Constructive Category Theory and Tilting Equivalences via Strong Exceptional Sequences

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Summary

In this thesis we establish a constructive framework for homological algebra with a special focus on (complete) strong exceptional sequences in bounded homotopy categories and their induced exact equivalences, alongside with a CAP [GSP22] and `homalg` [hom22] based implementation of this framework in the computer algebra system `GAP` [GAP21].

First, we assemble the key concepts in homological algebra in a constructive style that is suitable for a direct computer implementation. This includes constructing bounded complexes, homotopy and derived categories in which we can perform computations like projective and injective resolutions of bounded complexes and derived functors. Then, we set the stage for performing computations in triangulated categories. This is accomplished by stating all the existential quantifiers and disjunctions in the defining axioms of a triangulated category as concrete algorithms. The two primary examples of a triangulated category in this thesis are the stable category of a Frobenius category and the bounded homotopy category of an additive category.

Given a field k, a k-linear Hom-finite additive category $\mathscr{C}$ and a strong exceptional sequence $\mathscr{E}$ in the bounded homotopy category $\mathcal{K}^b(\mathscr{C})$, we develop an algorithm to check the membership of objects in the triangulated hull $\mathscr{E}^{\triangle} \subseteq \mathcal{K}^b(\mathscr{C})$. In particular, if $\mathcal{K}^b(\mathscr{C})$ is finitely generated as a triangulated category, one can employ that to algorithmically decide the completeness of the strong exceptional sequence $\mathscr{E}$, i.e., decide whether $\mathscr{E}^{\triangle} = \mathcal{K}^b(\mathscr{C})$. For a complete strong exceptional sequence $\mathscr{E}$, we use the so-called Postnikov systems to provide an explicit construction of exact equivalences

$$\mathcal{D}^b(\operatorname{End} T_{\mathscr{E}}) \simeq \mathcal{K}^b(\mathscr{E}^{\oplus}) \simeq \mathcal{K}^b(\mathscr{C})$$

where $T_{\mathscr{E}} := \bigoplus_{E \in \mathscr{E}} E$, $\mathcal{D}^b(\operatorname{End} T_{\mathscr{E}})$ denotes the bounded derived category of the category $\operatorname{End} T_{\mathscr{E}}$-**mod** of finitely generated $\operatorname{End} T_{\mathscr{E}}$-modules, and $\mathscr{E}^{\oplus}$ is the universal additive closure category of $\mathscr{E}$.

These techniques enable us to make the following special case of Happel's theorem for derived equivalences constructive: Let $\mathbb{A}$ be a finite dimensional k-algebra and T a tilting $\mathbb{A}$-module whose indecomposable summands form a complete strong exceptional sequence in $\mathbb{A}$-**mod**. Then we can compute the induced adjoint derived equivalences

$$- \otimes^{\mathbb{L}} T \colon \mathcal{D}^b(\operatorname{End} T) \rightleftarrows \mathcal{D}^b(\mathbb{A}) : \mathbb{R}\operatorname{Hom}(T, -).$$

The categorical framework along with all algorithms presented in this thesis are implemented in the `GAP` meta-package `HigherHomologicalAlgebra` [Sal21a].

Zusammenfassung

In dieser Arbeit wird ein konstruktiver Zugang für homologische Algebra mit einem Fokus auf (vollständige) stark-exzeptionelle Sequenzen in beschränkten Homotopiekategorien und ihre induzierten exakten Äquivalenzen entwickelt. Dieser Zugang wurde basierend auf CAP [GSP22] und homalg [hom22] im Computeralgebrasystem GAP [GAP21] implementiert.

Zuerst werden die zentralen klassischen Konzepte der homologischen Algebra in einem konstruktiven Rahmen entwickelt, der sich für eine direkte Computerimplementierung eignet. Diese beinhalten die Konstruktion von beschränkten Komplexen, Homotopie- und derivierten Kategorien, in denen zum Beispiel Berechnungen von projektiven und injektiven Auflösungen von beschränkten Komplexen, derivierten Funktoren durchgeführt werden können. Danach wird die Grundlagen für die Durchführung von Berechnungen in triangulierten Kategorien entwickelt. Dies geschieht, indem alle Existenzquantoren und Disjunktionen in den Definitionsaxiomen einer triangulierten Kategorie als konkrete Algorithmen spezifiziert werden. Damit sind wir in der Lage, die stabile Kategorie einer Frobenius-Kategorie und die beschränkte Homotopiekategorie einer additiven Kategorie auf dem Computer zu konstruieren.

Gegeben sei ein Körper k, eine k-lineare Hom-endliche additive Kategorie $\mathscr{C}$ und eine stark-exzeptionelle Sequenz $\mathscr{E}$ in der beschränkten Homotopiekategorie $\mathcal{K}^b(\mathscr{C})$. In der Arbeit wird ein Algorithmus entwickelt, um die Mitgliedschaft von Objekten in der triangulierten Hülle $\mathscr{E}^\triangle \subseteq \mathcal{K}^b(\mathscr{C})$ zu entscheiden. Und falls $\mathcal{K}^b(\mathscr{C})$ als triangulierte Kategorie endlich erzeugt ist, kann man insbesondere damit die Vollständigkeit der stark-exzeptionellen Sequenz $\mathscr{E}$ algorithmisch entscheiden, d.h. entscheiden, ob $\mathscr{E}^\triangle = \mathcal{K}^b(\mathscr{C})$. Für eine vollständige stark-exzeptionelle Sequenz $\mathscr{E}$ benutzen wir sogenannte Postnikov-Systeme, um folgende exakte Äquivalenzen auf dem Computer explizit zu realisieren

$$\mathcal{D}^b(\operatorname{End} T_{\mathscr{E}}) \simeq \mathcal{K}^b(\mathscr{E}^\oplus) \simeq \mathcal{K}^b(\mathscr{C}),$$

wobei $T_{\mathscr{E}} := \bigoplus_{E\in\mathscr{E}} E$, $\mathcal{D}^b(\operatorname{End} T_{\mathscr{E}})$ die beschränkte derivierte Kategorie der Kategorie $\operatorname{End} T_{\mathscr{E}}$-**mod** der endlich erzeugten $\operatorname{End} T_{\mathscr{E}}$-Moduln und $\mathscr{E}^\oplus$ ist die universelle additive Abschlusskategorie von $\mathscr{E}$ bezeichnen.

Diese Methoden ermöglichen den folgenden Spezialfall des Happel'schen Satzes für derivierte Äquivalenzen konstruktiv zu machen: Sei $\mathbb{A}$ eine endlichdimensionale k-Algebra und T ein Tilting $\mathbb{A}$-Modul, dessen unzerlegbare Summanden eine vollständige stark-exzeptionelle Sequenz in $\mathbb{A}$-**mod** bilden. Dann können die in dieser Arbeit entwickelten Software die induzierten adjungierten derivierten Äquivalenzen explizit ausrechnen:

$$- \otimes^{\mathbb{L}} T \colon \mathcal{D}^b(\operatorname{End} T) \rightleftarrows \mathcal{D}^b(\mathbb{A}) : \mathbb{R}\operatorname{Hom}(T, -).$$

Der konstruktiv-kategorielle Rahmen sowie alle in dieser Arbeit vorgestellten Algorithmen wurden im GAP-Metapaket HigherHomologicalAlgebra [Sal21a] implementiert.

Contents

Introduction and Scope

The Main Goal. Let k be a field, $\mathscr{C}$ a k-linear Hom-finite additive category, $\mathcal{K}^b(\mathscr{C})$ the bounded homotopy category of $\mathscr{C}$ and $\mathscr{E} \subset \mathcal{K}^b(\mathscr{C})$ a finite full subcategory consisting of the n objects $E_1, \dots, E_n$. Denote by $\mathscr{E}^{\oplus}$ the additive closure of $\mathscr{E}$ and by $\mathcal{D}^b(\operatorname{End} T_{\mathscr{E}})$ the bounded derived category of k-finite dimensional left $\operatorname{End} T_{\mathscr{E}}$-modules[1] where $T_{\mathscr{E}} := \oplus_1^n E_i$.

The main goal of this thesis is to design and implement an algorithmic framework to tackle the following two questions:

(1) Can we check whether $\mathscr{E}$ defines a complete[2] strong exceptional sequence in $\mathcal{K}^b(\mathscr{C})$? And if so,

(2) Can we construct exact equivalences

$$\mathcal{D}^b(\operatorname{End} T_{\mathscr{E}}) \simeq \mathcal{K}^b\left(\mathscr{E}^{\oplus}\right) \simeq \mathcal{K}^b(\mathscr{C})?$$

We will, among other things, prove:

Theorem 1 (Lemmas 6.77 and 6.79 and Corollary 6.81)**.** *Let k be a field, $\mathscr{C}$ a k-linear* Hom*-finite additive category and $\mathscr{E}$ a strong exceptional sequence in $\mathcal{K}^b(\mathscr{C})$. Then*

(1) $\mathscr{E}$ induces a fully faithful exact functor

$$F\colon \mathcal{K}^b\left(\mathscr{E}^{\oplus}\right) \to \mathcal{K}^b(\mathscr{C})$$

whose essential image is the triangulated hull $\mathscr{E}^{\triangle}$ of $\mathscr{E}$. Moreover, we can algorithmically decide whether an object A in $\mathcal{K}^b(\mathscr{C})$ belongs to $\mathscr{E}^{\triangle}$ or not. In particular, if $\mathcal{K}^b(\mathscr{C})$ is finitely generated as a triangulated category, then, using an explicit set of generating objects, we can decide the completeness of the strong exceptional sequence $\mathscr{E}$.

(2) If $\mathscr{E}$ is complete, then F has a right adjoint functor G giving rise to quasi-inverse adjoint exact equivalences

$$F : \mathcal{K}^b\left(\mathscr{E}^{\oplus}\right) \rightleftarrows \mathcal{K}^b(\mathscr{C}) : G.$$

Motivation. Although this thesis does not require any knowledge in algebraic geometry, it was sparked by a question that arose in algebraic geometry. The story started by the

[1] The multiplication in $\operatorname{End} T_{\mathscr{E}}$ is defined by the pre-composition "$\cdot$" of morphisms, i.e., $fg := f \cdot g := g \circ f$.

[2] We will use the notion "complete" for what is sometimes called "full". We do this to avoid confusion with the notion of a full subcategory.

celebrated "resolution of the diagonal" theorem of Beilinson (cf. [Beĭ78]) which states that the full subcategories Ω and $\mathcal{O}$ generated by the sequences

$$\left(\Omega^n_{\mathbb{P}^n_k}(n), \dots, \Omega^0_{\mathbb{P}^n_k}(0)\right)$$

and

$$\left(\mathcal{O}_{\mathbb{P}^n_k}, \dots, \mathcal{O}_{\mathbb{P}^n_k}(n)\right),$$

respectively, are complete strong exceptional sequences in the category $\mathfrak{Coh}\ \mathbb{P}^n_k$ of coherent sheaves over the projective space $\mathbb{P}^n_k$, where $\Omega^i_{\mathbb{P}^n_k}(i), i = 0, \dots, n$ are the twisted cotangent bundles over $\mathbb{P}^n_k$ and $\mathcal{O}_{\mathbb{P}^n_k}(i), i = 0, \dots, n$ are twists of structure sheaf $\mathcal{O}_{\mathbb{P}^n_k}$. That is, the associated tilting sheaves $T_\Omega \coloneqq \bigoplus_{i=0}^n \Omega^i_{\mathbb{P}^n_k}(i)$ and $T_\mathcal{O} \coloneqq \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n_k}(i)$ induce exact equivalences

$$\mathcal{D}^b(\operatorname{End} T_\Omega) \simeq \mathcal{K}^b\left(\Omega^\oplus\right) \simeq \mathcal{D}^b(\mathbb{P}^n_k) \simeq \mathcal{K}^b\left(\mathcal{O}^\oplus\right) \simeq \mathcal{D}^b(\operatorname{End} T_\mathcal{O}),$$

where

- $\mathcal{D}^b(\mathbb{P}^n_k)$ is the bounded derived category of $\mathfrak{Coh}\ \mathbb{P}^n_k$;
- $\Omega^\oplus$ and $\mathcal{O}^\oplus$ and are the additive closure categories of the full subcategories of $\mathfrak{Coh}\ \mathbb{P}^n_k$ generated by Ω resp. $\mathcal{O}$;
- $\mathcal{K}^b(\Omega^\oplus)$ and $\mathcal{K}^b(\mathcal{O}^\oplus)$ are the bounded homotopy categories of $\Omega^\oplus$ resp. $\mathcal{O}^\oplus$;
- $\mathcal{D}^b(\operatorname{End} T_\Omega)$ and $\mathcal{D}^b(\operatorname{End} T_\mathcal{O})$ are the bounded derived categories of the categories of the finite dimensional left modules over the endomorphism k-algebras $\operatorname{End} T_\Omega$ resp. $\operatorname{End} T_\mathcal{O}$.

In particular, any bounded complex over $\mathfrak{Coh}\ \mathbb{P}^n_k$ can be resolved (up to a quasi-isomorphism[3]) in terms of a complex which consists only of direct sums of objects in $\mathcal{O}$; and also can be resolved (up to a quasi-isomorphism) in terms of a complex which consists only of direct sums of objects in Ω.

This raises the following question: Suppose we know only how to resolve the objects of Ω and the morphisms between them in terms of objects and morphisms in $\mathcal{K}^b(\mathcal{O}^\oplus)$, i.e., we are given the full embedding $\Omega \xhookrightarrow{\iota} \mathcal{K}^b(\mathcal{O}^\oplus)$ whose image is a complete strong exceptional sequence in $\mathcal{K}^b(\mathcal{O}^\oplus)$. Can we algorithmically extend it to exact equivalences

$$\mathcal{D}^b(\operatorname{End} T_\Omega) \simeq \mathcal{K}^b\left(\Omega^\oplus\right) \simeq \mathcal{K}^b\left(\mathcal{O}^\oplus\right)$$

without using geometric methods, i.e., without needing to pass through $\mathcal{D}^b(\mathbb{P}^n_k)$?

Theorem 1 answers this question affirmatively with $\Omega \cong \iota(\Omega)$ and $\iota(\Omega) \subseteq \mathcal{K}^b(\mathcal{O}^\oplus)$ is a complete strong exceptional sequence. (cf. Chapter 1).

Another application of Theorem 1 originates from the representation theory of finite dimensional k-algebras. Given a finite dimensional k-algebra $\mathbb{A}$ over some field k and a finite full subcategory $\mathscr{E} = (E_1, \dots, E_n)$ in $\mathcal{K}^b(\mathbb{A}\text{-}\mathbf{proj})$ where $\mathbb{A}\text{-}\mathbf{proj}$ is the category of finitely generated projective left $\mathbb{A}$-modules. If we have $\operatorname{Hom}_{\mathcal{K}^b(\mathbb{A}\text{-}\mathbf{proj})}(E_i, E_i) \simeq k$ for $i = 1, \dots, n$ and $\operatorname{Hom}_{\mathcal{K}^b(\mathbb{A}\text{-}\mathbf{proj})}(E_i, E_j) = 0$ for $j > i$, then checking whether $\mathscr{E}$ is a complete

[3] Cf. Definition 3.4

strong exceptional sequence in $\mathcal{K}^b(\mathbb{A}\text{-}\mathbf{proj})$ is equivalent to checking whether $T_{\mathscr{E}} := \oplus_1^n E_i$ is a **tilting complex** in $\mathcal{K}^b(\mathbb{A}\text{-}\mathbf{proj})$, i.e., whether

(1) $\mathrm{Hom}_{\mathcal{K}^b(\mathbb{A}\text{-}\mathbf{proj})}(T_{\mathscr{E}}, \Sigma^i(T_{\mathscr{E}})) = 0$ for all $i \neq 0$ where Σ is the shift automorphism on $\mathcal{K}^b(\mathbb{A}\text{-}\mathbf{proj})$,
(2) The smallest thick triangulated subcategory in $\mathcal{K}^b(\mathbb{A}\text{-}\mathbf{proj})$ that contains $T_{\mathscr{E}}$ is $\mathcal{K}^b(\mathbb{A}\text{-}\mathbf{proj})$ itself.

In the affirmative case this means that $\mathbb{A}$ and $\mathrm{End}\, T_{\mathscr{E}}$ are derived equivalent. This follows as a special case of the celebrated "Morita theorem for derived categories" of Rickard [**Ric89**],[**Kel07**],[**KZ98**], which states that if R and S are two rings, then the following conditions are equivalent:

(1) R and S are derived equivalent[4];
(2) $\mathcal{K}^b(R\text{-}\mathbf{proj}) \simeq \mathcal{K}^b(S\text{-}\mathbf{proj})$ as triangulated categories;
(3) There exists a tilting complex T in $\mathcal{K}^b(R\text{-}\mathbf{proj})$ such that $S \cong \mathrm{End}\, T$.

The Proof Strategy. Our proof of Theorem 1 mainly relies on constructing a fully faithful exact functor

$$F\colon \mathcal{K}^b\Big(\mathscr{E}^{\oplus}\Big) \to \mathcal{K}^b(\mathscr{C}),$$

and then proving that the essential image of F is the triangulated hull $\mathscr{E}^{\triangle} \subseteq \mathcal{K}^b(\mathscr{C})$. After this, it is evident how to construct the desired right adjoint functor

$$G\colon \mathcal{K}^b(\mathscr{C}) \to \mathcal{K}^b\Big(\mathscr{E}^{\oplus}\Big)$$

of F in case the strong exceptional sequence $\mathscr{E}$ is complete, i.e., in case $\mathscr{E}^{\triangle} = \mathcal{K}^b(\mathscr{C})$.

