}{\mathrm{tr}[E_x\varrho]}. \tag{1.12}$$

Moreover, this implies that any POVM can be implemented by a unitary interaction with an ancilla and a projective measurement on the ancilla. This is also called *Naimark extention*.

To conclude the discussion we note that one can equally well describe dynamics by transforming measurements instead of transforming states. The latter one is called *Schrödinger picture* and the former is called *Heisenberg picture*. To change between both representations one simply replaces all linear maps Λ with their adjoints $\Lambda^\dagger$, defined to be the unique operator that is defined by $\mathrm{tr}[X\Lambda[Y]] = \mathrm{tr}\left[\Lambda^\dagger[X]Y\right]$ for all test operators X, Y. The conditions, e.g., for such a map to be a channel changes from being trace preserving, to being *unital*, i.e., preserving the identity.

1.4 Coherence

The possibility of preparing a quantum system in a superposition of basis states is referred to as *quantum coherence*. For instance, if we consider a qubit with two degrees of freedom our classical intuition would suggest that the two possible pure states of the system should be perfectly distinguishable, e.g. $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$. This would lead to the conclusion that any possible *classical* state of a system should be in the convex hull of those two points. These states do not contain any coherence and are thus called *incoherent*. Consequently, any state that is not in this convex hull contains at least one coherent state in any possible convex decomposition and thus is called *coherent*.

More generally, we can make the following definition. Given any orthonormal basis $\{|a_i\rangle\}_{i=0}^{d-1}$ any pure state can be written as $|\psi\rangle = \sum_i \lambda_i |a_i\rangle$; such a basis may be singled out by the physical context, for example it could be the eigenbasis of the Hamiltonian of a system, or of some other specific and relevant observable. The number of non-zero λ_i's is called the *coherence rank*, and if the coherence rank is one, the state is called incoherent, or simply classical. For density matrices this concept can be generalized by the so-called *coherence number*. The set of states having coherence number k or less is denoted by $\mathcal{C}_k$ and it is the convex hull of all pure states with coherence rank k or less. For a fixed dimension d one finds the inclusion relation $\mathcal{C}_1 \subsetneq \mathcal{C}_2 \subsetneq \cdots \subsetneq \mathcal{C}_d$. The set $\mathcal{C}_1$ contains all incoherent states and we will abbreviate it as I. This definition makes coherence a basis dependent concept that is defined for a single system. Such a single system may be composite in nature, and this is the case we will consider later.

Recently, a lot of effort has been put into studying quantum coherence and its quantification and manipulation (see, e.g., Ref [33] for a recent review). One reason is certainly the fact that all quantum mechanical effects rely on the possibility of preparing a system in a coherent state and maintaining those coherences during the processing of quantum information. Examples include, but are not limited to, high precision metrology [34, 35] (e.g., in high precision clocks or magnetometry) or quantum information processing (such as algorithms [36] and memories [37]). The fact that incoherent state are rather useless, in the sense that they cannot be used for any truly quantum information processing, leads to the concept of *resource theory* of coherence.[4] The idea of this concept is that one divides the state space into two disjoint sets, namely the set of incoherent states, that are deemed *resourceless* and those states, namely the coherent ones, that contain some resource that can be used to achieve some non-trivial task [39].

In a resource theory there also exists—bedsides the notion of resourceless states—the notion of free operations, which are those operations that do not create a resource state from a resourceless state. The largest class of free operations is called the *maximally incoherent operations* [40], which are defined by $\Lambda[I] \subseteq I$. Other possible choices of free operations include *incoherent operations*, for which $K_n \varrho K_n^\dagger / \operatorname{tr}\left[K_n \varrho K_n^\dagger\right] \in I$ for all Kraus operators n—these are the operations that do not even generate coherence probabilistically—, and *strictly incoherent operations*, where all Kraus operators are incoherent themselves. The definition of free operations, however, is not unique and many different choices can be considered [33]. Clearly, the largest class of unitary operations that do not create coherence are the ones that only permute the incoherent basis, those are of the form $U = \sum_j e^{i\varphi_j} |\pi(j)\rangle\langle j|$, i.e., they can be written as a phase gate and a permutation π of the incoherent basis.

1.4.1 Measures of coherence

One important problem is the quantification of resources, which can be done by defining appropriate coherence measures. One requires the following: (i) $\mathscr{C}(\varrho) = 0$ if and only if $\varrho \in I$, otherwise $\mathscr{C}(\varrho) > 0$, (ii) monotonicity under cptp maps $\mathscr{C}(\Lambda[\varrho]) \leq \mathscr{C}(\varrho)$, (iii) nonincreasing on average $\sum_i q_i \mathscr{C}(\sigma_i) \leq \mathscr{C}(\varrho)$, where $\sigma_i = K_i \varrho K_i^\dagger / q_i$, with $q_i = \operatorname{tr}[K_i \varrho K_i]$, and K_i incoherent Kraus operators, and (iv) convexity $\sum_i p_i \mathscr{C}(\varrho_i) \geq$

[4] We wish to empathize that the notion of resource theory is not tied to quantum coherence only, but rather a more general concept. See, e.g., Ref. [38] for a recent review on the general structure of quantum resource theories.

$\mathscr{C}(\sum_i p_i \varrho_i)$.

Measures based on matrix norms. The simplest strategy to quantify coherence is by considering matrix norms, such as the l_p norms $\|\cdot\|_{l_p}$ or the Schatten p-norms $\|\cdot\|_p$, which are defined by

$$\|M\|_{l_p} = \left(\sum_{i,j} |M_{ij}|^p\right)^{1/p} \quad \text{and} \quad \|M\|_p = \left(\mathrm{tr}[(M^\dagger M)^{p/2}]\right)^{1/p}, \tag{1.13}$$

with $p \geq 1$. For $p = 1$ the l_1 *norm of coherence* can be defined by

$$\mathscr{C}_{l_1}(\varrho) = \min_{\sigma \in I} \|\varrho - \sigma\|_{l_1} = \sum_{i \neq j} |\varrho_{ij}| \tag{1.14}$$

and it satisfies all the conditions (i) to (iv) for incoherent operations, but for larger p it violates condition (iii) for certain cases of free operations.

Relative entropy of coherence. As a quantifier of coherence we will later use the *relative entropy of coherence* [41], which is defined as

$$\mathscr{C}(\varrho) := \min_{\sigma \in I} S(\varrho \parallel \sigma). \tag{1.15}$$

Here, $S(\varrho \parallel \sigma) := \mathrm{tr}[\varrho \log(\varrho)] - \mathrm{tr}[\varrho \log(\sigma)]$ is the relative entropy, and the minimization is over all incoherent states σ. This quantifier fulfills conditions (i), (ii), and (iv), as well as (iii) in the case of incoherent operations. A very useful property of the relative entropy of coherence is that it is additive on tensor products,

$$\mathscr{C}(\varrho_A \otimes \varrho_B) = \mathscr{C}(\varrho_A) + \mathscr{C}(\varrho_B). \tag{1.16}$$

In addition, an analytic solution to the minimization problem is known [41]: the relative entropy of coherence can be expressed as

$$\mathscr{C}(\varrho) \quad = \quad S(\varrho^d) - S(\varrho), \tag{1.17}$$

where $\varrho^d = \sum_i \langle i| \varrho |i\rangle |i\rangle\langle i|$ is the totally decohered version of the state ϱ. It is immediate to see that the relative entropy of coherence is invariant under the action of incoherent unitary transformations, which makes it a good coherence quantifier [41].

1.5 Entanglement

The tensor product structure of quantum mechanics allows for a system to be in a state that, as such, has no classical counter part and is thus sometimes seen as a feature that clearly distinguishes classical mechanics from quantum mechanics. If one wishes, one can also see entanglement as a manifestation of coherence on the level of distributed systems. In the bipartite case entanglement has a rather simple structure in the sense that there is only one type of entanglement, whereas in the multipartite scenario entanglement possess a much richer structure. It is therefore natural to start with the definition of entanglement in the bipartite scenario and then discuss multipartite entanglement[5].

1.5.1 Bipartite entanglement

A pure bipartite state is called *separable* if it is of the form $|\psi_{AB}\rangle = |a\rangle \otimes |b\rangle$, otherwise the state is called *entangled*. Important examples of entangled states of two qubits are the *Bell states*

$$|\phi^{\pm}\rangle = \frac{1}{\sqrt{2}}[|00\rangle \pm |11\rangle] \quad \text{and} \quad |\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}[|01\rangle \pm |10\rangle]. \tag{1.18}$$

For pure bipartite states their entanglement can be decided by the so-called *Schmidt decomposition*, which is simply a restatement of the singular value decomposition. Namely, any state can be written in terms of its Schmidt basis as

$$|\psi_{AB}\rangle = \sum_{i=0}^{\min\{d_A,d_B\}-1} \sqrt{\lambda_i}\, |a_i\rangle\, |b_i\rangle\,, \tag{1.19}$$

where λ_i are the Schmidt coefficients and the number of non-zero Schmidt coefficients is called the *Schmidt rank*.[6] Then, a pure state is separable if its Schmidt rank is one, and entangled otherwise.

For mixed bipartite states the problem becomes much harder. A density matrix is called separable if the following decomposition exists [44]

$$\varrho = \sum_i p_i \varrho_i^A \otimes \varrho_i^B, \tag{1.20}$$

where p_i is a probability distribution, otherwise we call the state entangled. We will denote the set of separable states by SEP.[7] To decide if a state admits a separable decomposition by means of Eq. (1.20) is called the *separability problem* and it is known to be NP-hard [45]. However, for 2×2 and 2×3 the separability problem can be solved completely by means of the *positive partial transpose*. To be more precise, the Peres-Horodecki criterion (usually referred to as the PPT criterion) [46] states that

$$\varrho \in \text{SEP} \Rightarrow \varrho^{T_B} \geq 0, \tag{1.21}$$

[5] Again, we will only cover the things that are necessary to understand the content of this thesis. The interested reader is therefore advised to consult Refs. [42,43] for more details.

[6] To see this take an arbitrary state as in Eq. (1.6) and perform a singular value decomposition of the coefficient matrix $\lambda_{ij} = U_{ik} D_{kk} V_{kj}$. Applying the unitaries U and $V^\dagger$ to Alices and Bobs basis respectively proves the statement.

[7] Note that in the case of finite dimensions the Carathéodory theorem guarantees, that a separable state can always be decomposed into at most $rk(\varrho)^2$ pure states, since these states are the extreme points of SEP.

where ϱ^{T_B} denotes the partial transposition. For a bipartite state $\rho = \sum_{ijkl} p^{ij}_{kl} |i\rangle\langle j| \otimes |k\rangle\langle l|$ its partial transposition is defined by $\rho = \sum_{ijkl} p^{ij}_{lk} |i\rangle\langle j| \otimes |k\rangle\langle l|$[8]. For 2×2 and 2×3 systems this criterion is necessary and sufficient [47], the reason for this being that in these cases every positive map can be decomposed as $\Lambda = \Lambda_1 + \Lambda_2 \circ T$, where Λ_1, Λ_2 are completely positive maps. In fact, there is a deeper connection between the separability problem and positive maps, which was proven in Ref. [47]. Namely, a state ϱ is separable if and only if it remains positive under local application of any positive linear map, i.e.,

$$\varrho \in \text{SEP} \Leftrightarrow (\Lambda \otimes \mathbb{1})[\varrho] \geq 0, \quad \forall \Lambda \text{ positive.} \tag{1.22}$$

The proof of this results relies on the Hahn-Banach theorem. An important corollary of this the Hahn-Banach theorem is the *separating hyperplane theorem*, which states that for any point outside a convex compact set, there exists a hyperplane that separates this point from the convex set. This leads us to the notion of *entanglement witnesses*. An entanglement witness W is a Hermitian operator that has positive expectation values on all separable states and a negative expectation value for at least one entangled state, i.e.,

$$\text{tr}[\varrho W] \geq 0 \quad \forall \varrho \in \text{SEP and } \exists\, \sigma \notin \text{SEP for which } \text{tr}[\sigma W] < 0. \tag{1.23}$$

The state σ is said to be detected by the witness W. One possibility to derive entanglement witnesses is to construct projector-based witnesses. Given an entangled state $|\psi\rangle$ consider the operator $W = \lambda \mathbb{1} - |\psi\rangle\langle\psi|$, and

$$\lambda = \max_{\varrho \in SEP} \text{tr}[\varrho\, |\psi\rangle\langle\psi|] = \max_{|\varphi\rangle \in SEP} |\langle\varphi|\psi\rangle|^2, \tag{1.24}$$

where the last equality is due to the fact that the maximum of a convex function on a convex domain is obtained on its extreme points, which are in this case the pure product states. This clearly defines an entanglement witness since it is non-negative on all separable states and negative on states close to $|\psi\rangle$ by construction.

Another important observation is that if one extends the CJ isomorphism to Hermitian operators (witnesses) and one finds the correspondence

$$\Lambda \text{ is positive but not completely positive} \quad \Leftrightarrow \quad \varrho^{\Lambda}_{AB} \text{ is an entanglement witness.} \tag{1.25}$$

This has an interesting consequence, namely, characterizing all positive but not completely positive maps is equivalent to characterizing all entangled states.

[8]The proof is simple. If ϱ is separable it can be decomposed as $\varrho = \sum_i p_i \varrho^A_i \otimes \varrho^B_i$, and its partial transpose it given by $\varrho^{T_B} = \sum_i p_i \varrho^A_i \otimes (\varrho^B_i)^{T_B}$, which is still a non-negative sum of positive semidefinite operators, and hence, its partial transpose is positive semidefinite.

1.5.2 Multipartite entanglement

In the multipartite case, entanglement has a much richer structure. For simplicity we will only discuss the tripartite case. As in the bipartite case different types of multipartite entanglement are defined by what they are not. A state is called *fully separable* if it can be written as

$$\varrho_{fsep} = \sum_i p_i \varrho_i^A \otimes \varrho_i^B \otimes \varrho_i^C. \tag{1.26}$$

Similarly, a state is called *biseparable* if it is not fully separable and can be written as a mixture of states that are separable with respect to a fixed bipartition, i.e.,

$$\varrho_{bisep} = p_1 \varrho^{A|BC} + p_2 \varrho^{B|AC} + p_3 \varrho^{C|AB}. \tag{1.27}$$

Although biseparable states contain some entanglement, they do not show the strongest form of multipartite entanglement. Finally, a state is called *genuinely multipartite entangled* if it is not biseparable. The two most prominent examples of such states are the Greenberger-Horne-Zeilinger (GHZ) state [48] and the W state [49]

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \quad \text{and} \quad |W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle). \tag{1.28}$$

The reason why these two states are so important is because they represent two distinct types of genuine multipartite entanglement in the following sense: Given two states $|\psi\rangle$ and $|\varphi\rangle$ one can ask if one state can be transformed into the other by the use of local operations assisted by classical communication (LOCC) or even if these transformation can be achieved statistically (SLOCC). It was shown in Ref. [49] that such a transformation exists if and only if

$$|\varphi\rangle = A \otimes B \otimes C \, |\psi\rangle \,, \tag{1.29}$$

where A, B and C are invertible local operators. It was also proven that such a transformation does not exists between states of the two SLOCC classes represented by the GHZ state and the W state. To complete the discussion on tripartite entanglement the structure of tripartite entanglement is summarized in Fig. 1.2a.

1.5.3 Quantification of entanglement

There are several strategies to quantify the amount of entanglement that is provided by a given quantum state. The first one are so-called *entanglement measures* and they are required to fulfill the following conditions: (i) $E(\varrho) = 0$ for all separable states ϱ, (ii) it should be invariant under local unitary transformations $E(\varrho) = E(U \otimes V \varrho U^\dagger \otimes V^\dagger)$, (iii) it does not increase under LOCC transformations $E(\varrho) \geq E(\Lambda_{LOCC}[\varrho])$[9], (iv) it

[9] This is sometimes replaced with the stronger condition that the measure should not increase on average under SLOCC transformations.

decreases under mixing, i.e., it is convex. There are also further properties that are sometimes required, e.g., additivity on multiple copies or even more generally on tensor products.

Next we will focus on some particular cases of such measures, but we will only cover a small fraction of the topic and refer the interested reader to, e.g., Refs. [43, 50]. The *concurrence* [51, 52] is defined for pure states by $C(|\psi\rangle) = \sqrt{2[1 - \mathrm{tr}[\varrho_A]^2]}$. For mixed states this is generalized by the so-called convex roof construction, where one minimizes the measure over all decompositions into pure states, namely

$$E(\varrho) = \min_{\varrho=\sum_i p_i |\psi_i\rangle\langle\psi_i|} \sum_i p_i E(|\psi_i\rangle). \tag{1.30}$$

For most entanglement measures this is hard to compute but for the case of the concurrence of two qubits, this expression can be evaluated analytically [53]. One obtains $C(\varrho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$, where the λ_i are the eigenvalues of the operator $\sqrt{\sqrt{\varrho}\,(\sigma_y \otimes \sigma_y)\,\varrho^*\,(\sigma_y \otimes \sigma_y)\,\sqrt{\varrho}}$ in decreasing order. Another important property of the concurrence is the fact that is reveals an important property of entanglement, namely its *monogamy*. Entanglement monogamy refers to the fact that entanglement cannot be shared arbitrarily between multiple parties. It was shown in Ref. [54] that

$$\mathcal{C}^2_{AB}(\varrho_{AB}) + \mathcal{C}^2_{AC}(\varrho_{AC}) \leqslant \mathcal{C}^2_{A|BC}(|\psi_{ABC}\rangle), \tag{1.31}$$

which is usually referred to as the Coffman-Kundu-Wootters (CKW) inequality.

The violation of the PPT criterion also opens a possibility to measure entanglement. The *negativity* [55, 56] is defined by the violation of the PPT criterion, namely

$$N(\varrho) = \frac{1}{2}(\|\varrho^{T_B}\|_{tr} - 1), \tag{1.32}$$

where $\|\cdot\|_{tr}$ denoted the trace norm.

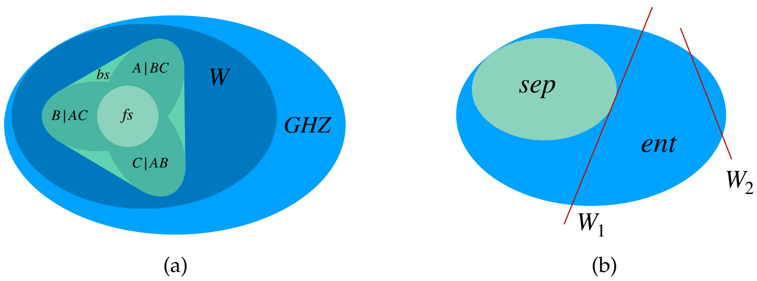


Figure 1.2: (a) This figure shows the entanglement structure of a tripartite system. (b) This figure illustrates the concept of entanglement witnesses. The witness W_1 detects more states than the witness W_2, and the witness W_1 is called optimal.

Some entanglement measures are based in more information theoretic quantities. One such example is the so-called *squashed entanglement*, which is defined by

$$E_{sq}(\varrho_{AB}) = \inf\{\frac{1}{2}I(AB|E)|\operatorname{tr}_E[\varrho_{ABE}] = \varrho_{AB}\}, \tag{1.33}$$

where $I(A;B|E) := S(AE) + S(BE) - S(ABE) - S(E)$ is the quantum conditional mutual information [57], and we note that the dimension of the system E is in principle unbounded. This measure was first studied in Ref. [58]. Later, it was proven in Ref. [59] that it is indeed an entanglement monotone (convex and LOCC non-increasing) as well as additive on tensor products.

Another possibility to quantify entanglement is by means of their distance to the set of separable states. For instance one can ask how much white noise can be added to an entangled states such that it becomes separable and call this the *white noise robustness* $R_r(\varrho)$. More mathematically this can be expressed as the following optimization problem

$$R_r(\varrho) = \max\{\lambda \geq 0|(1-\lambda)\varrho + \lambda \mathbb{1}/d = \sigma;\, \sigma \in \text{SEP}\}. \tag{1.34}$$

Computing such a quantity is in general very difficult due to the separability constraint. In a similar spirit one can consider other types of noise that can be added to make an entangled state separable. By transforming $\lambda \mapsto \frac{t}{1+t}$ and choosing $\tau \in$ SEP one obtains

$$R_s(\varrho) = \min\left\{t \geq 0 \middle| \frac{1}{1+t}\varrho + \frac{t}{1+t}\tau = \sigma;\, \sigma, \tau \in \text{SEP}\right\}, \tag{1.35}$$

the so called *robustness of entanglement* [60]. If τ is just an arbitrary state one obtains the *generalized robustness* $R_g(\varrho)$ [61].

The generalized robustness of entanglement has the interesting property that it can be related to the max-relative entropy, or more precisely the two are equivalent up to a shift by one and a logarithm. The max-relative entropy was first defined by Renner in Ref. [62]. It can be defined as

$$D_{\max}(\rho\|\sigma) = \log\min\{\lambda|\rho \leq \lambda\sigma\}, \tag{1.36}$$

where ϱ is a state and σ a positive semidefinite operator. This quantity is well behaved if $\operatorname{supp}\varrho \subseteq \operatorname{supp}\sigma$. Later, it was proven by Datta in Ref. [63] that for bipartite states this entropy can be used to construct a proper entanglement monotone by considering the quantity

$$E_{\max}(\rho) = \min_{\sigma\in SEP} D_{\max}(\rho\|\sigma), \tag{1.37}$$

and later that this quantity is actually the generalized robustness, i.e., $E_{\max}(\rho) = \log(1 + R_g(\varrho))$ [64].

1.5.4 Quantum marginal problem

The *quantum marginal problem* is concerned with the following question: Given a set of reduced states of a large composite system, does there exist a global state of this system, that has those states as its marginals? More mathematically, given a collection $\{\varrho_{J_k}\}_{k=1}^{|k|}$ of states, where $J_k \subset I$ and $I = \{1, \ldots, n\}$, does there exist a global state $\varrho_I \in \mathcal{S}(\mathcal{H}_I)$, such that $\varrho_{J_k} = \mathrm{tr}_{I \setminus J_k}[\varrho_I]$ for all k. For example, given two bipartite states ϱ_{AB_1} and ϱ_{AB_2} the problem is to find a tripartite state $\varrho_{AB_1B_2}$ which has the correct marginals.

Partial results on the quantum marginal problem are known [65–69], and many different scenarios have been considered. In addition to asking for the existence of a compatible global state one can put restrictions on the properties of the global state, e.g., purity or certain symmetries. The latter case includes the *N-representability problem* [70] in quantum chemistry and the former one includes the problem of existence of *absolutely maximally entangled states* [71]. A general solution to the quantum marginal problem is unlikely to exist due to the fact that it in computationally hard. Indeed, it was shown in Ref. [72] that it is QMA complete.

A special instance of the quantum marginal problem is known as *symmetric extendibility*. A bipartite state ϱ_{AB} is said to have n symmetric extensions if there exists a state $\varrho_{AB_1 \ldots B_n}$ such that $\varrho_{AB} = \mathrm{tr}_{\setminus AB_k}[\varrho_{AB_1 \ldots B_n}]$ for all k. The set of states having n symmetric extensions for all $n \geq 2$ coincides with the set of separable states [73–76].

1.6 Bell nonlocality

So far we have only discussed entanglement as a type of nonclassical correlation in quantum mechanics. However, different notions of classicality can be considered. Another notion was put forward by Bell in Ref. [22] in order to address an argument by Albert Einstein, Boris Podolsky, and Nathan Rosen (EPR) against quantum theory [21].[10]

The EPR argument was aimed to show that the quantum mechanical description of nature provided by the wave function was *incomplete* in the sense that in a complete theory any element of the physical reality must be represented in the theory [21]. In their view, an element of reality is a property of a system that can be predicted with certainty without disturbing the system. From this it follows that from the following two statements at least one must at least one be true: (i) The wave function does not provide a complete description of the physical reality, and (ii) properties that correspond to non-commuting observables are not part of the physical reality at the same time. They argued to have shown that if (i) is false so must be (ii), and hence, (i) must be true.

[10] For a recent review on the topic see, e.g., Ref. [77].

A modified version of their argument proposed by David Bohm [78] goes as follows: Consider the state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}(|x^+x^-\rangle - |x^-x^+\rangle)$, where the last inequality is due to the $U \otimes U$ invariance of the singlet state. Thus, if Alice measures σ_z or σ_x she can predict the state of Bobs particle with certainty, which makes it an element of reality on Bobs side. The contradiction comes from the fact that not both directions can be elements of the physical reality simultaneously, due to the non-commutativity of σ_z and σ_x.

However, one can show that quantum mechanics cannot be completed under some very natural assumptions, this is the statement of Bells theorem [22]. A simple strategy to complete quantum theory is by considering additional variables λ that, if they were known, would make quantum theory deterministic. One assumes that the value of such a variable is not accessible, hence we can only assume that they are distributed according to some distribution $p(\lambda)$. Hence, if we perform a measurement the outcomes are determined by $p(a|A) = \int \mathrm{d}\lambda\, p(\lambda)\chi(a|A,\lambda)$, where we can assume w.l.o.g that $\chi(a|A,\lambda)$ is a deterministic probability distribution[11]. Now we can consider again the bipartite scenario described by EPR, and, in addition, make the following three very natural assumptions.

B1 Locality: The probabilities for the outcomes observed by Alice do not depend on the choice of measurement direction of Bob, that is $p(a_i|A_i) = \sum_{b_j=\pm 1} p(a_i b_j|A_i B_j)$.

B2 Realism: Physical properties have a defined value, regardless of whether we measure them or not. In particular, the outcomes of every possible measurement is predetermined by the hidden variable.

B3 Freedom of choice: Both parties are free in choosing the measurement direction they want to measure in each round. This is not predetermined by the hidden variable λ.

Under these assumptions one can show that the observed probability distributions admit a *local hidden variable model*, which is of the form

$$p(a_i b_j|A_i B_j) = \int \mathrm{d}\lambda\, p(\lambda)\chi(a_i|A_i,\lambda)\chi(b_j|B_j,\lambda). \tag{1.38}$$

Now the question arises how one can decide if a given family of probability distributions admits such a LHV model or not. This leads us to the concept of *Bell inequalities*. Conceptually, the idea is quite simple. From Eq. (1.38) it is evident, that the set $\mathscr{L}$ of *behaviors* $\{p(ab|xy)\}_{a,b,x,y}$ admitting such a model is convex and compact in the space of all behaviors that can be obtained in quantum mechanics $\mathscr{Q}$. From the separating hyperplane theorem it follows that, whenever a probability distribution is outside of

[11] In case that λ does not lead to a deterministic model one can simply introduce additional hidden variables to obtain a deterministic model.

the local set, there exists a hyperplane that separates it from the local set. Any such hyperplane is a Bell inequality and it is of the form

$$\sum_{abxy} b_{xy}^{ab} p(ab|xy) \leq c \tag{1.39}$$

for all $\{p(ab|xy)\}_{a,b,x,y} \in \mathscr{L}$, and which is violated by at least one point outside of $\mathscr{L}$. It is known that the set $\mathscr{L}$ is a polytope, with the local deterministic behaviors as its extreme points. It is well known in polyhedra theory that, instead of characterizing a polytope by all its extreme points, one can equally well describe it by the intersection of finitely many halfspaces. Then it is clear that any such halfspace defines a Bell inequality. Those are of fundamental importance because they are tight, and they completely characterize the local polytope. Unfortunately, if one starts from the description by extreme points, the problem of finding all facets is very hard. However, for the simplest non-trivial case of two measurements per party and two outcomes per measurement it is known that there is only a single relevant Bell inequality, which is called the CHSH inequality. It reads

$$\langle \mathcal{B} \rangle = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2. \tag{1.40}$$

We wish to stress that Bell inequalities do not rely on the mathematical structure of quantum theory, as they do not assume that the probabilities in Eq. (1.38) have a quantum realization. Hence, observing a violation of a Bell inequality shows that nature itself is incompatible with the assumptions above, and not only quantum mechanics.

Indeed, in quantum mechanics this bound can be violated up to $2\sqrt{2}$, e.g., by measuring $A_0 = \sigma_x$, $A_1 = \sigma_z$ and $B_0 = \frac{\sigma_x+\sigma_z}{2}$ and $B_1 = \frac{\sigma_x-\sigma_z}{2}$ on a $|\phi^+\rangle$ state. The violation of the CHSH inequality can directly be linked to the commutator of the measurements that are performed. In fact, if Alice and Bob both measure the same observables A_0 and A_1, one can show that the value of the CHSH operator is simply given by $|\langle \mathcal{B} \rangle| = 2\sqrt{1 + \frac{1}{4}\|[A_0, A_1]\|^2}$, where $[X, Y] = XY - YX$ is the commutator [79]. In quantum mechanics two observables are called *compatible* if their commutator vanishes, since then the two Hermitian operators have the same eigenvectors. From this it is straightforward to see that no commuting set of observables can violate the CHSH inequality. In the next chapter we will take a closer look at incompatible measurements in quantum mechanics.

1.7 Incompatible quantum devices

In this section we will introduce different notions of incompatibility in quantum mechanics, namely the incompatibility of measurement and the incompatibility of channels.

1.7.1 Measurement incompatibility

The concept of *measurement incompatibility* has turned out to be a crucial property in the study of quantum correlations, which is due to the fact that without incompatible measurements experimental procedures can be explained classically and thus to not provide any quantum mechanical advantage. Such examples include the violation of Bell inequalities [79–81], EPR steering [80, 82] (cf. Sec. 1.8), as well as the security of cryptography protocols such as BB84 [5, 83]. For a recent review on the topic see, e.g., Ref. [84].

In order to introduce the concept of incompatible measurements for POVMs we first need the notion of simulability. Given two POVMs $\{M_a\}$ and $\{N_a\}$ we say that $\{M_a\}$ simulates $\{N_a\}$ if the outcomes of $\{N_a\}$ can be classically post-processed from a measurement of $\{M_a\}$ for any state ϱ. More precisely, we have the operator identity $N_a = \sum_\lambda p(a|\lambda) M_\lambda$.

Taking the idea of simulation even further we can define simulability for a set of measurements by a single measurement. This leads to the notion of *compatible* measurements, which generalized the concept of commuting observables to the realm of POVMs. Consider a set of measurements $\{M_{a|x}\}$, the set is called compatible if there exists a joint observable $\{G_\lambda\}$ such that

$$M_{a|x} = \sum_\lambda p(a|x,\lambda) G_\lambda, \tag{1.41}$$

and otherwise they are called *incompatible*. Operationally, compatibility means that the statistics of all the measurements $\{M_{a|x}\}$ can be obtained by performing a single measurement and post-processing the data according to some probability distributions $p(a|x,\lambda)$. Clearly, POVMs with commuting effect operators are compatible, but the converse is not necessarily true, as it can be seen from the following example. Consider the two POVMs $\{M_{i|1}\} = \{\frac{1}{2}(\mathbb{1} \pm \mu\sigma_x)\}$, and $\{M_{j|2}\} = \{\frac{1}{2}(\mathbb{1} \pm \mu\sigma_z)\}$, which are the noisy versions of the Pauli measurements σ_x and σ_z. One can define the POVM $\{G_{i,j}\} = \{\frac{1}{4}(\mathbb{1} + \mu(i\sigma_x + j\sigma_z))\}_{i,j=\pm 1}$, and verify that indeed $\sum_j G_{ij} = M_{i|1}$ and $\sum_i G_{ij} = M_{j|2}$. However, this is a POVM only for $0 \leq \mu \leq \frac{1}{\sqrt{2}}$, due to the positivity constraint. This shows that indeed, two POVMs can be compatible while at the same time their corresponding effect operators do not commute.

Moreover, we wish to stress that in this case the post-processing is deterministic, and one can show that an equivalent condition for compatibility can be obtained by requiring that

$$M_{a_x|x} = \sum_{/a_x} G_{a_1 \dots a_n}, \tag{1.42}$$

i.e., the post-processing can always be assumed to be deterministic [84]. This fact opens an interesting possibility to decide if a given set of POVMs is compatible or not. Consider the following task

$$\text{given } \{M_{a|x}\}_{a,x}, \{D(a|x,\lambda)\}_\lambda \tag{1.43}$$
$$\text{find } \{G_\lambda\}_\lambda \tag{1.44}$$
$$\text{s.t. } \sum_\lambda D(a|x,\lambda) G_\lambda = M_{a|x} \quad \forall a, x \tag{1.45}$$
$$G_\lambda \geq 0 \quad \sum_\lambda G_\lambda = \mathbb{1}. \tag{1.46}$$

This is called a *feasibility problem* and it is an instance of a *semidefinite program* (SDP), which a type of optimization problem (cf. Sec. 1.9). We note that such a feasibility problem only asks for the existence of operators such that the constraints are fulfilled. For now the only important point is that any such problem can be turned into an actual optimization problem, which in this case reads

$$\text{given } \{M_{a|x}\}_{a,x}, \{D(a|x,\lambda)\}_\lambda \tag{1.47}$$
$$\max_{\{G_\lambda\}} \mu \tag{1.48}$$
$$\text{s.t. } \sum_\lambda D(a|x,\lambda) G_\lambda = M_{a|x} \quad \forall a, x \tag{1.49}$$
$$G_\lambda \geq \mu \mathbb{1} \quad \sum_\lambda G_\lambda = \mathbb{1}. \tag{1.50}$$

Clearly, if this optimization results in a value of λ strictly less than zero the positivity constraint on the joint observable cannot be fulfilled, which proves incompatibility. Otherwise, a joint observable is found which proves compatibility.

In order to quantify the amount of incompatibility several quantities have been proposed, mainly focusing on certain types of noise robustnesses. Examples include the incompatibility random robustness [85], incompatibility robustness [86], and the incompatibility weight [87]. All those are based on semidefinite programs.

Besides the numerical results, also analytical results are known for certain cases, see, e.g., Refs. [88–91]. To give a simple example, consider two dichotomic unbiased qubit POVMs $E_\pm = \frac{1}{2}(\mathbb{1} \pm \vec{e} \cdot \vec{\sigma})$, and $F_\pm = \frac{1}{2}(\mathbb{1} \pm \vec{f} \cdot \vec{\sigma})$. One can show that they are compatible if and only if [88]

$$\|\vec{e} + \vec{f}\| + \|\vec{e} - \vec{f}\| \leq 2. \tag{1.51}$$

An interesting question is the relation between incompatible measurements and the violation of Bell inequalities. It is straightforward to verify that in any bipartite Bell experiment, whenever one of the parties (say Alice) performs a set of compatible measurements, the statistics admit a LHV model, regardless of the shared state and the measurements of Bob. To see this, note that

$$p(ab|xy) = \sum_\lambda p(a|x,\lambda) \operatorname{tr}\left[G_\lambda \otimes N_{b|y} \varrho_{AB}\right] = \sum_\lambda p_\lambda p(a|x,\lambda) p(b|y,\lambda). \tag{1.52}$$

In the case of the simplest Bell scenario, the CHSH scenario, the converse is also true. In Ref. [79] it was shown that whenever the two dichotomic POVMs defined by the effect operators P and Q are incompatible, there exists a state and measurements for Bob, such that the CHSH inequality is violated. Moreover, the amount of violation is quantified by an appropriate incompatibility measure. In the general case the connection between incompatibility and Bell violation is still an open problem, but for some cases results are known. E.g., in Ref. [81] it was shown that already for a certain triplet of dichotomic measurements that is incompatible, the resulting statistics do not violate any Bell inequality, regardless of the shared state or Bobs measurements. A more direct connection between incompatible measurements and quantum correlations can be established in the case of EPR steering, that we will discuss in the next section.

1.7.2 Channel incompatibility

Since POVMs can be seen as channels that implement a certain POVM and write the outcome into orthogonal quantum states, it is natural to extend the notion of compatibility to the realm of quantum channels. To introduce *channel compatibility* consider two channels $\Phi_{A\to B_1}$ and $\Phi_{A\to B_2}$. These channels are called *compatible* if there exists a broadcasting channel $\Phi_{A\to B_1B_2}$ from which they can be obtained as marginals [92]. More precisely, $\Phi_{A\to B_1}(\varrho) = \mathrm{tr}_{B_2}[\Phi_{A\to B_1B_2}(\varrho)]$ and $\Phi_{A\to B_2}(\varrho) = \mathrm{tr}_{B_1}[\Phi_{A\to B_1B_2}(\varrho)]$ for all input states ϱ.