A Problem. The full embedding $\mathscr{E} \hookrightarrow \mathcal{K}^b(\mathscr{C})$ extends to a full embedding $\mathscr{E}^{\oplus} \hookrightarrow \mathcal{K}^b(\mathscr{C})$, which also extends to a full embedding $\mathcal{K}^b(\mathscr{E}^{\oplus}) \hookrightarrow \mathcal{K}^b\Big(\mathcal{K}^b(\mathscr{C})\Big)$. That is, F should map some of the objects of $\mathcal{K}^b\Big(\mathcal{K}^b(\mathscr{C})\Big)$ to objects in $\mathcal{K}^b(\mathscr{C})$. Hence, we need a construction similar to the classical *total complex* construction which maps the objects of $\mathcal{C}^b\Big(\mathcal{C}^b(\mathscr{C})\Big)$ to objects in $\mathcal{C}^b(\mathscr{C})$. The objects of $\mathcal{K}^b\Big(\mathcal{K}^b(\mathscr{C})\Big)$ can not always be considered as objects in $\mathcal{C}^b\Big(\mathcal{C}^b(\mathscr{C})\Big)$, hence the brute-force application of the total complex construction on the objects of $\mathcal{K}^b\Big(\mathcal{K}^b(\mathscr{C})\Big)$ might not produce well-defined outputs in $\mathcal{K}^b(\mathscr{C})$.

The Solution. Our construction of F is based on the notion of a **Postnikov system**, a construction which associates to a bounded complex U over a triangulated category $\mathfrak{T}$ a set of objects in $\mathfrak{T}$, usually called the set of "totalizations" of U. This construction is a priori not functorial for arbitrary triangulated categories (cf. Section 6.3). However, we will be able to circumvente this limitation in case $\mathfrak{T}$ is a bounded homotopy category. The proposed technique relies on computing **chain-homotopies** witnessing the equality of morphisms in $\mathcal{K}^b(\mathscr{C})$, i.e., for two equal morphisms $\alpha, \beta\colon A \to B$ in $\mathcal{K}^b(\mathscr{C})$, we must be able to compute a family $(h^i\colon A^i \to B^{i-1})_{i\in\mathbb{Z}}$ of morphisms with

$$\partial_A^i \cdot h^{i+1} + h^i \cdot \partial_B^{i-1} = \alpha^i - \beta^i$$

[4] I.e., their bounded derived categories of modules are equivalent as triangulated categories.

for all $i \in \mathbb{Z}$. Computing such a witness amounts to solving **a system of two-sided inhomogeneous linear equations**:

$$\partial_A^i \bullet \chi^{i+1} + \chi^i \bullet \partial_B^{i-1} = \alpha^i - \beta^i$$

of morphisms in $\mathscr{C}$ for unknown morphisms χ^i where i takes a finite number of values dependent on the lower and upper bounds of A and B (cf. Corollary 3.26). The method we will use to solve these systems will be explained later in this introduction (cf. Chapter 4).

The essential image of F. For each object A in $\mathcal{K}^b(\mathscr{C})$, we construct an object R in $\mathcal{K}^b(\mathscr{E}^{\oplus})$ and then prove that $A \cong F(R)$ if and only if $A \in \mathscr{E}^{\triangle}$ (cf. Lemma 6.77). The computation of R is based on an iterative construction which terminates because it relies on computing what we call **$\mathscr{E}$-covers** of objects in $\mathcal{K}^b(\mathscr{C})$ (cf. Definition 6.65). The functor

$$\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, -) \colon \mathcal{K}^b(\mathscr{C}) \to \mathrm{End}\, T\text{-}\mathbf{mod}$$

reduces the computation of $\mathscr{E}$-covers in $\mathcal{K}^b(\mathscr{C})$ to the computation **projective covers** in the Abelian category $\mathrm{End}\, T_{\mathscr{E}}$-**mod** (cf. Remark 6.71). The computation of the aforementioned isomorphism $A \cong F(R)$ is mainly based on applying the **octahedral axiom** (**TR′ 4.**) (cf. Section 5.2) in the triangulated category $\mathcal{K}^b(\mathscr{C})$.

Constructing $\mathcal{D}^b(\mathrm{End}\, T_{\mathscr{E}})$. In Lemma 6.33, we provide an algorithm to compute a finite acyclic quiver $\mathfrak{q}_{\mathscr{E}}$ and an admissible set of k-relations ρ such that

$$\mathscr{E} \cong \mathbf{A}_{\mathscr{E}}$$

where $\mathbf{A}_{\mathscr{E}}$ is the finitely presented k-linear category defined by $\mathfrak{q}_{\mathscr{E}}$ subject to the set of relations ρ. We call the category $\mathbf{A}_{\mathscr{E}}$ the *abstraction k-algebroid* of $\mathscr{E}$. In fact, $\mathrm{End}\, T_{\mathscr{E}}$ can be recovered as the endomorphism k-algebra of $\mathbf{A}_{\mathscr{E}}$ (cf. Definition A.29).

According to the theory of quiver representations, a k-finite dimensional right (resp. left) module over $\mathrm{End}\, T_{\mathscr{E}}$ is nothing but a k-linear functor from $\mathbf{A}_{\mathscr{E}}$ (resp. $\mathbf{A}_{\mathscr{E}}^{\mathrm{op}}$) to the category k-**mat** of matrices[5] over k (cf. Theorem 2.70). That is

$$\mathrm{End}\, T_{\mathscr{E}}\text{-}\mathbf{mod} \cong \mathbf{A}_{\mathscr{E}}\text{-}\mathbf{mod} \coloneqq [\mathbf{A}_{\mathscr{E}}^{\mathrm{op}}, k\text{-}\mathbf{mat}]$$

where $[\mathbf{A}_{\mathscr{E}}^{\mathrm{op}}, k\text{-}\mathbf{mat}]$ denotes the Abelian category of k-linear functors from $\mathbf{A}_{\mathscr{E}}^{\mathrm{op}}$ to k-**mat**. We denote by $\mathbf{A}_{\mathscr{E}}$-**proj** the full subcategory of $\mathbf{A}_{\mathscr{E}}$-**mod** generated by the projective objects.

Since $\mathfrak{q}$ is acyclic and $\mathbf{A}_{\mathscr{E}}$ is Hom-finite we can extend the Yoneda embedding

$$\mathbf{A}_{\mathscr{E}} \hookrightarrow \mathbf{A}_{\mathscr{E}}\text{-}\mathbf{mod}$$

to the following exact equivalences (cf. Section 6.2):

$$\mathcal{K}^b\left(\mathscr{E}^{\oplus}\right) \cong \mathcal{K}^b\left(\mathbf{A}_{\mathscr{E}}^{\oplus}\right) \simeq \mathcal{K}^b(\mathbf{A}_{\mathscr{E}}\text{-}\mathbf{proj}) \simeq \mathcal{D}^b(\mathbf{A}_{\mathscr{E}})$$

where $\mathcal{D}^b(\mathbf{A}_{\mathscr{E}})$ is the bounded derived category of $\mathbf{A}_{\mathscr{E}}$-**mod**.

The role of homomorphism structures. We have already found that the functor F is based on computing chain-homotopies witnessing equalities of morphisms in $\mathcal{K}^b(\mathscr{C})$, i.e., on solving systems of two-sided inhomogeneous linear equations in the category $\mathscr{C}$. So, how can we solve such systems? The concept of homomorphism structures provides a very good

[5] The category k-**mat** is equivalent to the category $\mathbf{vec}_k$ of finite dimensional k-vector spaces (cf. Definition 2.11).

answer. This concept was first formulated by Posur in [**Pos21a**] as a common generalization of the external Hom functor and the internal Hom functor in a closed symmetric monoidal categories. Let $\mathscr{D}$ be a category. A $\mathscr{D}$-homomorphism structure on a category $\mathscr{C}$ consists of an object $1 \in \mathscr{D}$, a bifunctor

$$H(-,-) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{D}$$

and a natural isomorphism

$$\nu \colon \operatorname{Hom}_{\mathscr{C}}(-,-) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{D}}(1, H(-,-)).$$

The naturality of ν translates to the equality

$$\nu_{B,C}(\chi) \bullet H(\alpha,\beta) = \nu_{A,D}(\alpha \bullet \chi \bullet \beta)$$

for all triples of morphisms $A \xrightarrow{\alpha} B \xrightarrow{\chi} C \xrightarrow{\beta} D$ in $\mathscr{C}$ (cf. Chapter 4).

The main computational advantage of having a $\mathscr{D}$-homomorphism structure is the ability to convert any two-sided inhomogeneous equation

$$\alpha \bullet \chi \bullet \beta = \gamma$$

in $\mathscr{C}$ for given morphisms α, β, γ and an unknown morphism χ to a one-sided inhomogeneous equation

$$\chi' \bullet H(\alpha,\beta) = \nu_{A,D}(\gamma)$$

in $\mathscr{D}$. A solution χ can be recovered from a solution χ' as $\chi = \nu^{-1}_{B,C}(\chi')$. If $\mathscr{C}$ and $\mathscr{D}$ are additive, then we can extend this advantage to convert any system of two-sided inhomogeneous linear equations over $\mathscr{C}$ to a one-sided linear equation over $\mathscr{D}$ (cf. Theorem 4.17).

We also employ the $\mathscr{D}$-homomorphism structure of a category $\mathscr{C}$ to compute the external Hom bifunctor $\operatorname{Hom}_{\mathscr{C}}(-,-)$ itself. In particular, the homomorphism structure reduces computing a generating set of $\operatorname{Hom}_{\mathscr{C}}(A,B)$ to computing a generating set of $\operatorname{Hom}_{\mathscr{D}}(1, H(A,B))$, which is usually much easier to perform. As a matter of fact, the majority of triangulated categories considered in this thesis are k-linear, Hom-finite and equipped with a (k-**mat**)-homomorphism structure for some field k. For instance, we use this technique to compute the aforementioned functors:

- The Yoneda embedding (cf. Corollary 2.89)

$$\mathbf{A}_{\mathscr{E}} \hookrightarrow \mathbf{A}_{\mathscr{E}}\textbf{-mod};$$

- The $\operatorname{Hom}(T_{\mathscr{E}}, -)$ functor (cf. Lemma 6.35)

$$\operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, -) \colon \mathcal{K}^b(\mathscr{C}) \to \operatorname{End} T_{\mathscr{E}}\textbf{-mod}.$$

The Computer Implementation. All the proposed techniques and methods of this thesis are implemented in the `GAP` meta-package `HigherHomologicalAlgebra` [**Sal21a**] which is mainly based on the CAP-project, a software project written in `GAP` for constructive category theory [**GSP22**], [**GP19**] and [**GPS18**].

From the viewpoint of CAP, every category is specified by data types for its objects and morphisms together with two algorithms to compute identities of objects and compose morphisms. We refer to a category $\mathscr{C}$ as *computable* if the mathematical equality of the morphisms is realized by an algorithm. More generally, we say that $\mathscr{C}$ is *computable as an instance of a doctrine*[6] $\mathcal{D}$ if all existential quantifiers and disjunctions in the defining axioms of the doctrine $\mathcal{D}$ are realized by algorithms. The Appendix A provides a brief introduction to CAP.

In CAP, categories are constructed using so-called **category constructors**. The are special functions which are applied to data structures or already existing categories. The category constructor uses the data of the input to derive all algorithms required by the specified doctrines of the output category. For example, the category constructor **`MatrixCategory`**(-) takes as input a field k and returns the Abelian category of matrices k-**mat** over k whose objects are the non-negative integers and whose morphisms are the finite dimensional matrices over k. The skeletal category k-**mat** is computer-friendly and is equivalent to the category of finite dimensional k-vector spaces. The category constructors used in this thesis are covered in details in Chapter 2 and Chapter 3. For instance, we discuss how the bounded homotopy category $\mathcal{K}^b(\mathscr{C})$ can be constructed as a stable category of the category of bounded complexes $\mathcal{C}^b(\mathscr{C})$, and how End $T_{\mathscr{E}}$-**mod** can be constructed as a functor category.

The `HigherHomologicalAlgebra` meta-package [**Sal21a**] is intended to provide an accessible computing environment for performing all of the constructions proposed in this thesis. In the subsequent chapter, we present a software demonstration and thereby explain the syntax. It consists of various packages, the most important of which are

(1) `TriangulatedCategories` [**Sal21f**] provides a framework for triangulated categories. Our basic examples for triangulated categories are bounded homotopy categories and stable categories of Frobenius categories.
(2) `StableCategories` [**Sal21e**] provides a framework for stable categories associated to classes of lifting or colifting objects.
(3) `ComplexesCategories` [**Sal21b**] provides a framework for categories of complexes. If the $\mathscr{C}$ is an Abelian category with enough projectives or injectives, then we can compute projective and injective resolutions of complexes in $\mathcal{C}^b(\mathscr{C})$. It enables us to perform many homological constructions such as computing derived functors and total complexes of double complexes.
(4) `HomotopyCategories` [**Sal21d**] provides a framework for bounded homotopy categories. A homotopy category $\mathcal{K}^b(\mathscr{C})$ is constructed as a stable category of $\mathcal{C}^b(\mathscr{C})$ with respect to the class of contractible objects. It also provides an implementation of the triangulated structure of $\mathcal{K}^b(\mathscr{C})$. The computation of Postnikov systems and their associated convolutions are also performed by this package.
(5) `DerivedCategories` [**Sal21c**] provides a framework for constructing derived categories of Abelian categories with enough projectives or injectives. It also contains all implementations related to strong exceptional sequences and their associated exact equivalences.

Outline. This thesis is organized as follows. Chapter 1 gives a software demonstration to show how our framework solves the question raised at the beginning of this introduction: How to extend a given full embedding $\iota\colon \Omega \hookrightarrow \mathcal{K}^b(\mathcal{O}^{\oplus})$ to an exact equivalence $\mathcal{K}^b(\Omega^{\oplus}) \simeq \mathcal{K}^b(\mathcal{O}^{\oplus})$. Chapter 2 discuss in detail constructing the majority of preadditive, additive and Abelian categories that are used in Chapter 1, e.g., finitely presented categories (cf. Section 2.2.5) and their (Abelian) functor categories (cf. Section 2.2.7); and introduces also other categories for a later use, e.g.,

[6]We use the term "doctrine" to describe a class of categories with specified additional properties or structures, e.g., additive, Abelian, monoidal, etc.

Freyd categories (cf. Section 2.2.3). Chapter 3 summarizes in a constructive style the key concepts in homological algebra that needed in this thesis. e.g., constructing bounded homotopy (cf. Section 3.2) or derived categories (cf. Section 3.4), computing projective and injective resolutions of complexes (cf. Section 3.3), computing derived functors (cf. Examples 3.67 and 3.71), etc. Chapter 4 reviews the basic facts and examples on homomorphism structures (cf. Section 4.1) and investigates creating new homomorphism structures from already existing onces. For example, we discuss how to lift a homomorphism structure on a category $\mathscr{C}$ to the bounded homotopy category $\mathcal{K}^b(\mathscr{C})$ (cf. Section 4.5) and the functor category $[\mathscr{A},\mathscr{C}]$ for some finitely presented category $\mathscr{A}$ (cf. Section 4.2). Chapter 5 provides a constructive framework in which we can perform computations in triangulated categories. We provide two main examples: The bounded homotopy category of an additive category (cf. Section 5.2) and the stable category of a Frobenius category (cf. Section 5.3). Chapter 6 provides a constructive framework to perform computations on strong exceptional sequences such as computing the abstraction k-algebroids and then lastly computing their adjoint exact equivalences: The aforementioned adjunction $F \dashv G$. The Appendix is meant to provide a software demonstration for the computational goals of this thesis.

CHAPTER 1

A Demo for a Tilting Equivalence Using HigherHomologicalAlgebra

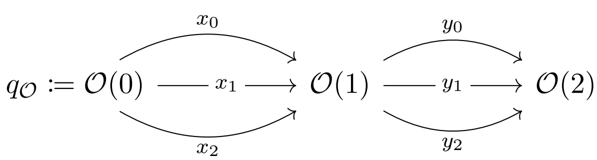
The following is a software-demonstration for creating a complete strong exceptional sequence in a bounded homotopy category. We use the associated tilting equivalences to resolve objects in the homotopy category in terms of the objects of the sequence.

We use the HIGHER HOMOLOGICAL ALGEBRA meta-package [**Sal21a**] which is mainly based on the `homalg` [**hom22**] and CAP [**GSP22**] software projects.

The JULIA [**BEKS17**] package `CapAndHomalg` [**CAP21a**] provides an interface to the above and many other required `GAP` packages. We start by loading `CapAndHomalg` and the `GAP` package `DerivedCategories` [**Sal21c**]:

```
julia> using CapAndHomalg
CapAndHomalg v1.4.0
Imported OSCAR's components GAP and Singular_jll
Type: ?CapAndHomalg for more information
julia> LoadPackage( "DerivedCategories" )
```

As our running example, we consider the **Beilinson k-algebroid** over a field[1] k. We construct it as the finitely presented k-linear category $\mathbf{A}_{\mathcal{O}}$ generated by the Beilinson quiver



subject to the relations $\rho_{\mathcal{O}} = \{x_i y_j - y_j x_i | i, j = 0, 1, 2\}$. This can be done in three steps:

(1) Construct the free category $\mathbf{F}_{\mathcal{O}}$ generated by the Beilinson quiver $q_{\mathcal{O}}$.
(2) Construct the k-linear closure category $k[\mathbf{F}_{\mathcal{O}}]$ of $\mathbf{F}_{\mathcal{O}}$.
(3) Construct the quotient category $\mathbf{A}_{\mathcal{O}}$ of $k[\mathbf{F}_{\mathcal{O}}]$ modulo the two-sided ideal generated by the relations $\rho_{\mathcal{O}}$.

The name and the labels of the quiver reflect its geometric origin. The Beilinson algebroid $\mathbf{A}_{\mathcal{O}}$ is an isomorphic to the full subcategory of the category $\mathfrak{Coh}\ \mathbb{P}^2_k$ of coherent sheaves over the projective space $\mathbb{P}^2_k$ generated (as a k-linear subcategory) by the structure sheaf $\mathcal{O}_{\mathbb{P}^2_k} = \mathcal{O} = \mathcal{O}(0)$ and two further twists $\mathcal{O}_{\mathbb{P}^2_k}(i) = \mathcal{O}(i)$ for $i = 1, 2$ (see, e.g., [**Beĭ78**]). However, none of this is relevant to the following demonstration.

The package `DerivedCategories` uses the package `QPA2` [**Qt21**] which provides the needed infrastructure for quivers with relations:

[1] or a nonzero commutative unital ring.

```
julia> q_𝒪 = RightQuiver(
"q_𝒪(𝒪0,𝒪1,𝒪2)[x0:𝒪0->𝒪1,x1:𝒪0->𝒪1,x2:𝒪0->𝒪1,y0:𝒪1->𝒪2,y1:𝒪1->𝒪2,y2:𝒪1->𝒪2]");

julia> SetLabelsAsLaTeXStrings( q_𝒪,
       [ "\\mathcal{O}(0)", "\\mathcal{O}(1)", "\\mathcal{O}(2)" ],
       [ "x_0", "x_1", "x_2", "y_0", "y_1", "y_2" ] )

julia> 𝐅_𝒪 = FreeCategory( q_𝒪 )
GAP: Category freely generated by the right quiver
q_𝒪(𝒪0,𝒪1,𝒪2)[x0:𝒪0->𝒪1,x1:𝒪0->𝒪1,x2:𝒪0->𝒪1,y0:𝒪1->𝒪2,y1:𝒪1->𝒪2,y2:𝒪1->𝒪2]

julia> ℚ = HomalgFieldOfRationals( )
GAP: Q

julia> 𝑘 = ℚ
GAP: Q

julia> 𝑘𝐅_𝒪 = 𝑘[𝐅_𝒪]
GAP: Algebroid( Q * q_𝒪 )

julia> ρ_𝒪 = [ PreCompose( 𝑘𝐅_𝒪.x0, 𝑘𝐅_𝒪.y1 ) - PreCompose( 𝑘𝐅_𝒪.x1, 𝑘𝐅_𝒪.y0 ),
               PreCompose( 𝑘𝐅_𝒪.x0, 𝑘𝐅_𝒪.y2 ) - PreCompose( 𝑘𝐅_𝒪.x2, 𝑘𝐅_𝒪.y0 ),
               PreCompose( 𝑘𝐅_𝒪.x1, 𝑘𝐅_𝒪.y2 ) - PreCompose( 𝑘𝐅_𝒪.x2, 𝑘𝐅_𝒪.y1 ) ];

julia> 𝐀_𝒪 = 𝑘𝐅_𝒪 / ρ_𝒪
GAP: Algebroid( (Q * q_𝒪) /
     [ -1*(x1*y0) + 1*(x0*y1), -1*(x2*y0) + 1*(x0*y2), -1*(x2*y1) + 1*(x1*y2) ] )

julia> InfoOfInstalledOperationsOfCategory( 𝐀_𝒪 )
27 primitive operations were used to derive 74 operations for this category which
* IsLinearCategoryOverCommutativeRing
* IsEquippedWithHomomorphismStructure

julia> ListInstalledOperationsOfCategory( 𝐀_𝒪 )
GAP: [ AdditionForMorphisms, AdditiveInverseForMorphisms, BasisOfExternalHom, ... ]

julia> 2 * 𝐀_𝒪."x0" + 3 * 𝐀_𝒪."x1" - 𝐀_𝒪."x2"
GAP: (𝒪0)-[{ -1*(x2) + 3*(x1) + 2*(x0) }]->(𝒪1)

julia> BasisOfExternalHom( 𝐀_𝒪."𝒪0", 𝐀_𝒪."𝒪1" )
GAP: [ (𝒪0)-[{ 1*(x0) }]->(𝒪1), (𝒪0)-[{ 1*(x1) }]->(𝒪1), (𝒪0)-[{ 1*(x2) }]->(𝒪1) ]
```

Since the relations $\rho_{\mathcal{O}} = \{x_i y_j = y_j x_i | i, j = 0, 1, 2\}$ of the Beilinson quiver are categorical, i.e., they can be expressed without reference to any coefficients ring k, one can equally construct the associated k-algebroid $\mathbf{A}_{\mathcal{O}}$ as the k-linear closure category of a finitely presented category $\mathbf{B}_{\mathcal{O}}$, which we call the Beilinson category:

```
julia> ρ_𝒪 = [ [ PreCompose( 𝐅_𝒪.x0, 𝐅_𝒪.y1 ), PreCompose( 𝐅_𝒪.x1, 𝐅_𝒪.y0 ) ],
```

```
        [ PreCompose( F_𝒪.x0, F_𝒪.y2 ), PreCompose( F_𝒪.x2, F_𝒪.y0 ) ],
        [ PreCompose( F_𝒪.x1, F_𝒪.y2 ), PreCompose( F_𝒪.x2, F_𝒪.y1 ) ] ];

julia> B_𝒪 = F_𝒪 / ρ_𝒪
GAP: Category generated by the right quiver
  q_𝒪(𝒪0,𝒪1,𝒪2)[x0:𝒪0->𝒪1,x1:𝒪0->𝒪1,x2:𝒪0->𝒪1,y0:𝒪1->𝒪2,y1:𝒪1->𝒪2,y2:𝒪1->𝒪2]
  with relations [ x0*y1 = x1*y0, x0*y2 = x2*y0, x1*y2 = x2*y1 ]

julia> InfoOfInstalledOperationsOfCategory( B_𝒪 )
7 primitive operations were used to derive 13 operations for this category which
* IsFinitelyPresentedCategory

julia> A_𝒪 = k[B_𝒪]
GAP: Algebroid( (Q * q_𝒪) /
     [ -1*(x1*y0) + 1*(x0*y1), -1*(x2*y0) + 1*(x0*y2), -1*(x2*y1) + 1*(x1*y2) ] )
```

Since $\mathbf{A}_{\mathcal{O}}$ is Hom-finite (i.e., has finite k-dimensional Hom-spaces), we get the Yoneda embedding

$$Y : \mathbf{A}_{\mathcal{O}} \hookrightarrow \mathrm{Hom}(\mathbf{A}_{\mathcal{O}}^{\mathrm{op}}, k\text{-}\mathbf{mat})$$

where k-**mat** is the category of matrices over k, which yields a full embedding of $\mathbf{A}_{\mathcal{O}}$ into a *k-linear Abelian category with enough injectives and projectives*:

$$\mathbf{A}_{\mathcal{O}} \cong Y(\mathbf{A}_{\mathcal{O}}).$$

For this k-Abelian functor category we use the notation

$$\mathbf{A}_{\mathcal{O}}\text{-}\mathbf{mod} \coloneqq \mathrm{Hom}(\mathbf{A}_{\mathcal{O}}^{\mathrm{op}}, k\text{-}\mathbf{mat})$$

and call it the *category finite k-dimensional $\mathbf{A}_{\mathcal{O}}$-modules.*

```
julia> A_𝒪_op = OppositeAlgebroid( A_𝒪 )
GAP: Algebroid( (Q * q_𝒪_op) /
     [ 1*(y1*x0) - 1*(y0*x1), 1*(y2*x0) - 1*(y0*x2), 1*(y2*x1) - 1*(y1*x2) ] )

julia> q_𝒪_op = UnderlyingQuiver( A_𝒪_op )
GAP: q_𝒪_op(𝒪0,𝒪1,𝒪2)[x0:𝒪1->𝒪0,x1:𝒪1->𝒪0,x2:𝒪1->𝒪0,y0:𝒪2->𝒪1,y1:𝒪2->𝒪1,y2:𝒪2->
    𝒪1]

julia> SetLabelsAsLaTeXStrings(
    q_𝒪_op,
    [ "\\mathcal{O}(0)", "\\mathcal{O}(1)", "\\mathcal{O}(2)" ],
    [ "x_1", "x_2", "x_3", "y_0", "y_1", "y_2" ] )

julia> kmat = MatrixCategory( k )
GAP: Category of matrices over Q

julia> A_𝒪mod = FunctorCategory( A_𝒪_op, kmat )
GAP: FunctorCategory(
  Algebroid( (Q * q_𝒪_op) / [ 1*(y1*x0) - 1*(y0*x1), 1*(y2*x0) - 1*(y0*x2), 1*(y2*x1) -
     1*(y1*x2) ] ) -> Category of matrices over Q )
```

```
julia> InfoOfInstalledOperationsOfCategory( A_𝒪mod )
103 primitive operations were used to derive 365 operations for this category which
* IsLinearCategoryOverCommutativeRing
* IsAbelianCategoryWithEnoughInjectives
* IsAbelianCategoryWithEnoughProjectives
* IsEquippedWithHomomorphismStructure
```

We denote by

$$\mathcal{K}^b(\mathbf{A}_{\mathcal{O}}) \coloneqq \mathcal{K}^b(\mathbf{A}_{\mathcal{O}}\text{-}\mathbf{mod})$$

and

$$\mathcal{D}^b(\mathbf{A}_{\mathcal{O}}) \coloneqq \mathcal{D}^b(\mathbf{A}_{\mathcal{O}}\text{-}\mathbf{mod})$$

the bounded homotopy resp. derived categories of $\mathbf{A}_{\mathcal{O}}$-**mod**. Since the quiver is acyclic and the relations are admissible, the category $\mathbf{A}_{\mathcal{O}}$-**mod** has finite global dimension[2]. And because $\mathbf{A}_{\mathcal{O}}$-**mod** has enough projectives (and injectives) we can *decide equality of morphisms*[3] in $\mathcal{D}^b(\mathbf{A}_{\mathcal{O}})$. We denote by

$$L\colon \mathcal{K}^b(\mathbf{A}_{\mathcal{O}}) \to \mathcal{D}^b(\mathbf{A}_{\mathcal{O}})$$

the natural localization functor which maps quasi-isomorphisms in $\mathcal{K}^b(\mathbf{A}_{\mathcal{O}})$ to isomorphisms in $\mathcal{D}^b(\mathbf{A}_{\mathcal{O}})$.

```
julia> KA_𝒪 = HomotopyCategoryByCochains( A_𝒪mod )
GAP: Homotopy category( FunctorCategory( Algebroid( (Q * q_𝒪_op) / [ 1*(y1*x0) - 1*(y0*
    x1), 1*(y2*x0) - 1*(y0*x2), 1*(y2*x1) - 1*(y1*x2) ] ) -> Category of matrices over
    Q ) )

julia> DA_𝒪 = DerivedCategoryByCochains( A_𝒪mod )
GAP: Derived category( FunctorCategory( Algebroid( (Q * q_𝒪_op) / [ 1*(y1*x0) - 1*(y0*
    x1), 1*(y2*x0) - 1*(y0*x2), 1*(y2*x1) - 1*(y1*x2) ] ) -> Category of matrices over
    Q ) )

julia> CanCompute( DA_𝒪, "IsCongruentForMorphisms" )
true

julia> KnownFunctors( KA_𝒪, DA_𝒪 )
1: The natural localization functor

julia> L = Functor( KA_𝒪, DA_𝒪, 1 )
GAP: Localization functor onto bounded derived category
```

Since $\mathbf{A}_{\mathcal{O}}$ is admissible[4] the YONEDA embedding Y identifies $\mathbf{A}_{\mathcal{O}}$ with a skeletal model for the full subcategory of $\mathbf{A}_{\mathcal{O}}$-**mod** generated by the *indecomposable* projective objects. Hence, the additive closure category $\mathbf{A}_{\mathcal{O}}^{\oplus}$ is equivalent to the additive full subcategory $\mathbf{A}_{\mathcal{O}}$-**proj** of projective

[2]bounded by the number of vertices in the generating quiver.

[3]Referred to as `IsCongruentForMorphisms` in the code below.

[4]i.e., the relations of the quiver involve only paths of length at least two.

objects in $\mathbf{A}_{\mathcal{O}}$-**mod**. The objects in $\mathbf{A}_{\mathcal{O}}^{\oplus}$ are lists of objects[5] in $\mathbf{A}_{\mathcal{O}}$ and morphisms are matrices of morphisms in $\mathbf{A}_{\mathcal{O}}$:

```
julia> A_𝒪add = AdditiveClosure( A_𝒪 )
GAP: Additive closure( Algebroid( (Q * q_𝒪) / [ -1*(x1*y0) + 1*(x0*y1), -1*(x2*y0) +
    1*(x0*y2), -1*(x2*y1) + 1*(x1*y2) ] ) )

julia> A_𝒪proj = FullSubcategoryGeneratedByProjectiveObjects( A_𝒪mod )
GAP: Full additive subcategory generated by projective objects( FunctorCategory(
    Algebroid( (Q * q_𝒪_op) / [ 1*(y1*x0) - 1*(y0*x1), 1*(y2*x0) - 1*(y0*x2), 1*(y2*x1)
     - 1*(y1*x2) ] ) -> Category of matrices over Q ) )

julia> KnownFunctors( A_𝒪add, A_𝒪proj )
1: Yoneda embedding

julia> Yadd = Functor( A_𝒪add, A_𝒪proj, 1 )
GAP: Yoneda embedding

julia> KnownFunctors( A_𝒪proj, A_𝒪add )
1: Decomposition of projective objects

julia> Dec = Functor( A_𝒪proj, A_𝒪add, 1 )
GAP: Decomposition of projective objects
```

The above equivalences can be extended to the equivalences

$$\mathcal{K}^b\left(\mathbf{A}_{\mathcal{O}}^{\oplus}\right) \simeq \mathcal{K}^b(\mathbf{A}_{\mathcal{O}}\text{-}\mathbf{proj}).$$

```
julia> KA_𝒪add = HomotopyCategoryByCochains( A_𝒪add )
GAP: Homotopy category( Additive closure( Algebroid( (Q * q_𝒪) / [ -1*(x1*y0) + 1*(x0*
    y1), -1*(x2*y0) + 1*(x0*y2), -1*(x2*y1) + 1*(x1*y2) ] ) ) )

julia> InfoOfInstalledOperationsOfCategory( KA_𝒪add )
61 primitive operations were used to derive 185 operations for this category which
* IsLinearCategoryOverCommutativeRing
* IsAdditiveCategory
* IsTriangulatedCategory
* IsEquippedWithHomomorphismStructure

julia> KA_𝒪proj = HomotopyCategoryByCochains( A_𝒪proj )
GAP: Homotopy category( Full additive subcategory generated by projective objects(
    FunctorCategory( Algebroid( (Q * q_𝒪_op) / [ 1*(y1*x0) - 1*(y0*x1), 1*(y2*x0) - 1*(
    y0*x2), 1*(y2*x1) - 1*(y1*x2) ] ) -> Category of matrices over Q ) ) )

julia> KYadd = ExtendFunctorToHomotopyCategoriesByCochains( Yadd )
GAP: Extension of ( Yoneda embedding ) to homotopy categories

julia> KDec = ExtendFunctorToHomotopyCategoriesByCochains( Dec )
```

[5]The lists represent formal direct sums of objects in $\mathbf{A}_{\mathcal{O}}$.

```
GAP: Extension of ( Decomposition of projective objects ) to homotopy categories
```

Since $\mathbf{A}_{\mathcal{O}}$-**mod** has a finite global dimension, the composition

$$\mathcal{K}^b(\mathbf{A}_{\mathcal{O}}\text{-}\mathbf{proj}) \hookrightarrow \mathcal{K}^b(\mathbf{A}_{\mathcal{O}}) \xrightarrow{L} \mathcal{D}^b(\mathbf{A}_{\mathcal{O}})$$

of the natural embedding functor with the standard localization functor defines an equivalence:

$$\mathcal{K}^b(\mathbf{A}_{\mathcal{O}}\text{-}\mathbf{proj}) \simeq \mathcal{D}^b(\mathbf{A}_{\mathcal{O}}).$$

```
julia> KnownFunctors( KA_𝒪proj, DA_𝒪 )
1: PreComposition of the following two functors:
     * Apply ExtendFunctorToHomotopyCategoriesByCochains on ( The inclusion functor )
     * The natural localization functor

julia> V = Functor( KA_𝒪proj, DA_𝒪, 1 )
GAP: Composition of Extension of ( The inclusion functor ) to homotopy categories and
    Localization functor in derived category

julia> KnownFunctors( DA_𝒪, KA_𝒪proj )
1: Universal functor from derived category

julia> U = Functor( DA_𝒪, KA_𝒪proj, 1 )
GAP: Universal functor from derived category onto a localization category
```

That is, we get an equivalence

$$J : \mathcal{K}^b\left(\mathbf{A}_{\mathcal{O}}^{\oplus}\right) \xrightarrow{\sim} \mathcal{D}^b(\mathbf{A}_{\mathcal{O}}).$$

```
julia> J = PreCompose( KYadd, V );

julia> Display( J )
Composition of Extension of ( Yoneda embedding ) to homotopy categories and Composition
     of Extension of ( The inclusion functor ) to homotopy categories and Localization
    functor in derived category:

Homotopy category( Additive closure( Algebroid( (Q * q_𝒪) / [ -1*(x1*y0) + 1*(x0*y1),
    -1*(x2*y0) + 1*(x0*y2), -1*(x2*y1) + 1*(x1*y2) ] ) ) )
  |
  V
Derived category( FunctorCategory( Algebroid( (Q * q_𝒪_op) / [ 1*(y1*x0) - 1*(y0*x1),
    1*(y2*x0) - 1*(y0*x2), 1*(y2*x1) - 1*(y1*x2) ] ) -> Category of matrices over Q ) )
```

We could have computed this equivalence as follows:

```
julia> J = EquivalenceOntoDerivedCategory( KA_𝒪add )
GAP: Equivalence functor from homotopy category onto derived category
```