This formulation can be directly generalized to sets of channels: Consider the set $I = \{A, B_1, \dots, B_n\}$ and its subsets $J_k = \{A, B_k\}$, each associated with a channel $\Phi_{J_k} = \Phi_{A\to B_k}$. These channels are compatible if there is a channel Φ_I with the input system A and the output Hilbert space $\mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_n}$ such that $\Phi_{J_k}(\varrho) = \mathrm{tr}_{I\setminus J_k}[\Phi_I(\varrho)]$ for all input states ϱ.

We emphasize that channel compatibility is a generalization of measurement compatibility. This follows directly from identifying measurements as channels with orthogonal outputs and noting that the broadcast channel corresponds to the simultaneous readout. Besides being more general, channel compatibility differs from measurement compatibility in the fact that channels can be incompatible with themselves. We say that a channel $\Phi_{A\to B}$ is *n-self-compatible* if it can be broadcasted n times by some channel. We call 2-self-compatible channels simply self-compatible. A channel is self-compatible if and only if it is a post-processing of its conjugate channel [92]. Such channels are also called *antidegradable*. Channels that are n-self-compatible for any $n \geq 2$ are exactly the *entanglement breaking channels* (or measure-and-prepare channels) [92].

1.8 Einstein-Podolsky-Rosen steering

The concept of quantum steering was first brought up by Einstein several years after the EPR argument, which was mainly due to the fact that Einstein seemed to prefer a different formulation of the original EPR argument [93]. Consider a free particle with wave function $|\psi\rangle$. In Einsteins view there are the following two possibilities:

E1 The system indeed has a fixed position and momentum but they cannot be measured at the same time. In this sense the wave function is an incomplete description of the physical state of the system.

E2 The system does have neither a predetermined position nor momentum. The wave function is a complete description in the sense that it allows to calculate probabilities and the actual physical state of the system is definite only after the measurement has occurred. In this way, two different wave functions also represent two different physical states of the system.

This also extends to multipartite systems, where one needs to additionally assume locality, meaning that in the spirit of E2 the wave function is a complete description of the physical state of the system and two different wave functions represent to different physical realities. Now, instead of discussing his original formulation of the argument we will consider a pair of qubits to keep everything as simple as possible. Considering a pair of qubits in the state $|\psi^-\rangle$, one can easily verify that

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|z_+z_-\rangle - |z_-z_+\rangle) = \frac{1}{\sqrt{2}}(|x_+x_-\rangle - |x_-x_+\rangle). \tag{1.53}$$

If one now chooses to measure the qubit on Alice in the σ_z direction the wave function on Bob changes to $|\varphi\rangle = |z_\pm\rangle$ and if σ_x is measures the state on Bob changes to $|\varphi\rangle = |x_\pm\rangle$. In this way, the same physical reality is described by different wave functions which in Einsteins view contradicts E2 and thus the description is incomplete. One should note that his argument does not show a possible violation of special relativity in a sense that Alice can send an instantaneous signal to Bob by choosing her measurement. Instead it shows a different kind of action at a distance, which was also studied by Schrödinger, who called this effect "rather discomforting" [94, 95].

Much later the effect of quantum steering was made more formal by Wiseman, Jones and Doherty [27] where the concept of a *local hidden state* (LHS) model was introduced. To be more precise, consider the following experiment. Alice and Bob share a bipartite state ϱ_{AB} and Alice performs a set of POVMs $\{A_{a|x}\}$ with label x and outcomes a. Then, Bobs conditional states are simply given by $\varrho^B_{a|x} = \mathrm{tr}_A[(A_{a|x} \otimes \mathbb{1})\varrho_{AB}]$. We note that the condition $\varrho_B = \sum_a \varrho^B_{a|x}$ holds for all settings x. This corresponds to the non-signaling condition. We say that the states $\varrho^B_{a|x}$ admit a LHS model if they can be written as

$$\varrho_{a|x}^{B} = \sum_{\lambda} p_{\lambda} p(a|x,\lambda)\sigma_{\lambda}. \tag{1.54}$$

This equation can be interpreted as the following simulation task: Instead of measuring her observables on the state ϱ_{AB} Alice prepares the state σ_λ with probability p_λ and announces to Bob the result a depending on the value of the hidden variable λ and the setting x. Therefore, if such a decomposition exists Bob does not have to believe that Alice indeed measured on an entangled state.[12] Conversely, if such a decomposition does not exist, Bob indeed has to believe Alice that she can remotely steer his system into different ensembles, and, hence, there is some kind of action at a distance. If one formulates the LHS model on the level of probability distributions it reads

$$p(ab|xy) = \sum_{\lambda} p(\lambda) p(a|x,\lambda) p_Q(b|B_y,\lambda), \tag{1.55}$$

where we require that $p_Q(b|B_y,\lambda)$ is given by the Born rule, i.e., it originates from Bob measuring on his system. By comparing it to the definition of separability in Eq. 1.20 and the LHV model in Eq. 1.38 we can deduce that steering is an intermediate concept between entanglement and Bell nonlocality. From the definition it is thus evident that Bell nonlocality implies steering, and steering implies entanglement, but not the other way around, see, e.g., Ref. [96]. In contrast to entanglement and Bell nonlocality, which are symmetric in nature, steering is a directed property in a sense that it can happen that Alice can steer Bob, but not the other way around [97].

From the definition of the LHS model in Eq. (1.54) it also becomes evident that there is a strong connection to measurement incompatibility [82]. In fact, one can easily show that if Alices measurements are compatible, a LHS model exists regardless of the shared state, i.e., $\varrho_{a|x} = \mathrm{tr}_A[(A_{a|x} \otimes \mathbb{1})\varrho_{AB}] = \sum_\lambda p(a|x,\lambda)\,\mathrm{tr}_A[(G_\lambda \otimes \mathbb{1})\varrho_{AB}] = \sum_\lambda p(\lambda)p(a|x,\lambda)\sigma_\lambda$. Moreover, whenever Alices measurements are incompatible, there exists a state that reveals steering. One can write $\varrho_{a|x} = \mathrm{tr}_A[(A_{a|x} \otimes \mathbb{1})\,|\psi^+\rangle\langle\psi^+|] = A_{a|x}^T/d$, which does not have a LHS model by the definition of $\{A_{a|x}\}$ not being jointly measurable.

Besides the connection to the compatibility properties of Alices measurements there is also a connection to the compatibility properties of Bobs steering equivalent observables. For any assemblage on Bob $\varrho_{a|x}$, one can define a POVM via the transformation $B_{a|x} = \varrho_B^{-1/2}\varrho_{a|x}\varrho_B^{-1/2}$, where $\varrho_B^{-1/2}$ denotes the pseudo-inverse of the total state of Bob. One can easily verify that these operators fulfill the definition of a POVM, since

[12] It is worth mentioning that Eq. (1.54) can be also written as $\varrho_{a|x}^B = p(a|x)\sum_\lambda p(\lambda|a,x)\sigma_\lambda$, by using that $p(x,\lambda) = p(x)p(\lambda)$. This can be interpreted as Bob having always the state $\varrho_B = \sum_\lambda p(\lambda)\sigma_\lambda$ and his knowledge of the state is simply updated by the additional information that he receives from Alice, hence $p(\lambda) \mapsto p(\lambda|a,x)$, and $p(a|x)$ is the probability that Alice observes a when measured setting x. Also in this case Bob does not need to assume any action at a distance by Alice.

they are positive semidefinite and due to the no-signaling constraint they sum up to the identity operator. From this it becomes clear that this transformation establishes a one-to-one connection between steering and incompatibility that can be used to translate results between both problems [86].

There are many techniques known to detect steering. First, the existence of a LHS model can be decided by a semidefinite program, that is similar to the one in Eq. (1.47), up to the normalization constraint. Other possibilities include steering robustnesses, steering inequalities, and entropic criteria based on uncertainty relations. For a more detailed discussion of the problem see Refs. [98, 99], and references therein.

1.9 Semidefinite and conic optimization

Many problems in quantum information theory, such as Eq. (1.47), can be cast as an optimization problem of a linear function over the set of semidefinite matrices, subjected to linear constraints. Such an optimization problem is called a semidefinite program[13] and it is of the form

$$\begin{aligned} p^* = \max_X \quad & \mathrm{tr}[AX] \\ \text{s.t.:} \quad & \Phi(X) = B \\ & X \geq 0, \end{aligned} \tag{1.56}$$

where $\Phi(X)$ is a hermicity-preserving map. Such optimization problems are usually treated using the method of Lagrange multipliers. The Lagrangian function of this optimization problem reads

$$\mathcal{L}(X, Y, Z) = \quad \mathrm{tr}[AX] + \mathrm{tr}[Y(B - \Phi(X))] + \mathrm{tr}[ZX] \tag{1.57}$$

where Y, Z are the Lagrange multipliers. The problem in Eq. (1.56) can then be phrased as $p^* = \max_X \min_{Y,Z\geq 0} \mathcal{L}(X, Y, Z)$, since

$$\min_{Y,Z\geq 0} \mathcal{L}(X, Y, Z) = \begin{cases} \mathrm{tr}[AX], & \text{if } X \geq 0, \text{ and } B - \Phi(X) = 0 \\ -\infty, & \text{otherwise.} \end{cases} \tag{1.58}$$

From the minimax inequality one obtains $p^* \leq d^* = \min_{Y,Z\geq 0} \max_X \mathcal{L}(X, Y, Z) = \min_{Y,Z\geq 0} g(Y, Z)$. The function $g(Y, Z)$ is called the dual objective function and rewriting $\mathcal{L}(X, Y, Z) = \mathrm{tr}\big[(A - \Phi^\dagger(Y) + Z)X\big] + \mathrm{tr}[YB]$ one finds

$$g(Y, Z) = \max_X \mathcal{L}(X, Y, Z) = \begin{cases} \mathrm{tr}[YB], & \text{if } A - \Phi^\dagger(Y) + Z = 0 \\ \infty, & \text{otherwise.} \end{cases} \tag{1.59}$$

[13] Excellent references on the mathematics of semidefinite programming are the books by Boyd and Vandenberghe [100], and Gärtner and Matoušek [101]. From a more physical point of view we refer to the book by Watrous [102] and the steering review by Cavalcanti and Skrzypczyk [98]

Thus, we are left with $d^* = \min_{Y,Z\geq 0} g(Y,Z)$, s.t.: $A - \Phi^\dagger(Y) + Z = 0$. The variable Z is called a slack variable and it can be eliminated by replacing the constraints by $\Phi^\dagger(Y) \geq A$. Thus, we arrive at the so-called *dual problem*

$$\begin{aligned} d^* = \min_Y \quad & \mathrm{tr}[YB] \\ \text{s.t.:} \quad & \Phi^\dagger(Y) \geq A \end{aligned} \tag{1.60}$$

From the minimax inequality it automatically follows that $p^* \leq d^*$, which is known as *weak duality*. The more important case is when $p^* = d^*$, which is called *strong duality*. This is obviously the case when $\mathcal{L}(X,Y,Z)$ has a saddle point. But there is a more direct way to decide if strong duality holds by using *Slaters condtions*. The condition is that strong duality holds if either the primal or the dual problem is strictly feasible, i.e., there exists $X > 0$ with $\Phi(X) = B$ or there exists Y with $\Phi^\dagger(Y) > A$.

As we have already discussed the positive semidefinite matrices form a cone in the space of Hermitian matrices (cf. Sec. 1.1). Instead of optimizing over positive semidefinite constraints one could introduce constraints that involve more abstract cones, e.g. the cone spanned by separable states that cannot be characterized by a finite number of positive semidefinite constraints. Such an optimization problem is called *conic program*. Let K and L be convex compact cones. The primal form of a conic program is of the form

$$\begin{aligned} p^* = \max_X \quad & \mathrm{tr}[AX] \\ \text{s.t.:} \quad & B - \Lambda[X] \in K, \\ & X \in L, \end{aligned} \tag{1.61}$$

where Λ is a linear operator. To obtain the dual cone program one uses Lagrange multipliers in the exact same way as before, one only needs to replace positive semidefinite by conic constraints. One arrives at the following dual cone program

$$\begin{aligned} d^* = \min_X \quad & \mathrm{tr}[BY] \\ \text{s.t.:} \quad & \Lambda^\dagger[Y] - A \in L^*, \\ & Y \in K^*. \end{aligned} \tag{1.62}$$

Similar as in the case of SDPs the Slater condition provide a sufficient criterion for strong duality. The Slater condition requires an interior point that is feasible. To be more precise we need a point x such that

$$X \in L, \text{ and } B - \Lambda[X] \in K, \tag{1.63}$$

such that $X \in \mathrm{int}(L)$ if $K = \{0\}$, and $B - \Lambda[X] \in \mathrm{int}(L)$ otherwise. Here, $\mathrm{int}(\cdot)$ denotes the interior of a cone.[14]

[14] The property of X being an element of a cone C is sometimes denoted by $X \succeq C$. Whenever X is in the interior of the cone C we write $X \succ C$. For the positive semidefinite cone this represents to the fact that being in the interior of the positive semidefinite cone is equivalent to not being of full rank.

2 Genuine correlated coherence

In this chapter, we will discuss coherence properties of distributed systems. More precisely, we will provide a general framework for the quantification of what we call *genuinely correlated coherence* (GCC). This chapter is based on publication [H].

2.1 Motivation

The idea of genuinely correlated coherence is based on the notion of localizing coherence on individual systems by means of incoherent unitaries, i.e., unitaries that neither create nor destroy coherence of the global state in a given reference basis. The amount of coherence that cannot be localized in the reduced states by such unitaries is then deemed to be genuinely correlated. Such an approach is inspired by Ref. [103], where the authors tackled the issue of localizing high dimensional entanglement on lower dimensional systems by means of unitaries that are free in the resource-theoretic approach to entanglement, that is, by means of local unitaries. Similar as in Ref. [103] this approach is in general related to studies where unitary orbits of some relevant quantum functionals are considered, see, e.g., Refs. [104, 105].

We wish to highlight that our approach is different from that of, e.g., Refs. [106–111]. In the latter references the authors consider the given distribution of local and multipartite coherence. We instead consider the reversible manipulation of coherence under the class of unitaries that, in a resource-theoretic approach to coherence, are deemed to be free operations, since they preserve the coherence that is present in the global state. Most importantly, while these operations maintain the amount of global coherence, they may allow to focus it on local sites. We emphasize that our notion of localization is different from the assisted distillation of coherence [112]. Also, our notion of genuinely correlated coherence is not related to the notion of genuine coherence, with the latter being the resource in a theory of coherence based on the notion of genuine incoherent operation [113]. Finally our genuine correlated coherence is not the same as the notion of intrinsic coherence [108].

2.2 Coherence of distributed systems

Let now $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a $(d_A \times d_B)$-dimensional composite Hilbert space, used to describe the state of Alice and Bob's joint system. We define the local reference basis $\{|i\rangle_A\}_{i=0}^{d_A-1}$ for Alice and similarly $\{|j\rangle_B\}_{j=0}^{d_B-1}$ for Bob. These are the local incoherent bases. The joint incoherent basis is then simply given by the tensor product of the local incoherent bases $\{|ij\rangle_{AB} := |i\rangle_A \otimes |j\rangle_B\}_{i,j}$. Recall, that a bipartite pure state $|\psi\rangle_{AB} = \sum_{ij} \psi_{ij} |ij\rangle_{AB}$ is incoherent if and only if exactly one of the ψ_{ij} is non-zero. Otherwise, the state is coherent and the number of non-zero coefficients is, as we mentioned already in the single-system case, the coherence rank. A pure state is maximally coherent if all the coefficients ψ_{ij} are non-zero and equal in modulus, that is, $|\psi_{ij}| = (d_A d_B)^{-1/2}$ for all i, j. Thus, a maximally coherent state of a bipartite system has the form

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{d_A d_B}} \sum_{ij} e^{i\varphi_{ij}} |i\rangle_A |j\rangle_B . \tag{2.1}$$

We will focus in particular on unitary transformations that leave coherence invariant. In the bipartite setting, incoherent unitary operations are of the form

$$U = \sum_{ij} e^{i\varphi_{ij}} |\pi(ij)\rangle\langle ij| , \tag{2.2}$$

where π is now a permutation of the *pairs* (i, j).

2.3 A first look at correlated coherence

In multipartite systems one can distinguish between different manifestations of coherence, going beyond simply detecting and quantifying coherence in the joint incoherent basis $\{|ij\rangle\}$ (see also [108–111, 114–116]). What we are mostly interested in in this chapter is the relation between global coherence—that is, the coherence of the global state—and the local coherence—the coherence exhibited by the local reduced states. In the simplest case the systems are uncorrelated and their state does not contain any coherence at all, such as in the case of

$$|0\rangle |0\rangle . \tag{2.3}$$

Then, there exist coherent, yet uncorrelated states. Consider the state

$$|+\rangle |+\rangle = \frac{1}{2} \sum_{i,j=0}^{1} |ij\rangle , \tag{2.4}$$

with $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. Here, not only the global state is coherent, but also its marginals are. In fact, the amount of local coherence is equal to the amount of global coherence, in the sense that $\mathscr{C}(\varrho_{AB}) = \mathscr{C}(\varrho_A) + \mathscr{C}(\varrho_B)$ for, e.g., the relative entropy measure of coherence.

A more interesting class of states are those that are globally coherent, but, due to the fact that they are entangled, have incoherent marginals. Nevertheless, in some of these cases the coherence can be concentrated on the subsystems by applying incoherent unitary operations such that the global coherence is preserved, but converted to local coherence. Consider the maximally entangled state in dimension $d \times d$

$$|\psi_d\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle \,. \tag{2.5}$$

This state has coherence rank d and its coherence is a property of the bipartite system since both marginals are maximally mixed and, thus, incoherent. Interestingly, all the coherence can be concentrated on one of the subsystems, say Alice, by applying an incoherent unitary operation. Indeed,

$$|\psi_d\rangle = \mathrm{CNOT}\left[\frac{1}{\sqrt{d}} \sum_i |i\rangle\,|0\rangle\right], \tag{2.6}$$

where CNOT is the generalized controlled-not gate (more precisely, a controlled shift) acting as $\mathrm{CNOT}\,|i\rangle\,|j\rangle = |i\rangle\,|j \oplus i\rangle$, where the addition $\oplus$ is modulo d_B. Notice that the coherence in the state inside the square brackets is located in Alice's system.

2.4 A quantifier of genuine correlated coherence

Our goal is to study coherence in multipartite systems by considering entropic quantifiers to measure to what extent coherence is spread across the subsystems and to what extent it can be concentrated on the individual systems by means of incoherent unitary operations. While we will focus on the bipartite case for the sake of clarity and simplicity, essentially all of the basic definitions extend naturally to the multipartite case.

2.4.1 Correlated coherence

Let us start with quantifying to what extent the coherence of a state is a property of the bipartite state and not only of its marginals. We will adopt the entropic correlated coherence (CC) [106, 107] to characterize the multipartite (as opposite to localized) coherence of a state. We define

$$\mathscr{C}_{\mathrm{CC}}(\varrho_{AB}) := \mathscr{C}(\varrho_{AB}) - \mathscr{C}(\varrho_A \otimes \varrho_B) = \mathscr{C}(\varrho_{AB}) - [\mathscr{C}(\varrho_A) + \mathscr{C}(\varrho_B)]. \tag{2.7}$$

This definition can be easily extended for multipartite systems in a straightforward way. Explicitly,

$$\mathscr{C}_{\mathrm{CC}}(\varrho_{A_1A_2\ldots A_n}) = \mathscr{C}(\varrho_{A_1A_2\ldots A_n}) - \mathscr{C}(\otimes\varrho_{A_i}) = \mathscr{C}(\varrho_{A_1A_2\ldots A_n}) - \sum_i \mathscr{C}(\varrho_{A_i}). \tag{2.8}$$

We remark that the fact that such CC quantifier is equal both to the difference between global coherence and the sum of the local coherences, and to the difference between global coherence and the coherence of the product of the marginals, is a consequence of the additivity of the relative entropy of coherence on tensor products in Eq. (1.16).

Recall that there is an analytic expression that can be used to express the quantifier in Eq. (2.7) in terms of entropies of the original state ϱ_{AB}, of its marginals, and of their decohered versions. Inserting the expression from Eq. (1.17), one obtains

$$\begin{aligned}\mathscr{C}_{\mathrm{CC}}(\varrho_{AB}) &= S(\varrho^d_{AB}) - S(\varrho_{AB}) - \left[S(\varrho^d_A) - S(\varrho_A) + S(\varrho^d_B) - S(\varrho_B)\right] \\ &= I_\varrho(A:B) - I_{\varrho^d}(A:B) \\ &:= \Delta I_\varrho(A:B),\end{aligned} \tag{2.9}$$

where $I_\varrho(A:B) = S(\varrho_A) + S(\varrho_B) - S(\varrho_{AB}) = S(\varrho_{AB} \| \varrho_A \otimes \varrho_B)$ is the mutual information. As observed in Refs. [107, 117], the fact that this difference is not negative comes from the data-processing inequality [118], related to strong-subadditivity of the von Neumann entropy, which ensures that mutual information $I_\varrho(A:B)$ decreases under local operations, and in particular under local projective measurements.

One recognizes (cf. Refs. [107, 117]) $\Delta I_\varrho(A:B)$ as a basis-dependent version of a discord quantifier based on the notion of local projective measurements, meant to capture the quantumness of correlations [119, 120]. One obtains a basis-*in*dependent discord quantifier—what is normally referred to as discord quantifier—by minimizing the difference $\Delta I_\varrho(A:B)$ over the choice of local bases [120, 121], i.e., by optimizing over local unitaries [1]. As we will see later, in this section we go down another path, optimizing over arbitrary incoherent unitaries that can be local and global.

Ref. [107] considered the problem of when $\Delta I_\varrho(A:B)$ vanishes (see also [114]). This happens if and only if

$$\sum_{a,b} P^A_a \otimes P^B_b \varrho_{AB} P^A_a \otimes P^B_b = \varrho_{AB}, \tag{2.10}$$

for some local orthogonal projective measurements $\{P^A_a\}$ and $\{P^B_b\}$ which are diagonal in the respective local incoherent bases, and such that

$$P^A_a \otimes P^B_b \varrho_{AB} P^A_a \otimes P^B_b$$

is uncorrelated for all a and b.

In the following we will focus on pure states. It is clear that in the case of a pure state $\varrho_{AB} = |\psi\rangle\langle\psi|_{AB}$ the conditions above can only be satisfied by a product state $|\psi\rangle_{AB} = |\alpha\rangle_A |\beta\rangle_B$. This is because Eq. (2.10) implies that $P^A_a \otimes P^B_b |\psi\rangle_{AB}$ must be proportional to $|\psi\rangle_{AB}$, besides also being uncorrelated.

[1] We remark that, in the case where one is interested in the quantumness of correlations, a further option is that of optimizing over general local measurements to "extract" the largest possible amount of classical correlation [122].

2.4.2 Genuinely correlated coherence

We introduce the concept of genuinely correlated coherence by taking into consideration that, in the framework of incoherent operations introduced by Baumgratz et al. [41], the coherence present in a distributed system is invariant under incoherent unitaries [123], which are considered as "free operations", even in the case where they are non-local. That is, (global) incoherent unitaries play the same role in coherence theory as local unitaries play in entanglement theory, at least, as mentioned, in the framework of Ref. [41].

Taking this idea seriously, as done previously in, e.g., Refs. [39, 123], in this section we focus on the amount of multipartite coherence that remains after a minimization of $\mathscr{C}_{\mathrm{CC}}$ over all incoherent unitaries defined in Eq. (2.2). This leads to the following definition of genuinely correlated coherence (with straightforward generalization to the multipartite case).

Definition 1. *For a bipartite state ϱ_{AB} the genuinely correlated coherence is defined by*

$$\begin{aligned}\mathscr{C}_{\mathrm{GCC}}(\varrho_{AB}) &= \min_{U_I} \left[\mathscr{C}(\xi_{AB}) - \mathscr{C}(\xi_A \otimes \xi_B)\right]_{\xi = U_I \varrho_{AB} U_I^\dagger} \\ &= \min_{U_I} \Delta I_\xi(A:B)\big|_{\xi = U_I \varrho_{AB} U_I^\dagger}. \end{aligned} \tag{2.11}$$

After having defined our concepts and quantifiers in a general way—that is, for mixed multipartite states—in the previous sections, in the next section we focus on pure bipartite states.

2.5 Genuine correlated coherence for pure bipartite states

We have argued that the only pure bipartite state with vanishing correlated coherence $\mathscr{C}_{\mathrm{CC}}$ are factorized states. This implies that the only pure bipartie states $|\psi\rangle_{AB}$ with vanishing *genuine* correlated coherence $\mathscr{C}_{\mathrm{GCC}}$ are those that can be decorrelated by means of an incoherent unitary. We now derive necessary and sufficient conditions for this to be possible.

Given a pure state $|\psi\rangle$, we can expand it in the incoherent basis,

$$|\psi\rangle = \sum_{ij} \psi_{ij} |ij\rangle = \sum_{ij} |\psi_{ij}| e^{i\varphi_{ij}} |ij\rangle\,, \tag{2.12}$$

where $\mathbb{C} \ni \psi_{ij} = |\psi_{ij}| e^{i\varphi_{ij}}$. Then, a state $|\psi\rangle$ can be decorrelated by incoherent unitaries U_I if and only if $\max_{U_I, |ab\rangle} |\langle ab| U_I |\psi\rangle| = 1$, where $|ab\rangle = \sum_{ij} a_i b_j |ij\rangle = \sum_{ij} |a_i||b_j| e^{i(\alpha_i + \beta_j)} |ij\rangle$. Recall that incoherent unitaries can be written as a combination of a phase gate and a permutation in the incoherent basis (see Eq. (2.2)). Thus, one has

$$|\psi'\rangle = U_I |\psi\rangle = \sum_{ij} |\psi_{\pi(ij)}|\, e^{i\varphi'_{ij}} |ij\rangle\,. \tag{2.13}$$

We remark that, thanks to the freedom in the phases of the incoherent unitary, the phases φ'_{ij} can be chosen arbitrarily, when optimizing over U_I. One therefore has,

$$\begin{aligned}\max_{U_I} |\langle ab| U_I |\psi\rangle| &= \max_{\pi,\varphi'_{ij}} \left| \sum_{ij} |\psi_{\pi(ij)}| \, |a_i||b_j| e^{i(-\alpha_i - \beta_j + \varphi'_{ij})} \right| \\ &= \max_{\pi} \sum_{ij} |\psi_{\pi(ij)}| \, |a_i||b_j|\end{aligned} \tag{2.14}$$

where the last equality comes from the triangle inequality, which can be saturated by a suitable choice of phases φ'_{ij}, specifically $\varphi'_{ij} = \alpha_i + \beta_j$. Then we are left with optimizing

$$\max_{\pi,|ab\rangle} \sum_{ij} |\psi_{\pi(ij)}| \, |a_i||b_j| = \max_{\pi} \left[\|\Psi^{\text{abs}}_{\pi}\|_\infty \right], \tag{2.15}$$

where $\Psi^{\text{abs}}_{\pi} = [|\psi_{\pi(ij)}|]$ is the matrix of the moduli of the coefficients ψ_{ij}, rearranged according to the permutation π, and $\|\cdot\|_\infty$ indicates the largest singular value. From this the following observation follows.

Observation 2. *A bipartite pure state $|\psi\rangle$ of dimension $d_A \times d_B$ with coefficients $\psi_{ij} \in \mathbb{C}$ has $\mathscr{C}_{\text{GCC}}(|\psi\rangle\langle\psi|) = 0$ if and only if $\max_\pi \left[\|\Psi^{abs}_{\pi}\|_\infty\right] = 1$, where $\Psi^{abs}_{\pi} = [|\psi_{\pi(ij)}|]$, and the maximization is over all permutations of the pairs (i, j). An equivalent condition is that there is a permutation π such that Ψ^{abs}_{π} has rank equal to one.*

For two-qubit maximally entangled states one can simply apply the previous observation to obtain the following.

Observation 3. *Any two-qubit maximally entangled state $|\psi\rangle$ has vanishing genuine correlated coherence. This is clear, once one considers that the matrix of coefficients $\Psi = [\psi_{ij}]$ is in such a case proportional to a unitary matrix, whose rows and columns are orthogonal vectors, so that necessarily $\psi^*_{00}\psi_{01} = -\psi^*_{10}\psi_{11}$, and hence, $|\psi_{00}||\psi_{01}| - |\psi_{10}||\psi_{11}| = 0$; this proves that there is a permutation π such that Ψ^{abs}_{π} has rank equal to one.*

We see that for two qubits, maximal entanglement is not compatible with the presence of genuine correlated coherence. Is this the case for all maximally entangled states in any local dimension? The following proves that it is not.

Any pure state $|\psi\rangle$ such that Ψ has a number of non-vanishing entries equal to a prime number strictly larger than $\max\{d_A, d_B\}$ has non-zero genuine correlated coherence. This is because, for Ψ^{abs}_{π} to have rank one, that is, to be of the form $|a\rangle\langle b|$, it must be that the number of its non-zero entries is either less or equal to $\max\{d_A, d_B\}$, or not a prime number. The two-qutrit maximally entangled state

$$|\psi\rangle = \frac{1}{\sqrt{3}} \left(|+\rangle |+\rangle + i |-\rangle |-\rangle + |2\rangle |2\rangle \right) \tag{2.16}$$

with $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, has genuinely correlated coherence, since it has five non-vanishing coefficient when expressed in the standard othonormal basis $\{|i\rangle |j\rangle\}$.

Despite the fact that computing the maximal singular value of the matrix Ψ_π^{abs} is rather easy, there still remains the problem of optimizing over the permutations of the indices. An upper bound on the number of arrangements of the coefficients that could potentially lead to different singular values is given by

$$N = \frac{(d_A \times d_B)!}{\prod_{i,j}(i+j-1)}. \tag{2.17}$$

If one is only interested in whether or not there is an arrangement, such that $\text{rank}(\Psi_\pi) = 1$, the number of arrangements that one has to test is at most

$$N' = \frac{(d_A + d_B - 2)!}{(d_A - 1)! \times (d_B - 1)!}. \tag{2.18}$$

As mentioned, our approach is insipired by the problem of characterizing high-dimensional entanglement tackled in Ref. [103]; a detailed proof and discussion of Eqs. (2.17) and (2.18) can be found therein.

For the case of two qubits the optimization over the permutations of coefficients can, however, easily be performed. Observe that if Ψ_π^{abs} has rank one, it can be written as $|a\rangle\langle b|$ for $|a\rangle = (a_0, a_1)$ (similarly for $|b\rangle$), and we can assume without loss of generality that $a_0 \geq a_1$ and $b_0 \geq b_1$; this is due to the freedom of absorbing the local permutations $\psi_{0j} \leftrightarrow \psi_{1j}$ and $\psi_{i0} \leftrightarrow \psi_{i1}$, which cannot change $\mathscr{C}_{\text{CC}}$, since they preserve both the global and the local coherences. Hence, it is optimal to permute the largest element in the upper left entry and the smallest in the lower right entry. The position of the intermediate values does not matter, since the rank is invariant under transposition. Hence we arrive at the following observation.

Observation 4. *A generic two-qubit pure state $|\psi\rangle = \psi_{00}|00\rangle + \psi_{01}|01\rangle + \psi_{10}|10\rangle + \psi_{11}|11\rangle$ has zero genuine multipartite coherence in the standard computational basis corresponding to this expansion if and only if*

$$\det \begin{bmatrix} |\psi_{\max}| & |\psi_1| \\ |\psi_2| & |\psi_{\min}| \end{bmatrix} = |\psi_{\max}||\psi_{\min}| - |\psi_1||\psi_2| = 0, \tag{2.19}$$

where $\psi_{\max}$ is the largest coefficient, $\psi_{\min}$ the smallest, and $\psi_{1,2}$ are the remaining two coefficients.

2.5.1 Correlated coherence of pure two-qubit states

In this subsection we illustrate the concept of correlated coherence of Sec. 2.4 by evaluating its quantifier $\mathscr{C}_{\text{CC}}$ [107] for generic two-qubit pure states. Again, we consider the generic form of a pure state $|\psi\rangle = \sum_{ij} \psi_{ij} |ij\rangle$. As noticed before, $\mathscr{C}_{\text{CC}}$ coincides with the difference in the mutual information, given by $\Delta I(A:B) = S(\varrho_A) + S(\varrho_B) - S(\varrho_{AB}) - S(\varrho_A^d) - S(\varrho_B^d) + S(\varrho_{AB}^d)$. First, note that $S(\varrho_{AB}) = 0$, since the global state is pure. For the other entropies one obtains

$$\begin{aligned} S(\varrho_A) + S(\varrho_B) &= 2\left[-\mu^+ \log_2(\mu^+) - \mu^- \log_2(\mu^-)\right] \\ &= 2h(\mu^+), \end{aligned} \tag{2.20}$$

where $\mu_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - |\det(\Psi)|^2}$, and

$$\begin{aligned} S(\varrho_A^d) + S(\varrho_B^d) =& -(|\psi_{00}|^2 + |\psi_{01}|^2) \log_2(|\psi_{00}|^2 + |\psi_{01}|^2) \\ & -(|\psi_{10}|^2 + |\psi_{11}|^2) \log_2(|\psi_{10}|^2 + |\psi_{11}|^2) \\ & -(|\psi_{00}|^2 + |\psi_{10}|^2) \log_2(|\psi_{00}|^2 + |\psi_{10}|^2) \\ & -(|\psi_{01}|^2 + |\psi_{11}|^2) \log_2(|\psi_{01}|^2 + |\psi_{11}|^2) \end{aligned} \tag{2.21}$$

$$S(\varrho_{AB}^d) = -\sum_{ij} |\psi_{ij}|^2 \log_2(|\psi_{ij}|^2), \tag{2.22}$$

where, we recall,

$$\Psi = \begin{bmatrix} \psi_{00} & \psi_{01} \\ \psi_{10} & \psi_{11} \end{bmatrix} \tag{2.23}$$

is the matrix of coefficients in the computational (incoherent) basis, and $h(p) := -p \log p - (1-p) \log_2(1-p)$ is the *binary entropy*.

The above constitutes a generic expression of $\mathscr{C}_{\mathrm{CC}}$ for any two-qubit pure state. It simplifies substantially for, e.g., a maximally entangled state. In the latter case, as mentioned already in Observation 3, the matrix of coefficients $\Psi = [\psi_{ij}]$ is proportional to a unitary, more precisely $\Psi = U/\sqrt{2}$, so that $|\det(\Psi)| = 1/2$, and the reduced states are maximally mixed. Thus $\Delta I(A:B) = S(\varrho_{AB}^d)$. Since $\Psi = U/\sqrt{2}$, with the columns and rows of U orthonormal, we have that

$$\begin{aligned} -\sum_{ij} |\psi_{ij}|^2 \log_2(|\psi_{ij}|^2) &= -2(|\psi_{00}|^2 \log_2 |\psi_{00}|^2 + |\psi_{01}|^2 \log_2 |\psi_{01}|^2) \\ &= -2\left(\frac{p}{2} \log_2 \frac{p}{2} + \frac{1-p}{2} \log_2 \frac{1-p}{2}\right) \\ &= 1 + (-p \log_2 p - (1-p) \log_2(1-p)) \\ &= 1 + h(p), \end{aligned}$$

with $p = 2|\psi_{00}|^2 = 2|\psi_{11}|^2$.

Thus the maximal amount of coherence for a maximally entangled state can simply be computed. It turns out that a state like

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle\,|+\rangle + |1\rangle\,|-\rangle), \tag{2.24}$$

which is a two-qubit graph state (see Ref. [124]) corresponding to the graph •–•, has the maximum amount of correlated coherence, $\Delta I(A:B) = 2$. Such a state was already pointed out in Ref. [117] as being at the same time maximally coherent and maximally entangled. Nonetheless, as we have seen in Observation 3, a maximally entangled two-qubit state has zero genuine correlated coherence because it can be decorrelated by an incoherent unitary; indeed, in this specific case, by applying a controlled-σ_z gate the state $|+\rangle\,|+\rangle$ is obtained.