Now consider the three objects $\Omega^2(2), \Omega^1(1)$, and $\Omega^0(0)$ in $\mathcal{K}^b\left(\mathbf{A}_{\mathcal{O}}^{\oplus}\right)$ defined by

$$\Omega^2(2) := \quad 0 \longrightarrow \mathcal{O}(0)^3 \xrightarrow[0]{\begin{pmatrix} x_1 & -x_0 & 0 \\ x_2 & 0 & -x_0 \\ 0 & x_2 & -x_1 \end{pmatrix}} \mathcal{O}(1)^3 \xrightarrow{\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}} \mathcal{O}(2) \longrightarrow 0,$$

$$\Omega^1(1) := \quad 0 \longrightarrow \mathcal{O}(0)^3 \xrightarrow[0]{\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}} \mathcal{O}(1) \longrightarrow 0,$$

$$\Omega^0(0) := \quad 0 \longrightarrow \mathcal{O}(0) \xrightarrow[0]{} 0$$

The labels of the objects reflect their geometric origin. They represent the twisted cotangent bundles $\Omega^i_{\mathbb{P}^2_k}(i) = \Omega^i(i)$, $i = 0, 1, 2$ in $\mathfrak{Coh}\,\mathbb{P}^2_k$ (see, e.g., [Beĭ78]). Again, this interpretation is irrelevant to the computations below.

```
julia> Ω2_0 = [ A_𝒪."𝒪0", A_𝒪."𝒪0", A_𝒪."𝒪0" ] / A_𝒪add;

julia> Ω2_1 = [ A_𝒪."𝒪1", A_𝒪."𝒪1", A_𝒪."𝒪1" ] / A_𝒪add;

julia> Ω2_2 = [ A_𝒪."𝒪2" ] / A_𝒪add
GAP: <An object in Additive closure( Algebroid( (Q * q_𝒪) / [ -1*(x1*y0) + 1*(x0*y1),
    -1*(x2*y0) + 1*(x0*y2), -1*(x2*y1) + 1*(x1*y2) ] ) ) defined by 1 underlying
    objects>

julia> ∂_0 = AdditiveClosureMorphism(
              Ω2_0,
              [ [ A_𝒪."x1", -A_𝒪."x0", ZeroMorphism(A_𝒪."𝒪0", A_𝒪."𝒪1") ],
                [ A_𝒪."x2", ZeroMorphism(A_𝒪."𝒪0", A_𝒪."𝒪1"), -A_𝒪."x0" ],
                [ ZeroMorphism(A_𝒪."𝒪0", A_𝒪."𝒪1"),  A_𝒪."x2", -A_𝒪."x1" ] ],
              Ω2_1 );

julia> ∂_1 = AdditiveClosureMorphism(
              Ω2_1,
              [ [ A_𝒪."y0" ],
                [ A_𝒪."y1" ],
                [ A_𝒪."y2" ] ],
              Ω2_2 )
GAP: <A morphism in Additive closure( Algebroid( (Q * q_𝒪) / [ -1*(x1*y0) + 1*(x0*y1),
    -1*(x2*y0) + 1*(x0*y2), -1*(x2*y1) + 1*(x1*y2) ] ) ) defined by a 3 x 1 matrix of
    underlying morphisms>

julia> Ω2 = [ [ ∂_0, ∂_1 ], 0 ] / KA_𝒪add
GAP: <An object in Homotopy category( Additive closure( Algebroid( (Q * q_𝒪) / [ -1*(x1
    *y0) + 1*(x0*y1), -1*(x2*y0) + 1*(x0*y2), -1*(x2*y1) + 1*(x1*y2) ] ) ) ) with
    active lower bound 0 and active upper bound 2>

julia> Ω1_0 = [ A_𝒪."𝒪0", A_𝒪."𝒪0", A_𝒪."𝒪0" ] / A_𝒪add;
julia> Ω1_1 = [ A_𝒪."𝒪1" ] / A_𝒪add;
```

```
julia> ∂_0 = AdditiveClosureMorphism(
             Ω1_0,
             [ [ A_𝒪."x0" ],
               [ A_𝒪."x1" ],
               [ A_𝒪."x2" ] ],
             Ω1_1 );

julia> Ω1 = [ [ ∂_0 ], 0 ] / KA_𝒪add
GAP: <An object in Homotopy category( Additive closure( Algebroid( (Q * q_𝒪) / [ -1*(x1
    *y0) + 1*(x0*y1), -1*(x2*y0) + 1*(x0*y2), -1*(x2*y1) + 1*(x1*y2) ] ) ) ) with
    active lower bound 0 and active upper bound 1>

julia> Ω0 = [ A_𝒪."𝒪0" ] / A_𝒪add / KA_𝒪add
GAP: <An object in Homotopy category( Additive closure( Algebroid( (Q * q_𝒪) / [ -1*(x1
    *y0) + 1*(x0*y1), -1*(x2*y0) + 1*(x0*y2), -1*(x2*y1) + 1*(x1*y2) ] ) ) ) with
    active lower bound 0 and active upper bound 0>
```

In the following we use our software to perform the following computations:

- Verify that $\Omega := (\Omega^2(2), \Omega^1(1), \Omega^0(0))$ is a complete strong exceptional sequence in $\mathcal{K}^b\left(\mathbf{A}_{\mathcal{O}}^{\oplus}\right)$ (see Definition 6.14).
- Verify that not all objects in $J(\Omega)$ belong to the standard Abelian heart $\mathbf{A}_{\mathcal{O}}$-**mod** of $\mathcal{D}^b(\mathbf{A}_{\mathcal{O}})$.
- Compute an abstract k-algebroid $\mathbf{A}_\Omega$ associated to Ω, i.e., a finite presentation (given by a quiver with relations) of the k-linear full subcategory of $\mathcal{K}^b\left(\mathbf{A}_{\mathcal{O}}^{\oplus}\right)$ generated by Ω.
- Construct the exact equivalences
$$\mathcal{D}^b(\mathbf{A}_{\mathcal{O}}) \simeq \mathcal{K}^b\left(\mathbf{A}_{\mathcal{O}}^{\oplus}\right) \simeq \mathcal{K}^b\left(\mathbf{A}_{\Omega}^{\oplus}\right) \simeq \mathcal{D}^b(\mathbf{A}_\Omega).$$
- Verify that the images of the objects $(\mathcal{O}(0), \mathcal{O}(1), \mathcal{O}(2))$ in $\mathcal{D}^b(\mathbf{A}_\Omega)$ live in the standard Abelian heart $\mathbf{A}_\Omega$-**mod**.

We can now create the strong exceptional sequence Ω. The last two arguments are optional and serve for a better accessibility and visibility as we will see later:

```
julia> Ω = CreateStrongExceptionalCollection(
            [ Ω2, Ω1, Ω0 ],
            [ "Ω2", "Ω1", "Ω0" ],
            [ "\\Omega^2(2)", "\\Omega^1(1)", "\\Omega^0(0)" ] )
GAP: <A strong exceptional sequence defined by the objects of the full subcategory {Ω2,
     Ω1, Ω0}>

julia> T_Ω = DirectSum( Ω2, Ω1, Ω0 )
GAP: <An object in Homotopy category( Additive closure( Algebroid( (Q * q_𝒪) / [ -1*(x1
    *y0) + 1*(x0*y1), -1*(x2*y0) + 1*(x0*y2), -1*(x2*y1) + 1*(x1*y2) ] ) ) ) with
    active lower bound 0 and active upper bound 2>

julia> Show( T_Ω )
```

$$
\begin{array}{c}
\mathcal{O}(2) \\
\uparrow \\
\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \cdot \end{pmatrix} \\
|_1 \\
\mathcal{O}(1)^{\oplus 4} \\
\uparrow \\
\begin{pmatrix}
x_1 & -x_0 & \cdot & \cdot \\
x_2 & \cdot & -x_0 & \cdot \\
\cdot & x_2 & -x_1 & \cdot \\
\cdot & \cdot & \cdot & x_0 \\
\cdot & \cdot & \cdot & x_1 \\
\cdot & \cdot & \cdot & x_2 \\
\cdot & \cdot & \cdot & \cdot
\end{pmatrix} \\
|_0 \\
O^{\oplus 7}
\end{array}
$$

By Definition 6.14, the sequence $\Omega = \big(\Omega^2(2), \Omega^1(1), \Omega^0(0)\big)$ is strong exceptional if

- $\mathrm{Hom}\big(\Omega^i(i), \Omega^i(i)\big) \simeq k$ for $i = 0, 1, 2$.
- $\mathrm{Hom}\big(\Omega^i(i), \Omega^j(j)\big) = 0$ for $i < j$.
- $\mathrm{Hom}(T_\Omega, \Sigma^r(T_\Omega)) = 0$ for all $r \neq 0$ where Σ is the standard shift automorphism of $\mathcal{K}^b\Big(\mathbf{A}_{\mathcal{O}}^{\oplus}\Big)$. Due to the lower and upper bounds of T_Ω, it is sufficient to verify this requirement only for $r \in \{-2, -1, 1, 2\}$:

```
julia> Dimension( HomStructure( Ω2, Ω2 ) ) == 1 &&
       Dimension( HomStructure( Ω1, Ω1 ) ) == 1 &&
       Dimension( HomStructure( Ω0, Ω0 ) ) == 1
true

julia> IsZero( HomStructure( Ω0, Ω1 ) ) &&
       IsZero( HomStructure( Ω1, Ω2 ) ) &&
       IsZero( HomStructure( Ω0, Ω2 ) )
true

julia> IsZero( HomStructure( T_Ω, Shift( T_Ω, -2 ) ) ) &&
       IsZero( HomStructure( T_Ω, Shift( T_Ω, -1 ) ) ) &&
       IsZero( HomStructure( T_Ω, Shift( T_Ω, 1 ) ) )  &&
       IsZero( HomStructure( T_Ω, Shift( T_Ω, 2 ) ) )
true
```

Of course, we can use the same operation to compute the dimension of $\mathrm{End}\, T_\Omega$:

```
julia> HomStructure( T_Ω, T_Ω )
GAP: <A vector space object over Q of dimension 12>
```

By applying the equivalence $J : \mathcal{K}^b\left(\mathbf{A}_{\mathcal{O}}^{\oplus}\right) \xrightarrow{\sim} \mathcal{D}^b(\mathbf{A}_{\mathcal{O}})$ on the objects of Ω and computing the cohomology support we can verify which of the images $J(\Omega^i(i)), i = 0, 1, 2$ belongs to the standard Abelian heart of $\mathcal{D}^b(\mathbf{A}_{\mathcal{O}})$ (cf. [**GM03**, §5]):

```
julia> JΩ2 = ApplyFunctor( J, Ω2 )
GAP: <An object in Derived category( FunctorCategory( Algebroid( (Q * q_𝒪_op) / [ 1*(y1
    *x0) - 1*(y0*x1), 1*(y2*x0) - 1*(y0*x2), 1*(y2*x1) - 1*(y1*x2) ] ) -> Category of
    matrices over Q ) ) with active lower bound 0 and active upper bound 2>

julia> CohomologySupport( JΩ2 )
GAP: [ 2 ]

julia> H2 = CohomologyAt( JΩ2, 2 )
GAP: <(𝒪0)->0, (𝒪1)->0, (𝒪2)->1; (x0)->0x0, (x1)->0x0, (x2)->0x0, (y0)->1x0, (y1)->1x0
    , (y2)->1x0>
```

That is, $J(\Omega^2(2))$ does not belong to $\mathbf{A}_{\mathcal{O}}$-**mod** $\subset \mathcal{D}^b(\mathbf{A}_{\mathcal{O}})$.

```
julia> JΩ1 = ApplyFunctor( J, Ω1 )
GAP: <An object in Derived category( FunctorCategory( Algebroid( (Q * q_𝒪_op) / [ 1*(y1
    *x0) - 1*(y0*x1), 1*(y2*x0) - 1*(y0*x2), 1*(y2*x1) - 1*(y1*x2) ] ) -> Category of
    matrices over Q ) ) with active lower bound 0 and active upper bound 1>

julia> H1 = CohomologySupport( JΩ1 )
GAP: [ 1 ]

julia> CohomologyAt( JΩ1, 1 )
GAP: <(𝒪0)->0, (𝒪1)->1, (𝒪2)->0; (x0)->1x0, (x1)->1x0, (x2)->1x0, (y0)->0x1, (y1)->0x1
    , (y2)->0x1>
```

That is, also $J(\Omega^1(1))$ does not belong to $\mathbf{A}_{\mathcal{O}}$-**mod** $\subset \mathcal{D}^b(\mathbf{A}_{\mathcal{O}})$.

```
julia> JΩ0 = ApplyFunctor( J, Ω0 )
GAP: <An object in Derived category( FunctorCategory( Algebroid( (Q * q_𝒪_op) / [ 1*(y1
    *x0) - 1*(y0*x1), 1*(y2*x0) - 1*(y0*x2), 1*(y2*x1) - 1*(y1*x2) ] ) -> Category of
    matrices over Q ) ) with active lower bound 0 and active upper bound 0>

julia> CohomologySupport( JΩ0 )
GAP: [ 0 ]

julia> H0 = CohomologyAt( JΩ1, 0 )
GAP: <(𝒪0)->1, (𝒪1)->0, (𝒪2)->0; (x0)->0x1, (x1)->0x1, (x2)->0x1, (y0)->0x0, (y1)->0x0
    , (y2)->0x0>
```

That is, only $J(\Omega^0(0))$ belongs to $\mathbf{A}_{\mathcal{O}}$-**mod** $\subset \mathcal{D}^b(\mathbf{A}_{\mathcal{O}})$.

As we will show later, the strong exceptional sequence Ω is even full. Since J is an equivalence, the above computation shows that $\mathcal{D}^b(\mathbf{A}_{\mathcal{O}})$ is generated by the three nonisomorphic simple objects $H(J(\Omega)) := (H^i(J(\Omega^i(i))) \mid i = 2,1,0)$ of $\mathbf{A}_{\mathcal{O}}$**-mod**. However, $H(J(\Omega))$ is not a strong exceptional sequence in $\mathcal{D}^b(\mathbf{A}_{\mathcal{O}})$. Still, the above computation shows that the sequence $(H^i(J(\Omega^i(i)))[-i] \mid i = 2,1,0) \cong J(\Omega)$ in $\mathcal{D}^b(\mathbf{A}_{\mathcal{O}})$ is strong exceptional.

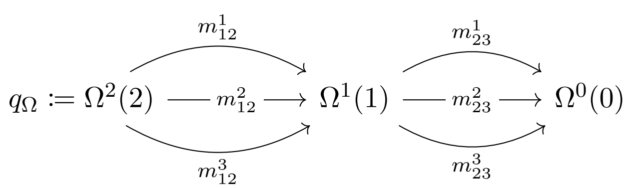
One of the main constructions associated to a strong exceptional sequence is its *abstraction algebroid*. The abstraction algebroid is a k-linear finitely presented category $\mathbf{A}_\Omega$ which is isomorphic to the full subcategory $\mathbf{C}_\Omega \subset \mathcal{K}^b\left(\mathbf{A}_{\mathcal{O}}^{\oplus}\right)$ generated by Ω. In particular, it exhibits the structure of Ω in terms of a quiver q_Ω and a set of relations ρ_Ω.

```
julia> A_Ω = Algebroid( Ω )
GAP: Algebroid( end( Ω2 ⊕ Ω1 ⊕ Ω0 ) )

julia> q_Ω = UnderlyingQuiver( A_Ω )
GAP: quiver(Ω2,Ω1,Ω0)[m1_2_1:Ω2->Ω1,m1_2_2:Ω2->Ω1,m1_2_3:Ω2->Ω1,m2_3_1:Ω1->Ω0,
    m2_3_2:Ω1->Ω0,m2_3_3:Ω1->Ω0]

julia> ρ_Ω = RelationsOfAlgebroid( A_Ω )
GAP: [ (Ω2)-[1*(m1_2_1*m2_3_1)]->(Ω0),
       (Ω2)-[1*(m1_2_2*m2_3_1) + 1*(m1_2_1*m2_3_2)]->(Ω0),
       (Ω2)-[1*(m1_2_2*m2_3_2)]->(Ω0),
       (Ω2)-[1*(m1_2_3*m2_3_1) + 1*(m1_2_1*m2_3_3)]->(Ω0),
       (Ω2)-[1*(m1_2_3*m2_3_2) + 1*(m1_2_2*m2_3_3)]->(Ω0),
       (Ω2)-[1*(m1_2_3*m2_3_3)]->(Ω0) ]
```

That is, the algebroid $\mathbf{A}_\Omega$ is defined by the following quiver



subject to the relations $\{m^i_{12}m^i_{23} | i = 1,2,3\} \cup \{m^i_{12}m^j_{23} + m^j_{12}m^i_{23} | i,j = 1,2,3, i \neq j\}$. The arrows of the quiver correspond to the irreducible morphisms of $\mathbf{C}_\Omega$. We can translate back and forth via the abstraction and relatization functors:

$$\text{abs} \ : \ \mathbf{C}_\Omega \rightleftarrows \mathbf{A}_\Omega \ : \ \text{rel}$$

```
julia> C_Ω = FullSubcategory( Ω )
GAP: The full subcategory { Ω2, Ω1, Ω0 }

julia> abs = IsomorphismOntoAlgebroid( Ω )
GAP: Abstraction isomorphism

julia> rel = IsomorphismFromAlgebroid( Ω )
GAP: Realization isomorphism
```

```
julia> m = 𝐀_Ω."m1_2_1"
GAP: (Ω2)-[{ 1*(m1_2_1) }]->(Ω1)

julia> rel_m = ApplyFunctor( rel, m )
GAP: A morphism in full subcategory given by: <A morphism in Homotopy category(
    Additive closure( Algebroid( (Q * q_𝒪) / [ -1*(x1*y0) + 1*(x0*y1), -1*(x2*y0) + 1*(
    x0*y2), -1*(x2*y1) + 1*(x1*y2) ] ) ) ) with active lower bound 0 and active upper
    bound 1>

julia> rel_m == IrreducibleMorphisms( Ω, 1, 2 )[ 1 ]
true

julia> Show( rel_m )
```

$$
\begin{array}{ccccc}
\mathcal{O}(2) & - & () & \rightarrow & 0 \\
\uparrow & & & & \uparrow \\
\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} & & & & () \\
|_1 & & & & |_1 \\
\mathcal{O}(1)^{\oplus 3} & - & \begin{pmatrix} \mathcal{O}(1) \\ \cdot \\ \cdot \end{pmatrix} & \rightarrow & \mathcal{O}(1) \\
\uparrow & & & & \uparrow \\
\begin{pmatrix} x_1 & -x_0 & \cdot \\ x_2 & \cdot & -x_0 \\ \cdot & x_2 & -x_1 \end{pmatrix} & & & & \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \\
|_0 & & & & |_0 \\
\mathcal{O}(0)^{\oplus 3} & - & \begin{pmatrix} \cdot & \mathcal{O}(0) & \cdot \\ \cdot & \cdot & \mathcal{O}(0) \\ \cdot & \cdot & \cdot \end{pmatrix} & \rightarrow & \mathcal{O}(0)^{\oplus 3}
\end{array}
$$