2.5.2 Genuine correlated coherence of pure two-qubit states

We now tackle the calculation of $\mathscr{C}_{\mathrm{GCC}}$ of a given two-qubit pure state, obtained by minimizing $\mathscr{C}_{\mathrm{CC}}$ over incoherent unitaries. Since the coherence $\mathscr{C}(\rho_{AB})$ is invariant under the action of the incoherent unitary operation, we are left with maximizing the sum of the local coherences, that is, with calculating

$$\max[\mathscr{C}(\varrho_A)+\mathscr{C}(\varrho_B)] = \max\left\{S(\varrho_A^d)+S(\varrho_B^d)-[S(\varrho_A)+S(\varrho_B)]\right\}. \tag{2.25}$$

First, note that only the maximization of the local coherences of ϱ_A and ϱ_B depends on the phases of the coefficients ψ_{ij}. The maximization of these terms is equivalent to the minimization of the square of the absolute value of the determinant of Ψ,

$$|\det(\Psi)|^2 = |\psi_{00}\psi_{11}-\psi_{01}\psi_{10}|^2 = |\psi_{00}|^2|\psi_{11}|^2+|\psi_{01}|^2|\psi_{10}|^2-2\,\mathrm{Re}\,\{\psi_{00}\psi_{01}^*\psi_{10}^*\psi_{11}\}.$$

Here, the minimum is obtained if all the phases in the last term cancel, i.e., if the product is rotated to the positive real axis. Therefore it is justified to assume that all the ψ_{ij} are real and positive, that is, to work with Ψ^{abs} rather than Ψ; indeed, we can achieve this by means of the phase freedom in the incoherent unitary.

Having optimized over the phases of the incoherent unitary, we now need to consider the optimization over permutations π of the pairs (i,j). Given the expressions in Eqs. (2.22), it is immediate to realize that it is sufficient to consider only the permutations given by the trivial permutation, by $(0,1)\leftrightarrow(1,1)$, and by $(1,0)\leftrightarrow(1,1)$. That is, the three arrangements of coefficients that could potentially lead to different values of the quantifier are the following:

$$\Psi=\begin{bmatrix}\psi_{00} & \psi_{01}\\ \psi_{10} & \psi_{11}\end{bmatrix},\quad \Psi'=\begin{bmatrix}\psi_{00} & \psi_{11}\\ \psi_{10} & \psi_{01}\end{bmatrix},\quad \Psi''=\begin{bmatrix}\psi_{00} & \psi_{01}\\ \psi_{11} & \psi_{10}\end{bmatrix}. \tag{2.26}$$

Thus, we have found that, for any given two-qubit pure state, one can compute the value of $\mathscr{C}_{\mathrm{GCC}}$ by evaluating the quantities in (2.22) with the use of the absolute values of the amplitudes, and for all the rearrangements (2.26), then picking the arrangement that realizes (2.25).

By optimizing numerically[2] over the amplitudes ψ_{ij} for a two-qubit pure state, we observe that the largest amount of genuine correlated coherence is achieved by pure

[2] We used the function NMaximize in Wolfram Mathematica 11.0.1.0.

states with coherence rank equal to three, rather than maximal (that is, four). More precisely, we find that a state with the largest genuine correlated coherence is

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle) = \frac{1}{\sqrt{3}}(\sqrt{2}\,|0\rangle\,|+\rangle + |1\rangle\,|0\rangle), \tag{2.27}$$

which has reduced states

$$\rho_A = \rho_B = \frac{2}{3}\,|+\rangle\langle+| + \frac{1}{3}\,|1\rangle\langle 1|\,.$$

Such a state has global coherence $\mathscr{C}(|\psi\rangle\langle\psi|) = \log_2 3$ and local coherences $\mathscr{C}(\rho_A) = \mathscr{C}(\rho_B) = h(1/3) - h((3+\sqrt{5})/6)$, so that it has correlated coherence

$$\log_2 3 - 2(h(1/3) - h((3+\sqrt{5})/6)) \approx 0.8485,$$

which can not be further decreased by incoherent unitaries, as evident from Eqs. (2.22) and from the discussion in this subsection. States that have the same amplitudes as $|\psi\rangle$, up to phases and to relabelling of the elements of the incoherent basis, have the same genuine correlated coherence and even the same correlated coherence.

More in general, taking into account our discussion on the optimization of phases, so that only real positive ψ_{ij} need to be considered to find a maximum for $\mathscr{C}_{\mathrm{GCC}}$, one is led to consider the class of rank-three states characterized by points in the first octant on the three-dimensional unit sphere, which can be written using spherical coordinates:

$$|\psi\rangle = \sin(\theta)\cos(\phi)\,|00\rangle + \sin(\theta)\sin(\phi)\,|01\rangle + \cos(\theta)\,|10\rangle\,, \tag{2.28}$$

where $\theta, \phi \in [0, \pi/2]$. In Fig. 2.1 we have plotted the genuine correlated coherence $\mathscr{C}_{\mathrm{GCC}}$ as a function of θ and ϕ, which shows graphically how, within this class, the state in Eq. (2.27) is optimal.

It is worth remarking that the state Eq. (2.27) that has the largest amount of genuinely correlated coherence has a structure similar to that of the four-qubit state

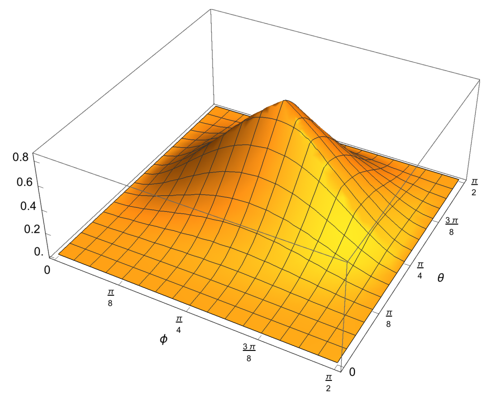


Figure 2.1: Genuine correlated coherence $\mathscr{C}_{\mathrm{GCC}}$ for the class of states described in Eq. (2.28).

that was shown to have the largest amount of genuine multilevel entanglement (see Ref. [103], in particular Observation 2):

$$|\xi\rangle_{AB} = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle). \tag{2.29}$$

This state cannot be reproduced by two pairs of (potentially entangled) qubits together with arbitrary local unitary operations on Alice's and Bob's qubits respectively; that is,

$$|\xi\rangle_{AB} \neq U_{A_1A_2} \otimes V_{B_1B_2} |\psi_1\rangle_{A_1B_1} |\psi_2\rangle_{A_2B_2},$$

for any two qubit states $|\psi_1\rangle$ and $|\psi_2\rangle$, and any two-qubit unitaries U and V. Interestingly, however, the state $|\xi\rangle_{AB}$ can be produced from the state in Eq. (2.27). Think of the latter as being the state of the two qubits held by Alice, and let such qubits each interact independently with one qubit of Bob, initially prepared in the state $|0\rangle$, via a CNOT, so one obtains

$$\begin{aligned}
&(\mathrm{CNOT}_{A_1B_1} \otimes \mathrm{CNOT}_{A_2B_2})\left[\frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)_{A_1A_2} \otimes |00\rangle_{B_1B_2}\right] \\
&= \frac{1}{\sqrt{3}}(|0000\rangle + |0101\rangle + |1010\rangle)_{A_1A_2B_1B_2} \\
&= \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)_{AB}.
\end{aligned}$$

with the identification/relabeling $|0\rangle_A = |00\rangle_{A_1A_2}$, $|1\rangle_A = |01\rangle_{A_1A_2}$, $|2\rangle_A = |10\rangle_{A_1A_2}$, and $|3\rangle_A = |11\rangle_{A_1A_2}$ (similarly for Bob's systems). We find this to be an additional indication of the similarity existing between the theory of coherence and the theory of entanglement, and of the role that the (generalized) CNOT plays in the mapping between coherence and entanglement [123, 125, 126] as well as between general quantumness (of correlations) and entanglement [127–129]

2.6 Conclusions

We have introduced a quantifier of genuine correlated coherence for multipartite systems. It is based on the combination of a quantifier of correlated coherence—the difference between global and local coherences—together with a minimization of such a quantifier over all possible global incoherent unitaries. This is justified by the fact that in principle, in the framework established by [41], and considering the natural choice of global incoherent basis as product of the local incoherent bases, incoherent unitaries that permute, up to phases, elements of such a global basis are free. We note that there is an on-going debate about the right class of incoherent operations that should be considered as free, in particular taking into account that one can distinguish between speakable and unspeakable notions of coherence [33, 130, 131]. The class of unitaries we consider as free makes the theory developed in this section be about speakable coherence. Nonetheless, the starting quantifier $\mathscr{C}_{\mathrm{CC}}$ of correlated coherence is well-defined also in other frameworks, and one could define alternative measures of genuine correlated coherence minimizing over other meaningful classes of unitaries. Given that the class of unitaries in Eq. (2.2) is the most general that preserves incoherent states, our quantifier is more likely to play the role of lower bound to quantifiers in other resource-theoretic frameworks. For that matter, we wish to emphasize that our approach to quantify genuine correlated coherence may contribute to the discussion about the validity and consistency of alternative resource-theoretic frameworks.

Finally, there is the open question of what kind of applications may be related to genuine correlated coherence, as well as of the means to detect such form of coherence, e.g., by means of suitably defined witnesses, like it can be done for coherence and multilevel coherence [132–134]. Moreover, it would be interesting to study the connection between genuine correlated coherence and genuine multilevel entanglement in more detail. Given the fact that the symmetric potion of the correlated coherence always defines a valid measure of entanglement, see Ref. [135], it could be that that the same holds for genuine correlated coherence in the sense that its symmetric portion always defines a valid measure of genuine multilevel entanglement.

3 Monogamy relations of quantum coherence between orthogonal subspaces

In this chapter, we will discuss the limitations on coherence that can exist between orthogonal subspaces. Such limitations arise from the positivity constraint on the density matrix. This leads to a trade-off, and hence, monogamy relations for coherence on a single system. This chapter is based on publication [C].

3.1 Motivation

An interesting feature of entanglement is the fact that it cannot be distributed arbitrarily amongst multiple parties. For instance, if two parties share a maximally entangled state, no entanglement can exist between one of those parties and a third one (cf. Sec. 1.5.3). The constraint that leads to such a bound is the positivity of the quantum state, which is apart from the normalization the only constraint. It is therefore quite natural to ask how the positivity of the quantum state limits the coherence properties of a state. On the level of multipartite systems this leads to monogamy of coherence that can be shared between multiple systems [108, 136] and, thus, limits the amount of entanglement that can be shared between multiple parties. In this chapter we will discuss the constraints that positivity places on the coherence that can exist between orthogonal subspaces of a single party system, by deriving trade-off relations in the coherence that can be shared between one subspace and all other orthogonal subspaces.

The fact that coherence cannot be shared arbitrarily leads to limitations on how well quantum states can be distinguished when the measurements that are being performed cannot access all subspaces. Suppose we are given a system in an initial state ϱ_0 that evolves under a Hamiltonian $H = \sum_i E_i P_i$, where P_i denotes the projector onto the eigenspace corresponding to the energy E_i. At time t the system has evolved into the state $\varrho(t) = U(t)\varrho_0 U^\dagger(t)$, where $U(t) = \exp(-iHt)$. Given that the state exhibits some coherence in the energy-eigenbasis of the Hamiltonian H, it will evolve in time and become more orthogonal, i.e., more distinguishable, to the initial state ϱ_0. To experimentally probe the the distinguishability we might be limited to perform only measurements in a subspace $\mathcal{H}_{E_i} \oplus \mathcal{H}_{E_j}$, where $\mathcal{H}_{E_i}$ denotes a potentially degenerate subspace corresponding to the energy E_i. The maximum distinguishability D_{max} is given by

$$\begin{aligned}
D_{max} &= \max_{0 \leq M_{E_0 E_j} \leq \mathbb{1}_{E_0 E_j}} \left| \mathrm{tr}\left[M_{E_0 E_j} \oplus \mathbb{1}_{\mathrm{rest}} (\varrho(t) - \varrho_0) \right] \right| \\
&= \max_{M_{E_0 E_j}} \left| \mathrm{tr}\left[M_{E_0 E_j} \left\{ \begin{pmatrix} \varrho_{00}(t) & \varrho_{0j}(t) \\ \varrho_{j0}(t) & \varrho_{jj}(t) \end{pmatrix} - \begin{pmatrix} \varrho_{00}^0 & \varrho_{0j}^0 \\ \varrho_{j0}^0 & \varrho_{jj}^0 \end{pmatrix} \right\} \right] \right| \\
&= \max_{M_{E_0 E_j}} \left| \mathrm{tr}\left[M_{E_0 E_j} \begin{pmatrix} 0 & X \\ X^\dagger & 0 \end{pmatrix} \right] \right| \\
&= \|X\|_1,
\end{aligned} \tag{3.1}$$

where $X = \varrho_{0j}(t) - \varrho_{0j}^0$. Thus we find

$$\begin{aligned}
D_{max} &= \|\varrho_{0j}(t) - \varrho_{0j}^0\|_1 = (\exp(-i(E_0 - E_j)t) - 1) \|\varrho_{0j}^0\|_1 \\
&= 2 \left| \sin\left(\frac{E_0 - E_j}{2} t \right) \right| \|\varrho_{0j}^0\|_1
\end{aligned} \tag{3.2}$$

We observe that the distinguishability oscillates with a frequency that depends on the difference in energy. The maximal value is determined by the amount of coherence between these two subspaces, quantified by the trace norm of the block ϱ_{0j}^0. In general, it holds that $\|\varrho_{0j}^0\|_1 \leq 1/2$ for any state. We will see later, that if more than two subspaces are involved these bounds get tighter, since not arbitrarily much coherence can be shared between each pair of subspaces, i.e., the total amount of coherence is bounded.

In Section 3.2 we will set the scenario and recall some important results about Schatten norms. In Section 3.3 we will state the main results, namely we will derive bounds on the amount of coherence that can be shared between one and all the other orthogonal subspaces, based on trace norm, Hilbert-Schmidt norm and the von Neumann relative entropy. In Section 3.4 we will illustrate our results by applying them to the case of a single qutrit. In Section 3.5 we will show how our approach can be used to detect genuine multisubspace coherence.

3.2 Matrix norms

We start with a complete set $\{P_i\}_{i=0}^N$ of orthogonal projectors, each acting like the identity on the orthogonal subspaces $\mathcal{H}_i$ of the Hilbert space $\mathcal{H}$. This imposes the structure $\bigoplus_{i=0}^N \mathcal{H}_i = \mathcal{H}$. Any state ϱ acting on $\mathcal{H}$ can decomposed into its block components with respect to the set $\{P_i\}$, namely $\varrho = \left[\varrho_{ij}\right]_{i,j=0}^N = \left[P_i \varrho P_j\right]_{i,j=0}^N$. For the diagonal blocks it holds that $\varrho_{ii}^\dagger = \varrho_{ii}$ and for the off-diagonal blocks we have that $\varrho_{ij}^\dagger = \varrho_{ji}$ for

$i \neq j$. Whenever we consider the components of the state ϱ with respect to two specific subspaces, we adopt the following shorthand notation

$$\varrho^{(ij)} = \begin{bmatrix} \varrho_{ii} & \varrho_{ij} \\ \varrho_{ij}^{\dagger} & \varrho_{jj} \end{bmatrix}.$$

It is worth noting how this relates to the theory of block-coherence. This was studied in Ref. [40] and more recently in Refs. [137, 138]. A state is called block incoherent if it can be written as

$$\varrho = \mathrm{diag}(\varrho_{ii}), \tag{3.3}$$

otherwise the state is deemed to be coherent with respect to $\{P_i\}$.

The coherence that exists between two different subspaces i and j is encoded in the off-diagonal block ϱ_{ij}. The amount of coherence between these blocks does not depend on the choice of basis within these blocks, that is, its amount should be invariant under block-diagonal unitary transformations of the form $U = \bigoplus_i U_i$. Such a unitary transformation changes the off-diagonal blocks to $\varrho_{ij} \mapsto U_i^{\dagger} \varrho_{ij} U_j$. Hence, to quantify the amount of coherence that is shared between different subspaces we take a unitarily invariant matrix norm of the off-diagonal blocks, such as the Schatten p-norms. Given a matrix $M \in \mathcal{M}_{n,m}$ the Schatten p-norms are defined on the vector of singular values $\sigma(M)$ of M by

$$\|M\|_p = \left(\sum_{i=1}^{\min\{m,n\}} \sigma_i(M)^p \right)^{1/p}.$$

Here $p \in [1, \infty]$ and the singular values are non-negative and assumed to be ordered such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min\{m,n\}} \geq 0$. They are the eigenvalues of the Hermitian operator $|M| = \sqrt{M^{\dagger}M}$ and thus the Schatten p-norms can be expressed as $\|M\|_p = \mathrm{tr}\big(M^{\dagger}M\big)^{\frac{p}{2}}$. For $p = 1$ the norm $\|\cdot\|_1 = \|\cdot\|_{tr}$ is called the trace norm, for $p = 2$ the norm $\|\cdot\|_2 = \|\cdot\|_{HS}$ is called the Hilbert-Schmidt norm and for $p = \infty$ (defined via a limit procedure) the norm $\|\cdot\|_{\infty}$ is called the operator norm, and equal to the largest singular value of M. One of the main properties of Schatten norms is their isometric invariance, i.e., $\|UMV\|_p = \|M\|_p$ for all isometries U, V.

3.3 Results

Our aim is to capture the trade-off in coherence that can be shared between one specific subspace (without loss of generality, the first one, $\mathcal{H}_0$) and and all the other N subspaces. First, we quantify the amount of coherence by taking the sum over the trace-norms and Hilbert-Schmidt norms of all the blocks that contain information about the coherence between the first and all other N subspaces. Then, we will derive trade-off relations in terms of the von Neumann relative entropy.

3.3.1 Trace norm

Let us start by defining the following quantity.

Definition 5. *Given a complete set of projectors $\{P_i\}$ on $\mathcal{H}$ and a state $\varrho = [\varrho_{ij}]_{i,j=0}^{N}$, the amount of coherence that is shared between the first and all other N subspaces is quantified by*

$$\mathscr{C}^{tr}_{0:1\dots N}(\varrho) = \sum_{k=1}^{N} \|\varrho_{0k}\|_{tr}.$$

Our first aim is to derive an upper bound on this quantity which results in a trade-off relation between the shared coherences. First, note that for any positive semidefinite matrix $M \geq 0$ which is of the form

$$M = \begin{pmatrix} A & X \\ X^{\dagger} & B \end{pmatrix}, \tag{3.4}$$

with $A, B \geq 0$, the off-diagonal blocks can be written as

$$X = A^{1/2} K B^{1/2}, \tag{3.5}$$

where K is a *contraction*, i.e., $K^{\dagger}K \leq \mathbb{1}$ [139]. It was shown in Ref. [140] that for matrices of the form of Eq. (3.4)

$$\left\| |X|^q \right\|^2 \leq \|A^q\| \|B^q\| \tag{3.6}$$

holds for all unitarily invariant norms $\|\cdot\|$ and all $q > 0$. In particular, this inequality holds for the trace norm. By choosing $q = 1$ we obtain

$$\mathscr{C}^{tr}_{0:1\dots N}(\varrho) = \sum_{k=1}^{N} \|\varrho_{0k}\|_{tr} \leq \sum_{k=1}^{N} \sqrt{\|\varrho_{00}\|_{tr}} \sqrt{\|\varrho_{kk}\|_{tr}} = \sqrt{\operatorname{tr}\varrho_{00}} \sum_{k=1}^{N} \sqrt{\operatorname{tr}\varrho_{kk}}. \tag{3.7}$$

Then, by the inequality between the arithmetic and quadratic mean, we obtain

$$\sqrt{\operatorname{tr}\varrho_{00}} \sum_{k=1}^{N} \sqrt{\operatorname{tr}\varrho_{kk}} \leq \sqrt{N} \sqrt{\operatorname{tr}\varrho_{00}} \sqrt{\sum_{k=1}^{N} \operatorname{tr}\varrho_{kk}} = \sqrt{N} \sqrt{\operatorname{tr}\varrho_{00}(1 - \operatorname{tr}\varrho_{00})} \tag{3.8}$$

Thus, we arrive at the following observation.

Observation 6. *The coherence measured by the trace norm between the first and all other N subspaces is bounded by*

$$\mathscr{C}^{tr}_{0:1\dots N}(\varrho) = \sum_{k=1}^{N} \|\varrho_{0k}\|_{tr} \leq \sqrt{N} \sqrt{\operatorname{tr}\varrho_{00}(1 - \operatorname{tr}\varrho_{00})}. \tag{3.9}$$

Let us discuss this result. First, this bound leads to a trade-off in coherence between the subspaces and the bound only depends on the number of subspaces involved and the accumulated probability in the first block. Second, the bound provided in Eq. (3.9) is tight, meaning that there always exists a state saturating the inequality. Consider a block ϱ_{00}, we can define the state

$$\sigma = |\varphi\rangle\langle\varphi| \otimes \hat{\varrho}_{00}, \tag{3.10}$$

with $|\varphi\rangle = \left(\sqrt{\operatorname{tr}\varrho_{00}}, \sqrt{(1-\operatorname{tr}\varrho_{00})/N}, \dots, \sqrt{(1-\operatorname{tr}\varrho_{00})/N}\right)$ and $\hat{\varrho}_{00} = \varrho_{00}/\operatorname{tr}\varrho_{00}$. It is straightforward to show that this state saturates the bound. Note that if all the blocks are one dimensional, ϱ_{00} is a probability, and hence, the saturating state is pure. But this is not true in general. Furthermore, not every pure state saturates the bound.

The bound in Observation 6 still depends on the number N of subspaces. Next we consider a slight variation in the quantifier, and derive a bound that is quadratic in the trace norm of the off-diagonal blocks. A calculation similar to the previous one leads to the following.

Observation 7. *Given a state $\varrho = [\varrho_{ij}]_{i,j=0}^N$ using Eq. (3.6) we obtain the following bound*

$$\begin{aligned}
\mathscr{C}^{tr,2}_{0:1\dots N}(\varrho) &\equiv \sum_{i=1}^N \|\varrho_{0i}\|_{tr}^2 \\
&\leq \sum_{i=1}^N \|\varrho_{00}\|_{tr}\|\varrho_{ii}\|_{tr} \\
&= \|\varrho_{00}\|_{tr}\sum_{i=1}^N \|\varrho_{ii}\|_{tr} \\
&= \operatorname{tr}[\varrho_{00}](1-\operatorname{tr}[\varrho_{00}]).
\end{aligned}$$

Again, this bound is tight in a sense that there always exist a state that saturates the bound. Specifically, it is straightforward to see that the state $\sigma = |\varphi\rangle\langle\varphi| \otimes \hat{\varrho}_{00}$, with $\hat{\varrho}_{00} = \varrho_{00}/\operatorname{tr}\varrho_{00}$ and $|\varphi\rangle = (\sqrt{\operatorname{tr}\varrho_{00}}, \sqrt{\operatorname{tr}\varrho_{11}}, \dots, \sqrt{\operatorname{tr}\varrho_{NN}})$, saturates the bound. In general, a pure state $|\psi\rangle$ always saturate the inequality, independently of the dimensions of the subspaces $\mathcal{H}_i$. Indeed, for a pure state $|\psi\rangle$, one has $\|\rho_{0i}\|_{tr}^2 = \|P_0\,|\psi\rangle\langle\psi|\,P_i\|_{tr}^2 = \|P_0\,|\psi\rangle\,\|_\infty^2\|P_i\,|\psi\rangle\,\|_\infty^2 = \|P_0\,|\psi\rangle\langle\psi|\,P_0\|_{tr}\|P_i\,|\psi\rangle\langle\psi|\,P_i\|_{tr} = \|\rho_{00}\|_{tr}\|\rho_{ii}\|_{tr}$. In particular we observe that in general Observation 6 follows from Observation 7. To see this, take Eq. (3.11) and use the inequality between arithmetic and quadratic mean. One obtains $1/N\left(\sum_{i=1}^N \|\varrho_{0i}\|_{tr}\right)^2 \leq \sum_{i=1}^N \|\varrho_{0i}\|_{tr}^2$, from which Observation 6 follows. In that sense Observation 7 is the stronger one.

3.3.2 Hilbert-Schmidt norm

Next, we will quantify the coherence by means of the Hilbert-Schmidt norm of the off-diagonal blocks. We make the following Observation.

Observation 8. *Let $\varrho = [\varrho_{ij}]_{i,j=0}^N$ be a state. Then*

$$\mathscr{C}^{HS,2}_{0:1\dots N}(\varrho) \equiv \sum_{i=1}^N \left\|\varrho_{0i}^\dagger\varrho_{0i}\right\|_{tr} \leq \operatorname{tr}\varrho_{00}(1-\operatorname{tr}\varrho_{00}), \tag{3.11}$$

where equality holds, if and only if the state ϱ is pure. Note that in general $\left\|X^\dagger X\right\|_{tr} = \|X\|_{HS}^2$.

Proof. First, we prove the inequality. Consider the submatrices $\varrho^{(0i)}$ where the diagonal blocks are Hermitian, $\varrho_{00}^\dagger = \varrho_{00}$ and $\varrho_{ii}^\dagger = \varrho_{ii}$, and positive semidefinite, $\varrho_{00} \geq 0$ and $\varrho_{ii} \geq 0$. Then, Eq. (3.6) reads $\left\| |\varrho_{0i}|^q \right\|^2 \leq \left\|\varrho_{00}^q\right\| \left\|\varrho_{ii}^q\right\|$ for any unitarily invariant norm, with $|\varrho_{0i}| = (\varrho_{0i}^\dagger \varrho_{0i})^{\frac{1}{2}}$. Choosing the trace norm and $q = 2$, we obtain

$$\sum_{i=1}^{N} \left\|\varrho_{0i}^\dagger \varrho_{0i}\right\|_{tr} \leq \sqrt{\operatorname{tr} \varrho_{00}^2} \sum_{i=1}^{N} \sqrt{\operatorname{tr} \varrho_{ii}^2}. \tag{3.12}$$

For (non-normalised) states it holds that $\operatorname{tr} \varrho^2 \leq (\operatorname{tr} \varrho)^2$, with equality if and only if $\operatorname{rk} \varrho = 1$, i.e., the state is pure. Hence

$$\sqrt{\operatorname{tr} \varrho_{00}^2} \sum_{i=1}^{N} \sqrt{\operatorname{tr} \varrho_{ii}^2} \leq \operatorname{tr} \varrho_{00} \sum_{i=1}^{N} \operatorname{tr} \varrho_{ii} = \operatorname{tr} \varrho_{00}(1 - \operatorname{tr} \varrho_{00}), \tag{3.13}$$

which proves the inequality. Next, we prove that the inequality is saturated by pure, and pure states only. First assume that the inequality is saturated. Then it follows from Eq. (3.13) that all the blocks must be pure, hence we can write $\varrho_{00} = \operatorname{tr}[\varrho_{00}] |\hat{\varphi}_0\rangle\langle\hat{\varphi}_0|$ and $\varrho_{ii} = \operatorname{tr}[\varrho_{ii}] |\hat{\varphi}_i\rangle\langle\hat{\varphi}_i|$, with $\operatorname{tr} |\hat{\varphi}_0\rangle\langle\hat{\varphi}_0| = \operatorname{tr} |\hat{\varphi}_i\rangle\langle\hat{\varphi}_i| = 1$. Then, there exists a contraction C_{0i}, with $C_{0i}^\dagger C_{0i} \leq \mathbb{1}$, such that the off-diagonal blocks can be written as $\varrho_{0i} = \varrho_{00}^{1/2} C_{0i} \varrho_{ii}^{1/2}$. Using this and the previous result we can evaluate the left hand-side of Eq. (3.13). We obtain

$$\begin{aligned} \left\|\varrho_{0i}^\dagger \varrho_{0i}\right\|_{tr} &= \operatorname{tr}\left[C_{0i} \varrho_{ii} C_{0i}^\dagger \varrho_{00}\right] \\ &= \operatorname{tr}[\varrho_{00}] \operatorname{tr}[\varrho_{ii}] \operatorname{tr}\left[C_{0i} |\hat{\varphi}_i\rangle\langle\hat{\varphi}_i| C_{0i}^\dagger |\hat{\varphi}_0\rangle\langle\hat{\varphi}_0|\right] \\ &= \operatorname{tr}[\varrho_{00}] \operatorname{tr}[\varrho_{ii}] |\langle\hat{\varphi}_0| C_{0i} |\hat{\varphi}_i\rangle|^2. \end{aligned} \tag{3.14}$$

Then, if equality in Eq. (3.12) holds, we must have that $\langle\hat{\varphi}_0| C_{0i} |\hat{\varphi}_i\rangle = e^{i\varphi_{0i}}$, for some real phase φ_{0i}, and the off-diagonal blocks take the following form:

$$\begin{aligned} \varrho_{0i} &= \sqrt{\operatorname{tr} \varrho_{00} \operatorname{tr} \varrho_{ii}}\, |\hat{\varphi}_0\rangle\langle\hat{\varphi}_0| C_{0i} |\hat{\varphi}_i\rangle\langle\hat{\varphi}_i| \\ &= \sqrt{\operatorname{tr} \varrho_{00} \operatorname{tr} \varrho_{ii}}\; e^{i\varphi_{0i}} |\hat{\varphi}_0\rangle\langle\hat{\varphi}_i| \\ &= \sqrt{\operatorname{tr} \varrho_{00} \operatorname{tr} \varrho_{ii}}\; e^{i\varphi_{0i}} \operatorname{diag}(\varphi_0) |+\rangle\langle+| \operatorname{diag}(\varphi_i)^\dagger, \end{aligned} \tag{3.15}$$

where $|+\rangle = (1, 1, \dots, 1)$. Using Eq. (3.5) it also follows that all the other blocks are of rank one. We find

$$\begin{aligned} \varrho_{kl} &= \sqrt{\operatorname{tr} \varrho_{kk} \operatorname{tr} \varrho_{ll}}\, |\hat{\varphi}_k\rangle\langle\hat{\varphi}_k| C_{kl} |\hat{\varphi}_l\rangle\langle\hat{\varphi}_l| \\ &= \sqrt{\operatorname{tr} \varrho_{kk} \operatorname{tr} \varrho_{ll}}\, c_{kl} e^{i\varphi_{kl}} |\hat{\varphi}_k\rangle\langle\hat{\varphi}_l| \\ &= \sqrt{\operatorname{tr} \varrho_{kk} \operatorname{tr} \varrho_{ll}}\, c_{kl} e^{i\varphi_{kl}} \operatorname{diag}(\varphi_k) |+\rangle\langle+| \operatorname{diag}(\varphi_l)^\dagger. \end{aligned} \tag{3.16}$$

So far we have shown that if the inequality is saturated, it follows that all the blocks are pure and we know the structure of the first row and the diagonal. What remains to be proven is that all the other blocks have a structure such that the overall state is pure. So far we have $\varrho = F(\tilde{\varrho} \otimes |+\rangle\langle+|)F^\dagger$, where F is a filter (i.e., invertible) of the form $F = \mathrm{diag}(\sqrt{\mathrm{tr}\,\varrho_{ii}}\mathrm{diag}(\varphi_i))$. Furthermore we have that $\tilde{\varrho} = \left[c_{ij}e^{i\varphi_{ij}}\right]_{i,j=0}^{N}$, where $c_{ii}e^{i\varphi_{ii}} = 1$ and $c_{0j} = c_{i0} = 1$. Then, the state ϱ must be positive semidefinite, which is the case if and only if $\tilde{\varrho}$ is positive semidefinite. Now, consider the vector $|\psi\rangle = \frac{1}{\sqrt{2}}(1-N, e^{-i\varphi_{01}}, \ldots, e^{-i\varphi_{0N}})$. For this vector we obtain

$$\begin{aligned}\langle\psi|\,\varrho\,|\psi\rangle &= -\frac{(N-2)(N-1)}{2} \\ &+ \sum_{1\le i<j\le N} c_{ij}\frac{e^{i(\varphi_{ij}-\varphi_{0j}+\varphi_{0i})}+e^{-i(\varphi_{ij}-\varphi_{0j}+\varphi_{0i})}}{2} \\ &\ge 0.\end{aligned}$$

Since the last term in the sum is $\cos\big(\varphi_{ij} - (\varphi_{0j} - \varphi_{0i})\big)$ and there are exactly $(N-2)(N-1)/2$ terms in the sum it follows that $c_{ij} = 1$ and $\varphi_{ij} = \varphi_{0j} - \varphi_{0i}$ for all i, j. Thus the state is of the form $\varrho = |\tilde{\psi}\rangle\langle\tilde{\psi}|$, with

$$|\tilde{\psi}\rangle = \bigoplus_{i=0}^{N} \sqrt{\mathrm{tr}\,\varrho_{ii}}\,|\hat{\varphi}_i\rangle\, e^{i\varphi_{0i}}. \tag{3.17}$$

This proves that the global state is necessarily pure. The converse it easy to prove, since any pure state admits a decomposition in the form of Eq. (3.17). □

3.3.3 Relative entropy of coherence

Another common quantifier of coherence is the von Neumann relative entropy [141]. It captures the increase of entropy when going from a state ϱ to its decohered version ϱ_d. Here we define $\varrho^{(0i)} = \sum_{k,l=0,i} P_k\varrho P_l$ to be a trimmed version of ϱ and $\varrho^{(0i)} = \sum_{k=0,i} P_k\varrho P_k$ its decohered version. We have

$$\sum_{i=1}^{N} S(\varrho^{(0i)}||\varrho_d^{(0i)}) = \sum_{i=1}^{N}\left[S(\varrho_d^{(0i)}) - S(\varrho^{(0i)})\right]. \tag{3.18}$$

For each of the terms in the sum we have

$$S(\varrho_d^{(0i)}) - S(\varrho^{(0i)}) = \mathrm{tr}\,\varrho^{(0i)}\left[S(\hat{\varrho}_d^{(0i)}) - S(\hat{\varrho}^{(0i)})\right], \tag{3.19}$$

where $\hat{\varrho} = \varrho/\,\mathrm{tr}[\varrho]$. Next, we expand the first term in the sum using basic properties of the von Neumann entropy [142]. We obtain

$$\begin{aligned}&\mathrm{tr}\,\varrho^{(0i)}\left[H\left(\left\{\frac{\mathrm{tr}\,\varrho_{00}}{\mathrm{tr}\,\varrho^{(0i)}}, \frac{\mathrm{tr}\,\varrho_{ii}}{\mathrm{tr}\,\varrho^{(0i)}}\right\}\right)\right. \\ &\qquad\left. + \frac{\mathrm{tr}\,\varrho_{00}}{\mathrm{tr}\,\varrho^{(0i)}}S(\hat{\varrho}_{00}) + \frac{\mathrm{tr}\,\varrho_{ii}}{\mathrm{tr}\,\varrho^{(0i)}}S(\hat{\varrho}_{ii}) - S(\hat{\varrho}^{(0i)})\right] \\ &\le \mathrm{tr}\left(\varrho^{(0i)}\right)h_2\left(\frac{\mathrm{tr}(\varrho_{00})}{\mathrm{tr}\left(\varrho^{(0i)}\right)}\right),\end{aligned} \tag{3.20}$$

where the inequality holds since $S(\varrho) \geq \sum_k p_k S(\varrho_k)$, where ϱ_k is the normalised post-measurement state corresponding to outcome k of a projective measurement and p_k are the outcome probabilities. This follows from the fact that when a projective measurement is performed and the outcome is recorded the uncertainty about the state does not increase on average, see Ref. [143]. Then the sum over the right-hand side of Eq. (3.20) can be further upper bounded by

$$\begin{aligned} &\sum_{i=1}^{N} \operatorname{tr}\Big(\varrho^{(0i)}\Big) h_2\left(\frac{\operatorname{tr}\varrho_{00}}{\operatorname{tr}\varrho^{(0i)}}\right) \\ &= \sum_{k=1}^{N} \operatorname{tr}\varrho^{(0k)} \sum_{i=1}^{N} \frac{\operatorname{tr}\varrho^{(0i)}}{\sum_{k=1}^{N}\operatorname{tr}\varrho^{(0k)}} h_2\left(\frac{\operatorname{tr}\varrho_{00}}{\operatorname{tr}\varrho^{(0i)}}\right) \\ &\leq [(N-1)\operatorname{tr}\varrho_{00}+1] h_2\left(\frac{N\operatorname{tr}\varrho_{00}}{(N-1)\operatorname{tr}\varrho_{00}+1}\right), \end{aligned} \tag{3.21}$$

where we have used that $\sum_{k=1}^{N}\operatorname{tr}\varrho^{(0k)} = (N-1)\operatorname{tr}\varrho_{00}+1$ and the concavity of the binary entropy, namely that $\sum_i p_i h_2(x_i) \leq h(\sum_i p_i x_i)$. We arrive at the following observation.