The category $\mathcal{K}^b\left(\mathbf{A}_{\mathcal{O}}^{\oplus}\right)$ is generated by $\mathcal{O}(0), \mathcal{O}(1)$ and $\mathcal{O}(2)$, hence Ω is complete if and only if $\mathcal{O}(0), \mathcal{O}(1)$ and $\mathcal{O}(2)$ are contained in the triangulated subcategory $\mathcal{T}_\Omega := \Omega^\triangle \subseteq \mathcal{K}^b\left(\mathbf{A}_{\mathcal{O}}^{\oplus}\right)$.

```
julia> TΩ = TriangulatedSubcategory( Ω )
GAP: The triangulated subcategory generated by {Ω2, Ω1, Ω0}

julia> 𝒪0 = [ 𝐀_𝒪."𝒪0" ] / 𝐀_𝒪add / K𝐀_𝒪add
GAP: <An object in Homotopy category( Additive closure( Algebroid( (Q * q_𝒪) / [ -1*(x1
    *y0) + 1*(x0*y1), -1*(x2*y0) + 1*(x0*y2), -1*(x2*y1) + 1*(x1*y2) ] ) ) ) with
    active lower bound 0 and active upper bound 0>
julia> IsWellDefined( 𝒪0 / TΩ )
true

julia> 𝒪1 = [ 𝐀_𝒪."𝒪1" ] / 𝐀_𝒪add / K𝐀_𝒪add;
julia> IsWellDefined( 𝒪1 / TΩ )
true
```

```
julia> 𝒪2 = [ A_𝒪."𝒪2" ] / A_𝒪add / KA_𝒪add;
julia> IsWellDefined( 𝒪2 / TΩ )
true
```

That is, $\mathcal{O}(0), \mathcal{O}(1)$ and $\mathcal{O}(2)$ considered as objects in $\mathcal{T}_\Omega$ are well-defined, hence they belong to $\mathcal{T}_\Omega$ and Ω is indeed a complete strong exceptional sequence in $\mathcal{K}^b\Big(\mathbf{A}_\mathcal{O}^\oplus\Big)$.

Since $\mathbf{A}_\mathcal{O}^\oplus$ is k-linear Hom-finite additive category, the complete strong exceptional sequence Ω induces a pair of exact quasi-inverses

$$G : \mathcal{K}^b\Big(\mathbf{A}_\mathcal{O}^\oplus\Big) \rightleftarrows \mathcal{K}^b\Big(\mathbf{C}_\Omega^\oplus\Big) : F$$

which we call *exceptional replacement* resp. *convolution* functors.

```
julia> G = ReplacementFunctor( Ω )
GAP: Replacement functor
julia> Display( G )
Replacement functor:

Homotopy category( Additive closure( Algebroid( (Q * q_𝒪) / [ -1*(x1*y0) + 1*(x0*y1),
    -1*(x2*y0) + 1*(x0*y2), -1*(x2*y1) + 1*(x1*y2) ] ) ) )
  |
  V
Homotopy category( Additive closure( The full subcategory {Ω2, Ω1, Ω0} ) )

julia> F = ConvolutionFunctor( Ω )
GAP: Convolution functor
julia> Display( F )
Convolution functor:

Homotopy category( Additive closure( The full subcategory {Ω2, Ω1, Ω0} ) )
  |
  V
Homotopy category( Additive closure( Algebroid( (Q * q_𝒪) / [ -1*(x1*y0) + 1*(x0*y1),
    -1*(x2*y0) + 1*(x0*y2), -1*(x2*y1) + 1*(x1*y2) ] ) ) )
```

Applying the functor G on the objects $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$ returns a representation of them in $\mathcal{K}^b(\Omega^\oplus)$:

```
julia> G_𝒪0 = ApplyFunctor( G, 𝒪0 )
GAP: <An object in Homotopy category( Additive closure( The full subcategory {Ω2, Ω1,
    Ω0} ) ) with active lower bound 0 and active upper bound 0>

julia> G_𝒪1 = ApplyFunctor( G, 𝒪1 )
GAP: <An object in Homotopy category( Additive closure( The full subcategory {Ω2, Ω1,
    Ω0} ) ) with active lower bound -1 and active upper bound 0>

julia> G_𝒪2 = ApplyFunctor( G, 𝒪2 )
GAP: <An object in Homotopy category( Additive closure( The full subcategory {Ω2, Ω1,
    Ω0} ) ) with active lower bound -2 and active upper bound 0>
```

For a better visualization of $G(\mathcal{O}(0)), G(\mathcal{O}(1))$ and $G(\mathcal{O}(2))$, we translate the results via the extension of abs: $\mathbf{C}_\Omega \xrightarrow{\sim} \mathbf{A}_\Omega$ to an isomorphism

$$\mathcal{K}^b\left(\mathbf{C}_\Omega^\oplus\right) \xrightarrow{\sim} \mathcal{K}^b\left(\mathbf{A}_\Omega^\oplus\right).$$

```
julia> abs = ExtendFunctorToAdditiveClosures( abs );
julia> abs = ExtendFunctorToHomotopyCategoriesByCochains( abs );
julia> Show( ApplyFunctor( abs, G_𝒪0 ) )
```

$$\Omega^0(0)$$

```
julia> Show( ApplyFunctor( abs, G_𝒪1 ) )
```

$$\begin{array}{c} \Omega^0(0)^{\oplus 3} \\ \uparrow \\ \begin{pmatrix} -m_{2,3}^1 & -m_{2,3}^2 & -m_{2,3}^3 \end{pmatrix} \\ |_{-1} \\ \Omega^1(1) \end{array}$$

```
julia> Show( ApplyFunctor( abs, G_𝒪2 ) )
```

$$\begin{array}{c} \Omega^0(0)^{\oplus 6} \\ \uparrow \\ \begin{pmatrix} -m_{2,3}^1 & -m_{2,3}^2 & -m_{2,3}^3 & \cdot & \cdot & \cdot \\ \cdot & -m_{2,3}^1 & \cdot & -m_{2,3}^2 & -m_{2,3}^3 & \cdot \\ \cdot & \cdot & -m_{2,3}^1 & \cdot & -m_{2,3}^2 & -m_{2,3}^3 \end{pmatrix} \\ |_{-1} \\ \Omega^1(1)^{\oplus 3} \\ \uparrow \\ \begin{pmatrix} -m_{1,2}^1 & -m_{1,2}^2 & -m_{1,2}^3 \end{pmatrix} \\ |_{-2} \\ \Omega^2(2) \end{array}$$

Let us apply the comonad $F \circ G\colon \mathcal{K}^b\left(\mathbf{A}_\mathcal{O}^\oplus\right) \xrightarrow{\sim} \mathcal{K}^b\left(\mathbf{A}_\mathcal{O}^\oplus\right)$ on the object $\mathcal{O}(2)$:

```
julia> FG_𝒪2 = ApplyFunctor( F, G_𝒪2 )
GAP: <An object in Homotopy category( Additive closure( Algebroid( (Q * q_𝒪) / [ -1*(x1
    *y0) + 1*(x0*y1), -1*(x2*y0) + 1*(x0*y2), -1*(x2*y1) + 1*(x1*y2) ] ) ) ) with
    active lower bound -2 and active upper bound 1>

julia> Show( FG_𝒪2 )
```

$$
\begin{array}{c}
0 \\
\uparrow \\
() \\
|_{0} \\
\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(0)^{\oplus 6} \\
\uparrow \\
\begin{pmatrix}
y_0 & \mathcal{O}(1) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
y_1 & \cdot & \mathcal{O}(1) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
y_2 & \cdot & \cdot & \mathcal{O}(1) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & -x_0 & \cdot & \cdot & -\mathcal{O}(0) & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & -x_1 & \cdot & \cdot & \cdot & -\mathcal{O}(0) & \cdot & \cdot & \cdot & \cdot \\
\cdot & -x_2 & \cdot & \cdot & \cdot & \cdot & -\mathcal{O}(0) & \cdot & \cdot & \cdot \\
\cdot & \cdot & -x_0 & \cdot & \cdot & -\mathcal{O}(0) & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -x_1 & \cdot & \cdot & \cdot & \cdot & -\mathcal{O}(0) & \cdot & \cdot \\
\cdot & \cdot & -x_2 & \cdot & \cdot & \cdot & \cdot & \cdot & -\mathcal{O}(0) & \cdot \\
\cdot & \cdot & \cdot & -x_0 & \cdot & \cdot & -\mathcal{O}(0) & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -x_1 & \cdot & \cdot & \cdot & \cdot & -\mathcal{O}(0) & \cdot \\
\cdot & \cdot & \cdot & -x_2 & \cdot & \cdot & \cdot & \cdot & \cdot & -\mathcal{O}(0)
\end{pmatrix} \\
|_{-1} \\
\mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(0)^{\oplus 9} \\
\uparrow \\
\begin{pmatrix}
x_1 & -x_0 & \cdot & \cdot & \mathcal{O}(0) & \cdot & -\mathcal{O}(0) & \cdot & \cdot & \cdot & \cdot & \cdot \\
x_2 & \cdot & -x_0 & \cdot & \cdot & \mathcal{O}(0) & \cdot & \cdot & \cdot & -\mathcal{O}(0) & \cdot & \cdot \\
\cdot & x_2 & -x_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \mathcal{O}(0) & \cdot & -\mathcal{O}(0) & \cdot
\end{pmatrix} \\
|_{-2} \\
\mathcal{O}(0)^{\oplus 3}
\end{array}
$$

Since the comonad is an autoequivalence, the objects $\mathcal{O}(2)$ and $(F \circ G)(\mathcal{O}(2))$ should be isomorphic in $\mathcal{K}^b\left(\mathbf{A}_{\mathcal{O}}^{\oplus}\right)$. Such an isomorphism can be computed by applying the counit $\epsilon\colon F \circ G \to \mathrm{id}_{\mathcal{K}^b\left(\mathbf{A}_{\mathcal{O}}^{\oplus}\right)}$ on $\mathcal{O}(2)$:

```
julia> ϵ = CounitOfConvolutionReplacementAdjunction( Ω )
GAP: F( G( - ) ) ⇒ Id
julia> ϵ_𝒪2 = ϵ( 𝒪2 )
<A morphism in Homotopy category( Additive closure( Algebroid( (Q * q_𝒪) / [ -1*(x1*y0)
     + 1*(x0*y1), -1*(x2*y0) + 1*(x0*y2), -1*(x2*y1) + 1*(x1*y2) ] ) ) ) with active
    lower bound 0 and active upper bound 0>
julia> IsWellDefined( ϵ_𝒪2 )
true
julia> Show( MorphismAt( ϵ_𝒪2, 0 ) )
```

$$\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(0)^{\oplus 6} \xrightarrow{\begin{pmatrix} \mathcal{O}(2) \\ -y_0 \\ -y_1 \\ -y_2 \\ x_0y_0 \\ x_0y_1 \\ x_0y_2 \\ x_1y_1 \\ x_1y_2 \\ x_2y_2 \end{pmatrix}} \mathcal{O}(2)$$

Let us check that $\epsilon(\mathcal{O}(2))$ is an isomorphism and then compute its inverse:

```
julia> IsIsomorphism( ϵ_𝒪2 )
true
julia> inv_ϵ_𝒪2 = InverseForMorphisms( ϵ_𝒪2 )
<A morphism in Homotopy category( Additive closure( Algebroid( (Q * q_𝒪) / [ -1*(x1*y0)
     + 1*(x0*y1), -1*(x2*y0) + 1*(x0*y2), -1*(x2*y1) + 1*(x1*y2) ] ) ) ) with active
    lower bound 0 and active upper bound 0>
julia> Show( MorphismAt( inv_ϵ_𝒪2, 0 ) )
```

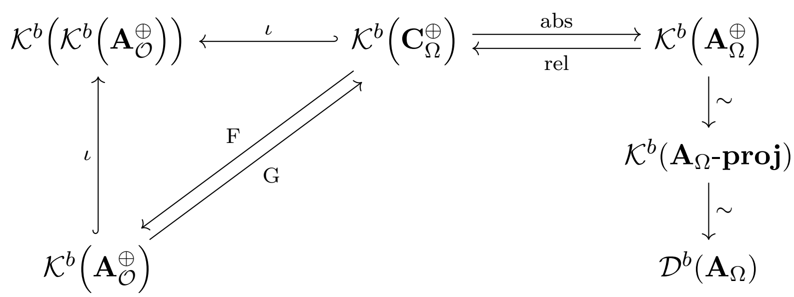
$$\mathcal{O}(2) \xrightarrow{\begin{pmatrix} \mathcal{O}(2) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}} \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(0)^{\oplus 6}$$

We can also verify that $\epsilon(\mathcal{O}(2))$ is an isomorphism by checking whether $J(\epsilon(\mathcal{O}(2)))$ is an isomorphism in $\mathcal{D}^b(\mathbf{A}_{\mathcal{O}})$. In fact, this is usually faster because checking whether a morphism φ in the derived category is isomorphism amounts to checking whether the numerator of its defining roof is a quasi-isomorphism.

```
julia> J_ϵ_𝒪2 = ApplyFunctor( J, ϵ_𝒪2 )
GAP: <A morphism in Derived category( FunctorCategory( Algebroid( (Q * q_𝒪_op) / [ 1*(
    y1*x0) - 1*(y0*x1), 1*(y2*x0) - 1*(y0*x2), 1*(y2*x1) - 1*(y1*x2) ] ) -> Category of
     matrices over Q ) )>
julia> IsIsomorphism( J_ϵ_𝒪2 )
true
```

Since $\mathcal{K}^b\left(\mathbf{C}_{\Omega}^{\oplus}\right) \cong \mathcal{K}^b\left(\mathbf{A}_{\Omega}^{\oplus}\right)$ via the abstraction functor and $\mathcal{K}^b\left(\mathbf{A}_{\Omega}^{\oplus}\right) \simeq \mathcal{D}^b(\mathbf{A}_{\Omega})$ via the Yoneda embedding, we get an equivalence

$$I : \mathcal{K}^b\left(\mathbf{C}_{\Omega}^{\oplus}\right) \xrightarrow{\sim} \mathcal{D}^b(\mathbf{A}_{\Omega}).$$



Let us check that the images of $\mathcal{O}(0), \mathcal{O}(1), \mathcal{O}(2)$ under $I \circ G$ belong to the standard Abelian heart $\mathbf{A}_{\Omega}$-**mod** $\subset \mathcal{D}^b(\mathbf{A}_{\Omega})$.

```
julia> I = EquivalenceOntoDerivedCategory( Ω )
GAP: Equivalence functor onto derived category of endomorphism algebra

julia> Display( I )
Equivalence functor onto derived category of endomorphism algebra:
Homotopy category( Additive closure( The full subcategory {Ω2, Ω1, Ω0} ) )
  |
  V
Derived category( The category of functors: Algebroid( End( Ω2 ⊕ Ω1 ⊕ Ω0 ) ) ->
    Category of matrices over Q )

julia> IG_𝒪0 = ApplyFunctor( I, G_𝒪0 )
GAP: <An object in Derived category( The category of functors: Algebroid( End( Ω2 ⊕ Ω1
     ⊕ Ω0 ) ) -> Category of matrices over Q ) with active lower bound 0 and active
    upper bound 0>

julia> CohomologySupport( IG_𝒪0 )
GAP: [ 0 ]

julia> IG_𝒪1 = ApplyFunctor( I, G_𝒪1 )
GAP: <An object in Derived category( The category of functors: Algebroid( End( Ω2 ⊕ Ω1
     ⊕ Ω0 ) ) -> Category of matrices over Q ) with active lower bound -1 and active
    upper bound 0>

julia> CohomologySupport( IG_𝒪1 )
GAP: [ 0 ]

julia> IG_𝒪2 = ApplyFunctor( I, G_𝒪2 )
GAP: <An object in Derived category( The category of functors: Algebroid( End( Ω2 ⊕ Ω1
     ⊕ Ω0 ) ) -> Category of matrices over Q ) with active lower bound -2 and active
    upper bound 0>

julia> CohomologySupport( IG_𝒪2 )
GAP: [ 0 ]
```

Since the cohomology is concentrated in degree 0, the objects $H^0((I \circ G)(\mathcal{O}(i))), i = 0, 1, 2$ live in the standard Abelian heart of $\mathcal{D}^b(\mathbf{A}_\Omega)$. In fact, their direct sum, say U, is a generalized tilting object in $\mathbf{A}_\Omega$**-mod**. By Happel's theorem the derived functors

$$- \otimes^{\mathbb{L}} U \;:\; \mathcal{D}^b(\operatorname{End} U) \rightleftarrows \mathcal{D}^b(\mathbf{A}_\Omega) \;:\; \mathbb{R}\mathrm{Hom}(U, -)$$

define an adjoint pair of exact equivalences. We refer to the Appendix E for a demonstration to Happel's theorem.

CHAPTER 2

Category Constructors

One of the main design features of CAP [**GSP22**] is its support for:

- defining *categorical doctrines*;
- building *category constructors* which create instances of such doctrines.

We use the term *doctrine*, as already mentioned in the introduction, in a loose sense to describe categories with specified additional properties or structures, e.g., additive, Abelian, monoidal, or closed monoidal categories (cf. Appendix A).

A category constructor, as the name suggests, is a procedure which outputs a category as a particular instance of a specific doctrine. Further we distinguish between

- *doctrine-based* category constructors;
- *primitive* category constructors.

A **doctrine-based** category constructor takes as input one or several categories $\mathscr{C}_1, \ldots, \mathscr{C}_n$ of specific doctrines $\mathcal{D}_1, \ldots, \mathcal{D}_n$ and outputs a category $\mathscr{A}$ as a particular instance of a specific doctrine $\mathcal{D}$ according to the two following rules: The constructor

- specifies the data structure of the objects and morphisms in $\mathscr{A}$ in terms of objects and morphisms of the input categories;
- expresses all defining categorical operations of the doctrine $\mathcal{D}$ (of its output category) as algorithms written in terms of the categorical operations supported by the doctrines $\mathcal{D}_1, \ldots, \mathcal{D}_n$ (of its input categories).

In particular, these algorithms do not depend on the instances $\mathscr{C}_1, \ldots, \mathscr{C}_n$ but only on their doctrines $\mathcal{D}_1, \ldots, \mathcal{D}_n$.

For example, the category constructor **`AdditiveClosure`** takes as input a preadditive category $\mathscr{C}$ and returns the additive closure $\mathscr{C}^{\oplus}$, along with an embedding of $\mathscr{C}$ in $\mathscr{C}^{\oplus}$ which is universal among all additive functors from $\mathscr{C}$ into an additive category (cf. Section 2.2.2). Another example is the FREYD category constructor **`FreydCategory`** which takes as input an additive category $\mathscr{C}$ and outputs another additive category $\mathcal{A}(\mathscr{C})$ along with an embedding of $\mathscr{C}$ in $\mathcal{A}(\mathscr{C})$ which is universal among all functors from $\mathscr{C}$ into a category with cokernels. In fact, $\mathcal{A}(\mathscr{C})$ is Abelian if $\mathscr{C}$ admits weak-kernels (cf. Section 2.2.3).

We refer to category constructors that are not doctrine-based as **primitive** category constructors. For example, the constructor **`FreeCategory`** takes as input a quiver `q` and outputs the free category $\mathcal{F}_{\mathtt{q}}$ defined by `q` (cf. Section 2.1.1). Another example is the category constructor **`RingAsCategory`** which takes as input a ring R and outputs the preadditive category $\mathscr{C}(R)$ consisting of a single object, say $*$, whose endomorphisms are the elements of R (cf. Section 2.1.2).

The attempt to implement doctrine-based constructors has led us to the development of new categorical concepts:

(1) Our desire to develop a doctrine-based constructor for certain stable categories which are described as quotients by two-sided ideals led us to the concept of classes of lifting

and colifting objects (cf. Section 2.2.6). The two-sided ideal of such a stable category consists of the morphisms that factor through an object in the class of lifting (reps. colifting) objects. The central point is that the membership in such a two-sided ideal is algorithmically decidable: Checking the membership translates to a lift or colift problem. This is essential for equipping the stable categories with a decidable equality of morphisms (cf. Lemma 2.53). We use this stability notion to provide a unified construction for stable module categories by projectives or injectives (cf. Example 2.60), stable categories of Frobenius categories (cf. Section 5.3) and for bounded homotopy categories (cf. Theorem 3.29)[1].

(2) Another example is the emergence of the concept of a $\mathscr{D}$-homomorphism structure of a category $\mathscr{C}$, which Posur proposed in his constructive approach to FREYD categories [**Pos21a**]. Equipping a category $\mathscr{C}$ with a $\mathscr{D}$-homomorphism structure enables us to translate two-sided equations in $\mathscr{C}$ into lift problems (i.e., one-sided equations[2]) in $\mathscr{D}$. If $\mathscr{C}$ is preadditive, then a $\mathscr{D}$-homomorphism structure enables us to reduce solving arbitrary two-sided linear systems[3] in $\mathscr{C}$ to computing lifts in $\mathscr{D}$ (cf. Theorem 4.17).

Solving two-sided linear systems is essential for several category constructors, primarily to ensure that the output category has decidable equality of morphisms. For instance, the equality of morphisms in the FREYD category $\mathcal{A}(\mathscr{C})$ translates to a lift problem in $\mathscr{C}$ (cf. Section 2.2.3); and the equality of morphisms in the bounded homotopy category $\mathcal{K}^b(\mathscr{C})$ translates to solving two-sided linear systems in $\mathscr{C}$, which finally reduces to computing lifts in the range $\mathscr{D}$ of the homomorphism structure on $\mathscr{C}$ (cf. Section 3.2). Teaching a category constructor how to equip its output category with a $\mathscr{D}$-homomorphism structure is essential for almost all of the proposed approaches in this thesis. This topic is covered in details in Chapter 4.

Meanwhile, the CAP universe[4] supports several categorical doctrines [**CAP21b**] and includes various category constructors [**CAP21c**]. One can now compose the category constructors to produce new instances of categories in which one can conduct the categorical operations supported by their doctrines. A category that is created by composing two or more category constructors will be referred to as a *tower* of categories. Such a tower necessarily starts by applying one or more primitive category constructors on one or more data structures (e.g., sets, rings, quivers, etc).

Let us illustrate the idea of towers by an explicit example: Let $\mathtt{q}$ a quiver (cf. Definition 2.1), k be a field and k-**mat** the category of matrices over k (cf. Section 2.1.3). Then

(1) $\mathcal{F}_{\mathtt{q}} \coloneqq$ `FreeCategory`($\mathtt{q}$) outputs the *universal free category* $\mathcal{F}_{\mathtt{q}}$ generated by $\mathtt{q}$. That is, this category constructor adds formal identity morphisms and enables us to compute composition of morphisms (cf. Section 2.1.1).

(2) $k\mathcal{F}_{\mathtt{q}} \coloneqq$ `LinearClosure`($k, \mathcal{F}_{\mathtt{q}}$) outputs the *universal k-linear closure* $k\mathcal{F}_{\mathtt{q}}$ of $\mathcal{F}_{\mathtt{q}}$. That is, this category constructor adds formal k-linear combinations of morphisms in $\mathcal{F}_{\mathtt{q}}$ (cf. Section 2.2.1).

Let ρ be finite set of morphisms in $k\mathcal{F}_{\mathtt{q}}$. Then

[1] and even FREYD categories (cf. Corollary 2.65).

[2] Any lift morphism $\alpha\colon A \to C$ along $\beta\colon B \to C$ is a solution to the one-sided equation $X \cdot \beta = \alpha$.

[3] Hence, a $\mathscr{D}$-homomorphism structure on a $\mathscr{C}$ can be used to *derive* methods for lifts and colifts operations in $\mathscr{C}$ as they are special linear systems.

[4] That is, all of the packages that are based on `CAP` [**CAP21d**].

(3) $\mathscr{A} := k\mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle :=$ `Algebroid`$(k\mathcal{F}_{\mathfrak{q}}, \rho)$ outputs the *k-linear finitely presented category* defined by $\mathfrak{q}$ subject to the relations ρ, i.e., the quotient category of $k\mathcal{F}_{\mathfrak{q}}$ modulo the two-sided ideal of morphisms generated by ρ (cf. Section 2.2.5).

(4) $\mathbf{mod}\text{-}\mathscr{A} := [\mathscr{A}, k\text{-}\mathbf{mat}] :=$ `FunctorCategory`$(\mathscr{A}, k\text{-}\mathbf{mat})$ outputs the category of k-linear functors from $\mathscr{A}$ to k-**mat**, which is k-linear Abelian and equipped with a (k-**mat**)-homomorphism structure (cf. Sections 2.2.7 and 4.2).

We can perform homological algebra computations over $\mathbf{mod}\text{-}\mathscr{A}$. For instance,

(5) $\mathcal{C}^b(\mathbf{mod}\text{-}\mathscr{A}) :=$ `ComplexCategoryByCochains`$(\mathbf{mod}\text{-}\mathscr{A})$ outputs the bounded cochain complex category of $\mathbf{mod}\text{-}\mathscr{A}$, which is Abelian (cf. Section 3.1) and is equipped with a (k-**mat**)-homomorphism structure (cf. Section 4.4).

(6) $\mathcal{K}^b(\mathbf{mod}\text{-}\mathscr{A}) :=$ `HomotopyCategoryByCochains`$(\mathbf{mod}\text{-}\mathscr{A})$ outputs the bounded homotopy category of $\mathbf{mod}\text{-}\mathscr{A}$, which is triangulated (cf. Section 3.2), has decidable equality of morphisms (cf. Corollary 3.26) and is equipped with a (k-**mat**)-homomorphism structure (cf. Section 4.5).

If $\mathscr{A}$ is admissible, then $\mathbf{mod}\text{-}\mathscr{A}$ is Abelian with enough projective and injective objects (cf. Corollary 2.96). If, furthermore, $\mathbf{mod}\text{-}\mathscr{A}$ has finite global dimension[5], then

(7) $\mathcal{D}^b(\mathbf{mod}\text{-}\mathscr{A}) :=$ `DerivedCategoryByCochains`$(\mathbf{mod}\text{-}\mathscr{A})$ outputs the bounded derived category of $\mathbf{mod}\text{-}\mathscr{A}$, which is triangulated (cf. Section 3.4), has decidable equality of morphisms (cf. Theorem 3.63) and is equipped with a (k-**mat**)-homomorphism structure (cf. Section 4.5). In particular, given two objects A, B in $\mathbf{mod}\text{-}\mathscr{A}$, we can employ the ($k$-**mat**)-homomorphism structure of $\mathcal{D}^b(\mathbf{mod}\text{-}\mathscr{A})$ to compute a basis of

$$\mathrm{Ext}^n(A, B) := \mathrm{Hom}_{\mathcal{D}^b(\mathbf{mod}\text{-}\mathscr{A})}(A, \Sigma^n(B)).$$

A categorical computation at the top category of a tower is usually accomplished by (1) incrementally unwrapping portions of the passed arguments until they are represented in terms of the algebraic or combinatorial data structure at the bottom of the tower, (2) performing the computation, and then, (3) wrapping the results all the way up to the top category to obtain the result. In almost all cases, the categorical operation must compute several intermediate values before returning the final result[6]. In these cases, the frequent wrapping and unwrapping adds overhead to the computations, slowing them down. This overhead can be avoided by reworking the algorithms of the top category to interact directly with the given data structure, bypassing the categories underneath. In other words, we "compile" the composition of category constructors forming the tower to a primitive category constructor. For instance, if we compile the preceding tower forming the category $\mathbf{mod}\text{-}\mathscr{A}$, we recover the category $\mathbf{rep}(\mathfrak{q}, \rho)$ of finite dimensional ρ-bounded quiver representations of $\mathfrak{q}$ (cf. Section 2.2.7). The `QPA2` [**Qt21**] provides a primitive category constructor[7] which creates this category. Another example for this idea is discussed in Section 2.2.3.

Reworking the algorithms of the top category allows us to take advantage of some of the features of the low-level data structure that would be inaccessible by a rigid categorical implementation. Yet, as category constructors get more complex, compiling them as primitive constructors becomes more cumbersome and error-prone, not to mention that the mathematics

[5]For example, if $\mathfrak{q}$ is acyclic (cf. Corollary 2.96).

[6]Especially if the operation has been *derived* from other *basic* operations (cf. [**Gut17**]).

[7]See `CategoryOfQuiverRepresentations`

behind the code becomes increasingly difficult to grasp, hence, making it harder to maintain the code.

The primary goal of the GAP package `CompilerForCAP` [**Zic22**] is to automate such conversions of high-level algorithms to low-level algorithms that operate directly on the provided data structures. Additionally, the user can assist the compiler by providing additional *rewriting rules* to enhance the generated code's quality. In other words, one may keep building doctrine-based implementations while the compiler handles the conversion of their composite to primitive constructors.

2.1. Primitive Category Constructors

In this section we list the primitive category constructors that are relevant to this thesis.

2.1.1. Free Categories Defined by Quivers. Finite dimensional k-algebras over some field k are often studied in terms of quivers (see e.g., [**DW17**] and [**ASS06**]). Quiver are essential for defining finitely presented categories (cf. Section 2.2.5) and their categories of functors (cf. Section 2.2.7), and to visualize strong exceptional sequences in k-linear triangulated categories (cf. Section 6.2).

Let us first state the definition of a quiver:

Definition 2.1. A **right quiver**[8] $\mathfrak{q}$ consists of the following data:

(1) A finite set $\mathfrak{q}_0$ (**vertices**).
(2) A finite set $\mathfrak{q}_1$ (**arrows**).
(3) Two maps $\mathfrak{s}, \mathfrak{r} \colon \mathfrak{q}_1 \to \mathfrak{q}_0$, called the **source** resp. **range** maps.

A **path** p of length $\ell \geq 1$ in a right quiver $\mathfrak{q}$ is a sequence $p = r_1 r_2 \dots r_\ell$ of arrows in $\mathfrak{q}_1$ such that $\mathfrak{r}(r_i) = \mathfrak{s}(r_{i+1})$ for $i = 1, \dots, \ell - 1$. We define $\mathfrak{s}(p)$ by $\mathfrak{s}(r_1)$ and $\mathfrak{r}(p)$ by $\mathfrak{r}(r_\ell)$. For each vertex $v \in \mathfrak{q}_0$, we define the **trivial path** e_v of length 0 with $\mathfrak{s}(e_v) = \mathfrak{r}(e_v) = v$. The quiver $\mathfrak{q}$ will be called an **acyclic** if $\mathfrak{s}(p) \neq \mathfrak{r}(p)$ for all nontrivial paths in $\mathfrak{q}$.

Given a quiver $\mathfrak{q}$, we can turn $\mathfrak{q}$ into a category by formally equipping all objects with identity morphisms and then defining the composition in term of a concatenation of paths:

Definition 2.2. Let $\mathfrak{q}$ be quiver. The **free category**[9] $\mathcal{F}_\mathfrak{q}$ generated by $\mathfrak{q}$ is defined by the following data:

(1) The object class is $\mathfrak{q}_0$.
(2) For two objects u, v, we define $\mathrm{Hom}_{\mathcal{F}_\mathfrak{q}}(u, v)$ by the set of all paths from u to v.
(3) Composition of morphisms is defined by the concatenation of the underlying paths.
(4) The identity morphism of an object u is the trivial path e_u.

2.1.2. (Graded) Ring as a Preadditive Category. Every ring can be interpreted as a preadditive category:

Definition 2.3. Let R be unital ring. The **ring category**[10] of R, denoted by $\mathscr{C}(R)$, is defined by the following data:

(1) The object class consists of a single object, say $*$.

[8]The term *right* is adapted from the GAP package `QPA2` [**Qt21**]. The distinction between right and left quivers only affects the definition of a path in the quiver.

[9]The associated category constructor is **`FreeCategory`**(-) [**BS21a**].

[10]The associated category constructor is **`RingAsCategory`** [**BP19a**].

(2) $\mathrm{Hom}_{\mathscr{C}(R)}(*, *) := R$.
(3) The composition of two morphisms $r \cdot s$ is defined by their multiplication rs as ring elements.
(4) The identity morphism of $*$ is defined by the unit of the ring.

Remark 2.4. The category $\mathscr{C}(R)$ has decidable equality of morphisms if and only if R is **constructable**, i.e., we have an algorithm to decide equality of elements in R.

Remark 2.5. With this interpretation in mind, we can think of a right (resp. left) R-module as an additive covariant (resp. contravariant) functor from $\mathscr{C}(R)$ to the category **Ab** of Abelian groups.

Remark 2.6. If R is k-algebra for some commutative ring k, then $\mathscr{C}(R)$ is a k-linear category.

The same can be done for graded rings:

Definition 2.7. Let G be an additively written Abelian group. A ring R is called G-graded if there is a subring $R_0 \subset R$ and for every $g \in G$ an R_0-submodule R_g such that

$$R = \bigoplus_{g \in G} R_g$$

and $R_g R_h \subseteq R_{g+h}$ for all $g, h \in G$. A nonzero element $x \in R_g$ is called **homogeneous of degree** g and we write $\deg x = g$.

Every graded ring defines a preadditive category:

Definition 2.8. Let $R = \bigoplus_{g \in G} R_g$ be G-graded ring. The **graded ring category**[11] of R, denoted by $\mathscr{C}(R, G)$, is defined by the following data:

(1) The object class of $\mathscr{C}(R, G)$ is given by G.
(2) For two objects g and h we define $\mathrm{Hom}_{\mathscr{C}(R,G)}(g, h) := R_{h-g}$.
(3) The identity morphism of an object $h \in G$ is given by $1 \in R_0$.
(4) The composition is inherited from the ring multiplication.

Remark 2.9. $\mathscr{C}(R, G)$ has decidable equality of morphisms if and only if R is constructable.

Remark 2.10. If R_0 is a commutative ring, then $\mathscr{C}(R, G)$ is an R_0-linear category.

2.1.3. Category of (Graded) Rows of a (Graded) Ring. The category of rows over a ring R provides a model for category of free row R-modules with finite rank. This category is useful because its FREYD category provides a computer friendly model for the category R-**fpmod** of finitely presented R-modules (cf. Section 2.2.3).