Observation 9. *The amount of coherence that is shared between the first and all other subspaces, quantified by the relative entropy of block coherence, is bounded by*

$$\begin{aligned} &\sum_{i=1}^{N} S(\varrho^{(0i)}||\varrho_d^{(0i)}) \\ &\leq [(N-1)\operatorname{tr}(\varrho_{00})+1] h_2\left(\frac{N\operatorname{tr}(\varrho_{00})}{(N-1)\operatorname{tr}(\varrho_{00})+1}\right). \end{aligned} \tag{3.22}$$

The bound in this inequality is also saturated by the state from Eq. (3.10). In the first inequality in Eq. (3.20) equality holds because $\hat{\sigma}_{00}$ and $\hat{\sigma}_{ii}$ have the same entropy as σ. Note, that we have $\sigma = |\varphi\rangle\langle\varphi| \otimes \hat{\varrho}_{00}$, $\sigma_{00} = (|0\rangle\langle 0| \otimes \mathbb{1})\sigma(|0\rangle\langle 0| \otimes \mathbb{1})$ as well as $\sigma_{ii} = (|i\rangle\langle i| \otimes \mathbb{1})\sigma(|i\rangle\langle i| \otimes \mathbb{1})$. In the second inequality on Eq. (3.21) $\operatorname{tr}(\sigma_{00}) / \operatorname{tr}(\sigma^{0i})$ is the same for all $i = 1, \dots, N$, and hence, equality holds due to the concavity of the binary entropy h_2.

3.4 Example of a single qutrit

Let us now discuss the results of the previous section in the case of a single qutrit and $N = 2$. We write ϱ as

$$\varrho = \begin{pmatrix} p_0 & a & b \\ \bar{a} & p_1 & \cdot \\ \bar{b} & \cdot & p_2 \end{pmatrix}, \tag{3.23}$$

$p_2 = 1 - p_0 - p_1$, where we have used dots for placeholders for entries that we do not directly consider. For a given value of p_0 the physical region is bounded by the

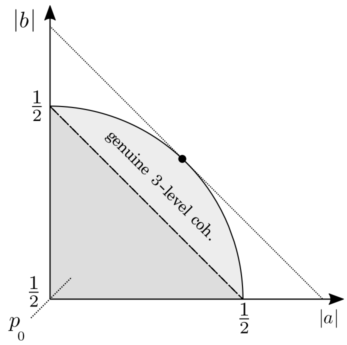


Figure 3.1: This figure shows the trade-off relations for a single qutrit state. The coherences between any two subspaces $|a|$ and $|b|$ are bounded by $\frac{1}{2}$. The bound from Observation 6 (dotted line) is tight, but does not completely characterize the set of physical realizable states (grey area). The quantifier from Observation 7 is also tight and it completely characterizes the set of physical realizable states (solid line). The dashed line is a witness for genuine three-level coherent states.

inequalities

$$|a| + |b| \leq \sqrt{2}\sqrt{p_0(1-p_0)} \tag{3.24}$$

and

$$|a|^2 + |b|^2 \leq p_0(1-p_0), \tag{3.25}$$

which are special cases of Observation 6 and Observation 7, respectively. The physical region, see Figure 3.1, corresponds to a quarter disk, where all pure states lie on the border defined by the quarter circle. The states lying on the a and b axis are two-level coherent states. By varying the parameter p_0 one sees, that the set of physical states is given by a quarter ball, where all pure states lie on the quarter sphere.

For the entropic trade-off relation in Observation 9 we find the following. For $N = 2$ the two entropies $S(\varrho^{(01)}||\varrho_d^{(01)}) = S_{01}$ and $S(\varrho^{(02)}||\varrho_d^{(02)}) = S_{02}$ are both bounded by one. For their sum we obtain the bound

$$S_{01} + S_{02} \leq (p_0+1)h_2\left(\frac{2p_0}{p_0+1}\right). \tag{3.26}$$

Computing the maximum value for this bound by maximizing over p_0 one obtains $S_{01} + S_{02} \leq \left(1 - \frac{2}{\sqrt{5}}\right) + \frac{1-\sqrt{5}}{\sqrt{5}} \log_2(3-\sqrt{5}) + \frac{1}{\sqrt{5}} \log_2(3+\sqrt{5}) \approx 1.3885$, for $p_0 = \frac{1}{\sqrt{5}}$.

3.5 Detection of genuine multisubspace coherence

Any pure state $|\psi\rangle$ can be written as $|\psi\rangle = \sum_{j=0}^{N} |\psi_j\rangle$, where $|\psi_j\rangle = P_j |\psi\rangle$. We say that $|\psi\rangle$ has block coherence rank $bcr(|\psi\rangle)$ equal to k if exactly k of the $|\psi_i\rangle$ do not vanish.

We denote the convex hull of all pure states with block coherence rank at most k by $\mathcal{BC}_k$. We say that a mixed state ρ has block-coherence number $bcn(\rho)$ equal to k if it is in $\mathcal{BC}_k$ but not in $\mathcal{BC}_{k-1}$. We are going to see that the quantifier $\mathscr{C}^{tr}_{0:1\ldots N}$ obeys a stricter bound for states with limited block coherence.

Observation 10. *It holds*

$$\mathscr{C}^{tr}_{0:1\ldots N}(\rho) \leq \sqrt{bcn(\varrho) - 1}\sqrt{\operatorname{tr}\varrho_{00}(1 - \operatorname{tr}\varrho_{00})}. \tag{3.27}$$

Proof. If a pure state $|\psi\rangle$ has block-coherence rank $bcr(|\psi\rangle)$, then

$$\begin{aligned}\mathscr{C}^{tr}_{0:1\ldots N}(|\psi\rangle\langle\psi|) \leq &\sqrt{bcr(|\psi\rangle) - 1} \\ &\times \sqrt{\langle\psi_0|\psi_0\rangle\,(1 - \langle\psi_0|\psi_0\rangle)}\end{aligned} \tag{3.28}$$

since there are at most $bcr(|\psi\rangle) - 1$ other blocks that are populated. A state ρ admits a pure-state ensemble decomposition $\rho = \sum_j p_j\, |\psi^j\rangle\langle\psi^j|$ with $bcr(\,|\psi^{(j)}\rangle) \leq bcn(\rho)$. Thus,

$$\mathscr{C}^{tr}_{0:1\ldots N}(\rho) \leq \sum_j p_j \mathscr{C}_{0:1\ldots N}(\,|\psi^j\rangle\langle\psi^j|) \tag{3.29}$$

$$\leq \sqrt{bcn(\rho) - 1}\sum_j p_j \sqrt{\,\langle\psi_0^j|\psi_0^j\rangle\,(1 -\,\langle\psi_0^j|\psi_0^j\rangle)} \tag{3.30}$$

$$\leq \sqrt{bcn(\rho) - 1}\sqrt{\sum_j p_j\,\langle\psi_0^j|\psi_0^j\rangle\,(1 - \sum_j p_j\,\langle\psi_0^j|\psi_0^j\rangle)} \tag{3.31}$$

$$= \sqrt{bcn(\rho) - 1}\sqrt{\operatorname{tr}\varrho_{00}(1 - \operatorname{tr}\varrho_{00})}. \tag{3.32}$$

The first inequality is due to the convexity of $\mathscr{C}^{tr}_{0:1\ldots N}$, the second inequality is due to the bound in Eq. (3.28), and the third inequality is due to the concavity in $0 \leq x \leq 1$ of $\sqrt{x(1-x)}$.

Notice that recover the always valid bound in Eq. (3.9) by considering that $bcn(\rho) \leq N + 1$. One benefit of the bound in Eq. (3.27) is that it allows one to certify genuine multisubspace coherence just by considering one block-row of the density matrix. □

For the qutrit example in Eq. (3.23) of Section 3.4 the inequality in Eq. (3.27) reads explicitly $\mathscr{C}^{tr}_{0:1\ldots N}(\rho) \leq \sqrt{\operatorname{tr}\varrho_{00}(1 - \operatorname{tr}\varrho_{00})}$ for any state with block-coherence number less or equal to two. Hence, any state of the form in Eq. (3.23) that violates this necessarily contains three-level coherence. See Figure 3.1 for $p_0 = 1/2$, i.e., $|a| + |b| \leq 1/2$.

3.6 Conclusions

In this chapter, we have derived trade-off relations for the coherence that can be shared between multiple subspaces, as a consequence of the positivity constraint on the density matrix. We formulated trade-off relations in terms of the trace norm and Hilbert-Schmidt norm as well as the von Neumann relative entropy. Furthermore, we found that out quantifier can be used to detect multisubspace coherence, i.e., that it obeys stricter bound when states with limited block coherence are considered. We further conclude that similar trade-offs also hold for other positive semidefinite matrices, e.g., covariance matrices and Choi matrices. In Ch. 5 we will see that monogamy relations can be used to characterize the topology of quantum networks, if one applies them to covariance matrices. For future research it would be interesting to see the implications of this trade-off in applications in which block coherence is important, e.g., in quantum clocks [144] or quantum metrology. Furthermore it would be interesting to study the connection to entanglement monogamy and to see to what extent this trade-off is relevant for genuine multipartite entanglement.

4 Entanglement in the triangle network

In this chapter, we will discuss correlations in quantum networks from the point of view of entanglement. More precisely, we will discuss how a network structure limits the entanglement of quantum states that can be prepared in such a network. This chapter is based on publication [B].

4.1 Motivation

Quantum networks are based on local quantum processors that receive entanglement from different sources, that is subsequently processed. In the spirit of entanglement swapping, the entanglement initially generated on the links of the networks can then be propagated to the entire network by performing entangled measurements at the nodes, see, e.g., [145–147]. The entanglement that has been produced can then be used to accomplish some quantum information processing tasks. Such tasks include, e.g., long-distance quantum communication [5, 148, 149], distributed quantum computation [150,151], and metrological tasks [152,153]. While the development of a large scale quantum network represents an outstanding technological challenge, recent works have already reported the implementation of basic quantum networks, based on physical platforms where light and matter interact [154–157].

Clearly, these developments raise important questions on the theoretical level and it is important to understand the quantum correlations that arise in such a quantum network. The problem of characterizing correlations in networks can be approached in two different ways. Namely, on the level of probability distributions, which is independent of any assumptions on the devices actually being quantum, or on the level of quantum states, where one explicitly assumes quantum mechanics. For a device-independent characterization of network correlations the concept of Bell inequalities [22] has been generalized to networks in Refs. [158–160]. An important assumption here lies in the fact that the different sources in the network distributing physical systems to the nodes are statistically independent from each other. This is fundamentally different from the usual notion of nonlocality. For instance, it is possible to detect quantum nonlocality in an experiment involving fixed measurements, i.e., a Bell inequality violation is observed without any input, see Refs. [160–163].

In this chapter, we want to explore the second possibility by studying quantum correlations from the more fundamental perspective of quantum entanglement. In the next chapter, we will turn our focus to the device-independent scenario. In the case of entanglement relatively little is known and the field of network entanglement is still in its infancy. Here, different scenarios can be considered, e.g., different types of classical correlations and shared randomness between the sources or the nodes, and different limitations on the possible actions of the local processors. We will discuss the production of entanglement by uncorrelated or classically correlated sources, where the local transformations are restricted to be local unitaries, i.e., choice of a different basis. Inspired from the developments above and recent developments in entanglement theory [103], we discuss the generation of multipartite entangled states in a network.

We focus our attention on the so-called *triangle network*. This simple network features three nodes, i.e., three quantum processors, each pair of nodes being connected by a bipartite quantum source, see Fig. 4.1. We explore the possibilities and limits for entanglement generation, given the constraints of the network topology. Crucially, it turns out that not all quantum states can be prepared under the network constraint. We discuss two scenarios, featuring independent or classically correlated quantum sources and unitaries, and derive general conditions for a quantum state to be preparable in the network. This allows us to show that important classes of multipartite quantum states cannot be prepared in the triangle network, including also some separable states in the case of independent sources. On the other hand, certain genuinely multipartite entangled states can be created in the network. This shows that the network structure imposes strong and non-trivial constraints on the set of possible quantum states. This represents a first step towards understanding quantum correlations in networks from the point of view of quantum states and their entanglement.

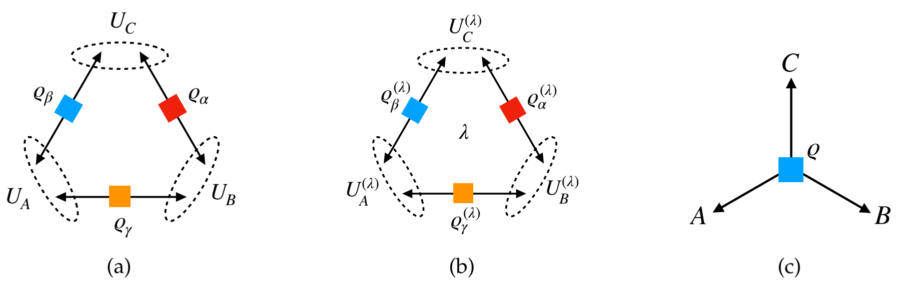


Figure 4.1: This figure shows the structure of the independent (a) and the correlated triangle network (b) compared to the usual scenario where entanglement is distributed by a single source (c).

4.2 Entanglement in the triangle network

The simplest non-trivial network, the triangle network, consists of three nodes A, B and C, which are connected pairwise by three sources. Each source produces a bipartite quantum state of arbitrary dimension $d \times d$ that is subsequently shared with the nodes. The state ϱ_α is shared by B and C, ϱ_β by A and C, and ϱ_γ by A and B. Thus, each party receives two d-dimensional quantum systems. Finally, each party can perform a local unitary transformation on their two-qudit systems, which we denote with U_A, U_B, and U_C. This results in a global state ϱ for the network, see Fig. 4.1a. Note that any state of the triangle network can be converted to a three qudit state with local dimension d^2. E.g., for $d = 2$ we use the map $|00\rangle \to |0\rangle$, $|01\rangle \to |1\rangle$, $|10\rangle \to |2\rangle$ and $|11\rangle \to |3\rangle$, and we will refer to this as the standard encoding.

Different scenarios can be considered. For instance, the sources could be assumed to be statistically independent, meaning that they do not share any common source or randomness. In contrast to that, the sources could also be assumed to be correlated, i.e., controlled by a common source of shared randomness. Since we will focus on the first case we will simply call this the triangular network and the latter one the correlated triangular network. Changing the way how entanglement is produced and distributed leads to fundamentally different notions of separability and entanglement compared to the usual scenario, where separable states are prepared by local operations and globally shared randomness, see Fig. 4.1c. To illustrate the difference consider for instance the so-called generalized Smolin state on six qubits [164], or equivalently, three ququarts. This state can be written as a statistical mixture of two-qubit Bell states. To be more precise, it is of the form $\varrho_6 \propto \mathbb{1}^{\otimes 6} + \sigma_x^{\otimes 6} + \sigma_y^{\otimes 6} + \sigma_z^{\otimes 6}$. Although the global six-qubit state is entangled it was proven in Ref. [165, Lemma 1] that this state is separable with respect to any $2:2:2$ partition, hence separable in the sense of Fig. 4.1c when two qubits are considered to belong to one party, see Fig. 4.2a. However, one can prove that the structure of this state is not compatible with the triangular network. This is due to the fact that the local ranks of certain reduced states are incompatible with the production in the triangular network, see Ref. [166] for a detailed proof. Nevertheless, one can argue that the state becomes compatible with the triangular network if one allows shared randomness between the three sources, see Fig. 4.2b. This highlights the different notions of entanglement that depend on the structure of the network. Characterizing the possible quantum states that can be prepared in these network scenarios is typically a very hard problem. This is due to the fact that the set of possible states preparable in the triangular network is not necessarily convex. In what follows, we will derive necessary conditions for a state to be preparable in the triangular network without shared randomness.

4.3 Triangle network with independent sources

Let us start our discussion by focusing on the scenario where the three sources are assumed to be statistically independent from each other. Statistical independence of the sources is a relatively natural assumption for practical quantum networks, where the sources are placed in distant labs that are operated independently. This network we call the *independent triangle network* (ITN). The set of all the states that can be prepared by the triangle network with independent sources we denote by $\triangle_I$.

A natural question that arises is which states can be prepared in such a network and which ones are incompatible with the preparation in this network. More formally, we have the following definition.

Definition 11 (ITN). *We say that a state is preparable in the triangle network and write $\varrho \in \triangle_I$ if it admits a decomposition of the form*

$$\rho = (U_A \otimes U_B \otimes U_C)(\varrho_\alpha \otimes \varrho_\beta \otimes \varrho_\gamma)(U_A^\dagger \otimes U_B^\dagger \otimes U_C^\dagger). \tag{4.1}$$

We note that one needs to be careful when reading this equation. The different unitaries and states overlap in a non-trivial way according to the connectivity of the triangle network as shown in Fig. 4.1a.

Clearly, there exist some states that are not of the form of Eq. 4.1, and thus cannot be produced in the triangle network. In fact, by simply counting the number of free parameters one can deduce that the set $\triangle_I$ represents only a subset of measure zero in the entire set of quantum states. For any compatible state $\rho \in \triangle_I$, we have $6(d^4 - 1)$ parameters, i.e., $d^4 - 1$ per state and per unitary, which is indeed much smaller than the $d^{12} - 1$ parameters for a general state. In the following we will discuss the characterization of the set $\triangle_I$ which is challenging, mainly due to the fact that it is a non-convex set, as we will see later.

In what follows we will present three different criteria that provide necessary condi-

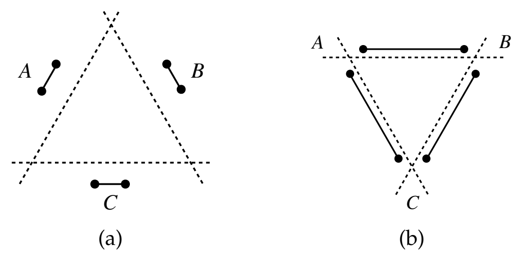


Figure 4.2: The Smolin state on six qubits is separable with respect to any $2:2:2$ partition. This proves that it is (a) separable in the sense of Fig. 4.1c and (b) preparable in the triangular network, if the sources are correlated. Dashed lines indicate separability.

tions for any state $\rho \in \triangle_I$. They follow from the limitations on classical and quantum correlations that can be prepared in the triangle network, as well as restriction on the possible ranks of states on certain marginals.

4.3.1 Constraints from tripartite mutual information

When one looks at the formal definition of the ITN, see Fig. 4.1a, one could intuitively state, on a very abstract level, that in the triangle network the total is simply the sum of its parts. This is due to the fact that the sources are bipartite and because of the absence of any globally shared classical randomness. Such an intuitive statement can be made more precise by the so-called *tripartite quantum mutual information* (TMI). This quantity was discovered independently by McGill in Ref. [167] and later by Ting in Ref. [168].[1] Since then it has enjoyed much attention both in classical information theory as well as quantum theory, where it was first defined in Ref. [170]. The TMI on the level of classical probability distributions is defined by

$$I_3(X:Y:Z) = I_2(X:Y) + I_2(X:Z) - I_2(X:YZ) \tag{4.2}$$

where $I_2(X:Y) = S(X) - S(X|Y) = S(X) + S(Y) - S(X,Y)$ is the bipartite mutual information, and $S(\cdot)$, $S(\cdot|\cdot)$, and $S(\cdot,\cdot)$ are the Shannon usual, conditional, and joint entropies, respectively. Whereas the bipartite mutual information is always positive, i.e., $I(X:Y) \geq 0$, this is no longer true for the TMI, which can be positive, zero and negative. To gain some intuition for the TMI let us discuss some examples for different values of the TMI. Let us start with a positive TMI. Consider a Markov chain $X \leftarrow Y \leftarrow Z$, we have that $I_2(X:YZ) = S(X) - S(X|YZ) = S(X) - S(X|Y) \geq 0$, where the last equality follows from the definition of a Markov process and the inequality is due to the positivity of the bipartite mutual information. Thus, it follows that for Markov chains we have that $I_3(X:Y:Z) = I_2(X:Z) \geq 0$. A negative TMI can be obtained in the case where $X \rightarrow Z \leftarrow Y$, and X, Y are independent variable (i.e., $I_2(X:Y) = 0$). An example of such a case is the XOR operation – if the inputs X and Y are statistically independent, one can still deduce correlations if the output Z is known. Then, a negative TMI follows from the monotonicity of the bipartite mutual information. This also motivates to write the TMI as

$$I_3(X:Y:Z) = I_2(X:Y) - I_2(X:Y|Z), \tag{4.3}$$

where $I_2(X:Y|Z) = S(X|Z) - S(X|YZ)$ is the conditional mutual information. The interpretation of this equation is that in the case of negative TMI the correlations between X and Y increase when conditioned on the knowledge of Z. From this way of writing the TMI it also becomes evident that the TMI is zero when the knowledge

[1] The TMI goes under many different names, which is sometimes a bit confusing. McGill originally called it the *interaction information*. Later also the name *co-information* appeared [169].

of any of the variables does not increase the correlations between the two remaining ones.

In the case of quantum states we simply have to replace the Shannon entropies by the von Neumann entropies. Then one can evaluate the TMI in Eq. (4.2) for a generic state of the ITN and make the following observation.

Observation 12. *For any state $\rho \in \triangle_I$ compatible with the triangle network we have*

$$I_3(A:B:C) = 0, \tag{4.4}$$

Proof. For a precise mathematical proof we can write the TMI in terms of von Neumann entropies and obtain

$$I_3(A:B:C) = S(ABC) + S(A) + S(B) + S(C) - S(AB) - S(AC) - S(BC). \tag{4.5}$$

Since the von Neumann entropy is invariant under unitary transformations and additive on tensor products, it follows form Eq. (4.1) that $S(\rho) = S(\varrho_\alpha) + S(\varrho_\beta) + S(\varrho_\gamma)$ for any $\rho \in \triangle_I$. Expanding the bipartite entropies as, e.g., $S(AB) = S(\mathrm{tr}_C\, \varrho_\beta) + S(\mathrm{tr}_C\, \varrho_\alpha) + S(\varrho_\gamma)$. and inserting it into Eq. (4.5) proves the claim. □

We note, that this criterion is not necessary and sufficient for states in $\triangle_I$ since it also vanishes on all pure states. However, the statement of this theorem can be refined by considering the application of local channels on the parties. We make the following observation.

Observation 13. *For any state $\varrho \in \triangle_I$ the application of local channels on either a single node, or on a pair of nodes, cannot increase the TMI.*

Proof. For the application of a single channel we start with a state $\varrho \in \triangle_I$ and remove the local unitaries. After applying the local channel, say on node A we can decompose the sum of the bipartite entropies as $S(AB) + S(BC) + S(AC) = S(A_1A_2B_1) + S(B_2) + S(B_1) + S(C_2) + S(B_2C_1) + S(C_1) + S(A_1A_2C_2)$ and the sum of the remaining entropies as $S(A) + S(B) + S(C) + S(ABC) = S(A_1A_2) + S(B_1) + S(B_2) + S(C_1) + S(C_2) + S(B_2C_1) + S(B_1A_1A_2C_2)$. Taking the difference results in $I(A:B:C) = S(A_1A_2) + S(B_1A_1A_2C_2) - S(A_1A_2B_1) - S(A_1A_2C_2) \leq 0$, where the last inequality is the *strong subadditivity* condition of the von Neumann entropy.

The proof for two channels is very similar. In this case the difference of the entropies reduces to $I_3(A:B:C) = I(A_1A_2 : B_1B_2) - I(A_1A_2C_2 : B_1B_2C_1) \leq 0$ due to the the monotonicity of the mutual information, which itself is equivalent to the strong subadditivity condition. □

It might be not too surprising that the statement for one and two channels follows from the strong subadditivity condition, simply because if one ignores the triangle structure and the unitary on the third system one is still basically working with a chain structure. The problem becomes more complicated if also a third channel is applied. In this case we were not able to prove that the TMI is still non-increasing, however a simple numerical test suggests that it is indeed non-increasing. To do so we generated ten thousand states, random in the Hilbert-Schmidt measure, and found that the TMI always decreased. Although, this is not a very strong evidence for the non-increasingness under three local channels, it still motivates to find a better algorithm to search more systematically for counterexamples, for which the TMI might increase.

4.3.2 Constraints from entanglement measures

Going beyond limitations on classical correlations, we can observe that also quantum correlations are limited by the network structure. Intuitively one can argue that the amount of entanglement in any bipartition of the type $A|BC$ cannot exceed the amount of entanglement that was produced by the connected sources, i.e., the amount of entanglement in the bipartitions $A|B$ and $A|C$. Hence if we consider an entanglement measure that is additive on tensor products and fulfills a monogamy relation of the type

$$E_{X|Y}[\sigma_{XY}] + E_{X|Z}[\sigma_{XZ}] \leq E_{X|YZ}[\sigma_{XYZ}], \tag{4.6}$$

we expect that equality should hold whenever a state can be prepared in the triangle network. An example of such an entanglement measure is the squashed entanglement [59, 171] that we discussed in Subsec. 1.5.3. We note, however, that not all entanglement measures fulfill the above property, see, e.g., Refs [43, 50] for more details. Thus, we arrive at the following observation.

Observation 14. *Let $E[\cdot]$ be an entanglement measure that is additive on tensor products and monogamous. For any $\rho \in \triangle_I$ we have that $E_{X|YZ}[\rho] = E_{X|Y}[\mathrm{tr}_Z\, \rho] + E_{X|Z}[\mathrm{tr}_Y\, \rho]$ holds for all the bipartitions $A|BC$, $B|AC$ and $C|AB$.*

Proof. To prove this intuition more formal we first observe that for states in $\triangle_I$ the local unitaries U_A, U_B and U_C can always be disregarded, since they do not change the amount of entanglement between the parties. Hence the right-hand side of Eq. (4.6) can be evaluated as $E_{A|BC} = E_{A_\beta A_\gamma|B_\alpha B_\gamma C_\alpha C_\beta} = E_{A_\gamma|B_\gamma} + E_{A_\beta|C_\beta} = E_{A|B} + E_{A|C}$, where A_β denotes the subsystem that A received from the source β, and similarly for the other subsystems, from which the observation follows. □

An example of a state that is excluded from being preprable in the triangle network by this criterion is the GHZ state. Clearly, it is entangled in the $A|BC$ bipartition, but all its two party reduced states are maximally mixed, and hence separable. This clearly violates the above condition. Furthermore, we note that this condition is clearly not sufficient because the condition in Obs. 14 is also satisfied for all fully separable states, some of which are outside of $\triangle_I$.

4.3.3 Local rank constraints

We have already seen in the beginning of our discussion of the Smolin state that the independent triangle network imposes constraints on the global and local ranks of the preparable states. We make the following observation.

Observation 15. *For any state that is preparable in the independent triangle network the following constraints hold for its global and local ranks.*

$$\mathrm{rk}(\rho) = r_\alpha r_\beta r_\gamma, \qquad \mathrm{rk}(\mathrm{tr}_A\,\rho) = r_\alpha r^C_\beta r^B_\gamma, \qquad \mathrm{rk}(\mathrm{tr}_{BC}\,\rho) = r^A_\beta r^A_\gamma, \tag{4.7}$$

$$\mathrm{rk}(\mathrm{tr}_B\,\rho) = r^C_\alpha r_\beta r^A_\gamma, \qquad \mathrm{rk}(\mathrm{tr}_{AC}\,\rho) = r^B_\alpha r^B_\gamma, \tag{4.8}$$

$$\mathrm{rk}(\mathrm{tr}_C\,\rho) = r^B_\alpha r^A_\beta r_\gamma, \qquad \mathrm{rk}(\mathrm{tr}_{AB}\,\rho) = r^C_\alpha r^C_\beta. \tag{4.9}$$

A proof of this statement can be found in Ref. [166]. It is worth noting that in the case of pure states $|\psi\rangle$ a prime tensor rank of the state does not exclude the possibility of the state $|\psi\rangle$ being preparable in the ITN. For multipartite pure states their degree of entanglement can be characterized by the so-called Schmidt measure that was introduced in Ref. [172]. This measure is equivalent to the tensor rank of the coefficient tensor of a pure state $|\psi\rangle$, i.e., it is the smallest number r of product vectors such that $|\psi\rangle = \sum_{i=1}^r \alpha_i\, |\psi^{(i)}_{A_1}\rangle \otimes |\psi^{(i)}_{A_2}\rangle \otimes \cdots \otimes |\psi^{(i)}_{A_n}\rangle$, where n is the number of parties. E.g., the GHZ state in Eq. (1.28) has a tensor rank equal to two and the W state has a tensor rank of three. For bipartite states this measure reduces to the Schmidt rank [43].

In Ref. [103] we considered a similar question of decomposing quantum states and it was proven that if a pure state has prime tensor rank, it cannot be decomposed into lower-dimensional states. E.g., the GHZ state on three ququarts can be decomposed into two three-qubit GHZ states and has a tensor rank of four, whereas the GHZ on three qutrits cannot be decomposed into lower-dimensional systems since its tensor rank is three. In the triangle network this is no longer true. A prime tensor rank of a pure state does not imply that it cannot be produced in the independent triangle network. To prove this consider the case where each source prepares a two-qubit maximally entangled state $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. This results in a state that corresponds to the tensor $T = \sum_{i,j,k=0}^1 (|i\rangle \otimes |j\rangle) \otimes (|j\rangle \otimes |k\rangle) \otimes (|k\rangle \otimes |i\rangle)$. In this decomposition the tensor can be represented as a sum of $2^3 = 8$ terms. However, it is known that this tensor has tensor rank seven [173].[2] From that we can conclude that also states with a prime tensor rank can be prepared in the independent triangle network, and hence, a prime tensor rank does not exclude a state to be compatible with the independent triangle network.

[2]This tensor is known as the two-by-two matrix multiplication tensor [174].

4.4 Further properties of $\triangle_I$

So far we have only discussed necessary conditions that must hold for any state that is preparable in the independent triangle network. But we can say much more about the general structure of this set. The first observation is that although the network structure is a strong constraint on the possible states it sill contains highly entangled states. As an example consider the six qubit ring cluster state$|\mathrm{RCl}_6\rangle$ [175]. This state can be prepared in the ITN by each source producing a maximally entangled two-qubit state, and each party applying a controlled-σ_z transformation. Although the state is preparable in the independent triangle network it nonetheless features entanglement in the strongest sense, i.e., it is genuine tripartite entangled [43]. The fact that this state is in the set $\triangle_I$ allows us to establish the following observation.

Observation 16. *The set $\triangle_I$ is not convex.*

Proof. Consider the mixed state that is obtained by mixing with equal probability two copies of the ring cluster state, one in the computational basis, and the other one in a slightly rotated basis. The resulting mixed state is by definition in the convex hull of the triangular statespace and has a global rank of two. It remains to be argued that such a state cannot be prepared in the triangular network. First, observe that if the global rank of the target state is of rank two, the only possibility to achieve that in the triangular network is by having two sources producing pure, i.e., rank-one, states and a third source preparing a rank-two state. Assume without loss of generality that the sources α and β prepare a pure state and the source γ prepares the rank-two state. Then the reduced state on A and B is at most of rank four. The rank of the reduced state of $|RCl_6\rangle$ is also four, but when one considers a mixture of two of those states in different bases the rank of the reduced state of the mixture can exceed four and, hence, this state cannot be reached in the triangular network. □

Another peculiarity of the set $\triangle_I$ is that it trivially contains the maximally mixed state, but at the same time there are states arbitrarily close to the maximally mixed state that are not compatible with the independent triangle network. As an example consider the GHZ state, which is not preparable in the network, mixed with the maximally mixed state. From Obs. 12, Obs. 14, and the fact that $\mathbb{1}/d^6 \in \triangle_I$ it follows that $\lambda\, |GHZ\rangle\langle GHZ| + (1-\lambda)\mathbb{1}/d^6 \in \triangle_I$ if and only if $\lambda = 0$.

4.5 Triangle network with classical correlations

Another relevant scenario is the one of a classically correlated network, where the three sources and the three nodes share a common source of classical randomness (see Fig. 4.1b). From a experimental point of view this can be motivated by the fact that sharing classical correlations is usually considered to be free of charge in a resource theoretic sense, while the distribution of entanglement might still be limited by the topology of the quantum network. From the theoretical point of view this configuration is interesting because it is simply the convex hull of the ITN. This scenario we will call the correlated triangle network (CTN), and the set of states that can be prepared by such a network we denote by $\triangle_C$. Then we have the following definition.

Definition 17. *Any state $\rho \in \triangle_C$ admits a decomposition of the form*

$$\rho = \sum_\lambda p_\lambda \rho_\lambda, \tag{4.10}$$

where $\rho_\lambda \in \triangle_I$, i.e., each ρ_λ admits a decomposition of the form of Eq. (4.1)*, and λ is a classical variable shared by all the nodes and sources, according to the distribution p_λ.*

Let us now discuss some properties of $\triangle_C$. While any state in $\triangle_I$ is also in $\triangle_C$, the converse is not true since $\triangle_I$ is non-convex, as we saw before. Although the set of possible states is clearly larger, still not every state can be obtained in the CTN. In particular, we make the following observation.

Observation 18. *No three-qubit genuine multipartite entangled state, embedded in larger dimensional systems, can be prepared in the CTN.*

Proof. First, we prove the statement for pure states. We note, that the rank of the global state is one and it is entangled along each possible bipartition, simply by the definition of multipartite entanglement. Hence, due to the Schmidt decomposition in Eq. (1.19), all single party reduced states have rank two. However, this is not possible in the ITN according to Obs. 15. Recall, that the local ranks are determined only by the sources. Furthermore we observe that, if one source prepares a two-qubit entangled state the local ranks at the connected nodes are two and the remaining one has rank one. Similarly, if two sources produce a two-qubit entangled state there is one reduced state which has rank four, which proves the claim. To conclude the proof, note that also no mixed three-qubit genuine multipartite entangled state can be prepared in the CTN. Such a mixed state necessarily has a pure three-qubit genuine multipartite entangled state in its range, i.e., any convex decomposition into pure states contains at least one pure three-qubit genuine multipartite entangled state, and thus, is not preparable in the CTN. □

Observation 18 in itself is already quite interesting, as it rules out a large and important class of states that are not preparable in the CTN. However, due to the convexity of the set we can characterize this set in a more refined way by means of witnesses [43]. To do so we borrow techniques from Ref. [103] which allows us to numerically compute such witnesses. Recall that any violation of such a witness proves that the state is not within the set $\triangle_C$.

state $\vert\psi\rangle$	GHZ_2	GHZ_3	GHZ_4	W	AME	AS_3
numerical estimate of μ^2	$\frac{1}{2}$	$\frac{4}{9}$	$\frac{1}{2}$	$\frac{6}{9}$	$\frac{1}{2}$	0.5362(5)

Table 4.1: This table shows the results of the see-saw algorithm that computes a lower bound on μ^2 given in Eq. (4.11), computed for different target states. The AME is the absolutely maximally entangled state of six qubits (or three ququarts) and AS_3 is the totally antisymmetric state on three qutrits [177, 178]. All states, except the AME state, are embedded into the triangle network by choosing local dimension $d^2 = 4$ and using the standard encoding.

As a starting point consider a target pure state $|\psi\rangle$, which is not in $\triangle_C$. Recall that the linear operator $W = \mu^2\mathbb{1} - |\psi\rangle\langle\psi|$ is a witness, where μ is the largest fidelity between $|\psi\rangle$ and any state in $\triangle_C$. The task is now to numerically estimate μ, i.e., to find the maximal overlap between $|\psi\rangle$ and any state $\rho \in \triangle_C$. Again, it is sufficient to consider pure states in this optimization, namely, $|\varphi\rangle = (U_A \otimes U_B \otimes U_C)\,|\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle$. Hence, our goal is to numerically solve the optimization problem

$$\mu = \max_{\substack{U_A,U_B,U_C\\ |\alpha\rangle,|\beta\rangle,|\gamma\rangle}} \left|\langle\alpha\beta\gamma|\,(U_A \otimes U_B \otimes U_C)\,|\psi\rangle\right|. \tag{4.11}$$

This can be done by a numerical see-saw optimization procedure. To be more precise, we start with random initial states $|\alpha\rangle$, $|\beta\rangle$, and $|\gamma\rangle$ and random unitaries U_A, U_B, and U_C, where the dimension d of the sources is chosen large enough so that the state $|\psi\rangle$ can be embedded into the space of local dimension d^2. Then, an optimization over all states and all unitaries is performed one by one, while keeping everything else fixed. The procedure is terminated when a fix point is reached. Although we are not guaranteed to end in the global maximum, we found that in practice the method works well for low dimensional systems. In Table 4.1 we show results for some interesting states. For more details on the algorithm and analytical proofs of the individual steps we refer to Refs. [166, 176].

Next, we discuss the possibility of obtaining analytical upper bounds on the overlap of a given pure state with pure states from the ITN. To that end, consider a bipartite system, where we wish to maximize the overlap between some fixed target state $|\psi\rangle$ and some state $|\tau\rangle$ which is constrained to be in some subset S. This implies that the Schmidt coefficients of the state $|\tau\rangle \in S$ obey some constraint $\{t_i\} \in \mathcal{S}$. If the target state has a Schmidt decomposition $|\psi\rangle = \sum_i s_i\,|ii\rangle$, then the overlap between the two states is bounded by

$$\sup_{|\tau\rangle\in S} |\langle\tau|\psi\rangle|^2 \leq \sup_{\{t_i\}\in\mathcal{S}} \Big|\sum_i t_i s_i\Big|^2, \tag{4.12}$$

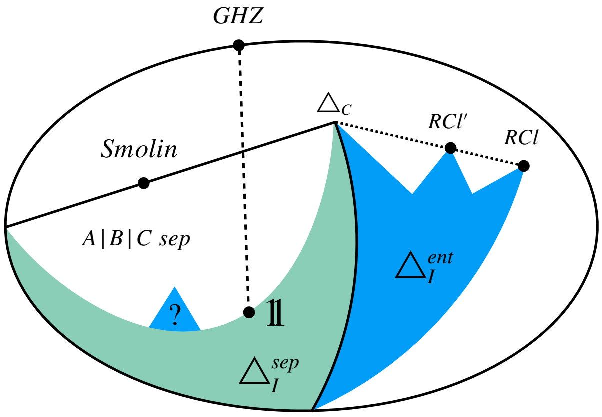


Figure 4.3: This figure summarizes the results of the previous discussion on entanglement in the triangle network. The set $\triangle_I$ is divided in the part that contains all ITN states that are produced with separable sources $\triangle_I^{sep}$ and the ones that require entangled sources $\triangle_I^{ent}$. The maximally mixed state is on the border of the set of triangle separable states, since it it compatible with the ITN, but there are incompatible states arbitrarily close to the maximally mixed state. One open question if the existence of states that are separable in the usual definition, but require entangled sources to be produced in the ITN (area indicated with question mark).

which is due to the von Neumann trace inequality. In case the subset S is the ITN, i.e., $S = \triangle_I$, and the global state is pure, we must also assume the sources to produce pure states, having the Schmidt coefficients $[\cos(a), \sin(a)]$, $[\cos(b), \sin(b)]$, and $[\cos(c), \sin(c)]$, respectively. In such a case one can simply consider the construction from above for all the bipartitions and take the smallest upper bound. To be more precise, let us consider the GHZ state from Eq. (1.28). Due to its permutational symmetry it has the same Schmidt coefficients for any possible bipartition. Due to this symmetry, we can furthermore assume the Schmidt coefficients of the state $|\psi\rangle \in \triangle_I$ to be constrained by $\pi/4 \geq a \geq b \geq c \geq 0$. Then, we find the following bound on the overlap

$$\sup_{|\tau\rangle \in \triangle_I} |\langle \tau | GHZ_2 \rangle|^2 \leq \frac{1}{2} \sup_{a,b,c} f(a,b,c)^2, \tag{4.13}$$

where

$$\begin{aligned} f(a,b,c) = \min\{ & \cos(a)\cos(b) + \sin(a)\cos(b), \\ & \cos(c)\cos(b) + \sin(b)\cos(c), \\ & \cos(a)\cos(c) + \sin(a)\cos(c)\}. \end{aligned} \tag{4.14}$$

From this we can obtain the analytical bound

$$\sup_{|\tau\rangle \in \triangle_I} |\langle \tau | GHZ_2 \rangle|^2 \leq \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4} \approx 0.8536, \tag{4.15}$$

which is, however, still larger than the value of $1/2$ that we have found numerically (cf. Tab. 4.1). We note, that similar calculations can be performed for other pure states as well.

4.6 Conclusions

In this chapter, we have discussed the structure of quantum entanglement in quantum networks, particularly focusing on the triangle network. First, we have considered the case where the sources do not share a common source of randomness, which results in the non-convexity of the set of possible states that can be prepared. We have derived necessary criteria for states to be compatible with the network structure based on the statistical independence of the sources, the monogamy of entanglement, and constraints on the local ranks. Considering the possibility of classically correlating the sources allowed us to construct witnesses for detecting states that are not preparable in the network, neither with nor without correlated sources. These results can be seen as a first step toward a theory of network entanglement, which is fundamentally different from the usual theory of multipartite entanglement. To highlight this difference, we have summarized our results in Fig. 4.3. Moreover, it would be interesting for future research to study the structure of network entanglement under different classes of transformations, e.g., finite-round LOCC transformations.

Finally, many questions remain open and many problems remain unsolved. First, one should clarify whether or not there exists a state that is $A|B|C$ separable, but requires entangled sources when produced in the triangle network. The existence of such a state would show, that the entanglement cost of preparing states in a network is different from the cost of preparing states with a single source. Moreover, one could try to show that, whenever one starts with a state that is preparable in the triangle and one applies local channels on all three nodes, the TMI can only decrease, i.e., it cannot become strictly positive.

5 Characterizing quantum networks using coherence theory

So far our discussion was only concerned with the entanglement properties of states that can be prepared in the triangle network. In this section we will study correlations in larger networks, but this time on the level of probability distributions. Therefore, it naturally follows that these results can also be applied to correlations originating from *classical networks*. This chapter is based on publication [A].

5.1 Motivation

Another possibility to characterize the correlations in quantum networks was put forward in Refs. [28, 179]. It was shown that the topology of the network imposes strong constraints on the structure of the covariance matrices that can arise form such networks. More precisely, the covariance matrix can be decomposed in a sum of positive matrices that have a certain block structure, corresponding to the connectivity of the sources. The verification of this structure is then an instance of a semidefinite program (cf. Sec. 1.9). In this chapter we demonstrate that the theory of quantum coherence provides powerful tools for analyzing correlations in quantum networks. We provide a direct link between the theory of multisubspace coherence [134, 180] and the approach to quantum networks using covariance matrices established in Refs. [28, 179]. This allows to solve analytically the criteria developed there for important cases; furthermore, some conjectures can be proven and, besides that, our methods can be applied to large networks for which tools based on numerical optimization are infeasible. We note that, since the covariance matrix approach is essentially a tool coming from classical causal models [179], our results demonstrate that results from the theory of *quantum* coherence are useful beyond the level of quantum states for the analysis of *classical* networks.

Let us begin with precisely defining the problem and adopt an appropriate notation. We consider networks that are build up by M sources, labeled by $m = 1, 2, \ldots, M$ that independently produce quantum states ϱ_m, which are then distributed to N nodes, labeled by $n = 1, 2, \ldots, N$. For every source m we denote by $\mathcal{C}_m$ the set of all connected nodes that have access to a part of the state ϱ_m.

After the entanglement is distributed, we perform a measurement at each node that is described by a POVM $A^{(n)} = \{A_x^{(n)}\}_x$. Thus, the observed probability distribution of the outcomes reads[1]

$$p(x_1 \dots x_N) = \mathrm{tr}\Big[(A_{x_1}^{(1)} \otimes \cdots \otimes A_{x_N}^{(N)})\varrho_1 \otimes \cdots \otimes \varrho_M\Big]. \tag{5.1}$$

The central question that we want to address in this section is the following: Given a certain network structure and an observed probability distribution, is this distribution compatible with the structure of the network, i.e., is the probability distribution of the form of Eq. (5.1)? The problem is, however, that the set of probability distributions that are compatible with a certain network structure is non-convex and thus, in general, hard to characterize. One possible way to overcome this problem was put forward in Ref. [28]. Their idea was the following: Take the non-convex set of compatible probability distributions and map it to the space of covariance matrices, and then build a convex relaxation of the problem. In this way it is possible to derive network witnesses based on SDP's. If such a witness is violated it implies that the probability distribution is incompatible with the structure of the network.

To be more precise, the strategy is as follows. First, a so-called feature map is defined that maps the outcomes x_n at a fixed vertex n to a vector $\mathbf{v}_{x_n}^{(n)} \in \mathcal{V}_n$. Here, the $\mathcal{V}_n$ are some orthogonal vector spaces. Combining all the feature maps, one obtains a random vector $\mathbf{v}$ with components $\mathbf{v}_{x_1\dots,x_N} = \mathbf{v}_{x_1}^{(1)} + \cdots + \mathbf{v}_{x_N}^{(N)}$. The covariance matrix is then defined as

$$\Gamma(\mathbf{v}) = E(\mathbf{v}\mathbf{v}^\dagger) - E(\mathbf{v})E(\mathbf{v})^\dagger \tag{5.2}$$

where

$$E(\mathbf{v}\mathbf{v}^\dagger) = \sum_{x_1,\dots,x_N} \mathbf{v}_{x_1,\dots,x_N}\mathbf{v}_{x_1,\dots,x_N}^\dagger P(x_1,\dots,x_N), \tag{5.3}$$

$$E(\mathbf{v}) = \sum_{x_1,\dots,x_N} \mathbf{v}_{x_1,\dots,x_N} P(x_1,\dots,x_N). \tag{5.4}$$

Due to the structure of $\mathbf{v}$, the covariance matrix naturally has a certain block structure. More precisely, Γ is an $N \times N$ block matrix with blocks $\Gamma_{\alpha\beta}$, where each block is a $r \times r$ matrix, with r being the dimension of $\mathcal{V}_n$.[2] Here, we will simply assume that the feature map maps the outcome x_n to $|x_n\rangle$. This is because for measurements with more than two outcomes the mean value contains much less information in comparison to the probability distribution.

[1] Again, one needs to be careful with the fact that the measurements and the states overlap non-trivially according to how the nodes are connected to the sources.

[2] The usual covariance matrix formalism that is used in quantum information is a special instance of this notion, where one assigns to the outcomes x_n just real numbers and hence takes the $\mathcal{V}_n$ to be one-dimensional.

5.2 Covariance matrices and coherence

The central result of Refs. [28, 179] is that the structure of the network imposes strong constraints on the structure of the covariance matrix. To be more precise, if the network has a certain structure the covariance matrix can be decomposed into a sum of positive matrices that have a certain block structure. The block structure depends on how the sources are connected to the nodes. The verification of such a structure is then an instance of an sdp [100, 101]. To gain some intuition into the construction we can consider for instance the triangle network in Fig. 5.1a. Here, the tree sources connect the nodes $\mathcal{C}_3 = (1,2)$, $\mathcal{C}_2 = (1,3)$, and $\mathcal{C}_1 = (2,3)$. Hence, the covariance matrix can be decomposed into three terms that correspond to these three sources, as $\Gamma(\mathbf{v}) = \Gamma_3 + \Gamma_2 + \Gamma_1$, where Γ_m corresponding to the source m has support in the blocks $\mathcal{C}_m$, i.e., in the subspaces that correspond to the connected nodes (see Fig. 5.1c for this decomposition).

In the general case, we will restrict our attention in the following to k-complete networks. This simply corresponds to the case where all sources distribute their states to $k < N$ parties and all possible k-partite sources are being used. Thus, we have $M = \binom{N}{k}$ (see also Fig. 5.1b for an example of a 3-complete network on four nodes).

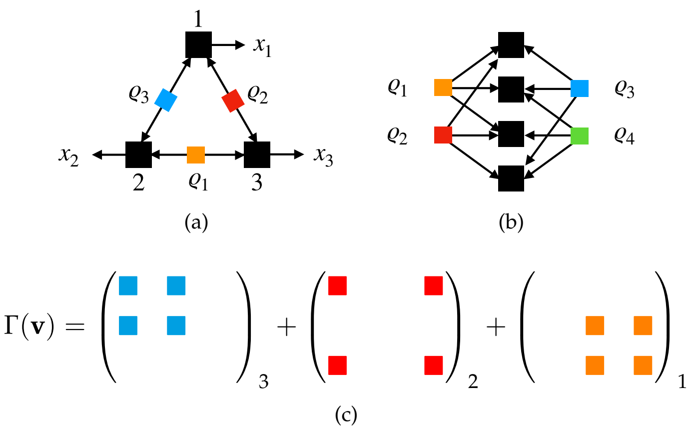


Figure 5.1: (a) We can again consider the triangle network as the simplest non-trivial network, featuring three sources that distribute bipartite entanglement that is shared amongst the nodes. In comparison to the previous scenario we now perform measurements at each of the three nodes that produce measurement outcomes $x_1, \ldots, x_3$. (b) A larger network consisting of four nodes that is 3-complete, i.e., it features four sources that distribute tripartite entanglement. (c) The covariance matrix of the triangle network has a 3×3 block structure and consists of three terms, where $(\square)_i$ denotes those blocks that are contributed by the source i.

Applying the criterion from Refs. [28, 179] we arrive at the following observation.

Observation 19. *Check if the covariance matrix $\Gamma(\mathbf{v})$ can be decomposed into blocks Y_m according to*

$$\text{find: } Y_m \geq 0 \tag{5.5}$$

$$\text{subject to: } Y_m = \Pi_m Y_m \Pi_m \text{ and } \Gamma(\mathbf{v}) = \sum_{m=1}^{M} Y_m, \tag{5.6}$$

where $\Pi_m = \sum_{i \in \mathcal{C}_m} P_i$, with P_i being the projector onto $\mathcal{V}_i$; so Π_m is effectively a projector onto all spaces affected by the source m. If such a decomposition does not exist, then the probability distribution is incompatible with the structure of the network.

Note, that from the existence of such a decomposition one cannot conclude compatibility with the structure of the network, since this was just an outer relaxation of the compatible covariance matrices. We furthermore note that the formulation in Eqs. (5.5, 5.6) is different from the formulation in Ref. [28], but certainly equivalent. The advantage of this reformulation is that it allows to more easily establish a connection to the theory of quantum coherence.

From our previous discussion on multisubspace coherence (cf. Sec. 3.5), it becomes evident that Eqs. (5.5, 5.6) are simply a reformulation of the notion of multisubspace coherence for the covariance matrix. Thus, we can make the following observation.

Observation 20. *If a covariance matrix $\Gamma(\mathbf{v})$ has block coherence number $k+1$, then it cannot have originated from a k-complete network.*

Since we have now established a connection between the coherence properties of the covariance matrix and the incompatibility with k-complete networks, we can continue with some more explicit examples.

5.3 Networks with dichotomic measurements

Let us begin by considering dichotomic measurements, that is, measurements with two outcomes only. In this case the covariance matrix can be simplified in the following sense.

Observation 21. *Consider a network of N vertices, where each node performs a dichotomic measurement. Then the covariance matrix $\Gamma(\mathbf{v})$ is of the form*

$$\Gamma(\mathbf{v}) = C \otimes (\mathbb{1} - \sigma_x), \tag{5.7}$$

where C is an $N \times N$ matrix.

Proof. We will prove the observation for the triangle ($M = 3$). The generalization to more parties is straightforward. We consider here the feature map x_j to $|x_j\rangle$, which implies that, e.g., $\mathbf{v}_{0,0,0} = (1,0,1,0,1,0)^T$ and $\mathbf{v}_{1,1,1} = (0,1,0,1,0,1)^T$. Using this, and the notation $\mathbf{v}^{(i)}_{r_i} = \sum_{x_j,x_k} p(x_i = r_i, x_j, x_k)$ and $\mathbf{v}^{(i,j)}_{r_i,r_j} = \sum_{x_k} p(x_i = r_i, x_j = r_j, x_k)$ for $r_i, r_j \in \{0,1\}$ and $\{i,j,k\} = \{1,2,3\}$, one can straightforwardly compute that

$$E(\mathbf{v}\mathbf{v}^\dagger) = \begin{pmatrix} \lambda_0^{(1)} & 0 & \lambda_{0,0}^{(1,2)} & \lambda_{0,1}^{(1,2)} & \lambda_{0,0}^{(1,3)} & \lambda_{0,1}^{(1,3)} \\ 0 & \lambda_1^{(1)} & \lambda_{1,0}^{(1,2)} & \lambda_{1,1}^{(1,2)} & \lambda_{1,0}^{(1,3)} & \lambda_{1,1}^{(1,3)} \\ \lambda_{0,0}^{(1,2)} & \lambda_{1,0}^{(1,2)} & \lambda_0^{(2)} & 0 & \lambda_{0,0}^{(2,3)} & \lambda_{0,1}^{(2,3)} \\ \lambda_{0,1}^{(1,2)} & \lambda_{1,1}^{(1,2)} & 0 & \lambda_1^{(2)} & \lambda_{1,0}^{(2,3)} & \lambda_{1,1}^{(2,3)} \\ \lambda_{0,0}^{(1,3)} & \lambda_{1,0}^{(1,3)} & \lambda_{0,0}^{(2,3)} & \lambda_{1,0}^{(2,3)} & \lambda_0^{(3)} & 0 \\ \lambda_{0,1}^{(1,3)} & \lambda_{1,1}^{(1,3)} & \lambda_{0,1}^{(2,3)} & \lambda_{1,1}^{(2,3)} & 0 & \lambda_1^{(3)} \end{pmatrix} \tag{5.8}$$

and

$$E(\mathbf{v}) = \left(\lambda_0^{(1)}, \lambda_1^{(1)}, \lambda_0^{(2)}, \lambda_1^{(2)}, \lambda_0^{(3)}, \lambda_1^{(3)}\right)^T. \tag{5.9}$$

Considering the diagonal 2×2 block-matrices of $\Gamma(\mathbf{v}) = E(\mathbf{v}\mathbf{v}^\dagger) - E(\mathbf{v})E(\mathbf{v})^\dagger$ one obtains

$$\begin{pmatrix} \lambda_0^{(i)} - (\lambda_0^{(i)})^2 & -\lambda_0^{(i)}\lambda_1^{(i)} \\ -\lambda_0^{(i)}\lambda_1^{(i)} & \lambda_1^{(i)} - (\lambda_1^{(i)})^2 \end{pmatrix} = \left[\lambda_0^{(i)}(1 - \lambda_0^{(i)})\right](\mathbb{1}_2 - \sigma_x), \tag{5.10}$$

where here we used that $\lambda_1^{(1)} = 1 - \lambda_0^{(1)}$, due to the normalization of the probabilities. The off-diagonal 2×2 block-matrices of $\Gamma(\mathbf{v})$ are of the form

$$\begin{pmatrix} \lambda_{0,0}^{(i,j)} - \lambda_0^{(i)}\lambda_0^{(j)} & \lambda_{0,1}^{(i,j)} - \lambda_0^{(i)}\lambda_1^{(j)} \\ \lambda_{1,0}^{(i,j)} - \lambda_1^{(i)}\lambda_0^{(j)} & \lambda_{1,1}^{(i,j)} - \lambda_1^{(i)}\lambda_1^{(j)} \end{pmatrix} = \left[\lambda_{0,0}^{(i,j)} - \lambda_0^{(i)}\lambda_0^{(j)}\right](\mathbb{1}_2 - \sigma_x), \tag{5.11}$$

with $i < j$. Here, we used that $\lambda_1^{(l)} = 1 - \lambda_0^{(l)}$ and $\lambda_0^{(l)} - \lambda_{0,1}^{(l,l')} = \lambda_{0,0}^{(l,l')}$ for $l, l' \in \{i,j\}$ and $l \neq l'$, as well as $\lambda_{1,1}^{(i,j)} + \lambda_0^{(i)} + \lambda_0^{(j)} = 1 + \lambda_{0,0}^{(i,j)}$. Using Eqs. (5.10) and (5.11), we see that $\Gamma(\mathbf{v})$ is of the form

$$\begin{pmatrix} \mu_1 & \mu_{1,2} & \mu_{1,3} \\ \mu_{1,2} & \mu_2 & \mu_{2,3} \\ \mu_{1,3} & \mu_{2,3} & \mu_3 \end{pmatrix} \otimes (\mathbb{1}_2 - \sigma_x), \tag{5.12}$$

where $\mu_i = \lambda_0^{(i)}(1 - \lambda_0^{(i)})$ and $\mu_{i,j} = \lambda_{0,0}^{(i,j)} - \lambda_0^{(i)}\lambda_0^{(j)}$, which proves the statement for $M = 3$. Note that for more parties the 2×2 block-matrices are also of the form given in Eqs. (5.10) and (5.11) and therefore the argument extends straightforwardly. □

From the previous observation we can deduce that in the case of dichotomic measurements the test for compatibility with a k-complete network is equivalent to checking the k-level coherence of the smaller covariance matrix C. While this is, in general, still a hard problem, the solution can be directly written down for the simplest nontrivial case of $k = 2$, see Ref. [134]. To be more precise, it is known that a matrix X has coherence number less than or equal to two if and only if the so-called comparison matrix $M(X)$ defined by

$$(M[X])_{ij} = \begin{cases} |X_{ii}| & \text{if } i = j \\ -|X_{ij}| & \text{if } i \neq j \end{cases} \tag{5.13}$$

is positive semidefinite. Thus, we arrive at the following observation.

Observation 22. *If the comparison matrix $M(C)$ of the smaller covariance matrix C has a negative eigenvalue, then the observed probability distribution is incompatible with a complete network of bipartite sources.*

Example of a GHZ-type distribution. Consider the following family of distributions that has previously been studied in Refs. [28, 181]

$$P(x_1, \ldots, x_N) = p\delta_0^{(N)} + q\delta_1^{(N)} + (1-p-q)\frac{1-\delta_0^{(N)}-\delta_1^{(N)}}{2^N-2}, \tag{5.14}$$

where $\delta_i^{(N)} = \prod_{j=1}^N \delta_{ix_j}$. We note that for $p = q = \frac{1}{2}$ this corresponds to measuring locally σ_z on an N-particle Greenberger-Horne-Zeilinger (GHZ) state $|GHZ\rangle = (|00\ldots0\rangle + |11\ldots1\rangle)/\sqrt{2}$. The covariance matrix for this distribution reads

$$C = \Delta\mathbb{1} + \chi\,|\mathbf{1}\rangle\langle\mathbf{1}|\,, \tag{5.15}$$

where $\Delta = 2^{N-2}(1-p-q)/(2^N-2)$, $\chi = \frac{1}{4}[1-(p-q)^2] - \Delta$ and $|\mathbf{1}\rangle = \sum_{n=1}^N |n\rangle$. From Obs. 22 it directly follows that C has coherence number less or equal two if and only if the matrix $M(C) = (\Delta + 2\chi)\mathbb{1} - \chi\,|\mathbf{1}\rangle\langle\mathbf{1}|$ is positive semidefinite. For the eigenvalues of this matrix one obtains $\lambda_1 = \Delta + 2\chi$ and $\lambda_2 = \Delta - (N-2)\chi$. Hence, we arrive at the condition that C is incompatible with a 2-complete network if

$$q > p + \kappa - \sqrt{4\kappa p + (\kappa-1)^2}, \tag{5.16}$$

with $\kappa = [(N-1)2^{N-2}]/[(N-2)(2^{N-1}-1)]$. We note that this condition was already observed in Ref. [28], however only numerical evidence for its optimality was provided. Indeed, from Observation 22 it also becomes evident that this condition is optimal for the GHZ-type distributions in Eq. (5.14) in the sense that there cannot be a better witness based on the multilevel properties of the covariance matrix. This follows from the fact that the criterion in Eq. (5.13) is necessary and sufficient to characterize two level coherence of an operator.

Although there does not exist a similar criterion for k-level coherence with $k > 2$, we can still learn more about the GHZ-type distribution. Due to the simple structure of the matrix C in Eq. (5.15) its multilevel coherence properties can be completely characterized. To be more concrete, one can obtain a complete family of optimal criteria for the GHZ-type distribution that characterizes its incompatibility with k-complete networks for arbitrary k. First, we need the concept of coherence witnesses. To that end, let us consider an arbitrary pure state $|\psi\rangle = \sum_{i=1}^{M} c_i |i\rangle$. A $(k+1)$-level coherence witness is given by [134]

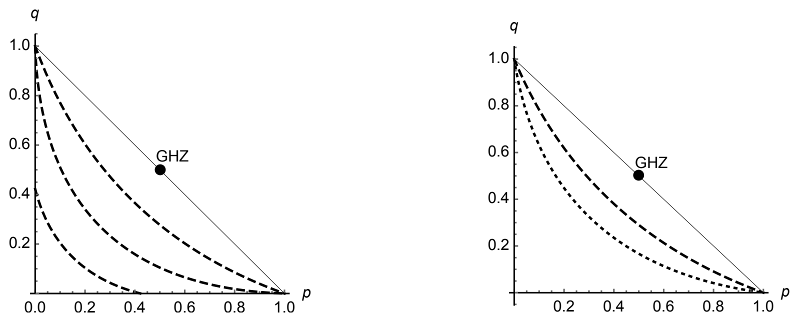
$$W_k = \mathbb{1} - \frac{1}{\sum_{i=1}^{k} |c_i^{\downarrow}|^2} |\psi\rangle\langle\psi| , \tag{5.17}$$

where $c_i^{\downarrow}$ denote the coefficients c_i reordered decreasingly according to their absolute values. From this it follows that $\mathrm{tr}[W_k \varrho] \geq 0$, if ϱ has coherence number k or less.

For the maximally coherent state $|\psi^+\rangle = (\sum_{i=1}^{N} |i\rangle)/\sqrt{N}$ a short calculation reveals that the corresponding witness is of the form $W_k = \mathbb{1} - |\mathbf{1}\rangle\langle\mathbf{1}| / k$. This witness can easily be proven to be optimal for the family of states $\varrho(\mu) = \mu |\psi^+\rangle\langle\psi^+| + (1-\mu)\frac{\mathbb{1}}{N}$ that obey the same symmetry. Such states are, up to normalization and suitable choice of the parameter μ, equivalent to the covariance matrix C of the GHZ-type distribution. Thus, applying the witness we obtain $\mathrm{tr}[W_k C] = (1-1/k)\Delta + (1-N/k)\chi$. From this, it directly follows that C is incompatible with a k-complete network, if

$$q > p + \eta - \sqrt{4\eta p + (\eta - 1)^2}, \tag{5.18}$$

with $\eta = (N-1)2^{N-2}/[(N-k)(2^{N-1}-1)]$. Now one can plot these conditions for, e.g., $N = 5$ and $k = 4, 3, 2$. The results are shown in Fig. 5.2a.



(a) (b)

Figure 5.2: (a) Complete family of criteria that exclude k-complete networks as possible explanations for the GHZ-type distribution in Eq. (5.14) using Eq. (5.18) for $N = 5$ and $k = 4, 3, 2$. Every point above the lines is detected to be incompatible with the respective network structure, and *GHZ* denotes the distribution that is obtained from measuring σ_z on the GHZ state. (b) Here, we compare the criterion in Eq. (5.16) (dotted line) to the monogamy criterion in Eq. (5.20) (dashed line) in the triangular network.

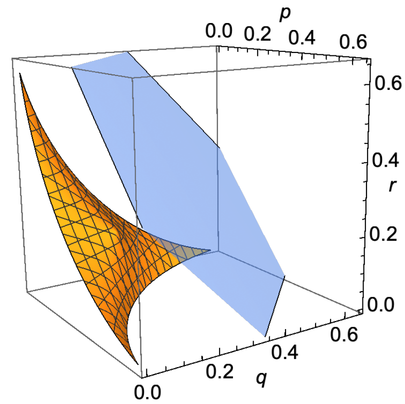


Figure 5.3: Analysis of the GHZ-type distribution with three outcomes per measurement in Eq. (5.19) using Observation 23. Everything above the orange surface is detected to be incompatible with the triangle network. The blue surface represents the normalization constraint.

5.4 Networks beyond dichotomic measurements

In case the measurements that are performed at the nodes have more than two outcomes, we need to check the multisubspace coherence properties of the covariance matrix. For 2-complete networks we can make the following observation.

Observation 23. *Let $\Gamma(\mathbf{v}) \in \mathcal{BC}_2$ be a covariance matrix with block coherence number two. Then, whenever the signs of some off-diagonal blocks are flipped such that the matrix remains symmetric, the resulting matrix remains positive semidefinite.*

Proof. First, note that any matrix with block coherence number two can be written as a convex combination of pure states with coherence rank two or less. That is, we can write $|\psi\rangle = P_i |\psi\rangle + P_j |\psi\rangle$. Then, for any such state, adding a minus sign in the off-diagonal blocks of the density matrix corresponds to the transformation $P_i |\psi\rangle + P_j |\psi\rangle \mapsto P_i |\psi\rangle - P_j |\psi\rangle$. This operation, however, leaves the density matrix positive semidefinite. □

To demonstrate the power of this Observation, let us consider again the GHZ-type distribution, but now with three outcomes per measurement,

$$\begin{aligned} P(x_1, x_2, x_3) =& p\delta_0^{(3)} + q\delta_1^{(3)} + r\delta_2^{(3)} \\ &+ (1-p-q-r)\frac{1-\delta_0^{(3)}-\delta_1^{(3)}-\delta_2^{(3)}}{3^3-3}. \end{aligned} \tag{5.19}$$

A straightforward calculation provides a regime where this is incompatible with the triangle network, see Fig. 5.3.

5.5 Characterizing networks with monogamy relations

Another possibility to characterize networks is to evaluate monogamy relations for the coherence between different subspaces [180]. The idea is that the amount of coherence that can be shared between one subspace and all other subspaces is limited if a certain block coherence number is imposed. To be more precise, for a trace one positive semidefinite block matrix $X = \left[X_{\alpha\beta}\right]_{\alpha,\beta=0}^{N}$ with block coherence number k it holds that $\sum_{\beta=1}^{N} \left\|X_{0\beta}\right\|_{tr} \leq \sqrt{k-1}\sqrt{\mathrm{tr}[X_{00}](1-\mathrm{tr}[X_{00}])}$. If we consider the normalized matrix matrix $\tilde{C} = C/\,\mathrm{tr}[C]$, evaluating such a monogamy relation provides a necessary criterion for C to have coherence number k. For the matrix in Eq. (5.15) this gives

$$\Delta - \left(\frac{\sqrt{N-1}}{\sqrt{k-1}} - 1\right)\chi \geq 0. \tag{5.20}$$

Hence, if this inequality is violated then the observed correlations are not compatible with a k-complete network. This is also shown in Fig. 5.2b for the triangle network. Although this test is in this case not as powerful as the analytical solution, it is easy to evaluate especially for large networks, since it requires only computing traces of smaller block matrices.

5.6 Further results

So far, we provided criteria to show that correlations are incompatible with a k-complete network. It would be interesting to derive also sufficient criteria for being compatible with a given network structure. In the framework of Ref. [28] this is not directly possible, as the criterion in Eqs. (5.5, 5.6) is a convex relaxation of the original problem. Still, coherence theory allows to identify scenarios where the covariance matrix can be certified to have a small block coherence number, so the covariance matrix approach must fail to prove incompatibility with a k-complete network.

Here we can make two small observations in this direction. The following results from Ref. [134] can be directly applied to networks with dichotomic outcomes. Namely, if we have for the normalized matrix $\tilde{C} \geq \frac{N-k}{N-1}\Lambda(\tilde{C})$, where Λ is the fully decohering map, mapping any matrix to its diagonal part, then $\tilde{C} \in \mathcal{BC}_k$, implying that the test in Eqs. (5.5, 5.6) for $(k+1)$-complete networks will fail. Furthermore we have that if $\mathrm{tr}\left[\tilde{C}^2\right]/\,\mathrm{tr}\left[\tilde{C}\right]^2 \leq 1/(N-1)$, then $\tilde{C}$ is two-level coherent, and thus, a test for 3-complete networks will fail.

A similar observation can be made in the general case.

Observation 24. *If $M_b(\Gamma) \geq 0$, where $M_b(\Gamma)$ is the block comparison matrix defined by*

$$(M_b[\Gamma])_{\alpha\beta} = \begin{cases} (\|\Gamma_{\alpha\alpha}^{-1}\|\,)^{-1} & \textit{for } \alpha = \beta \\ -\left\|\Gamma_{\alpha\beta}\right\| & \textit{for } \alpha \neq \beta, \end{cases} \tag{5.21}$$

with $\|X\|$ denoting the largest singular value of the block X, then $\Gamma \in \mathcal{BC}_2$.

To prove this observation let the block matrix $X = \left[X_{\alpha\beta}\right] > 0$, with $X_{\alpha\beta} \in \mathbb{C}^{d\times d}$, be partitioned as follows

$$X = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1K} \\ X_{21} & X_{22} & \cdots & X_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ X_{K1} & X_{K2} & \cdots & X_{KK} \end{bmatrix}. \tag{5.22}$$

Definition 25 (from Ref. [182])**.** *Let X be partitioned as in Eq.* (5.22)*. If the matrices $X_{\alpha\alpha}$ on the diagonal are non-singular, and if*

$$(\|X_{\alpha\alpha}^{-1}\|)^{-1} \geq \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{K} \|X_{\alpha\beta}\|, \tag{5.23}$$

then X is called block diagonally dominant. *Here, $\|Y\|$ denotes the largest singular value, so for the positive $X_{\alpha\alpha}$ the expression $\left\|X_{\alpha\alpha}^{-1}\right\|^{-1}$ is the smallest eigenvalue of $X_{\alpha\alpha}$.*

Observation 26. *If X is positive and block diagonally dominant, then the block coherence number is smaller or equal to two, $bcn(X) \leq 2$.*

Proof. Suppose X satisfies the hypothesis. Define 2×2 block matrices

$$G^{\alpha\beta} = \begin{bmatrix} |X_{\alpha\beta}| & X_{\alpha\beta} \\ X_{\alpha\beta}^{\dagger} & |X_{\alpha\beta}^{\dagger}| \end{bmatrix}, \tag{5.24}$$

where $\left|X_{\alpha\beta}\right| = \sqrt{X_{\alpha\beta}^{\dagger} X_{\alpha\beta}}$ and the support of $G^{\alpha\beta}$ is the subspace α, β. Clearly, the $G^{\alpha\beta}$ are positive semidefinite and have block coherence number two. Next, consider the matrix $D = X - \sum_{\alpha=1}^{K} \sum_{\beta>\alpha} G^{\alpha\beta}$. Since $X > 0$ it is also hermitian, and thus, $X_{\beta\alpha} = X_{\alpha\beta}^{\dagger}$, precisely as for $G^{\alpha\beta}$. From this we can conclude that the off-diagonal blocks of D vanish and the diagonal blocks are given by $D_{\alpha\alpha} = X_{\alpha\alpha} - \sum_{\beta=1,\beta\neq\alpha}^{K} \left|X_{\alpha\beta}\right|$. Furthermore, observe that $\lambda_{\min}(X_{\alpha\alpha}) \geq \sum_{\beta=1,\beta\neq\alpha}^{K} \lambda_{\max}(X_{\alpha\beta}) \geq \lambda_{\max}(\sum_{\beta=1,\beta\neq\alpha}^{K} X_{\alpha\beta})$, where the first inequality is due to Eq. (5.23) and the second inequality is straightforward. This proves that, besides being block diagonal, D is also positive semidefinite. Thus X can be written as a positive sum of a block incoherent matrix D and matrices $G^{\alpha\beta}$ of block coherence number two, from which the statement follows. □

The next concept that is needed is the so-called comparison matrix, which is defined as follows.

Definition 27 (from Ref. [183])**.** *Let X be partitioned as in Eq.* (5.22) *and $X_{\alpha\alpha}$ non-singular. Then the block comparison matrix $M_b[X]$ is defined by*

$$(M_b[X])_{\alpha\beta} = \begin{cases} (\|X_{\alpha\alpha}^{-1}\|)^{-1} & \textit{for } \alpha = \beta \\ -\|X_{\alpha\beta}\| & \textit{for } \alpha \neq \beta. \end{cases} \tag{5.25}$$

From this definition it is evident that if the comparison matrix $M_b[X]$ exists and is (strictly) diagonally dominant, then X itself is (strictly) block diagonally dominant.