Definition 2.11. Let R be a ring. The category R-**rows** of rows[12] over R is defined by the following data:

(1) The object class is $\mathbb{N}_0$.
(2) For objects m and n, we define $\mathrm{Hom}_{R\text{-}\mathbf{rows}}(m, n) := R^{m \times n}$.
(3) The composition is defined by the usual matrix multiplication.
(4) The identity morphism of an object $n \in \mathbb{N}_0$ is given by the identity matrix I_n over R.

[11] The associated category constructor is `GradedRingAsCategory` [**BP19a**].
[12] The associated category constructor is `CategoryOfRows` [**BP19a**].

Remark 2.12. Similarly, we can define the category of columns R-**cols**. It is easy to check that R-**cols** $\cong$ (R-**rows**)$^{\mathrm{op}}$.

Remark 2.13. Obviously, R-**rows** (resp. R-**cols**) has decidable equality of morphisms if and only if R is constructable.

Remark 2.14. If R is a nonzero commutative ring, then it has the invariant basis number property[13]. In this case, R-**rows** provides a skeletal model for the full subcategory of R-**mod**[14] that is generated the free R-modules of finite rank.

Remark 2.15. For an arbitrary ring R, the category R-**rows** is additive. If R is a k-algebra for a commutative ring k, then R-**rows** is a k-linear category.

Example 2.16. For a field k, the category k-**rows** is Abelian. The majority of the required algorithms (cf. Definition A.44) can be derived from GAUSSIAN ALGORITHM. It is obvious that k-**rows** is equivalent to the category vec_k of finite dimensional k-vector spaces via:

$$F\colon \begin{cases} k\text{-}\mathbf{rows} & \to \mathrm{vec}_k, \\ \quad m & \mapsto k^{1\times m}, \\ m \xrightarrow{A} n & \mapsto \begin{cases} k^{1\times m} & \to k^{1\times n}, \\ \quad x & \mapsto x\cdot A. \end{cases} \end{cases}$$

If we equip each object V in vec_k with an ordered basis $\mathcal{B}(V)$, then a morphism $f\colon V\to W$ in vec_k corresponds in k-**rows** to the morphism $A_f\colon \dim_k V \to \dim_k W$ where A_f is the matrix of f with respect to $\mathcal{B}(V)$ and $\mathcal{B}(W)$.

Notation 2.17. For a field k, we might use the notation k-**mat** (stands for **category of matrices over** k) instead of k-**rows**.

The category of graded rows R-**grrows** over a graded ring R provides a model for the full subcategory of R-**grmod**[15] that is generated by the graded free R-modules of finite rank.

Definition 2.18. Let R be a G-graded ring. The category R-**grrows** of graded rows[16] over R is defined by the following data:

(1) The object class is $\bigcup_{n\in\mathbb{N}_0} G^n$, i.e., the objects are the finite tuples of elements in G.
(2) A morphism from $\mathtt{d} = [\mathtt{d}_1,\dots,\mathtt{d}_n] \in G^n$ to $\mathtt{e} = [\mathtt{e}_1,\dots,\mathtt{e}_t] \in G^t$ is a matrix $\mathtt{F} \in R^{n\times t}$ such that $\mathtt{F}$ has homogeneous entries and $\mathtt{F}_{\mathtt{i},\mathtt{j}} = 0$ or $\deg \mathtt{F}_{\mathtt{i},\mathtt{j}} = \mathtt{e}_\mathtt{j} - \mathtt{d}_\mathtt{i}$ for all $j = 1,\dots,t$, $i = 1,\dots,n$. Two such morphisms $\mathtt{F},\mathtt{G}\colon \mathtt{d}\to\mathtt{e}$ are considered equal in R-**grrows** if they are equal as matrices.
(3) The identity morphism of an object $\mathtt{d}\in G^n$ is the identity matrix $\mathtt{I}_n$.
(4) The composition is given by the usual matrix multiplication.

Remark 2.19. R-**grrows** has decidable equality of morphisms if and only if R is constructable.

Remark 2.20. The category R-**grrows** is additive. Furthermore, if R_0 is commutative, then R-**grrows** is R_0-linear.

[13] A ring R has invariant basis number (IBN) if for all positive integers m and n, $R^{1\times m} \cong R^{1\times n}$ (as left R-modules) only if $m = n$.

[14] R-**mod** denotes the category of finitely generated R-modules.

[15] R-**grmod** denotes the category of finitely generated graded R-modules (cf. Remark 2.36).

[16] The associated category constructor is `CategoryOfGradedRows` [**BP19a**].

2.2. Doctrine-based Category Constructors

In this section we list the doctrine-based category constructors that can be applied on the categories introduced in Section 2.1.

2.2.1. Linear Closure Categories. Every category $\mathscr{C}$ can be embedded in a k-linear category for any commutative unital ring k.

Definition 2.21. Let k be a commutative unital ring. We define the k-**linear closure**[17] $k\mathscr{C}$ of a category $\mathscr{C}$ by the following data:

- The objects of $k\mathscr{C}$ are the objects of $\mathscr{C}$.
- For a pair A, B of objects in $k\mathscr{C}$ we define $\mathrm{Hom}_{k\mathscr{C}}(A,B)$ by the k-module freely generated by $\mathrm{Hom}_{\mathscr{C}}(A,B)$, i.e., morphisms in $k\mathscr{C}$ are finite formal k-linear combinations of morphisms in $\mathscr{C}$. The identity morphisms are inherited from $\mathscr{C}$.
- The composition is the k-bilinear extension of the composition in $\mathscr{C}$ to $k\mathscr{C}$.

Remark 2.22. There exists a natural embedding of $\mathscr{C}$ in $k\mathscr{C}$ is defined by

$$\iota\colon \begin{cases} \mathscr{C} & \to k\mathscr{C}, \\ A & \mapsto A, \\ \alpha\colon A \to B & \mapsto 1_k \cdot \alpha. \end{cases}$$

Furthermore, this embedding ι is universal among the functors from $\mathscr{C}$ to a k-linear category.

Example 2.23. The main instance for $\mathscr{C}$ we have in mind is the free category $\mathcal{F}_{\mathfrak{q}}$ defined by some quiver $\mathfrak{q}$ (cf. Section 2.1.1).

2.2.2. Additive Closure Categories. Every preadditive category can be embedded in an additive category.

Definition 2.24. Let $\mathscr{C}$ be a preadditive category. The **additive closure**[18] $\mathscr{C}^{\oplus}$ of $\mathscr{C}$ is defined by the following data:

a. An object in $\mathscr{C}^{\oplus}$ is given by an integer $m \geq 0$ and a list $(A_1, \dots, A_m)$ where A_i belongs to $\mathscr{C}$ for all $i = 1, \dots, m$.
b. A morphism from an object $(A_1, \dots, A_m)$ to another object $(B_1, \dots, B_n)$ is given by a matrix
$$\begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \dots & \alpha_{mn} \end{pmatrix}$$
consisting of morphisms $\alpha_{ij}\colon A_i \to B_j$ in $\mathscr{C}$.

(1) We define the composition by the usual formula for matrix multiplication.
(2) The identity morphism of an object $(A_1, \dots, A_m)$ is given by the diagonal matrix
$$\begin{pmatrix} \mathrm{id}_{A_1} & & 0 \\ & \ddots & \\ 0 & & \mathrm{id}_{A_m} \end{pmatrix}.$$
Equality for morphisms is checked entrywise.

[17]The associated category constructor is `LinearClosure` in [BP19a] resp. `Algebroid` in [BHP+21].
[18]The associated category constructor is `AdditiveClosure`(-) in [BS21a] resp. [BP19a].

Remark 2.25. Clearly, $\mathscr{C}^{\oplus}$ has decidable equality of morphisms if and only if $\mathscr{C}$ has decidable equality of morphisms.

Remark 2.26. There exists a natural embedding functor

$$\iota\colon \begin{cases} \mathscr{C} & \to \mathscr{C}^{\oplus}, \\ A \xrightarrow{\alpha} B & \mapsto (A) \xrightarrow{(\alpha)} (B). \end{cases}$$

Furthermore, this embedding is universal among the additive functors from $\mathscr{C}$ to an additive category.

Example 2.27. The categories of rows and graded rows are equivalent to additive closure categories:

- If R is a unital ring, then R-**rows** $\cong \mathscr{C}(R)^{\oplus}$ (cf. Sections 2.1.2 and 2.1.3).
- If R a G-graded ring for an Abelian group G, then R-**grrows** $\simeq \mathscr{C}(R,G)^{\oplus}$ (cf. Sections 2.1.2 and 2.1.3).

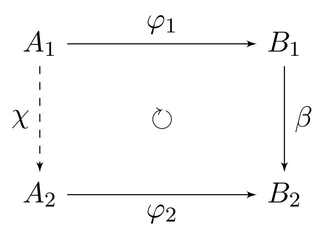
2.2.3. Freyd Categories and Finitely Presented (Graded) R-Modules.

The Freyd category constructor takes as input an additive category $\mathscr{C}$ and outputs a new additive category $\mathcal{A}(\mathscr{C})$ that is equipped with cokernels in a universal way. The category $\mathcal{A}(\mathscr{C})$ comes with a natural functor $\mathscr{C} \to \mathcal{A}(\mathscr{C})$ which is universal among all functors from $\mathscr{C}$ into a category admitting cokernels. Freyd categories can be used to model the category of finitely presented (graded) modules over coherent (graded) rings (cf. Definition 2.32). The original treatment can be found in e.g., [**Fre66**] and [**Bel00**], while [**Pos21a**] and [**Pos21b**] offer a constructive approach to these categories.

Let us first state the definition of Freyd categories:

Definition 2.28. Let $\mathscr{C}$ be an additive category. The **Freyd category**[19] $\mathcal{A}(\mathscr{C})$ consists of the following data:

(1) An object in $\mathcal{A}(\mathscr{C})$ is simply a morphism in $\mathscr{C}$.
(2) A morphism in $\mathcal{A}(\mathscr{C})$ from $\left(A_1 \xrightarrow{\varphi_1} B_1\right)$ to $\left(A_2 \xrightarrow{\varphi_2} B_2\right)$ is given by a morphism $B_1 \xrightarrow{\beta} B_2$ in $\mathscr{C}$ for which there exists a morphism $A_1 \xrightarrow{\chi} A_2$ rendering the diagram



commutative. We call β the **morphism datum** and χ a **morphism witness**. Two morphisms $B_1 \xrightarrow{\beta} B_2$, $B_1 \xrightarrow{\beta'} B_2$ from $\left(A_1 \xrightarrow{\varphi_1} B_1\right)$ to $\left(A_2 \xrightarrow{\varphi_2} B_2\right)$ are declared to be equal in $\mathcal{A}(\mathscr{C})$ if there exists a morphism $\lambda\colon B_1 \to A_2$ such that $\beta - \beta' = \lambda \cdot \varphi_2$.
(3) Composition and identities are directly inherited from $\mathscr{C}$.

Remark 2.29. Clearly, $\mathcal{A}(\mathscr{C})$ has decidable equality of morphisms if and only if $\mathscr{C}$ has decidable lifts (cf. Definition A.8).

[19] The associated category constructor is `FreydCategory` in [**BP19a**].

FREYD categories provide a universal way to equip an additive category with cokernels (cf. [**Pos21a**]). In order to equip it with kernels, we need to require more assumptions on $\mathscr{C}$.

Definition 2.30. Let $\varphi\colon A \to B$ be a morphism in $\mathscr{C}$. A **weak-kernel**[20] of φ consists of the following data:

1. an object K in $\mathscr{C}$ (**weak-kernel object**),
2. a morphism $\iota\colon K \to A$ such that $\iota \cdot \varphi = 0$ (**weak-kernel morphism**) and
3. for any morphism $\tau\colon T \to A$ with $\tau \cdot \varphi = 0$, a lift morphism $\lambda\colon T \to K$ of τ along ι (**weak-kernel lift**).

A category $\mathscr{C}$ is said to **have weak-kernels** if we have an algorithm which for a given morphism φ computes a weak-kernel of φ.

The following is the fundamental theorem in FREYD categories (see e.g., [**Fre66**] and [**Pos21a**, Corollary 3.16]).

Theorem 2.31. *Let $\mathscr{C}$ be an additive category. Then $\mathcal{A}(\mathscr{C})$ is Abelian if and only if $\mathscr{C}$ has weak-kernels.*

FREYD categories have a variety of applications (see e.g., [**Pos21a**] and [**Pos21b**]), but we are mainly interested in using them to construct finitely presented (graded) categories over so-called *left/right computable rings.* We will use FREYD categories in Section 5.3 to construct the following two (FROBENIUS) categories:

- The category E-**fpmod** of finitely presented left E-modules over an exterior k-algebra $E = k[e_0, \dots, e_n]$ for some field k,
- The category E-**fpgrmod** of finitely presented graded left E-modules over a $\mathbb{Z}$-graded exterior k-algebra $E = k[e_0, \dots, e_n]$ for some field k. If we assume $\deg e_0 = \deg e_1 = \cdots = \deg e_n = -1$, then the stable category of E-**fpgrmod** modulo projectives is equivalent to the bounded derived category $\mathcal{D}^b(\mathbb{P}^n_k)$ via the BGG correspondence [**BGG78**], [**EFS03**].

The following definition characterizes the rings whose categories of rows have weak-kernels.

Definition 2.32. Let R be a ring. Then

1. R is called **left coherent** if for any matrix $\mathtt{A}$ over R, we can compute a matrix $\mathtt{L}$ such that $\mathtt{LA} = 0$ and for any matrix $\mathtt{T}$ with $\mathtt{TA} = 0$, there exists a matrix $\mathtt{U}$ such that $\mathtt{UL} = \mathtt{T}$.
2. R **has decidable lifts** if there is an algorithm to decide solvability and construct a particular solution of linear systems $\mathtt{XA} = \mathtt{B}$ for given matrices over R.
3. R is called **left computable** if it is left coherent and has decidable lifts.
4. R is called **right computable** if R^{op} is left computable.
5. R is called **computable**[21] if it is left and right computable.

The following rings are (left) computable:

Example 2.33.
1. A constructive field k with the GAUSSian normal form algorithm, i.e., an algorithm to compute the row reduced echelon form (RREF).
2. An EUCLIDean ring with a Hermite normal form algorithm, e.g., $R = \mathbb{Z}$ or $R = k[x]$, where k is a constructive field.

[20]The dual notion is weak-cokernel.

[21]The original axiomatization of computable rings can be found in [**BLH11**].

(3) Any ring R with a GRÖBNER basis notion and equipped with an algorithm to compute reduced GRÖBNER bases, e.g., the polynomial ring $R = k[x_0, \dots, x_n]$ or the exterior k-algebra $R = k[e_0, \dots, e_n]$.

In particular, we get the following:

- $\mathcal{A}(R\textbf{-rows})$ is Abelian if and only if R is left coherent.
- $\mathcal{A}(R\textbf{-rows})$ is Abelian and has decidable equality of morphisms if and only if R is left computable.
- Let R be commutative and computable. Then $\mathcal{A}(R\textbf{-rows})$ is a closed monoidal Abelian category with enough projectives. It is also equipped with an $\mathcal{A}(R\textbf{-rows})$-homomorphism structure. In this case, the three functors
  - the external Hom functor on $\mathcal{A}(R\textbf{-rows})$,
  - the internal Hom functor of the closed monoidal structure on $\mathcal{A}(R\textbf{-rows})$ and
  - the bifunctor of the $\mathcal{A}(R\textbf{-rows})$-homomorphism structure on $\mathcal{A}(R\textbf{-rows})$

  are equivalent (see [**Pos21a**] and [**BP19b**]).

Example 2.34. Let R be a left computable ring and R-**rows** the category of rows over R (cf. Definition 2.11). We can construct the FREYD category $\mathcal{A}(R\textbf{-rows})$ as a tower of categories:

(1) R-**rows** $:=$ `CategoryOfRows`(R); ($\simeq$ `AdditiveClosure`(`RingAsCategory`(R)));
(2) $\mathcal{A}(R\textbf{-rows}) :=$ `FreydCategory`(R-**rows**).

If we *manually compile*[22] this tower we recover the definition of the category R-**fpres** of **finite left R-presentations**[23]. This category is used in [**hom22**] to model the category R-**fpmod** (cf. [**BLH11**], [**Pos17**] or [**DL06**]). In the following we state the definition of this category:

Definition 2.35. Let R be a ring. The category R-**fpres** is defined by the following data:

(1) An object is simply a finite dimensional matrix over R.
(2) A morphism from an object $\mathtt{M} \in R^{m\times n}$ to $\mathtt{N} \in R^{s\times t}$ is a matrix $\mathtt{F} \in R^{n\times t}$ for which the equation $\mathtt{MF} = \mathtt{XN}$ is solvable for $\mathtt{X}$. Two such morphisms $\mathtt{F}, \mathtt{G} \colon \mathtt{M} \to \mathtt{N}$ are considered equal if the equation $\mathtt{F} - \mathtt{G} = \mathtt{XN}$ is solvable.
(3) The identity morphism of $\mathtt{M} \in R^{m\times n}$ is the identity matrix $\mathtt{I}_n$.
(4) The composition is given by the usual matrix multiplication.

For a left computable ring R, the category R-**fpres** is Abelian and has decidable equality of morphisms. If, furthermore, R is commutative, then R-**fpres** is a closed symmetric monoidal category. See [**Gut17**] and [**Pos17**] for details.

Let us illustrate this category by a concrete example. Let $R := \mathbb{Q}[x, y]$ and consider the following two objects in R-**fpres**:

$$\mathtt{M} = \begin{pmatrix} -2y & -2x \\ x^2 - 2y & x \\ -y & -y^2 \end{pmatrix}, \quad \mathtt{N} = \begin{pmatrix} -x & -x^2 - x \\ -3x & 2x \\ x^2y - y & -y \end{pmatrix}.$$

Then the matrices

$$\mathtt{F} = \begin{pmatrix} 4 & 4x+4 \\ x & x^2 + x \end{pmatrix}, \quad \mathtt{G} = \begin{pmatrix} 32x^4 + 4 & 32x^5 + 32x^4 + 4x + 4 \\ y^4 + x & xy^4 + y^4 + x^2 + x \end{pmatrix}$$

[22] I.e., turn it to a primitive category constructor.