Definition 28 (M-matrix)**.** *Let the matrix $A = (a_{ij})$ be a real matrix such that $a_{ij} \leq 0$ for $i \neq j$. Then A is called a nonsingular M-matrix if and only if every real eigenvalue of A is positive.*

Definition 29 (Def. 3.2. in Ref. [183])**.** *If there exist nonsingular block diagonal matrices D and E such that $M_b[DXE]$ is a nonsingular M-matrix, then X is said to be a nonsingular block H-matrix.*

Lemma 30 (Lemma 4. in Ref. [183])**.** *If X is a nonsingular block H-matrix then there exist nonsingular block diagonal matrices D and E such that DXE is strictly block diagonally dominant.*

Now, we are ready to prove Obs. 24.

Proof. The proof follows the idea of Ref. [134]. First, define the operator $X_\epsilon = X + \epsilon\mathbb{1}$, for $\epsilon \geq 0$. Then, for $\epsilon > 0$ we have that $M_b[X_\epsilon] = M[X] + \epsilon\mathbb{1} > 0$. Evidently, since $M_b[X_\epsilon]$ is a real matrix with non-positive off-diagonal entries and furthermore has only strictly positive eigenvalues it is a nonsingular M-matrix, according to Def. 28. Then, according to Def. 29 X_ϵ is a nonsingular block H-matrix. From the proof of Lemma 30 in Ref. [183] we can conclude that there exists a block diagonal matrix $D > 0$ such that $D_\epsilon X_\epsilon D_\epsilon$ is strictly block diagonally dominant. Then it follows from Observation 26 that strictly block diagonally dominant matrices can have at most block coherence number two. We find that $bcn(X_\epsilon) = bcn(D_\epsilon X_\epsilon D_\epsilon) \leq 2$, and since the block coherence number is lower semi-continuous we have $bcn(X) = bcn(\lim_{\epsilon\to 0^+} X_\epsilon) \leq \lim_{\epsilon\to 0^+} bcn(X_\epsilon) \leq 2$. □

5.7 Conclusions

In this section, we have discussed a characterization of quantum (and classical) networks based on properties of the covariance matrix. Motivated by the work of Åberg et al. [28], we have shown that under certain assumptions, the block structure of the covariance matrix imposed by the topology of the network, is equivalent to its multisubspace coherence properties, thus establishing a connection between the theory of multisubspace coherence and the characterization of quantum networks. This provides a useful application of the resource theory of multisubspace coherence outside of the usual realm of quantum states. This allowed us to find a useful application of the coherence monogamy relations that we have introduced in Ch. 3. Furthermore, we note that this technique can be applied to large networks where an approach based on SDPs would become infeasible.

Finally, there are several interesting problems remaining for future work. First, it would be highly desirable to extend the covariance approach to the case where each node of the network can perform more than one measurement. Moreover, one could also consider the usual definition of covariance matrix in quantum mechanics to see if one could infer properties of the underlying quantum states and not only about the topology of the network. Second, it seems to be promising to study the coherence in networks on the level of the quantum state, and not the covariance matrix. This may shed light on the question which types of network correlations are useful for applications in quantum information processing. Especially, it would be interesting to study the limitations on, e.g., the correlated and genuinely correlated coherence, that we have studied in Ch. 2.

6 Quantifying measurement and state resources with conic programming

In this chapter, we will introduce an operational interpretation of certain robustness quantifiers in convex resource theories. Starting from measurement incompatibility we will show that for any measurement or state resource, their advantage in certain discrimination tasks can be quantified by an appropriately defined robustness measure. Notably, such an interpretation is independent of the type of resource that one considers. This chapter is based on publications [G], and [D].

6.1 Motivation

Measurement incompatibility is intimately tied to quantum steering (cf. the discussions in Sec. 1.7 and 1.8). In the case of steering, it was shown by Piani and Watrous in Ref. [184] that the steerability of quantum states can be quantified by their performance in a task called sub-channel discrimination with one-way LOCC measurements. In this task, one half of a bipartite state is sent through one of the sub-channels $\{\Lambda_a\}_a$ and subsequently undergoes a certain measurement strategy involving one way classical communication in order to make a guess on the label a. To be more precise, they showed that for any steerable state, there exists an instance of this discrimination task in which this state strictly outperforms any unsteerable state, thus, the task is acting as a witness. This immediately raises the question if a similar statement can be made for the case of incompatibility. In this chapter, we answer this question in the affirmative by providing a necessary and sufficient operational characterization of measurement incompatibility. Moreover, we will show how our characterization generalizes to any other possible convex and compact resource theory of measurements, states, and sets thereof.

We wish to emphasize that for single system resources similar results are known. In the case of coherence, it was shown in Ref. [132] that any coherent state provides an advantage in a phase discrimination game. Later, it was proven in Ref. [185] that in any convex and compact resource theory of states, a similar statement is true. Concerning the specific case of entanglement of bipartite states, it was shown in Ref. [186] that entanglement provides an advantage in minimum-error channel discrimination problems, and, more specifically, that this advantage increases with the Schmidt number [187].

For the case of single measurements, their robustness against mixing with trivial measurements, those that have all effects proportional to the identity operator, has been proven to have an information theoretic interpretation in terms of minimum accessible information [188]. By extending our results from measurement assemblages to state assemblages, which are, up to normalization, equivalent, and using the techniques from Ref. [184], we show that our framework extends naturally to other resource theories of states and sets thereof, and, thus, naturally recovers also what was previously known.

In addition we will consider another type of robustness, called the *convex weight*. We will show that, similar to the generalized robustness, this quantifier has an operational interpretation in terms of certain exclusion game. Those are games, where the aim is to guess a label that was not sent, rather than the label that was sent. Such games naturally occur, e.g., in the realm of the PBR argument [189, 190].

6.2 State discrimination with prior information

A fundamental task in quantum information is that of minimum-error state discrimination, where we aim at correctly guessing a state ϱ_a that is randomly drawn from an ensemble $\mathcal{E} = \{p_a, \varrho_a\}_a$, with prior probability p_a. To be more precise, upon receiving a state, we perform a measurement $\{M_a\}$, and guess the state to be ϱ_a whenever we observe the outcome a. Our success in correctly guessing the label a can be quantified by the *probability of success* $p_{succ}(\mathcal{E}, \{M_a\}) = \sum_a p_a \operatorname{tr}[M_a \varrho_a]$. The maximum probability of success is simply obtained by maximizing over all measurements, i.e. $p_{succ}(\mathcal{E}) = \max_{\{M_a\}} p_{succ}(\mathcal{E}, \{M_a\})$.

A typical instance of this task is state-discrimination with post-measurement information, that has been considered in, e.g., Refs. [191, 192]. Suppose that the ensemble $\mathcal{E} = \{p_a, \varrho_a\}_{a \in I}$ is partitioned in non-empty disjoint sets $\mathcal{E}_x = \{p_a, \varrho_a\}_{a \in I_x}$, where $\bigcup_x I_x = I$. Then, the label x is revealed to us after we have performed the measurement $\{M_a\}$. It is clear that this information cannot decrease our probability of guessing correctly. However, our probability of success can be increased if the label x is revealed to us prior to our measurement. This is because in this case, we can tailor a separate measurement to each label x individually. Thus, we arrive at the conclusion that $p_{\text{guess}}(\mathcal{E}) \leq p^{\text{post}}_{\text{guess}}(\mathcal{E}) \leq p^{\text{prior}}_{\text{guess}}(\mathcal{E})$. It was proven in Ref. [193] that $p^{\text{post}}_{\text{guess}}(\mathcal{E}) = p^{\text{prior}}_{\text{guess}}(\mathcal{E})$ if and only if there exist compatible measurements that maximize the probability of success. This raises the question if this result can be refined in a sense that, for any set of incompatible measurements, one can find a state discrimination task in which this incompatible set performs better than any compatible one.

This is the question that we want to address in the following.

6.3 Incompatibility provides an advantage in state discrimination with prior information

There are in general many ways to quantify the amount of incompatibly that is provided by a set of POVMs. A natural quantifiers for the incompatibility is based on the idea that any set of POVMs can be decomposed as

$$M_{a|x} = (1+t)O_{a|x} - tN_{a|x}, \tag{6.1}$$

where $\{O_{a|x}\}$ is jointly measurable and $\{N_{a|x}\}$ is an arbitrary set of POVMs. For a geometrical interpretation of such a decomposition, see Fig. 6.1a and note that the idea is similar as in the case of generalized robustness of entanglement that we discussed in Sec. 1.5.3. In fact, the minimum $t \geq 0$ for which such a decomposition exists is called the *incompatibility robustness* (IR). To relate the incompatibility robustness to state discrimination problems with prior information, we rewrite our partitioned state ensemble from the previous discussion as $\mathcal{E} = \{p(x)p(a|x), \varrho_{a|x}\}$, where $p(x)$ denotes the probability that the state comes from the sub-ensemble x and $p(a|x)$ is the conditional probability of receiving the label a from the sub-ensemble x. For an arbitrary set of measurements in Eq. (6.1), the probability of success in a state discrimination task with prior information reads

$$\begin{aligned} p_{succ}(M_{a|x}, \mathcal{E}) &= \sum_{a,x} p(a,x) \operatorname{tr}\Big[M_{a|x}\varrho_{a|x}\Big] && (6.2)\\ &\leq (1 + IR(M_{a|x})) \sum_{a,x} p(a,x) \operatorname{tr}\Big[O_{a|x}\varrho_{a|x}\Big] && (6.3)\\ &\leq (1 + IR(M_{a|x})) \max_{O_{a|x}\in JM} p_{succ}(O_{a|x}, \mathcal{E}). && (6.4) \end{aligned}$$

Rewriting this, we conclude that the incompatibility robustness provides an upper bound to the relative probability of success that can be achieved with the set of measurements $\{M_{a|x}\}$ normalized by the maximum probability of success that can be achieved with any compatible measurements, i.e.,

$$\frac{p_{succ}(M_{a|x}, \mathcal{E})}{\max_{O_{a|x}\in JM} p_{succ}(O_{a|x}, \mathcal{E})} \leq 1 + IR(M_{a|x}), \tag{6.5}$$

for all ensembles $\mathcal{E}$. Our goal is now to prove that this upper bound is tight, i.e., the robustness exactly quantifies the advantage that can be gained from a set of incompatible measurements over all compatible ones in a tailored discrimination task. Note that the incompatibility robustness can be cast as the optimization problem

$$IR(M_{a|x}) = \min\left\{ t \geq 0 \,\middle|\, \frac{M_{a|x} + tN_{a|x}}{1+t} = O_{a|x} \in JM \right\}, \tag{6.6}$$

where the optimization is performed over all POVMs $\{N_{a|x}\}$. Since $\{N_{a|x}\}$ is a set of POVMs, we have $(1+t)O_{a|x} - M_{a|x} \geq 0$ for all a, x. Furthermore, the set $\{O_{a|x}\}$ is jointly measurable, and hence, this condition can be rewritten using Eq. (1.41) as $(1+t)\sum_\lambda D(a|x,\lambda)G_\lambda \geq M_{a|x}$. Note that it is sufficient to consider only a finite number of deterministic post-processings [194].[1] By rescaling the variables $\tilde{G}_\lambda = (1+t)G_\lambda$, the incompatibility robustness can be cast as the SDP

$$1 + IR(M_{a|x}) = \min_{\tilde{G}_\lambda} \sum_\lambda \frac{\mathrm{tr}[\tilde{G}_\lambda]}{d} \tag{6.7}$$

$$\text{s. t.: } \sum_\lambda D(a|x,\lambda)\tilde{G}_\lambda \geq M_{a|x} \text{ for all } a, x, \tag{6.8}$$

$$\sum_\lambda \tilde{G}_\lambda = \frac{\mathbb{1}_d}{d} \sum_\lambda \mathrm{tr}[\tilde{G}_\lambda], \tag{6.9}$$

$$\tilde{G}_\lambda \geq 0, \tag{6.10}$$

where d is the dimension of the space of the G_λ such that $\sum_\lambda G_\lambda = \mathbb{1}_d$. The number of constraints in Eq. (6.8) is ax and in Eq. (6.10), it is a^x. Our next goal is to compute the dual of this SDP.

First, we note that this particular SDP has both, equalities and inequalities as con-

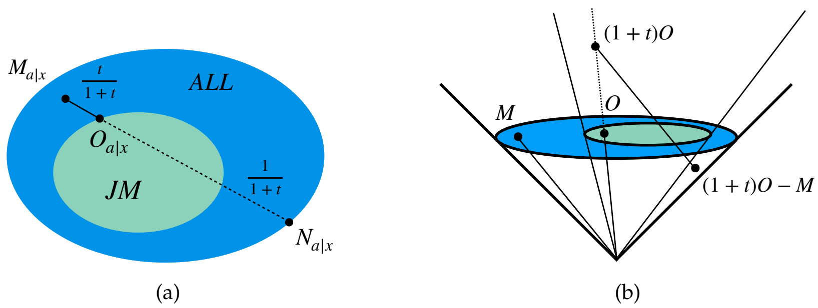


Figure 6.1: (a) This figure shows the construction of the incompatibility robustness in the space of sets of POVMs. (b) This figure shows the construction of the incompatibility robustness in the cone of sets of POVMs. A certain possible decomposition is shown, but it does not represent the minimum t for which such a decomposition is possible. To see this, note that by shifting the point O a little bit to the left, a smaller value of t can be realized without the point $(1+t)O - M$ leaving the outer cone, i.e., violating the positivity constraint.

[1] We emphasize that we omit the optimization over the finite number of deterministic post-processings, and simply assume that we are already given the optimal one.

straints and that it is of the general form

$$p^* = \min_X \operatorname{tr}[AX] \tag{6.11}$$

$$\text{s. t.: } \Phi(X) = B_1, \tag{6.12}$$

$$\Psi(X) \geq B_2, \tag{6.13}$$

$$X \geq 0, \tag{6.14}$$

where $A = \mathbb{1}_{d\cdot a^x}$ and $X = \operatorname{diag}\big[\tilde{G}_\lambda/d\big]_\lambda \in \mathcal{M}(\mathbb{C}^{d\cdot a^x})$, which is a block diagonal matrix with elements $\tilde{G}_\lambda/d$, the objective function reads $\operatorname{tr}[X] = \sum_\lambda \operatorname{tr}[X_\lambda]$. For the equality constraint we choose $B_1 = 0$ and a map $\Phi : \mathcal{M}(\mathbb{C}^{d\cdot a^x}) \to \mathcal{M}(\mathbb{C}^d)$ that is defined by $\Phi(X) = \mathbb{1}_d \operatorname{tr}[X] - d\sum_\lambda X_\lambda$. For the inequality constraint we have $B_2 = \operatorname{diag}[M_{a|x}]_{a,x}$ and $\Psi(X) : \mathcal{M}(\mathbb{C}^{d\cdot a^x}) \mapsto \mathcal{M}(\mathbb{C}^{d\cdot ax})$ which is defined by $\Psi(X) = \operatorname{diag}[d\sum_\lambda D(a|x,\lambda)X_\lambda]_{a,x}$. Hence, we have brought the incompatibility robustness SDP to the form of the SDP in Eq. (6.11). The dual of this SDP can be found in Ref. [102] and is simply given by

$$d^* = \max_{Y,Z} \operatorname{tr}[B_1 Z] + \operatorname{tr}[B_2 Y] \tag{6.15}$$

$$\text{s. t.: } \Phi^\dagger(Z) + \Psi^\dagger(Y) \leq A, \tag{6.16}$$

$$Z = Z^\dagger, \tag{6.17}$$

$$Y \geq 0. \tag{6.18}$$

To obtain the dual of the incompatibility robustness, the only remaining task is to compute the adjoint maps $\Psi^\dagger(Y)$ and $\Phi^\dagger(Z)$. The dual $\Psi^\dagger(Y)$ can be found in Ref. [184], where it reads

$$\Psi^\dagger(Y) = \operatorname{diag}\left[d\sum_{a,x} D(a|x,\lambda)Y^{a|x}\right]_\lambda . \tag{6.19}$$

To find $\Phi^\dagger(Z)$ we write

$$\operatorname{tr}[\Phi(X)Z] = \operatorname{tr}[\operatorname{tr}(X)Z] - \operatorname{tr}\left[d\sum_\lambda X_\lambda Z\right] \tag{6.20}$$

$$= \operatorname{tr}[X\{\operatorname{tr}(Z)\mathbb{1}_{d\cdot a^x} - d(Z \oplus Z \oplus \cdots \oplus Z)\}] \tag{6.21}$$

$$= \operatorname{tr}\Big[X\Phi^\dagger(Z)\Big]. \tag{6.22}$$

Since in the case of incompatibility robustness strong duality holds [86], we can write

IR in terms of the dual program, namely, we find

$$1 + IR(M_{a|x}) = \max_{Y,Z} \sum_{a,x} \operatorname{tr}\left[M_{a|x} Y^{a|x}\right] \tag{6.23}$$

$$\text{s. t.: } \operatorname{diag}\left[d \sum_{a,x} D(a|x,\lambda) Y^{a|x}\right]_\lambda$$

$$+ \operatorname{tr}(Z)\mathbb{1}_{d\cdot a^x} - d(Z \oplus Z \oplus \cdots \oplus Z) \leq \mathbb{1}_{d\cdot a^x}, \tag{6.24}$$

$$Z = Z^\dagger, \tag{6.25}$$

$$Y \geq 0. \tag{6.26}$$

From the optimal solution of the dual we can now construct a state discrimination task with prior information in the following way. First, observe that

$$\begin{aligned} Y^{a|x} &= \operatorname{tr}[Y] \frac{\sum_{a'} \operatorname{tr}[Y^{a'|x}]}{\operatorname{tr}[Y]} \frac{\operatorname{tr}\left[Y^{a|x}\right]}{\sum_{a'} \operatorname{tr}\left[Y^{a'|x}\right]} \frac{Y^{a|x}}{\operatorname{tr}\left[Y^{a|x}\right]} \\ &= \operatorname{tr}[Y] p(x) p(a|x) \varrho_{a|x}. \end{aligned} \tag{6.27}$$

Inserting this into the objective function of the dual in Eq. (6.23) yields

$$\begin{aligned} \sum_{a,x} \operatorname{tr}\left[M_{a|x} Y^{a|x}\right] &= \operatorname{tr}[Y] \sum_{a,x} p(x) p(a|x) \operatorname{tr}\left[M_{a|x} \varrho_{a|x}\right] \\ &= \operatorname{tr}[Y] \quad p_{\text{succ}}(M_{a|x}, \mathcal{E}), \end{aligned} \tag{6.28}$$

which is, up to normalization, the success probability in some discrimination task. Furthermore, we note that the first constraint of the dual in Eq. (6.24) is not really intuitive. Therefore, out next goal is to show that this simply corresponds to an upper bound on the objective function whenever a compatible set of measurements is used. More precisely, we insert a set of compatible measurements in the objective function and obtain

$$\sum_{a,x} \operatorname{tr}\left[O_{a|x} Y^{a|x}\right] = \sum_{a,x,\lambda} D(a|x,\lambda) \operatorname{tr}\left[J_\lambda Y^{a|x}\right] \tag{6.29}$$

$$=: \sum_\lambda \operatorname{tr}\left[J_\lambda \tilde{Y}^\lambda\right], \tag{6.30}$$

where $\{J_\lambda\}$ is the joint POVM for the compatible set $\{O_{a|x}\}$. From the first constraint of the dual program in Eq. (6.24), we obtain from each block labeled by λ, that $\tilde{Y}^\lambda \leq \frac{\mathbb{1}_d}{d}(1 - \operatorname{tr} Z) + Z$. This leads us to

$$\begin{aligned} \sum_\lambda \operatorname{tr}\left[J_\lambda \tilde{Y}^\lambda\right] &\leq \sum_\lambda \operatorname{tr}\left[J_\lambda \frac{\mathbb{1}_d}{d}(1 - \operatorname{tr} Z) + Z\right] \\ &= \operatorname{tr}\left[\frac{\mathbb{1}_d}{d}(1 - \operatorname{tr} Z) + Z\right] = 1. \end{aligned} \tag{6.31}$$

Hence, for any set of compatible measurements it holds that

$$\sum_{a,x} \mathrm{tr}\left[O_{a|x} Y^{a|x}\right] \leq 1. \tag{6.32}$$

Then, for the optimal solution $Y^{a|x_{opt}}$, the ratio of success probabilities in Eq. (6.5) is lower bounded by

$$\frac{p_{\text{succ}}(M_{a|x}, \mathcal{E}_{opt})}{\max_{O_{a|x} \in JM} p_{\text{succ}}(O_{a|x}, \mathcal{E}_{opt})} = \frac{\sum_{a,x} \mathrm{tr}\left[M_{a|x} Y^{a|x}_{opt}\right]}{\max_{O_{a|x} \in JM} \sum_{a,x} \mathrm{tr}\left[O_{a|x} Y^{a|x}_{opt}\right]} \tag{6.33}$$

$$\geq \sum_{a,x} \mathrm{tr}\left[M_{a|x} Y^{a|x}_{opt}\right] = 1 + IR. \tag{6.34}$$

The inequality follows from Eq. (6.32). Since the incompatibility robustness provides an upper bound, for all ensembles (see Eq. 6.5), and a lower bound, for the optimal ensemble constructed from the optimal $Y^{a|x}_{opt}$, on the ratio of success probabilities, they must be equal. Thus, we arrive at the following observation.

Observation 31. *For any set of incompatible measurements $\{M_{a|x}\}$, there exists a state discrimination task in which this set strictly outperforms any set of compatible measurements. The outperformance can be quantified by the incompatibility robustness $IR(M_{a|x})$ and we have*

$$\sup_{\mathcal{E}} \frac{p_{succ}(M_{a|x}, \mathcal{E})}{\max_{O_{a|x} \in JM} p_{succ}(O_{a|x}, \mathcal{E})} = 1 + IR(M_{a|x}). \tag{6.35}$$

The state discrimination task can be obtained from the optimal solution of the set of SDPs that calculates the incompatibility robustness.

This observation in itself is already quite interesting for several reasons. First, it generalizes the concept of witnesses to the case of incompatible measurements, where witnesses can be interpreted as state discrimination tasks with prior information. Conceptually, this is similar as in the case of entanglement, where witnesses are interpreted as Hermitian observables. Second, this result shows a way to witness incompatibility in a semi-device-independent way, namely, the incompatibility can be certified by only trusting the state preparation and not the actual measurement. Any violation of the incompatibility witness proves incompatibility, regardless of the actual measurement that was performed. Finally, it gives an operational interpretation in terms of state discrimination games of a mathematically very abstract quantity, the incompatibility robustness.

6.4 Any set of POVMs provides an advantage in a tailored discrimination game

So far, we were only concerned with distinguishing compatible and incompatible measurements. In this section, we will show that the connection that we found is quite general, namely, that the exact statement holds for any set of POVMs that is outside of any convex compact set F in the space of sets of POVMs. Let λ be the number of POVMs. Again, the first step is to define the robustness of a set $\{M_{a|x}\}$ of POVMs relative to the so-called *free set* F. The generalized robustness reads

$$R_F(M_{a|x}) = \min\left\{t \geq 0 \middle| \frac{M_{a|x} + tN_{a|x}}{1+t} = O_{a|x} \in F\right\}. \tag{6.36}$$

The difference to the case of incompatible measurements is that now the set F does not need to be characterized as in the case of joint measurability. The only assumption on F is that it is convex and compact. The first step is to rewrite this optimization problem as follows

$$\begin{aligned} R_F(M_{a|x}) = \min_t t & \qquad (6.37)\\ \text{s. t.: } t \geq 0, & \qquad (6.38)\\ \frac{M_{a|x} + tN_{a|x}}{1+t} = O_{a|x} \in F, & \qquad (6.39)\\ \{N_{a|x}\} \text{ is a POVM.} & \qquad (6.40)\end{aligned}$$

Solving the second constraint for $N_{a|x}$, one obtains $tN_{a|x} = (1+t)O_{a|x} - M_{a|x}$. The normalization of the $N_{a|x}$ is guaranteed by $M_{a|x}$ being a POVM and $O_{a|x}$ being required to be in F, which is a subset of all sets of POVMs. Thus, if we require the positivity of $\{N_{a|x}\}$, this changes to

$$\begin{aligned} 1 + R_F(M_{a|x}) = \min_t 1 + t & \qquad (6.41)\\ \text{s. t.: } t \geq 0, & \qquad (6.42)\\ (1+t)O_{a|x} - M_{a|x} \geq 0, & \qquad (6.43)\\ O_{a|x} \in F. & \qquad (6.44)\end{aligned}$$

This form of the robustness provides a very intuitive understanding of the construction of the generalized robustness, see Fig. 6.3. By redefining the variables as $\tilde{O}_{a|x} = (1 + t)O_{a|x}$, this can be written as

$$\begin{aligned} 1 + R_F(M_{a|x}) = \min_{\tilde{O}_{a|x}} \frac{1}{|x|} \sum_{a,x} \frac{\operatorname{tr}\left[\tilde{O}_{a|x}\right]}{d} & \qquad (6.45)\\ \text{s. t.: } \tilde{O}_{a|x} \geq M_{a|x} & \qquad (6.46)\\ \tilde{O}_{a|x} \in C_F, & \qquad (6.47)\end{aligned}$$

where C_F is the cone that is generated by the set F, and $|x|$ is the number of POVMs. This optimization problem is an instance of conic programming, see Eq. (1.61), where the involved cones are the positive semidefinite cone and the cone C_F. This can be brought into the form of Eq. (1.61) by choosing $A = -\mathbb{1}/|x|d$, $X = \mathrm{diag}(\tilde{O}_{a|x})_{a,x}$, $B = -\mathrm{diag}(M_{a|x})_{a,x}$, and $\Lambda = -id$. From this, we can simply derive the dual cone program which now reads[2]

$$1 + R_F(M_{a|x}) = \max_Y \sum_{a,x} \mathrm{tr}\left[M_{a|x} Y^{a|x}\right] \tag{6.48}$$

$$\text{s. t.:} \; -Y + \frac{1}{\lambda d}\mathbb{1} \in C_F^* \tag{6.49}$$

$$Y \geq 0. \tag{6.50}$$

The first constraint translates to $\langle \mathbb{1}/|x|d - Y|\tilde{T}\rangle \geq 0$,where $\tilde{T} \in C_F$, simply by the definition of the dual cone. Hence, $\mathrm{tr}\left[Y\tilde{T}\right] \leq \mathrm{tr}\left[\tilde{T}/|x|d\right]$ for all $\tilde{T} \in C_F$ or equivalently $\mathrm{tr}[YT] \leq 1$ for all $T \in F$. The final form of the dual then reads

$$1 + R_F(M_{a|x}) = \max_Y \sum_{a,x} \mathrm{tr}\left[M_{a|x} Y^{a|x}\right] \tag{6.51}$$

$$\text{s. t.:} \; Y \geq 0 \tag{6.52}$$

$$\mathrm{tr}[YT] \leq 1 \text{ for all } T \in F. \tag{6.53}$$

From this way of formulating the problem, one can see the connection to witnesses, since the constraint in Eq. (6.53) is a common constraint of a witness. A similar line of arguments as in the previous section leads us to the following observation.

Observation 32. *For any convex and compact set F, the robustness $R(M_{a|x})$ of a set of measurements $\{M_{a|x}\}$ exactly quantifies the outperformance of this set over all sets of measurements in F in a tailored discrimination task with prior information, i.e.,*

$$\sup_{\mathcal{E}} \frac{p_{succ}(M_{a|x}, \mathcal{E})}{\max_{O_{a|x} \in F} p_{succ}(O_{a|x}, \mathcal{E})} = 1 + R_F(M_{a|x}). \tag{6.54}$$

We note again that this technique provides a way to construct semi-device-independent witnesses for any type of measurement resource and gives it an operational interpretation in terms of a state discrimination task.

Moreover, for certain choices of the set F, one can obtain an analytic solution for the robustness. An example is the informativeness of measurements that was considered in Ref. [188], where the free set F is the set of measurements for which all POVM elements are proportional to the identity matrix. Those measurements produce outcomes completely independent of the input state and thus contain no information about the state. It is shown that in this case the robustness is simply given by $R(M_a) = \sum_a \|M_a\|_\infty - 1$ [188].

[2] Note, that strong duality holds if there exists an interior point in C_F for which the positive semidefinite constraint is strictly fulfilled. One can simply choose the identity and scale it up, which is a natural assumption for most choices of F.

6.5 State assemblages and sub-channel discrimination

In this section, we will discuss a generalization of the results on sets of measurements to state assemblages. Recall that in Ref. [184] it was shown that for any steerable state assemblage, there exists a one-way LOCC assisted subchannel discrimination task in which the assemblage outperforms all unsteerable ones. Here, we show that such kind of statement is not specific to the case of steering, but it is rather a generic feature of resource theories with convex and closed sets of assemblages.

In a subchannel discrimination task one wishes to discriminate between different elements of an instrument $\{\Lambda_a\}_a$, i.e., a collection of cp maps that sums up to a cptp map, with some POVM $\{N_a\}$. For a given quantum state ϱ, the success probability reads

$$p_{\text{succ}}(\varrho, \Lambda_x, N_a) = \sum_a \text{tr}[\Lambda_a(\varrho) N_a]. \tag{6.55}$$

An instance of this task is that of subchannel discrimination with one-way LOCC measurements from Ref. [184]. In such a task the instrument $\{\Lambda_a\}_a$ is applied locally on one half of a bipartite state, say Bob, and then a measurement $\{N_x\}$ is performed on Bob, who then tells his outcome to Alice. Alice subsequently performs one of the measurements from the set $\{M_{a|x}\}$ depending on Bobs outcome x to guess the label a. The success probability reads

$$p_{\text{succ}}(\varrho_{AB}, \Lambda_a, N_x, M_{a|x}) = \sum_{a,x} \text{tr}[M_{a|x} \otimes N_x (\mathbb{1} \otimes \Lambda_a)[\varrho_{AB}]]. \tag{6.56}$$

We emphasize that this can equivalently be written on the level of assemblages as

$$p_{\text{succ}}(\varrho_{a|x}, \Lambda_a, N_x) = \sum_{a,x} \text{tr}[\Lambda_a^\dagger[N_x] \varrho_{a|x}]. \tag{6.57}$$

Note that we assume Alice's measurements $\{M_{a|x}\}$ and the shared state ϱ_{AB} to be such that they prepare the assemblage $\varrho_{a|x}$.

In a similar way as before we can define the robustness of a state assemblage $R_F(\varrho_{a|x})$ relative to a convex and compact set F of free assemblages via Eq. (6.36), the only difference being the normalization of the operators. For a given F, an assemblage $\{\varrho_{a|x}\}$, and a subchannel discrimination task, defined by Λ_a and N_x, we have

$$\frac{p_{\text{succ}}(\varrho_{a|x}, \Lambda_a, N_x)}{\max_F p_{\text{succ}}(\sigma_{a|x}, \Lambda_a, N_x)} \leq 1 + R_F(\varrho_{a|x}), \tag{6.58}$$

for all Λ_a and N_x, which follows from a similar line of arguments as in Eqs. (6.2) to (6.4). For state assemblages we note that the primal problem for the robustness of an assemblage $\{\varrho_{a|x}\}_{a,x}$ with respect to the free set F of assemblages is given as

$$\begin{aligned} 1 + R_F(\varrho_{a|x}) = \min_{\tilde{\sigma}_{a|x}} & \frac{1}{|x|} \sum_{a,x} \text{tr}\left[\tilde{\sigma}_{a|x}\right] \\ \text{s. t.: } & \tilde{\sigma}_{a|x} \geq \varrho_{a|x}, \quad \tilde{\sigma}_{a|x} \in C_F, \end{aligned} \tag{6.59}$$

where $\tilde{\sigma}_{a|x} = (1+t)\sigma_{a|x}$. Hence, the dual program can be written as

$$1 + R_F(\varrho_{a|x}) = \max_{Y^{a|x}} \sum_{a,x} \operatorname{tr}\left[\varrho_{a|x} Y^{a|x}\right] \qquad (6.60)$$
$$\text{s. t.: } Y \geq 0, \quad \operatorname{tr}[TY] \leq 1 \, \forall T \in F.$$

We have again denoted by Y the direct sum of the operators $\{Y^{a|x}\}_{a,x}$. Note that Slater's conditions can be verified similarly to the case of measurements for the free sets we are interested in. The question that remains is how to interpret the optimal solution Y as a discrimination game. Using the construction introduced in Ref. [184] it is clear that any witness Y that is of the above form can be cast as a subchannel discrimination task with one-way LOCC measurements.

To that end, define subchannels by their duals as $\Lambda_a^\dagger(|x\rangle\langle x|) = \alpha Y^{a|x}$ and a POVM $N_x = |x\rangle\langle x|$, where $\{|x\rangle\}_x$ is an orthonormal basis. Note that the Λ_a form subchannels as long as $\sum_a \Lambda_a^\dagger(\mathbb{1}) = \sum_{a,x} \Lambda_a^\dagger(|x\rangle\langle x|) = \alpha \sum_{a,x} Y^{a|x} \leq \mathbb{1}$, which motives us to choose $\alpha = \| \sum_{a,x} Y^{a|x} \|_\infty^{-1}$.

Thus, the subchannels Λ_a act as $\Lambda_a(\varrho) = \alpha \sum_x \operatorname{tr}[Y^{a|x}\varrho] \, |x\rangle\langle x|$. If these subchannels do not form an instrument, i.e., $\sum_a \Lambda_a^\dagger(\mathbb{1}) \neq \mathbb{1}$, the set can be completed into an instrument by defining an extra subchannel as $\hat{\Lambda}(\varrho) = \operatorname{tr}\left[(\mathbb{1} - \sum_a \Lambda_a^\dagger(\mathbb{1}))\varrho\right]\sigma$, where σ is some quantum state in the subspace orthogonal to $\operatorname{span}\{|x\rangle\}$. One can verify that, indeed, these subchannels sum up to a cptp map, simply by construction. It is worth noting that we have one more subchannel in the discrimination problem than we have outputs. Since we only wish to discriminate the subchannels Λ_a, we simply cannot make a guess on the label a if we find the state in the orthogonal subspace. Thus, we find that the objective function can be written as

$$\sum_{a,x} \operatorname{tr}\left[\varrho_{a|x} Y^{a|x}\right] = \frac{1}{\alpha} \sum_{a,x} \operatorname{tr}\left[\Lambda_a^\dagger[N_x]\varrho_{a|x}\right] = \frac{1}{\alpha} p_{\text{succ}}(\varrho_{a|x}, \Lambda_a, N_x). \qquad (6.61)$$

We arrive at the following observation.

Observation 33. *Let F be a convex and compact set of state assemblages. For any state assemblage $\{\varrho_{a|x}\}_{a,x}$ that is not in F there exists an instance of one-way LOCC assisted subchannel discrimination, where $\{\varrho_{a|x}\}_{a,x}$ strictly outperforms all assemblages in F. The outperformance is exactly quantified by the generalized robustness with respect to F, i.e.,*

$$\sup_{\Lambda_a, N_x} \frac{p_{succ}(\varrho_{a|x}, \Lambda_a, N_x)}{\max_F p_{succ}(\sigma_{a|x}, \Lambda_a, N_x)} = 1 + R_F(\varrho_{a|x}). \qquad (6.62)$$

We note that the task Λ_a, N_x can be interpreted as a one-way LOCC assisted subchannel discrimination task using the Gisin theorem [95, 195], which states that for any non-signaling assemblage $\varrho_{a|x}$, there exist measurements $\{M_{a|x}\}$ and a global state ϱ_{AB}, such that they prepare the assemblage $\varrho_{a|x}$. Hence, the tasks consists in preparing the state ϱ_{AB} and applying the subchannels $\{\Lambda_a, \hat{\Lambda}\}$. Whenever the output of the subchannels is found in the orthogonal subspace, no guess is made on the label a, otherwise Bob measures N_x. Upon receiving the output x from Bob, Alice performs the measurement $\{M_{a|x}\}$ to guess the label a.

Let us give a physically motivated example of the free set F. Consider the set of assemblages that can be prepared using states with Schmidt number k or smaller. An SDP formulation for such scenario is not known. First, we observe that the set of assemblages is convex and compact.

Observation 34. *The set of assemblages that can be remotely prepared using a state with Schmidt number k or smaller is convex and compact.*

Proof. Convexity: Consider two assemblages preparable by states in $\mathcal{S}_k$, namely, $\varrho_{a|x} = \mathrm{tr}_A[(M_{a|x} \otimes \mathbb{1})\varrho^k_{AB}]$ and $\sigma_{a|x} = \mathrm{tr}_A[(N_{a|x} \otimes \mathbb{1})\sigma^k_{AB}]$. Then, their mixture can be prepared by increasing the dimension of Alice's system, i.e., by choosing $\xi = \lambda\, |0\rangle\langle 0| \otimes \varrho_k + (1-\lambda)\, |1\rangle\langle 1| \otimes \sigma_k$ and measurements $O_{a|x} = |0\rangle\langle 0| \otimes M_{a|x} + |1\rangle\langle 1| \otimes N_{a|x}$, from which convexity follows.

Compactness: The set of states with fixed Schmidt number $\mathcal{S}_k$ is compact [196], and so is the set of POVMs $\mathcal{M}$ on Alices side. The cartesian product of two compact sets is compact with respect to the product topology (Tychonoff's theorem), hence $\mathcal{M} \times \mathcal{S}_k$ is compact. Define a bilinear map $f : \mathcal{M} \times \mathcal{S}_k \to \mathcal{A}$ from the cartesian product to the set of possible assemblages by $f[\{M_{a|x}\}, \varrho^k_{AB}] = \mathrm{tr}_A[(M_{a|x} \otimes \mathbb{1})\varrho^k_{AB}]$. For a bilinear map between Banach spaces separate continuity is equivalent to full continuity, which follows from the Banach-Steinhaus theorem. Hence, the set of reachable assemblages is compact. This concludes the proof. □

Slater's conditions are also fulfilled since the cone includes a full rank point, i.e., the uniform assemblage. We furthermore emphasize that the free set of states with Schmidt number k or less is properly included into the set of states with Schmidt number $k+1$ [197]. This example is in the spirit of Ref. [187], where it was shown that higher Schmidt number provides an advantage in channel discrimination tasks. Moreover, the example provides a refined characterization of steerable assemblages in Ref. [184]. Moreover, it leads to a semi-device-independent verification of the Schmidt number.

State ensembles. By setting $|x| = 1$ in our previous discussion we arrive at the case of state ensembles $\mathcal{E} = \{p_a, \varrho_a\}$. This corresponds to the case where the probability of success is given by Eq. (6.55). To give a physically motivated example, we consider ensembles that are created through a given instrument $\{\Lambda_a\}$, which can be related to the robustness $R(\varrho)$ of a single state—this could be the robustness of, e.g., entanglement, coherence, asymmetry or coherence number.

To get the connection, we first map the entire state space $S(\mathcal{H})$ to state assemblages by means of the instrument $\{\Lambda_a\}$. Hence, we are dealing with elements in the set $\{\{\Lambda_a(\varrho)\}|\varrho \in \mathcal{S}(\mathcal{H})\}$ of ensembles. Within this set we can define the robustness of an ensemble $\{\varrho_a\}$ as $R(\varrho_a) = \min\{t \geq 0|\varrho_a + t\tau_a = (1+t)\sigma_a \in C_F\}$, where F is the set of ensembles preparable with the given instrument $\{\Lambda_a\}$ and a resourceless state, and $\{\tau_a\}$ is any ensemble preparable with the given instrument. The results presented in the previous discussion provide a subchannel discrimination problem as the dual of this robustness, with the dual variables being POVMs. We emphasize that in order to fulfill Slater's conditions we need a full rank point in F. Typical free sets include the maximally mixed state and thus typically also the maximally mixed ensemble, and hence, the set F has a full rank point.

The ensemble robustness is always less than or equal to the state robustness as one can input an optimal solution of the state robustness to the instrument. We have $p_{\text{succ}}(N_a, \varrho_a) \leq (1 + R(\varrho)) \max_F p_{\text{succ}}(N_a, \sigma_a)$, where $\{N_a\}$ is a POVM. Whenever the instrument is a bijection from the set of states to the set of ensembles, e.g., in phase discrimination, the ensemble robustness coincides with the corresponding state robustness. Therefore, maximizing over all instruments and POVMs saturates the bound. This recovers the result of Ref. [198] stating that robustnesses of state resources are connected to subchannel discrimination.

6.6 Convex weight and state exclusion

Instead of defining the generalized robustness, one can also consider the so-called *convex weight*. This quantifier is very well known, e.g., in the case of entanglement, it was first studied in Ref. [199], where it was termed the *best separable approximation*. The idea behind this quantifier originates from the question of the largest amount of a free resource, e.g., a separable state, that can be contained in a certain resource. This idea can be captured in the following construction, which, for simplicity, we will spell out only for sets of measurements. The rest goes similarly as in the earlier discussion.

For a convex and compact subset F of a given measurement resource one defines the convex weight $W_F(M_{a|x})$ of a set of measurements $\{M_{a|x}\}$ as

$$W_F(M_{a|x}) = \min \lambda \tag{6.63}$$

$$\text{s. t.: } M_{a|x} = (1-\lambda)O_{a|x} + \lambda N_{a|x}, \quad O_{a|x} \in F \tag{6.64}$$

where the optimization runs over all measurement assemblages $N_{a|x}$. Solving the constraint for $N_{a|x}$ and introducing new variables $\tilde{O}_{a|x} = (1-\lambda)O_{a|x}$ allows one to write the weight as

$$1 - W_F(M_{a|x}) = \max \frac{1}{|x|} \sum_{a,x} \frac{\mathrm{tr}[\tilde{O}_{a|x}]}{d} \tag{6.65}$$

$$\text{s.t.: } M_{a|x} \geq \tilde{O}_{i|x} \ \forall a, x, \tag{6.66}$$

$$\tilde{O}_{a|x} \in C_F,$$

where d is the dimension of the Hilbert space. The dual of this reads

$$1 - W_F(M_{a|x}) = \min_{Y \geq 0} \sum_{a,x} \mathrm{tr}\left[M_{a|x} Y_{a|x}\right] \tag{6.67}$$

$$\text{s.t.: } \sum_{a,x} \mathrm{tr}\left[T_{a|x} Y_{a|x}\right] \geq 1 \quad \forall T_{a|x} \in F \tag{6.68}$$

where $Y = \bigoplus_{i,x} Y_{i|x}$ is again a witness. With a similar argument as before, one checks that the Slater condition is fulfilled.

The next step is finding the correct task in which the objective function (6.67) acts as a success (or failure) quantifier. We note that we can decompose the witness as in Eq. (6.27) to obtain (up to a global factor) a partitioned state ensemble, where $Y_{a|x} = \mathrm{tr}[Y] p(x) p(a|x) \varrho_{a|x}$. Inserting this into the objective function, we observe that it is the probability of correctly guessing the label a of the state given x. Since we aim at minimizing this quantity, the game consists of obtaining the state $\varrho_{a|x}$ and guessing a label that was not sent, i.e., guess a label $b \neq a$. One wins the game if correctly guesses a label that was not send, otherwise the game is lost. Hence, minimizing the objective function corresponds to minimizing the failure probability in this game. For the case where there is only one index x, this game is called *state exclusion game* or *anti-distinguishability* [200]. In our case we will refer to it as *state exclusion game with prior information*. Such games were first formalized in Ref. [190] in the context of the PBR argument [189].

To obtain a connection between the failure probability and the convex weight, we insert the decomposed witness into the conic program in Eq. (6.67) which results in

$$1 - W_F(M_{a|x}) \geq \frac{p_{fail}(M_{a|x}, \mathcal{E})}{\min_{O_{a|x} \in F} p_{fail}(O_{a|x}, \mathcal{E})}. \tag{6.69}$$

At the same time, we also observe that

$$1 - W_F(M_{a|x}) \leq \frac{p_{fail}(M_{a|x}, \mathcal{E})}{\min_{O_{a|x} \in F} p_{fail}(O_{a|x}, \mathcal{E})}. \tag{6.70}$$

for an arbitrary discrimination game. Thus, we arrive at the following observation.

Observation 35. *Let F be a convex subset of sets of POVMs. For any set of POVMs $M_{a|x} \notin F$, there exists a state exclusion task where $M_{a|x}$ outperforms any set of POVMs in F. Moreover,* *the relative advantage is exactly quantified by the convex weight of $M_{a|x}$ with respect to F, namely*

$$1 - W_F(M_{a|x}) = \inf_{\mathcal{E}} \frac{p_{fail}(M_{a|x}, \mathcal{E})}{\min_{O_{a|x} \in F} p_{fail}(O_{a|x}, \mathcal{E})}, \tag{6.71}$$

where the optimization is performed over those sets $\{O_{a|x}\}$ for which the right hand side is finite.

As a simple example, one could consider, e.g., the case where F is the set of measurements that have a random output, or more precisely, that are of the form $\{p(a)\mathbb{1}\}$. This case was considered in Ref. [201], where it was also shown that for this case, one finds a simple analytical expression for the weight, namely $W_F(M_a) = 1 - \sum_a \lambda_{min}[M_a]$.

6.7 Conclusion

In this chapter we, have discussed an operational interpretation of the generalized robustness quantifier in convex resource theories. Motivated by the result from Piani and Watrous [184] on steering, we have shown that a similar operational interpretation can be given to incompatible measurements. Namely, for each set of incompatible measurements, there exists an instance of state discrimination with prior information in which this set strictly outperforms any compatible set, and the outperformance is quantified by the incompatibility robustness. This provides a way to semi-device independently witness measurement incompatibility. We wish to highlight that such witnesses are within the reach of current experiments, which has been demonstrated for steering [202] and coherence [203].

Moreover, we have shown that this characterization is not unique to measurement incompatibility, but it is rather a general property of any generalized robustness measure in convex resource theories of measurements, states, and sets thereof. This interpretation of the generalized robustness follows from the duality in conic optimization.

Furthermore, we have shown that a similar connection can be found for other robustness-type measures, i.e., the convex weight, where the task changed from a state discrimination to state exclusion.

For future research, it will be interesting to identify other properties than the ones discussed here as the free set. Moreover, an information theoretic interpretation, as in the case of informativeness of measurements [188], would be highly desirable also for other resources.

Finally, we note that similar results were obtained in Refs. [204–206] at the same time as our work.

7 Dynamical resources and input-output games

So far we were only concerned with "static" resources such as states or measurements. In this chapter, we will generalize our results to "dynamical" resources such as channels, instruments and general quantum processes. We will discuss examples of cases, where such a technique has, to our knowledge, not been applied. This chapter is based on publication [F].

7.1 Motivation

In the case of quantum memories, i.e., channels that are not measure and prepare channels, it is known that they can be faithfully verified in semiquantum signaling games, which was shown in Ref. [207]. The game can be described in the following way. Alice, which acts as the referee, draws a random state ξ_x from the ensemble $\mathcal{E}_X$ and sends the state to Bob. Bob stores this state in his quantum memory. After some time has passed, Alice sends another state ψ_y from a different ensemble $\mathcal{E}_Y$ to Bob. Upon receiving the second state, Bob is required to perform a measurement on both states in order to compute his reply b, which he tells to Alice. Alice can estimate the distribution $p(b|x,y)$ after sufficiently many rounds of the game. In each round of the game, Bob is rewarded according to a reward function, depending on Alice's inputs x and y, as well as his reply b. The reward function is publicly announced before the game so that Bob can choose his reply b such that he maximizes his payoff. One of the main results of Ref. [207] is that Alice can faithfully verify if Bobs memory is actually a quantum memory or just a measure-and-prepare channel. To be more precise, whenever Bobs memory is a quantum memory, there exists an instance of such a game, where his payoff is strictly larger than the maximum attainable value for any measure-and-prepare channel. The procedure is shown in Fig. 7.1a.

The relevant point is that such a game is constructed from an entanglement witness that detects the Choi state of any quantum memory. Recall, that under the Choi isomorphism the set of measure-and-prepare channels coincide with the separable Choi states, and the quantum memories coincide with the entangled Choi states. This raises the question, if similar results can be obtained for other relevant classes of quantum channels. In this chapter, we will show that this is indeed the case, by generalizing our approach from the previous chapter. Although the game that we consider is slightly different, is can be brought to the form disused above. Moreover, we will show that similar techniques can also be applied to instruments and collections thereof, and even higher-order dynamics. Finally, we will show that the generalized robustness is related to well-known quantifiers based on the max-relative entropy.

7.2 Input-output games

As in our previous discussion, we first need to define the task that we wish to use to quantify dynamical resources. The task that we consider is called an *input-output game*. This can be seen as a generalization of the state discrimination game discussed before.

Consider the three players Alice, Bob, and Charlie. Charlie claims that he has access to a collection of channels $\mathbf{\Lambda} = \{\Lambda_x^{A\to B}\}$ from Alice to Bob that have a certain property, and he wishes to prove to Alice and Bob, that this is indeed the case. Thus, they agree to play the following game. Before each round of the game they agree on a label $x = 1, \ldots, |x|$, according to $p(x)$, which determines which channel from the collection they want to use. Then, Alice randomly prepares a state from the state ensemble $\mathcal{E}_x = \{p(i|x), \varrho_{i|x}\}_x$, which might be different for each label x, and sends her state to Bob though the channel with label x. Subsequently, Bob performs a measurement with label x from the set of POVMs $\{M_{j|x}\}$. For each label x, and pair of preparation i and measurement result j the players are rewarded according to a function $\Omega = \{\omega_{ijx}\}$, where ω_{ijx} are real numbers. The tuple $\mathcal{G} = (\{p(x), \mathcal{E}_x\}, \{M_{j|x}\}, \Omega)$ defines an input-output game, and Charlies strategy is simply to use the correct channels that he claims to have access to. The quantifier of success then takes the form

$$P(\Lambda_x, \mathcal{G}) = \sum_{i,j,x} p(x)p(i|x)\omega_{ijx} \operatorname{tr}\left[\Lambda_x^{A\to B}(\varrho_{i|x})M_{j|x}\right]. \tag{7.1}$$

For the case of a single channel, i.e. $|x| = 1$, the game is illustrated in Fig. 7.1.

Any input-output game gives rise to a class of equivalent games obtained by scaling and shifting the payoff. In order to use such games to quantify dynamical resource it will be necessary to restrict ourselves to a class of *canonical* input-output games, for which $\min_{\Lambda_x} P(\Lambda_x, \mathcal{G}) = 0$ and $\max_{\Lambda_x} P(\Lambda_x, \mathcal{G}) = 1$. Note that any input-output game can be brought to this form, and hence, we will implicitly assume that all input-output games are canonical. In the next section we will show that, whenever Charlie has a collection of channels that have a certain property, there exists a game in which he can exceed a certain payoff, and thus, convince Alice and Bob that his claim was correct.

7.3 Sets of channels

We denote a convex and compact subset of collections of channels by F (which includes channels as trivial collections) and call this the free set. The robustness $R_F(\Lambda_x)$ of a collection Λ_x with respect to the free set F is defined as

$$R_F(\Lambda_x) = \min\left\{t \geq 0 \,\middle|\, \frac{\Lambda_x + t\Sigma_x}{1+t} = \Gamma_x \in F\right\}, \tag{7.2}$$

where $\{\Sigma_x\}$ is an arbitrary collection of channels. Similarly, as in Eq. (6.2) to (6.4), we can observe that

$$\frac{P(\Lambda_x, \mathcal{G})}{\max_{\Gamma_x \in F} P(\Gamma_x, \mathcal{G})} \leq 1 + R_F(\Lambda_x), \tag{7.3}$$

where the maximization is taken over all free collections $\Gamma_x \in F$. The channel robustness in Eq. (7.2) can be expressed as

$$1 + R_F(\Lambda_x) = \min_t 1 + t \tag{7.4}$$

$$\text{s.t.: } t \geq 0, \tag{7.5}$$

$$\frac{\Lambda_x + t\Sigma_x}{1+t} = \Gamma_x \in F, \tag{7.6}$$

$$\Sigma_x \text{ is a collection of channels.} \tag{7.7}$$

By solving Σ_x from the constraint in Eq. (7.6) one obtains $\Sigma_x = \frac{1}{t}\left[\tilde{\Gamma}_x - \Lambda_x\right]$, where $\tilde{\Gamma}_x = (1+t)\Gamma_x$ and $\Gamma_x \in F$.

Recall that using the Choi isomorphism we can map any channel Λ to a bipartite state $J_\Lambda = \frac{1}{d}\sum_{ij} |i\rangle\langle j| \otimes \Lambda[|i\rangle\langle j|]$ with a fixed marginal on the first system [29, 31] (cf. Sec. 1.3), where d is the dimension of the channel input. The fact that this mapping is one-to-one allows us to evaluate the robustness on a subset of bipartite quantum states with a fixed marginal. Hence, the optimization problem in Eq. (7.2) can be cast, using

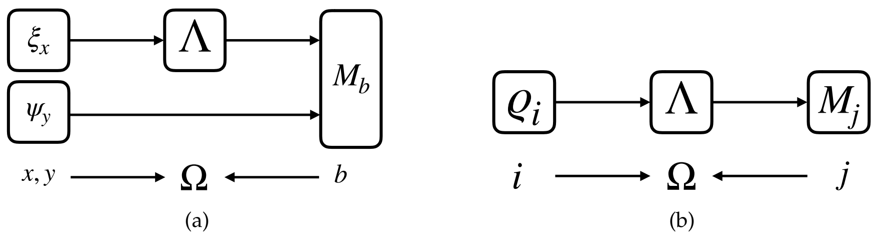


Figure 7.1: (a) This figure shows the verification strategy for quantum memories that was put forward in Ref. [207]. (b) This figure shows the verification procedure that we discuss in this chapter, but for a single channel, i.e., $|x| = 1$.

the Choi isomorphism, as

$$1 + R_F(\Lambda_x) = \min_{J_{\tilde{\Gamma}}} \frac{1}{|x|} \operatorname{tr}[J_{\tilde{\Gamma}}] \qquad (7.8)$$
$$\text{s.t.: } J_{\tilde{\Gamma}} - J_{\Lambda} \geq 0, \quad J_{\tilde{\Gamma}} \in C_{J_F},$$

where $J_{\tilde{\Gamma}} = \bigoplus_x J_{\tilde{\Gamma}_x}$, $J_{\Lambda} = \bigoplus_x J_{\Lambda_x}$, and C_{J_F} is the cone generated by the set of Choi states J_F that correspond to free collections of channels. This optimization problem is now in the form of the cone program (1.61). The dual cone program can be obtained from Eq. (1.62). We note that the dual cone constraint can be further simplified as in Eq. (6.60), such that the resulting dual program reads

$$1 + R_F(\Lambda_x) = \max_{Y} \operatorname{tr}[Y J_{\Lambda}] \qquad (7.9)$$
$$\text{s.t.: } Y \geq 0, \quad \operatorname{tr}[YT] \leq 1 \,\forall\, T \in J_F.$$

where $Y = \bigoplus_x Y_x$ constitutes a witness.[1] Again, the solutions of the primal and dual problems coincide if Slater's condition is fulfilled. In our case these conditions simply require that there exists an interior point in the free set such that positive semidefinite constraint in the primal problem can be satisfied in the strict form, i.e., $J_{\tilde{\Gamma}} - J_{\Lambda} > 0$. Under natural assumptions this is always the case as the maximally mixed state, which corresponds to the trivial channel, is in the free set of Choi states. Hence, one has a positive full rank point which can be scaled up to be strictly larger than a given J_{Λ}.

Now, we wish to find a way to implement a witness that comes form the optimal solution of the cone program in Eq. (7.9) in a way that resembles an input-output game, defined in Eq. (7.1). From the techniques developed in Ref. [207] and appropriate normalization, we can conclude that the operators Y_x can be decomposed as $Y_x = d \sum_{ij} p(i,x) \omega_{ijx} \varrho_{i|x}^T \otimes \eta_{j|x}$, where $\varrho_{i|x}$ are quantum states, $\eta_{j|x}$ are positive semidefinite operators satisfying $\sum_{j=1}^n \eta_{j|x} \leq \mathbb{1}$ for all x, $p(i,x)$ is a probability distribution and ω_{ijx} are real numbers. Note, that for every x the collection $\{\eta_{j|x}\}_{j=1}^n$ can be completed into a POVM by adding an element $\eta_{n+1|x} := \mathbb{1} - \sum_{j=1}^n \eta_{j|x}$ for which the reward function is taken to be zero.

Inserting this decomposition in the dual objective function and using the definition of the Choi state, as well as the fact that $(X \otimes \mathbb{1}) |\phi^+\rangle\langle\phi^+| = (\mathbb{1} \otimes X^T) |\phi^+\rangle\langle\phi^+|$, we find that

$$\operatorname{tr}[Y J_{\Lambda}] = d \sum_{x,i,j} p(i,x) \omega_{ijx} \operatorname{tr}\left[\varrho_{i|x}^T \otimes \eta_{j|x} (\mathbb{1} \otimes \Lambda_x) \left|\phi^+\right\rangle\!\left\langle\phi^+\right|\right] \qquad (7.10)$$

$$= \sum_{x,i,j} p(i,x) \omega_{ijx} \operatorname{tr}\left[\Lambda_x(\varrho_{i|x}) \eta_{j|x}\right] = P(\Lambda_x, \mathcal{G}). \qquad (7.11)$$

[1]We emphasize, that in the case of instruments one can simply follow the calculations above. The only difference is that each instrument element is treated as its own block, which changes the structure of the input-output game, as we will see later.

From this, we can conclude that an optimal witness corresponds to an input-output game up to the normalization, and the objective function defines the reward, cf. Eq. (7.1).

To see that the minimum value of the game is zero, one can solve Λ_x from Eq. (7.2) and obtain $\Lambda_x = [1 + R_F(\Lambda_x)]\Gamma_x - R_F(\Lambda_x)\Sigma_x$ where $\Gamma_x \in F$. Writing the expression to the Choi picture, and multiplying the result by an optimal witness Y, and taking the trace on both sides one arrives at $\text{tr}[YJ_{\mathbf{\Lambda}}] = [1 + R_F(\Lambda_x)]\,\text{tr}[YJ_{\mathbf{\Gamma}}] - R_F(\Lambda_x)\,\text{tr}[YJ_{\mathbf{\Sigma}}]$. The left hand side can be evaluated as $1 + R_F(\Lambda_x)$, and the first term in the right hand side is upper bounded by one, and thus, it follows that $\text{tr}[YJ_{\mathbf{\Sigma}}] = 0$. Noting that the normalization of a game does not affect the ratio in Eq. (7.3), we can combine Eq. (7.3) with Eq. (7.9) and write

$$\sup_{\mathcal{G}} \frac{P(\Lambda_x, \mathcal{G})}{\max_{\Gamma_x \in F} P(\Gamma_x, \mathcal{G})} = 1 + R_F(\Lambda_x), \tag{7.12}$$

where the supremum is taken over all canonical input-output games $\mathcal{G}$. This leaves us with the following observation.

Observation 36. *Let F be a convex and compact set of collections of channels. For any Λ_x outside of F there exists a tailored input-output game $\mathcal{G}$ for which Λ_x outperforms any element in F. Moreover, this outperformance is quantified by the generalized robustness according to Eq.* (7.12).

We emphasize that, besides our approach, one can also choose to interpret the witnesses derived from the robustness as a discrimination game on a bipartite system, which was done in Ref. [208]. We instead decompose the witness in a way that it can be implemented on a single system.

7.4 Applications

In this section, we want to discuss several scenarios in which the robustness quantifier, or more precisely a quantification via input-output games, could be applied or has already been applied, and thus, might lead to new insights.

7.4.1 Entanglement and incompatibility breaking channels

On the level of quantum channels one can define the notions of *entanglement (or incompatibility) breaking channels*, i.e., the set of channels that, for any possible input, destroy all the entanglement (or incompatibility). Entanglement breaking channels are also known to coincide with measure-and-prepare channels [209], i.e., channels that perform a measurement on the input and depending on that prepare some output state, whereas incompatibility breaking channels are so far lacking a simple operational characterization [85].

Both entanglement and incompatibility breaking channels form convex and compact subsets in the space of channels and, hence, using the framework that we have developed, one can define the corresponding robustnesses and deduce a task-oriented characterization of these sets. We emphasize, that for entanglement breaking channels our results complement the results that have been put forward in Ref. [207], that we have discussed in the beginning of this chapter. Our results provide a simple task-oriented quantifier for such quantum memories, that does not rely on a bipartite scenario, i.e., the preparation of two ensembles and a Bell measurement, as in Ref. [207]. This comes at a price, as we now have to make assumptions on Bobs measurements, i.e., our framework is not any longer measurement-device-independent. However, we wish to emphasize that our approach can be made measurement-device-independent simply by using teleportation to move all trust from Bobs measurements to Alices second system.

Our results can also be used to characterize interesting subsets of measure-and-prepare channels, such as those corresponding to POVMs, i.e., channels performing a measurement on the input and sending only a classical message. Mathematically, these channels can be written as $\Lambda^{A\to B}(\varrho) = \sum_a \mathrm{tr}[N_a \varrho] |a\rangle\langle a|$, where $\{|a\rangle\}$ is an orthonormal basis. Interestingly, this complements recent studies on semi-quantum games [210] and measure-and-prepare scenarios by providing an alternative operational quantifier for the advantage a channel sending a quantum message provides over all channels sending classical messages in a specific input-output game.

7.4.2 Channel incompatibility

A similar concept as that of measurement compatibility can be put forward on the level of quantum channels, which is called channel compatibility (cf. Sec. 1.7). An interesting special case of channel compatibility is obtained when one considers sets of compatible channels with classical outputs, i.e., each channel measures a POVM and writes the outcome in some orthonormal basis. In such a case it is easy to see that the compatibility of these channels corresponds to the compatibility of the POVMs that they implement. Motivated by recent developments on the connection between compatibility of measurements and communication tasks, see, e.g., Refs. [205,206,210–213], we will have a closer look at this example.

Let the set $\{A_{a|x}\}_{a,x}$ of POVMs be compatible, and let $\{G_\lambda\}_\lambda$ be its joint observable. A set $\{A_{a|x}\}_{a,x}$ of POVMs can be seen as a set of measure-and-prepare channels $\{\Lambda_x\}_x$ by defining $\Lambda_x(\varrho) = \sum_a \mathrm{tr}[A_{a|x}\varrho]|a\rangle\langle a|$, as mentioned before. The common channels are characterized as those that first measure the joint POVM $\{G_\lambda\}_\lambda$, produce a classical output λ and post-process the output according to some probability distribution $p(a|x,\lambda)$. This indeed provides a one-to-one correspondence between compatible sets of POVMs and compatible sets of classical-output channels [92]. This implies that the incompatibility of such channels can be witnessed through input-output games. To see this, we can insert the optimal witness into the objective function of the corresponding conic program and obtain

$$1 + R_F(\Lambda_x) = \sum_x \mathrm{tr}[Y_x J_{\Lambda_x}] = \sum_{a,i,j,x} \omega_{ijx} \, \mathrm{tr}\Big[A_{a|x}\varrho_{i|x}\Big] \langle a|\eta_{j|x}|a\rangle \tag{7.13}$$

$$= \sum_{a,i,x} \tilde{\omega}_{aix} \, \mathrm{tr}\Big[A_{a|x}\varrho_{i|x}\Big], \tag{7.14}$$

where $\tilde{\omega}_{aix} = \sum_j \omega_{ijx} \langle a|\eta_{j|x}|a\rangle$, which can be interpreted as the payoff in a discrimination game with prior information. In this way, the input-output game becomes a witness of the incompatibility of the measurements. This shows that in the formalism of input-output games, incompatible measurements can perform better than compatible ones in measure-and-prepare scenarios where only classical information is sent forward, c.f. Ref. [210] for a more detailed discussion on the connection between incompatibility and the quantumness of the sent message. Note that in the case of joint measurability, the explicit form of an optimal witness can be calculated via semidefinite programming.

7.4.3 G-covariant operations

Besides the cases discussed in the previous section, our methods can also be applied to other scenarios that have not yet been studied in the literature. The first example are so-called G-covariant operations.

Any transformation of a physical system requires a reference frame. For instance, a rotation of a qubit state on the Bloch sphere requires a notion of direction, i.e., asymmetry. On the contrary, lack of symmetry in the reference frame puts a restriction on what transformations can be implemented. Mathematically, the lack of symmetry is described by symmetry transformations [214, 215]. Denote by G the group of transformations that leaves the reference frame invariant and let $\mathcal{U}_g(\varrho) = U_g \varrho U_g^\dagger$ with $g \in G$ be a unitary representation of the group G. The *G-covariant* operations Λ that can be implemented under this restriction are those that commute with all symmetry transformations, i.e., $[\Lambda, \mathcal{U}_g] = 0$ for all $g \in G$. The set of all G-covariant operations is convex and compact and hence the asymmetry of a channel can be quantified using the methods we have developed.

7.5 Remarks on possible extensions

The technique that we have developed can also be applied to quantum instruments and collections of instruments. We define the robustness analogously to that in Eq. (7.2). As in the case of channels, the robustness is preserved under the Choi isomorphism.

As an example, a natural notion of compatibility for a set of instruments $\{I_{a|x}\}$ is defined by the existence of a common instrument together with classical post-processings such that $I_{a|x} = \sum_\lambda p(a|x,\lambda) I_\lambda$ [216]. This definition is equivalent to unsteerability of channels [217], i.e., the non-existence of an incoherent channel extension. Compatibility of sets of instruments clearly defines a convex set. Moreover, note that steering on the level of quantum states is a special case of channel steering, i.e., instruments with one-dimensional input systems.

The same game-theoretic approach can be also generalized to higher-order dynamics, e.g., transformations of channels. Such higher-order dynamics have become an increasingly active field of research, from their role in studying quantum causality [218] to their use as operations in resource theories [219], but thus far have been given no operational resource theoretic study.

Formally, higher-order dynamics are "supermaps" that map a set of channels to another channel [220,221]. For simplicity, we focus here on supermaps of two channels, but the following generalizes immediately to any number of channels. A supermap $\mathcal{S}$ thus transforms the channels Λ_C, Λ_D to $\Lambda^{A\to B} = \mathcal{S}(\Lambda_C, \Lambda_D)$. For $\mathcal{S}$ to be valid, (i) $\Lambda^{A\to B}$ must be a valid channel whenever Λ_C, Λ_D are channels, and (ii) $\mathcal{S}$ must give valid channels when applied locally to part of some bipartite channels, i.e., $\mathbb{1} \otimes \mathcal{S}$ must map the bipartite channels to channels [221–223].

The generalization to higher-order dynamics requires also a generalization of input-output games to collaborative games between several players. For more details we refer to Ref. [224].

7.6 Relation to max-relative entropy

In the case of entanglement it is known that the max relative entropy of entanglement is, up to a logarithm, equal to the generalized robustness (cf. Sec. 1.5.3). In this section, we will show that a similar quantifier for general convex resource theories is, by construction, also equal to the generalized robustness measure.

It is sufficient to derive the results for quantum states, since the results for channels generalize by the Choi isomorphism. The max-relative entropy is defined by

$$D_{max}(\varrho\|\sigma) = \log\min\{\lambda | \varrho \leq \lambda\sigma\}, \tag{7.15}$$

for positive operators $\varrho, \sigma \geq 0$, with $\mathrm{tr}[\varrho] = 1$, and $\mathrm{supp}(\varrho) \subseteq \mathrm{supp}(\sigma)$. Let F be the convex and compact set of free states, the *max-relative entropy* of ϱ with respect to the set F is defined as

$$E_{max}(\varrho) = \min_{\sigma\in F} D_{max}(\varrho\|\sigma). \tag{7.16}$$

Next, we will show that such quantities are naturally related to the generalized robustness.

Observation 37. *Let ϱ and σ be quantum states, then*

$$E_{max}(\varrho) = \log(1 + R_F(\rho)). \tag{7.17}$$

Proof. The proof goes similar as in Ref. [64]. First, observe that the robustness can be written as

$$\begin{aligned} R_F(\varrho) &= \min\{t \geq 0 | \varrho + t\tau = (1+t)\sigma, \sigma \in F\} &(7.18)\\ &= \min\{t \geq 0 | \varrho \leq (1+t)\sigma, \sigma \in F\}. &(7.19)\end{aligned}$$

Then one finds that

$$\begin{aligned} E_{max}(\varrho) &= \min_{\sigma \in F} \log \min \{\lambda \geq 1 | \varrho \leq \lambda\sigma\} = \log \min_{\sigma \in F} \min \{\lambda \geq 1 | \varrho \leq \lambda\sigma\} \\ &= \log \min \{\lambda \geq 1 | \varrho \leq \lambda\sigma, \sigma \in F\} = \log(1 + R_F(\varrho)), &(7.20)\end{aligned}$$

where the fist equality is due to the concavity of the logarithm and the last equality is obtained by $\lambda \mapsto 1 + t$. □

Since the generalized robustness and the max-relative entropy are equal up to a constant and a logarithm, we can conclude that the same operational interpretation holds for both quantifiers.

7.7 Conclusion

In this chapter, we have discussed an operational interpretation of the generalized robustness of quantum channels, and collections thereof.

Inspired by the results of Ref. [207] on the resource theory of quantum memories, we have shown that in any convex resource theory of channels, and collections thereof, the generalized robustness quantifies the relative advantage of any channel resource over the resourceless channels in input-output games. This approach can be generalized to instruments, and collections thereof, as well as higher-order dynamics, i.e., supermaps that transform channels to channels.

Moreover, we have shown that the generalized robustness measure is intimately related to the resource measures based on the max-relative entropy, which generalizes a well known result from entanglement theory [63, 64].

Finally, we have discussed several relevant scenarios to which our approach could be applied in future research. Particularly in the realm of higher-order dynamics there are still many interesting properties that so far have not been given a resource theoretic treatment. Another path that one could explore are multi-object resource theories, i.e., resource theories of collections of different types of resources. First steps in this direction have been reported in Ref. [225] for the case of state-measurement pairs.

8 Channel incompatibility and the quantum marginal problem

In this chapter, we will discuss the connection between channel incompatibility (cf. Sec. 1.7) and certain instances of the quantum marginal problem (cf. Sec. 1.5.4). This chapter is based on publication [E].

8.1 Motivation

One of the distinguishing features between classical mechanics and quantum theory is the existence of fundamentally incompatible objects, which include, e.g., states, measurements and channels. In this chapter we will show that two seemingly different notions of quantum incompatibility are intimately related. More precisely, we will show that the concept of channel incompatibility (cf. Sec. 1.7.2), which includes measurement incompatibility (cf. Sec. 1.7.1) and no-broadcasting as special cases, is connected to an instance of certain quantum marginal problems (cf. Sec. 1.5.4). In particular, since measurement incompatibility is a special instance of channel incompatibility, it also forms an instance of the quantum marginal problem.

We will use this connection to solve the marginal problem for pairs of two-qubit Bell diagonal states, as well as for pure states under depolarizing noise. Furthermore, we derive entropic criteria for channel compatibility, and put forward a converging hierarchy of SDPs to quantifying the strength of quantum memories.

8.2 Mapping between channel compatibility and the marginal problem

In this section, we focus on an instance of the quantum marginal problem, where all given marginals are bipartite and overlap on a single party. Namely, we search for a global state in the Hilbert space $\mathcal{H}_I$, where $I := \{A, B_1, \ldots, B_n\}$, given all the marginal states on $\mathcal{H}_{I_k}$, where $I_k = \{A, B_k\}$. One immediate necessary condition for the existence of a global state is a common marginal state on A, i.e., $\varrho_A = \mathrm{tr}_{B_k}[\varrho_{AB_k}]$ is the same for all k.

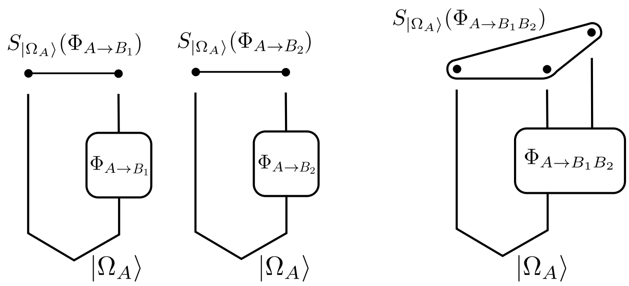


Figure 8.1: Channels $\Phi_{A\to B_1}$ and $\Phi_{A\to B_2}$ are compatible if they have a broadcasting channel $\Phi_{A\to B_1B_2}$ whose input system is the shared input system of $\Phi_{A\to B_1}$ and $\Phi_{A\to B_2}$ and which gives these channels after tracing out one of the output subsystems B_1 or B_2. Also the Choi states of the channels are depicted in this figure. Note that the states share the same reduced state on subsystem A.

A natural quantifier for such marginal problems is what we call the *consistent marginal robustness* or simply *consistent robustness*. In the simplest case, namely the tripartite case, one has a pair of bipartite states $\boldsymbol{\varrho} := (\varrho_{AB_1}, \varrho_{AB_2})$ sharing a common first marginal ϱ_A, and one asks for the existence of a global state $\varrho_{AB_1B_2}$, which has those two states as marginals. The consistent robustness is then defined as

$$R_F^c[\boldsymbol{\varrho}] = \min\left\{ t \geq 0 \middle| \frac{\boldsymbol{\varrho} + t\boldsymbol{\tau}}{1+t} \in F \right\}, \tag{8.1}$$

where the optimization is performed over all pairs of states $\boldsymbol{\tau} := (\tau_{AB_1}, \tau_{AB_2})$ having ϱ_A as the first marginal, and F denotes those pairs of states for which the marginal problem has a solution[1].

Similarly to the marginal problem, channel incompatibility (cf. Sec. 1.7) can be quantified by an appropriately chosen robustness. For a tuple of channels $\boldsymbol{\Lambda} = (\Lambda_1^{A\to B_1}, \Lambda_2^{A\to B_2})$ one defines the channel compatibility robustness $R_F^g(\boldsymbol{\Lambda})$ with respect to the set of compatible tuples F as

$$R_F^g(\boldsymbol{\Lambda}) = \min\left\{ t \geq 0 \middle| \frac{\boldsymbol{\Lambda} + t\boldsymbol{\Gamma}}{1+t} \in F \right\}, \tag{8.2}$$

where the optimization is taken over $\boldsymbol{\Gamma}$. In case of a single channel, one can take the set F to be the set of n-self-compatible channels.

In order to prove the connection, we first need to extend our definition of the Choi-Jamiołkowski isomorphism. Whenever $\varrho_A = \sum_n t_n |n\rangle\langle n|$ is a state on $\mathcal{H}_A$ having full-rank we denote its canonical purification (cf. Sec. 1.2) by $|\Omega_A\rangle = \sum_n \sqrt{t_n} |nn\rangle$. Then,

[1]The term consistent refers to the fact that we don't consider mixing with all pairs of states, but only the ones sharing common first marginals with the original pair.

for any channel $\Phi_{A\to B}$ we denote its $|\Omega_A\rangle$-*Choi state* as

$$J_{|\Omega_A\rangle}(\Phi_{A\to B}) := (\mathbb{1}\otimes\Phi_{A\to B})(|\Omega_A\rangle\langle\Omega_A|). \tag{8.3}$$

According to Refs. [226, 227], the mapping $J_{|\Omega_A\rangle}$ is a well defined bijection between the set of channels $\Phi_{A\to B}$ and the set of bipartite states ϱ_{AB} with the fixed marginal $\mathrm{tr}_B[\varrho_{AB}] = \varrho_A$ on A.

We can also define the inverse of the above mapping. When ϱ_{AB} is a state on $\mathcal{H}_A\otimes\mathcal{H}_B$, we can assume that the reduced state $\varrho_A = \mathrm{tr}_B[\varrho_{AB}]$ is of full rank by restricting the dimension of the subsystem A. By considering the canonical purification $|\Omega_A\rangle$ of ϱ_A, we call the unique channel $\Phi_{A\to B}(\varrho) := \mathrm{tr}_A[\varrho_{AB}(\varrho_A^{-1/2}\varrho^{T_A}\varrho_A^{-1/2}\otimes\mathbb{1}_{\mathcal{H}_B})]$ the $|\Omega_A\rangle$-*Choi channel of* ϱ_{AB}, where $\varrho\mapsto\varrho^{T_A}$ is the transpose defined w.r.t. the eigenbasis of ϱ_A.[2] It can be easily verified that this is indeed the inverse, namely

$$\begin{aligned}
&\mathrm{tr}_A[\varrho_{AB}(\varrho_A^{-1/2}\varrho^{T_A}\varrho_A^{-1/2}\otimes\mathbb{1}_B)]\\
=&\sum_{nm}\mathrm{tr}_A[(\mathbb{1}_A\otimes\Phi_B)\sqrt{t_nt_m}\,|nn\rangle\langle mm|\,(\varrho_A^{-1/2}\varrho^{T_A}\varrho_A^{-1/2}\otimes\mathbb{1}_B)]\\
=&\sum_{nm}\Phi(|n\rangle\langle m|)\,\mathrm{tr}_A[\sqrt{t_nt_m}\,|n\rangle\langle m|\,\varrho_A^{-1/2}\varrho^{T_A}\varrho_A^{-1/2}]\\
=&\sum_{nm}\Phi(|n\rangle\langle m|)\sqrt{t_nt_m}\,\langle m|\,\varrho_A^{-1/2}\varrho^{T_A}\varrho_A^{-1/2}\,|n\rangle\\
=&\Phi(\sum_{nm}\sqrt{t_nt_m}\,\langle n|\,\varrho_A^{-1/2}\varrho\varrho_A^{-1/2}\,|m\rangle\,|n\rangle\langle m|) = \Phi(\rho)
\end{aligned} \tag{8.4}$$

This inversion formula shows that the Choi channel of a bipartite state is unique, up to the choice of the purification of its first marginal. In contrast, the Choi state of a channel depends on the state ϱ_A of the input system and its purification chosen to set up the Choi-Jamiołkowski isomorphism.

Having an equal marginal on Alice's side is a necessary condition for the existence of a solution to the marginal problem. As this amounts to fixing the mapping in the Choi-Jamiołkowski isomorphism, we are ready to state the main result.

Observation 38. *Let* $I := \{A, B_1, \dots, B_n\}$, $I_k := \{A, B_k\}$, *and* $k = 1, \dots, n$. *The following statements are true.*

(i) Let ϱ_A *be a full-rank state on* $\mathcal{H}_A$ *and* $|\Omega_A\rangle$ *be a canonical purification of* ϱ_A. *The channels* $\Phi_{A\to B_k}$ *from* $\mathcal{H}_A$ *to* $\mathcal{H}_{B_k}$ *are compatible if and only if there is a state* ϱ *on* $\mathcal{H}_I$ *such that* $\mathrm{tr}_{I\setminus I_k}[\varrho] = J_{|\Omega_A\rangle}(\Phi_{A\to B_k})$ *for all k.*

(ii) Let ϱ_{AB_k} *be states for all* $k = 1, \dots, n$. *There is a state* ϱ *on* $\mathcal{H}_I$ *such that* $\mathrm{tr}_{I\setminus I_k}[\varrho] = \varrho_k$ *if and only if* $\mathrm{tr}_{B_k}[\varrho_k] = \varrho_A$ *is the same for all* $k = 1, \dots, n$ *and, upon assuming that* ϱ_A *is of full rank and picking a canonical purification* $|\Omega_A\rangle$ *for* ϱ_A, *the* $|\Omega_A\rangle$*-Choi channels of* ϱ_k *are compatible.*

[2] Not the partial transpose!

(iii) *Let $\Phi_{A\to B_k}$ be channels from $\mathcal{H}_A$ to $\mathcal{H}_{B_k}$ and ϱ_{AB_k} be states sharing the common marginal ϱ_A, and pick a canonical purification $|\Omega_A\rangle$ for ϱ_A. Whenever $\rho_k = J_{|\Omega_A\rangle}(\Phi_{A\to B_k})$, the incompatibility robustness of $(\Phi_{A\to B_1},\ldots,\Phi_{A\to B_n})$ coincides with the consistent marginal robustness of $(\varrho_1,\ldots,\varrho_n)$.*

Proof. Statement (i): Assume that $\{\Phi_{A\to B_k}\}$ are compatible and denote the joint channel by Φ. Recall that the input space of Φ is $\mathcal{H}_A$ and the output space is $\mathcal{H}_{B_1}\otimes\cdots\otimes\mathcal{H}_{B_n}$. Denote $\varrho_{AB_k} := J_{|\Omega_A\rangle}(\Phi_{A\to B_k})$, $k=1,\ldots,n$, and $\varrho := J_{|\Omega_A\rangle}(\Phi)$. Denote the identity operator on $\mathcal{H}_{B_k}$ by $\mathbb{1}_k$. First, note that for all bounded operators D on $\mathcal{H}_A$ and E on $\mathcal{H}_B$ we have

$$\begin{aligned}
\operatorname{tr}[\varrho_{AB}(D\otimes E)] &= \operatorname{tr}[(\mathbb{1}\otimes\Phi_{A\to B})(|\Omega_A\rangle\langle\Omega_A|)(D\otimes E)]\\
&= \langle\Omega_A|\, D\otimes\Phi^\dagger_{A\to B}(E)\,|\Omega_A\rangle \qquad (8.5)\\
&= \sum_{m,n}\sqrt{t_m t_n}\,\langle n|\,D^{T_A}\,|m\rangle\,\langle m|\,\Phi^\dagger_{A\to B}(E)\,|n\rangle\\
&= \operatorname{tr}\Big[\varrho_A^{1/2}D^{T_A}\varrho_A^{1/2}\Phi^\dagger_{A\to B}(E)\Big] = \operatorname{tr}\Big[\Phi_{A\to B}(\varrho_A^{1/2}D^{T_A}\varrho_A^{1/2})E\Big].
\end{aligned}$$

Using this, we find that

$$\begin{aligned}
\operatorname{tr}\big[\varrho_{AB_k}(D\otimes E)\big] &= \operatorname{tr}\Big[\Phi_{A\to B_k}(\varrho_A^{1/2}D^{T_A}\varrho_A^{1/2})E\Big]\\
&= \operatorname{tr}\big[\Phi(\varrho_A^{1/2}D^{T_A}\varrho_A^{1/2})(\mathbb{1}_{1,\ldots,k-1}\otimes E\otimes\mathbb{1}_{k+1,\ldots,n})\big]\\
&= \operatorname{tr}[\varrho(D\otimes\mathbb{1}_{1,\ldots,k-1}\otimes E\otimes\mathbb{1}_{k+1,\ldots,n})]\\
&= \operatorname{tr}\Big[\operatorname{tr}_{I\setminus I_k}[\varrho](D\otimes E)\Big]. \qquad (8.6)
\end{aligned}$$

for all bounded operators D on $\mathcal{H}_A$ and E on $\mathcal{H}_{B_k}$. Thus, $\varrho_{AB_k} = \operatorname{tr}_{I\setminus I_k}[\varrho]$, $k=1,\ldots,n$. The proof of the converse statement is contained in the proof of statement (ii).

Statement (ii): Note that, for the existence of a joint state ϱ of the claim, it is necessary that the marginals ϱ_A of the states ϱ_{AB_k}, $k=1,\ldots,n$, coincide. According to our earlier observation, we may freely assume that this shared marginal ϱ_A is of full rank and we may fix a canonical purification $|\Omega_A\rangle$ for it.

First, assume that there is ϱ such that $\operatorname{tr}_{I\setminus I_k}[\varrho] = \varrho_k$. Denote, for each $k=1,\ldots,n$, by $\Phi_{A\to B_k}$ the channel such that $J_{|\Omega_A\rangle}(\Phi_{A\to B_k}) = \varrho_k$ and by $\Phi = \Phi_{A\to B_1\ldots B_n}$ the channel such that $J_{|\Omega_A\rangle}(\Phi) = \varrho$. Denote, again, the identity operator on $\mathcal{H}_{B_i}$ by $\mathbb{1}_i$ and pick $k\in\{1,\ldots,n\}$. For any bounded D on $\mathcal{H}_A$ and E on $\mathcal{H}_{B_k}$,

$$\begin{aligned}
\operatorname{tr}\Big[\operatorname{tr}_{I\setminus I_k}[\Phi(\varrho_A^{1/2}D\varrho_A^{1/2})]E\Big] &= \operatorname{tr}\big[\Phi(\varrho_A^{1/2}D\varrho_A^{1/2})(\mathbb{1}_{1,\ldots,k-1}\otimes E\otimes\mathbb{1}_{k+1,\ldots,n})\big] \qquad (8.7)\\
&= \operatorname{tr}\Big[\varrho(D^{T_A}\otimes\mathbb{1}_{1,\ldots,k-1}\otimes E\otimes\mathbb{1}_{k+1,\ldots,n})\Big]\\
&= \operatorname{tr}\Big[\varrho_k(D^{T_A}\otimes E)\Big] = \operatorname{tr}\Big[\Phi_{A\to B_k}(\varrho_A^{1/2}D\varrho_A^{1/2})E\Big]. \qquad (8.8)
\end{aligned}$$

This implies $\Phi_{A\to B_k}(\varrho) = \operatorname{tr}_{I\setminus I_k}[\Phi(\varrho)]$ for all states ϱ on $\mathcal{H}_A$ and $k=1,\ldots,n$. The proof of the converse statement follows from the proof of item (i).

The item (iii) follows from the observation that the Choi-Jamiołkowski isomorphism $J_{|\Omega_A\rangle}$ is an affine bijection between the set of channels $\Phi_{A\to B}$ and the set of states ϱ_{AB} such that $\mathrm{tr}_B[\varrho_{AB}] = \varrho_A$ is fixed and of full rank. Thus, all the convex structures of these sets are mapped in a one-to-one fashion and, particularly, the two robustness measures coincide. □

8.3 Quantum memories

A natural way of quantifying the strength of quantum memories is by asking how close the corresponding quantum channel is to a measure-and-prepare channel. The generalized robustness with respect to measure-and-prepare channels gives a reasonable measure of such distance, as it has the basic properties one expects from a resource quantifier, i.e., faithfulness, monotonicity, convexity, and stability under tensor products [228]. The quantifier has also an operational meaning as the amount of advantage a quantum memory can give over measure-and-prepare channels in correlation tasks [224, 228].

The quantifier is the robustness of quantum memories (RoQM), that we have discussed already in the previous section to some extend. Although having many desired properties, RoQM has one drawback: general methods for its efficient evaluation remain unknown. We note that approximate methods were developed in Ref. [228], e.g., by relaxing the separability condition that defines the free cone in the Choi picture by the PPT condition, which leads to lower bounds on the actual RoQM.

Here, we provide a strategy for the efficient evaluation of the RoQM. Measure-and-prepare channels are closely related to separable states through the Choi-Jamiołkowski isomorphism, as we have discussed already earlier. As the set of separable states can be characterized with a converging hierarchy of SDPs [229, 230], Theorem 38 can be used to develop a hierarchy of SDPs converging to the RoQM. On the round n of the hierarchy, one calculates the robustness with respect to n-self-compatible channels. As n-self-compatible channels form a superset of $n+1$-self-compatible channels, every round of the hierarchy gives a lower bound on the next one. The hierarchy converges to the RoQM, as infinitely many times self-compatible channels coincide with measure-and-prepare channels [92].

Observation 39. *The robustness of quantum memories can be evaluated with a converging hierarchy of SDPs.*

Proof. The proof can be found in Ref. [231]. □

We note that every level of the hierarchy can be evaluated from input-output correlations [224, 228]. Indeed, similar techniques have been used for the experimental evaluation of the robustness of quantum steering and coherence [202, 203], and the RoQM [228].

8.4 From states to channels

In this section, we will translate some known results on the marginal problem to the compatibility of channels.

8.4.1 Entropic criteria

For states of low dimension the quantum marginal problem, and hence, compatibility of the corresponding channels, can be tackled with SDPs [100, 101]. However, for higher-dimensional cases this approach becomes computationally demanding. Nevertheless, one can give some approximate solutions by means of entropic inequalities. The von Neumann entropy is known to satisfy certain linear inequalities, namely, strong subadditivity [232] and weak monotonicity [233]. In Ref. [234] it was argued for two states ϱ_{AB_1}, and ϱ_{AB_2}, that if the common state $\varrho_{AB_1B_2}$ exists, then

$$S(AB_1) + S(AB_2) - S(B_1) - S(B_2) \geq 0, \tag{8.9}$$

which is known as weak monotonicity. Now we demonstrate the use of Obs. 38 by translating the basic entropic results into witnesses of channel incompatibility. To that end, define the $|\Omega_A\rangle$-entropy of a channel $\Phi_{A\to B}$ as the von Neumann entropy of its $|\Omega_A\rangle$-Choi state

$$S_{|\Omega_A\rangle}(\Phi_{A\to B}) = -\operatorname{tr}\Big[J_{|\Omega_A\rangle}(\Phi_{A\to B}) \log J_{|\Omega_A\rangle}(\Phi_{A\to B})\Big]. \tag{8.10}$$

Then we can make the following observation.

Observation 40. *For two channels $\Phi_{A\to B_1}$ and $\Phi_{A\to B_2}$ the condition in Eq.* (8.9) *takes the form*

$$S_{|\Omega_A\rangle}(\Phi_{A\to B_1}) + S_{|\Omega_A\rangle}(\Phi_{A\to B_2}) - S(\varrho_{B_1}) - S(\varrho_{B_2}) \geq 0. \tag{8.11}$$

In Fig. 8.2 we show the boundaries of the areas of compatibility of two depolarizing channels $\Phi_{A\to B_1}$ and $\Phi_{A\to B_2}$ defined as

$$\Phi_{A\to B_1}(\varrho) = (1-\mu)W(q,p)\varrho W(q,p)^\dagger + \mu\frac{1}{d}\mathbb{1}_d, \tag{8.12}$$

and

$$\Phi_{A\to B_2}(\varrho) = (1-\nu)W(r,s)\varrho W(r,s)^\dagger + \nu\frac{1}{d}\mathbb{1}_d, \tag{8.13}$$

with $W(q,p)\,|j\rangle = e^{i\frac{\pi}{d}(q+2j)p}\,|j+q\rangle$, for $q,\ p,\ r,\ s \in \mathbb{Z}_d$ for dimensions $d=2$ and $d=16$ and compare those with analytical results that can be found in Ref. [235] (see also

Eq. (8.17)). We note that for the above case we have chosen the state $|\Omega_A\rangle$ to be the maximally entangled state, which is optimal. Furthermore, we note that the entropic criterion in Eq. (8.11) is not the only one that can be translated and that the criteria may depend on the choice of Choi isomorphism.

It is clear that the aforementioned entropic constraints can also be applied to the problem of symmetric extendibility. Moreover, the symmetric extendibility of a bipartite qubit state has been fully resolved in Ref. [236] and this result readily characterizes self-compatibility and, hence, the antidegradability of any qubit-to-qubit channel.

8.4.2 Self-compatibility of channels

In this section we will discuss constraints on self-compatibility of channels, which is equivalent to the symmetric extendibility of the corresponding Choi state. Certain spectral constrains for symmetric extendibility of two-qubit states are known [236], which can be translated to the problem of self-compatibility of channels.

In order to identify the spectrum of the Choi state of a channel, recall that any channel can be written in the Kraus decomposition, i.e., $\Phi(\varrho) = \sum_i K_i \varrho K_i^\dagger$. For any state ϱ_A on $\mathcal{H}_A$, the Kraus operators K_i can be chosen such that $\mathrm{tr}\left[\varrho_A K_i^\dagger K_j\right] = 0$, whenever $i \neq j$; see Section 3.1 of Ref. [227]. In this case, we say that K_i are ϱ_A-orthogonal. Whenever ϱ_A is of full rank and $|\Omega_A\rangle$ is a canonical purification for ϱ_A, we have the spectral decomposition $J_{|\Omega_A\rangle}(\Phi) = \sum_i |w_i\rangle\langle w_i|$ where $|w_i\rangle = (\mathbb{1}_{\mathcal{H}} \otimes K_i)\,|\Omega_A\rangle$ for any ϱ_A-orthogonal set $\{K_i\}_i$ of Kraus operators for Φ [227, Proposition 1]. Thus, the spectrum of $J_{|\Omega_A\rangle}(\Phi)$ consists of the numbers

$$\lambda^{\Phi}_{\varrho_A}(i) := \mathrm{tr}\left[K_i \varrho_A K_i^\dagger\right], \tag{8.14}$$

and the vector $\boldsymbol{\lambda}^{\Phi}_{\varrho_A} := \left(\lambda^{\Phi}_{\varrho_A}(i)\right)_i$ is essentially independent of the particular ϱ_A-orthogonal set of Kraus operators for Φ.

Let us consider the case when A, B, and C are qubit systems. In Ref. [236] it was shown that a state ϱ_{AB} is symmetrically extendible, i.e., there is a three-qubit state ϱ_{ABC} such that $\varrho_{AB} = \varrho_{AC}$ if and only if $\mathrm{tr}\left[\mathrm{tr}_A[\varrho_{AB}]^2\right] \geq \mathrm{tr}\left[\varrho_{AB}^2\right] - 4\sqrt{\det(\varrho_{AB})}$. Let

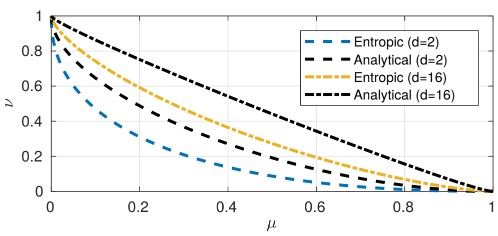


Figure 8.2: Areas of compatibility of two depolarizing channels with parameters μ and ν for two cases: $d = 2$ and $d = 16$.

$\Phi_{A\to B}$ be the Choi-channel of ϱ_{AB}, i.e., $\varrho_{AB} = J_{|\Omega_A\rangle}(\Phi_{A\to B})$ for a standard purification $|\Omega_A\rangle$ of $\mathrm{tr}_B[\varrho_{AB}]$. The right-hand side of the above inequality can be written entirely in terms of the spectrum of ϱ_{AB} and, thus, in terms of the probability vector $\boldsymbol{\lambda}_{\varrho_A}^{\Phi_{A\to B}}$. Moreover, $\mathrm{tr}_A[\varrho] = \Phi_{A\to B}(\varrho_A)$. Thus, we have the following observation.

Observation 41. *A qubit-to-qubit channel Φ is self-compatible if, for some, and hence for any, full-rank qubit state ϱ_A,*

$$\mathrm{tr}\Big[\Phi(\varrho_A)^2\Big] \geq \sum_i \lambda_{\varrho_A}^{\Phi}(i)^2 - 4\prod_i \sqrt{\lambda_{\varrho_A}^{\Phi}(i)}. \tag{8.15}$$

In particular, choosing $\varrho_A = \frac{1}{2}\mathbb{1}$ and a Hilbert-Schmidt-orthogonal set $\{K_i\}_{i=1}^R$ of Kraus operators for Φ, and $R \leq 4$ being the Kraus rank of Φ, the channel Φ is self-compatible if and only if

$$\mathrm{tr}\Big[\Phi(\mathbb{1})^2\Big] \geq \sum_i \|K_i\|_{HS}^4 - \frac{16}{2^{R/2}}\prod_i \|K_i\|_{HS} \tag{8.16}$$

where, for any qubit operator K, the Hilbert-Schmidt norm is defined by $\|K\|_{HS} = \sqrt{\mathrm{tr}[K^\dagger K]}$.

8.5 From channels to states

In this section we will translate some known results on channel compatibility to the marginal problem.

8.5.1 Depolarising noise

Consider two depolarizing channels $\Phi_{A\to B_1}$ and $\Phi_{A\to B_2}$ as defined earlier. The compatibility of these channels was completely characterized in Ref. [235]. Namely for $\mu, \nu \in [0,1]$, the channels $\Phi_{A\to B_1}$ and $\Phi_{A\to B_2}$ are compatible if and only if

$$\mu + \frac{2}{d}\sqrt{\mu\nu} + \nu \geq 1. \tag{8.17}$$

Using Obs. 38 we obtain the following result.

Observation 42. *Consider two pure states $|\varphi_{AB_1}\rangle$ and $|\varphi_{AB_2}\rangle$, such that the common marginal ϱ_A is of full rank. For $\mu, \nu \in [0,1]$, there is a tripartite state $\varrho_{AB_1B_2}$ such that $\varrho_{AB_1} = (1-\mu)\,|\varphi_{AB_1}\rangle\!\langle\varphi_{AB_1}| + \mu\frac{1}{d}\varrho_A\otimes\mathbb{1}_d$ and $\varrho_{AB_2} = (1-\nu)\,|\varphi_{AB_2}\rangle\!\langle\varphi_{AB_2}| + \nu\frac{1}{d}\varrho_A\otimes\mathbb{1}_d$ if and only if the inequality in Eq.* (8.17) *holds.*

Proof. Suppose that $|\Omega_A\rangle$ is the standard purification of ϱ_A. It follows that there are unitaries U_{B_1} and U_{B_2} such that $|\varphi_{AB_1}\rangle = (\mathbb{1}_d\otimes U_{B_1})\,|\Omega_A\rangle$ and $|\varphi_{AB_2}\rangle = (\mathbb{1}_d\otimes U_{B_2})\,|\Omega_A\rangle$, since all purifications are equivalent up to unitaries on the purifying system (cf. Sec. 1.2). Clearly, the states ϱ_{AB_1}, and ϱ_{AB_2} are marginals of a tripartite state $\varrho_{AB_1B_2}$ if and only if there is a tripartite state $\tilde{\varrho}_{AB_1B_2}$ such that $\tilde{\varrho}_{AB_1} = \varrho_\mu$ and $\tilde{\varrho}_{AB_2} = \varrho_\nu$, where $\varrho_\lambda = (1-\lambda)\,|\Omega_A\rangle\!\langle\Omega_A| + \lambda\frac{1}{d}\varrho_A \otimes \mathbb{1}_d$ for all $\lambda \in [0,1]$. Using the channel state dualism $J_{|\Omega_A\rangle}$, this problem is equivalent to finding those $\mu, \nu \in [0,1]$ such that the channels $\varrho \mapsto (1-\mu)\varrho + \mu\frac{1}{d}\mathbb{1}_d$ and $\varrho \mapsto (1-\nu)\varrho + \nu\frac{1}{d}\mathbb{1}_d$ are compatible. This happens, according to the above, if and only if the inequality (8.17) holds. □

8.5.2 Pairs of Bell-diagonal states

Another result is concerned with Pauli channels, which are defined by $\Phi_{\boldsymbol{p}}(\varrho) = p_0\varrho + p_x\sigma_x\varrho\sigma_x^\dagger + p_y\sigma_y\varrho\sigma_y^\dagger + p_z\sigma_z\varrho\sigma_z^\dagger$, where $\boldsymbol{p} = (p_0, p_x, p_y, p_z)$ is a vector of probabilities. According to Ref. [235], two Pauli channels with probability vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ are compatible if and only if there are $\lambda,\ \mu,\ \nu \in [0,1]$ such that $M_{\boldsymbol{p},\boldsymbol{q}}(\lambda,\mu,\nu) \geq 0$, where

$$M_{\boldsymbol{p},\boldsymbol{q}}(\lambda,\mu,\nu) = \begin{pmatrix} p_0 & \lambda & \mu & \langle\boldsymbol{q}\rangle_1 - \nu \\ \cdot & p_x & \nu & \langle\boldsymbol{q}\rangle_2 - \mu \\ \cdot & \cdot & p_y & \langle\boldsymbol{q}\rangle_3 - \lambda \\ \cdot & \cdot & \cdot & p_z \end{pmatrix}, \tag{8.18}$$

$\langle\boldsymbol{q}\rangle_1 = \frac{1}{2}(q_0 - q_x - q_y + q_z)$, $\langle\boldsymbol{q}\rangle_2 = \frac{1}{2}(q_0 - q_x + q_y - q_z)$, and $\langle\boldsymbol{q}\rangle_3 = \frac{1}{2}(q_0 + q_x - q_y - q_z)$.

The Choi states of Pauli channels are so-called *Bell-diagonal* states. A two-qubit state ϱ_{AB} is called Bell-diagonal if there exists a vector $\boldsymbol{p}$ of probabilities such that $\varrho_{AB} = \varrho_{\boldsymbol{p}} := p_0\,|\Omega_0\rangle\!\langle\Omega_0| + p_x\,|\Omega_x\rangle\!\langle\Omega_x| + p_y\,|\Omega_y\rangle\!\langle\Omega_y| + p_z\,|\Omega_z\rangle\!\langle\Omega_z|$, where $|\Omega_0\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $|\Omega_r\rangle := (\mathbb{1}_2 \otimes \sigma_r)\,|\Omega_0\rangle$ for $r = x,\ y,\ z$. The following observation follows from Obs. 38 and the above result on compatibility of Pauli channels. It provides a complete solution of the marginal problem of pairs of two-qubit Bell-diagonal states.

Observation 43. *For probability vectors $\boldsymbol{p}$ and $\boldsymbol{q}$, there is a three-qubit state $\varrho_{AB_1B_2}$ such that $\varrho_{AB_1} = \varrho_{\boldsymbol{p}}$ and $\varrho_{AB_2} = \varrho_{\boldsymbol{q}}$ if and only if there are $\lambda,\ \mu,\ \nu \in [-1,1]$ such that $M_{\boldsymbol{p},\boldsymbol{q}}(\lambda,\mu,\nu) \geq 0$.*

8.6 Conclusion

In this chapter, we have discussed a one-to-many correspondence between channel compatibility and certain instances of the quantum marginal problem. To be more precise, we have shown that a set of channels is compatible if and only if there exists a solution to the marginal problem of the corresponding Choi states. Subsequently, we have shown that for a set of bipartite states, which share a common marginal, the marginal problem has a solution if and only if their corresponding Choi channels are compatible. Moreover, both properties can be quantified by a robustness measure, and we have shown that their values coincide. For the case of self-compatible channels, the corresponding marginal problem is that of symmetric extendibility. Since the existence of symmetric extensions can be verified by SDPs, which in the limit proves separability, one can use this to find a converging hierarchy of SDPs that converges to the RoQM. This is a natural quantifier for the strength of a quantum memory.

The connection allowed us to translate various criteria from channel compatibility to the marginal problem, and vice versa. In particular, we have solved the problem of self-compatibility for qubit-to-qubit channels, and the marginal problem for pairs of two-qubit Bell-diagonal states. Together with the symmetric extendibility result of Ref. [236], Observation 43 can be taken as a first step towards characterizing all those pairs of two-qubit states which are marginals of a three-qubit state.

Some open questions remain for future research, e.g., it would be interesting to generalize the results to the case of Gaussian states and channels. Moreover, one could ask if other instances of the quantum marginal problem can be interpreted as more general types of compatibility problems of channels.

To conclude our discussion, we wish to emphasize that the connection discussed here was also discussed independently in Ref. [237], and in Ref. [238] under a different name.

Summary and outlook

This thesis was concerned with different aspects of quantum resources and, in particular, their quantification.

To summarize, we have first discussed the interplay between coherence and entanglement in the framework of the resource theory of coherence. More precisely, we have introduced a quantifier of genuine correlated coherence for multipartite systems. It is based on the combination of a quantifier of correlated coherence, i.e., the difference between global and local coherences, together with a minimization over all possible global incoherent unitaries. We have derived analytic expressions for this quantifier for two qubit pure states and have found a connection to genuine multilevel entanglement. Moreover, these results might contribute to an ongoing discussion on the possible free operations in a resource theory of multipartite coherence.

Then, we have addressed the question of how the positivity of the quantum state limits the amount of coherence that can be shared between multiple orthogonal subspaces. This led to monogamy relations that capture the trade-off in coherence between one and all other subspaces. We have discussed how such a trade-off puts limits on the distinguishability of quantum states under unitary evolution, when measurements are restricted to act only on subspaces. As an application, we have discussed the possibility to witness the block-coherence number of a quantum state. It would be interesting to see if such relations could find applications in, e.g., multi-parameter estimation under restricted measurements, which would be an interesting question for future research.

Another large part of this thesis was concerned with the characterization of quantum networks. We have shown how the structure of the network limits the distribution of entanglement, with particular emphasis on the triangle network. In the case of uncorrelated sources, we have shown that the set of compatible states is non convex. We have derived necessary criteria a state must fulfill to be compatible with the production in the triangle network, based on the statistical independence of the sources, the monogamy of entanglement, and certain constraints on the local ranks. Moreover, we were able to numerically construct witnesses that detect states that are not preparable in the triangle network, not even with shared randomness. These results show that the network structure imposes strong and non-trivial constraints on the states that can be prepared in a quantum network. Furthermore, this can be seen as a first step toward a theory of network entanglement. An important direction for future research would be to clarify whether there exists a state that is $A|B|C$ separable, but requires entangled sources when produced in the triangle network. The existence of such a state would show, that the entanglement cost of preparing states in a network is different from the cost of preparing states with a single source. Moreover, a proof of the non-increasingness of the TMI in the triangle network under local channels is still lacking. Another promising direction for future research could be to study the transformation of network states, e.g., by means of (finite round) LOCC transformations. This might lead to a resource theory of network entanglement.

Subsequently, we have considered a different approach based on the coherence properties of covariance matrices that arise from performing measurements on a network state. This was motivated by the work in Ref. [28], where it was shown that the topology of the network leads to a certain block structure of the covariance matrix. We have shown that the theory of coherence can be utilized to analyze this block structure, which allowed us to derive conditions that witness probability distributions that are incompatible with the structure of the network. Moreover, our results were applicable in scenarios where numerical approaches are infeasible due to the rapidly growing number of free parameters. For future projects, it would be desirable to extend the results to the usual definition of covariance matrix in quantum mechanics, or to study the coherence in networks on the level of quantum states. This may shed light on the question which types of network correlations are useful for applications in quantum information processing.

Motivated by the task-oriented characterization of quantum steering that was put forward by Piani and Watrous, we have shown that a similar result holds in the case of incompatible measurements. To be more precise, we have shown that for any set of incompatible measurements there exists an instance of state discrimination with prior information in which this set strictly outperforms any compatible set of measurements. Moreover, the outperformance can be quantified by the incompatibility robustness. Based on the duality in conic programming we have furthermore shown that a similar characterization is possible, whenever the set of free measurements is convex and compact, and that such structural results also extend to state assemblages. In addition we have shown that another quantifier, i.e., the convex weight, admits a similar operational interpretation in terms of performance in exclusion tasks.

Motivated by the resource theory of quantum memories that was developed by Rosset et al. in Ref. [207], we have shown that in other convex resource theories of channels, instruments and collections thereof, similar results can be proven. We emphasize that, although our scenario is not measurement-device-independent, it can be made measurement-device-independent by using a similar approach as in Ref. [207]. Moreover, we have discussed the similarity to other well known resource quantifiers, namely the max-relative entropy. A promising way for future research on robustnesses certainly is the generalization to quantum resources in inifinite-dimensional systems, where first steps have been taken in Ref. [239].

Finally, we have discussed a connection between the compatibility of channels and instances of the quantum marginal problem, which allowed us to translate many structural results between the two fields. For instance, we discussed an SDP hierarchy to compute the robustness of quantum memories and solved the quantum marginal problem for pairs of states under depolarizing noise, and pairs of two-qubit Bell-diagonal states. Moreover, the connection allowed us to solve the self-compatibility of qubit-to-qubit channels and to derive new conditions for channel incompatibility based on entropic criteria. An interesting open question is to what extend other instances of the quantum marginal problem can be interpreted as more general types of compatibility problems of channels.

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List of publications

[A, Ch. 5] Tristan Kraft, Cornelia Spee, Xiao-Dong Yu, Otfried Gühne
Characterizing Quantum Networks: Insights from Coherence Theory
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[B, Ch. 4] Tristan Kraft, Sébastien Designolle, Christina Ritz, Nicolas Brunner, Otfried Gühne, Marcus Huber
Quantum entanglement in the triangle network
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[C, Ch. 3] Tristan Kraft, Marco Piani
Monogamy relations of quantum coherence between multiple subspaces
arXiv:1911.10026

[D, Ch. 6] Roope Uola, Tom Bullock, Tristan Kraft, Juha-Pekka Pellonpää, Nicolas Brunner
All quantum resources provide an advantage in exclusion tasks
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[E, Ch. 8] Erkka Haapasalo, Tristan Kraft, Nikolai Miklin, Roope Uola
Quantum marginal problem and incompatibility
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[F, Ch. 7] Roope Uola, Tristan Kraft, Alastair A. Abbott
Quantification of quantum dynamics with input-output games
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[G, Ch. 6] Roope Uola, Tristan Kraft, Jiangwei Shang, Xiao-Dong Yu, Otfried Gühne
Quantifying quantum resources with conic programming
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[H, Ch. 2] Tristan Kraft, Marco Piani
Genuine Distributed Coherence
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[I] Tristan Kraft, Christina Ritz, Nicolas Brunner, Marcus Huber, Otfried Gühne
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