[23] The associated primitive category constructor is `LeftPresentations` in [**GP21b**].

define equal morphisms[24] $\mathtt{F}, \mathtt{G}\colon \mathtt{M} \to \mathtt{N}$ in R-**fpres** because $\mathtt{F} - \mathtt{G} = \mathtt{XN}$ for

$$\mathtt{X} = \begin{pmatrix} 32x^3 & \cdot & \cdot \\ \frac{2}{5}xy^4 & \frac{1}{5}x^2y^4 + \frac{1}{5}xy^4 - \frac{1}{5}y^4 & \frac{3}{5}xy^3 + y^3 \end{pmatrix};$$

while the matrix

$$\mathtt{H} = \begin{pmatrix} x & 1 \\ y & x - y \end{pmatrix}$$

does not define a morphism $\mathtt{H}\colon \mathtt{M} \to \mathtt{N}$ in R-**fpres** because the equation $\mathtt{MH} = \mathtt{XN}$ is not solvable[25] for $\mathtt{X}$ over R.

Let us take a closer look at the equivalence R-**fpmod** $\simeq$ R-**fpres**. For each module M in R-**fpmod** there exists $m, n \in \mathbb{Z}$, a matrix $\rho_M \in R^{m \times n}$ and an exact sequence

$$R^{1\times m} \xrightarrow{\rho_M} R^{1\times n} \xrightarrow{\pi_M} M.$$

The matrix ρ_M in the above sequence is called a **presentation matrix** of M. Let N be another module in R-**fpmod** with a presentation matrix $\rho_N \in R^{s\times t}$. Due to the fact that free modules are projective, any R-homomorphism $\varphi\colon M \to N$ induces two morphisms μ_φ and λ_φ which render the following diagram

$$\begin{array}{ccccc} R^{1\times m} & \xrightarrow{\rho_M} & R^{1\times n} & \xrightarrow{\pi_M} & M \\ \downarrow{\scriptstyle \lambda_\varphi} & \circlearrowleft & \downarrow{\scriptstyle \mu_\varphi} & \circlearrowleft & \downarrow{\scriptstyle \varphi} \\ R^{1\times s} & \xrightarrow{\rho_N} & R^{1\times t} & \xrightarrow{\pi_N} & N \end{array}$$

commutative. On the other hand, every pair of morphisms λ, μ with $\rho_M \cdot \mu = \lambda \cdot \rho_N$ gives rise to a morphism $\varphi_\mu\colon M \to N$ defined by the cokernel colift of $\mu \cdot \pi_N$ along π_M.

Denote by $\Omega_{M,N}$ the set of all pairs (λ, μ) of morphisms with $\rho_M \cdot \mu = \lambda \cdot \rho_N$. We define on $\Omega_{M,N}$ the equivalence relation $\sim_{M,N}$ as follows: $(\lambda, \mu) \sim_{M,N} (\lambda', \mu')$ if $\mu - \mu'$ lifts along ρ_N.

A straightforward verification shows that there exists a one-to-one correspondence between $\mathrm{Hom}_{R\text{-}\mathbf{fpmod}}(M, N)$ and $\Omega_{M,N}/\sim_{M,N}$. Furthermore, if (λ, μ) and (λ', μ') belong to $\Omega_{M,N}$ and $\mu = \mu'$, then $(\lambda, \mu) \sim_{M,N} (\lambda', \mu')$, i.e., the $\sim_{M,N}$-equivalence class of (λ, μ) is independent of the choice of λ. Hence, we can refine the above correspondence as follows: we define $\Omega_{M,N}$ by the set of all morphisms $\mu\colon R^{1\times n} \to R^{1\times t}$ such that $\rho_M \cdot \mu$ is liftable along ρ_N and declare two such morphisms μ and μ' as equivalent if $\mu - \mu'$ is liftable along ρ_N. Similarly, we can prove the existence of a one-to-one correspondence between $\mathrm{Hom}_{R\text{-}\mathbf{fpmod}}(M, N)$ and $\Omega_{M,N}/\sim_{M,N}$.

That is, the object

$$\mathtt{M} = \begin{pmatrix} -2y & -2x \\ x^2 - 2y & x \\ -y & -y^2 \end{pmatrix}$$

in R-**fpres** corresponds in R-**fpmod** to the R-module

$$\mathrm{coker}(\mathtt{M}) := \mathrm{coker}\Big(R^{1\times 3} \xrightarrow{\mathtt{M}} R^{1\times 2}\Big) \cong R^{1\times 2}/R^{1\times 3}\,\mathtt{M},$$

[24]They are syntactically different but semantically equal (cf. Remark A.5).

[25]This can be checked by the `RightDivide` operation in the GAP package `RingsForHomalg` [**BGK+21a**].

i.e., using the language of generators and relations, coker(`M`) has two generators m_1, m_2 subject to three relations

$$\{-2ym_1 - 2xm_2,\ (x^2-2y)m_1 + xm_2,\ -ym_1 - y^2m_2\}.$$

Similarly, coker(`N`) has two generators n_1, n_2 subject to three relations

$$\{-xn_1 - (x^2+x)n_2,\ -3xn_1 + 2xn_2,\ (x^2y - y)n_1 - yn_2\}.$$

Moreover, the morphisms $\mathtt{F}, \mathtt{G}\colon \mathtt{M} \to \mathtt{N}$ correspond in R-**fpmod** to the R-homomorphisms

$$f\colon \begin{cases} \mathrm{coker}(\mathtt{M}) & \to \mathrm{coker}(\mathtt{N}) \\ m_1 & \mapsto 4n_1 + (4x+4)n_2 \\ m_2 & \mapsto xn_1 + (x^2+x)n_2. \end{cases}$$

and

$$g\colon \begin{cases} \mathrm{coker}(\mathtt{M}) & \to \mathrm{coker}(\mathtt{N}) \\ m_1 & \mapsto (32x^4+4)n_1 + (32x^5 + 32x^4 + 4x + 4)n_2 \\ m_2 & \mapsto (y^4+x)n_1 + (xy^4 + y^4 + x^2 + x)n_2 \end{cases}$$

which are equal since

$$\mathtt{F} \cdot \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = (\mathtt{G} + \mathtt{XN}) \cdot \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \mathtt{G} \cdot \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}.$$

For an implementation of the above equivalent models for R-**fpmod** we refer to the `GAP` packages [**BP19a**], [**BS21c**] or [**GP21b**]. For a software demonstration we refer to their manuals and to Appendix C.

Remark 2.36. Let R be a G-graded ring. A left G-graded R-module M is a left R-module such that

$$M = \bigoplus_{h \in G} M_h$$

where every M_h is an additive subgroup of M, and for every $g, h \in G$ we have

$$R_g M_h \subseteq M_{g+h}.$$

Since $R_0 M_h \subseteq M_h$ we see that every M_h is an R_0-submodule of M. A nonzero element $x \in M_h$ is called homogeneous of degree h and we write $\deg x = h$.

Let M, N be two G-graded R-modules. An R-homomorphism $\varphi\colon M \to N$ in R-**Mod** is called **graded** of degree $d \in G$ if $\varphi(M_h) \subseteq N_{h+d}$ for all $h \in G$. The set of all graded morphisms of degree $d \in G$ will be denoted by $\mathrm{Hom}_d(M,N)$. Obviously, $\mathrm{Hom}_d(M,N)$ is a subgroup of $\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(M,N)$.

We define R-**grmod** by the subcategory of R-**mod** whose objects are the G-graded R-modules and whose morphisms from an object M to N are the graded R-homomorphisms of degree 0, i.e., $\mathrm{Hom}_{R\text{-}\mathbf{grmod}}(M,N) \coloneqq \mathrm{Hom}_0(M,N)$.

Let M be an object in R-**grmod** and $h \in G$. We denote by $M(h)$ the left G-graded R-module whose homogeneous parts $M(h)_g \coloneqq M_{h+g}$ for all $g \in G$. For instance, the element $1 \in R(h)$ is homogeneous of degree $-h$; and if $\varphi\colon R(g) \to R(h), r \mapsto rx$ is a homomorphism of left G-graded R-modules, then x is a homogeneous element with $\deg x = h - g$ or $x = 0$.

For instance, if R is the $\mathbb{Z}$-graded polynomial ring $\mathbb{Q}[x,y]$ with $\deg x = \deg y = 1$, then

$$\varphi\colon R(-2)\oplus R(-4)\oplus R(-3) \xrightarrow{\begin{pmatrix} x^2 \\ y^4 \\ xy^2 \end{pmatrix}} R(0)$$

is a homomorphism of left $\mathbb{Z}$-graded R-modules. Furthermore, $v = (xy\ \ 1\ \ y) \in R(-2)\oplus R(-4)\oplus R(-3)$ has degree 4 and $\varphi(v) = x^3y + y^4 + xy^3$ has also degree 4 in $R(0)$.

Example 2.37. Let R be a G-graded ring and R-**grrows** the category of graded rows over R (cf. Definition 2.7). Analogously to the above example, the *simplified version* of $\mathcal{A}(R\text{-}\mathbf{grrows})$ is called the category of **finite graded left R-presentations** and is denoted by R-**grfpres**. In the following we state the definition of this category:

Definition 2.38. Let R be a G-graded ring. The category R-**grfpres** is defined by the following data:

(1) An object is a tuple $M := (\mathtt{M},\mathtt{d}) \in R^{m\times n}\times G^n$ for $m,n \geq 0$ such that
  (a) $\mathtt{M}$ is a matrix with homogeneous entries and
  (b) $\mathtt{d_1} - \deg \mathtt{M_{i,1}} = \mathtt{d_2} - \deg \mathtt{M_{i,2}} = \cdots = \mathtt{d_n} - \deg \mathtt{M_{i,n}}$ for all $i = 1,\ldots,m$ and $\mathtt{M_{i,j}} \neq 0$.

(2) A morphism from $(\mathtt{M},\mathtt{d}) \in R^{m\times n}\times G^n$ to $(\mathtt{N},\mathtt{e}) \in R^{s\times t}\times G^t$ is a matrix $\mathtt{F} \in R^{n\times t}$ such that
  (a) $\mathtt{F}$ is a matrix with homogeneous entries,
  (b) the equation $\mathtt{MF} = \mathtt{XN}$ is solvable for $\mathtt{X}$ and
  (c) we have $\mathtt{e_j} - \deg \mathtt{F_{i,j}} = \mathtt{d_i}$ for all $j = 1,\ldots,t,\ i = 1,\ldots,n$ and $\mathtt{F_{i,j}} \neq 0$.

  Two such morphisms $\mathtt{F},\mathtt{G}$ are considered equal in R-**grfpres** if $\mathtt{F} - \mathtt{G} = \mathtt{XN}$ is solvable for $\mathtt{X}$.

(3) The identity morphism of an object $(\mathtt{M},\mathtt{d}) \in R^{m\times n}\times G^n$ is the identity matrix $\mathtt{I}_n$.

(4) The composition is given by the usual matrix multiplication.

Similar to the nongraded case, if R is left computable, then R-**grfpres** is Abelian and has decidable equality of morphisms. Furthermore, if R is commutative, then R-**grfpres** is a closed symmetric monoidal category. See [**Gut17**] and [**Pos17**] for details. Similarly, we obtain the equivalences

$$R\text{-}\mathbf{fpgrmod} \simeq R\text{-}\mathbf{grfpres} \cong \mathcal{A}(R\text{-}\mathbf{grrows})$$

where R-**fpgrmod** is the category of finitely presented graded R-modules, i.e., the full subcategory of R-**grmod** generated by the cokernels of morphisms between graded free R-modules of finite rank.

Let us illustrate the above model for the $\mathbb{Z}$-graded polynomial ring $R := \mathbb{Q}[x,y]$ with $\deg x = \deg y = 1$. The following tuples

$$(\mathtt{M},\mathtt{d}) := \left(\begin{pmatrix} 3xy+5y^2 & -3x+5y \\ -2x^2 & 3x \\ -x & \cdot \end{pmatrix}, [1,0] \right),$$

$$(\mathtt{N},\mathtt{e}) := \left(\begin{pmatrix} 9x^2-4xy & -18x^3+8x^2y \\ 14xy+15y^2 & -28x^2y-30xy^2 \end{pmatrix}, [2,3] \right)$$

are well-defined objects in R-**grfpres**, and the matrix

$$\mathtt{F} := \begin{pmatrix} -9x+4y & 18x^2-8xy \\ -3x^2-8xy-10y^2 & 6x^3+16x^2y+20xy^2 \end{pmatrix}$$

is a well-defined morphism from $(\mathtt{M}, \mathtt{d})$ to $(\mathtt{N}, \mathtt{e})$.

The object $(\mathtt{M}, \mathtt{d})$ corresponds in R-**fpgrmod** to

$$M \coloneqq \operatorname{coker}\Big(R(-1)^{\oplus 2} \oplus R(0) \xrightarrow{\ \mathtt{M}\ } R(1) \oplus R(0)\Big)$$

i.e., to a graded R-module generated by two elements m_1, m_2 with $\deg m_1 = -1$ and $\deg m_2 = 0$, subject to the relations

$$\{(3xy + 5y^2)m_1 + (-3x + 5y)m_2,\ -2x^2 m_1 + 3x m_2,\ -x m_1\}.$$

Similarly, $(\mathtt{N}, \mathtt{e})$ corresponds to

$$N \coloneqq \operatorname{coker}\Big(R(0)^{\oplus 2} \xrightarrow{\ \mathtt{N}\ } R(2) \oplus R(3)\Big)$$

which is generated by two elements n_1, n_2 with $\deg n_1 = -2$ and $\deg n_2 = -3$, subject to the relations

$$\{(9x^2 - 4xy)n_1 + (-18x^3 + 8x^2 y)n_2,\ (14xy + 15y^2)n_1 + (-28x^2 y - 30xy^2)n_2\}.$$

The morphism $\mathtt{F}\colon (\mathtt{M}, \mathtt{d}) \to (\mathtt{N}, \mathtt{e})$ corresponds in R-**fpgrmod** to the morphism

$$f\colon \begin{cases} M & \to N \\ m_1 & \mapsto (-9x + 4y)n_1 + (18x^2 - 8xy)n_2, \\ m_2 & \mapsto (-3x^2 - 8xy - 10y^2)n_1 + (6x^3 + 16x^2 y + 20xy^2)n_2. \end{cases}$$

An implementation of the above models of R-**grmod** can be found in the `GAP` packages [**BP19a**], [**BS21b**] and [**Gut21**].

2.2.4. Quotient Categories. Analogous to quotient groups, rings and modules, a quotient category can be obtained from a category by identifying sets of morphisms. Many important category constructors can be recovered as quotient categories:

- The finitely presented categories defined by quivers with relations as we will see in Definition 2.47;
- The (bounded) homotopy categories as we will see in Theorem 3.29;
- The Freyd categories as we will see in Corollary 2.65.

In the following we state the definition of quotient categories:

Definition 2.39. Let $\mathscr{C}$ be a category. A **congruence relation** $\sim$ on $\mathscr{C}$ is an equivalence relation $\sim$ on morphisms of $\mathscr{C}$ such that

- $\alpha \sim \beta$ implies that α and β have the same source and range.
- If $\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$, then $\alpha_1 \cdot \beta_1 \sim \alpha_2 \cdot \beta_2$.

For two object A and B in $\mathscr{C}$, we denote the restriction of $\sim$ to $\operatorname{Hom}_{\mathscr{C}}(A, B)$ by $\sim^{A,B}$. The equivalence class of a morphism $\alpha\colon A \to B$ will be denoted by $[\alpha]$. We define the **quotient category** $\mathscr{C}/\sim$ by the following data:

(1) $\operatorname{Obj}_{\mathscr{C}/\sim} \coloneqq \operatorname{Obj}_{\mathscr{C}}$.
(2) For two objects A and B in $\mathscr{C}/\sim$ we define

$$\operatorname{Hom}_{\mathscr{C}/\sim}(A, B) \coloneqq \operatorname{Hom}_{\mathscr{C}}(A, B)/\sim^{A,B},$$

i.e., the set of all equivalence classes in $\operatorname{Hom}_{\mathscr{C}}(A, B)$ with respect to $\sim^{A,B}$. Hence, a morphism in $\operatorname{Hom}_{\mathscr{C}/\sim}(A, B)$ is of the form $[\alpha]$ for some $\alpha\colon A \to B$ in $\mathscr{C}$.
(3) Composition and identity morphisms are directly inherited from $\mathscr{C}$.

Remark 2.40. The functor $[\,]\colon \mathscr{C} \to \mathscr{C}/\sim$, $\alpha \mapsto [\alpha]$ will be called the **quotient functor** associated to $\mathscr{C}/\sim$. It can be shown that any functor $Q\colon \mathscr{C} \to \mathscr{D}$ for which $\varphi \sim \psi$ implies $Q(\varphi) = Q(\psi)$ for all φ, ψ in $\mathscr{C}$ colifts uniquely along $[\,]\colon \mathscr{C} \to \mathscr{C}/\sim$ (see e.g., [**ML98**, Proposition II.8.1]).

The following is an immediate consequence of the definition:

Remark 2.41. The category $\mathscr{C}/\sim$ has decidable equality of morphisms if and only if we have an algorithm which for a given pair of morphisms $\alpha, \beta\colon A \to B$ decides whether $\alpha \sim^{A,B} \beta$.

Remark 2.42. Suppose $\mathscr{C}$ is a (pre)additive category equipped with a congruence relation $\sim$. We call $\sim$ **additive** if following holds: For given objects A, B in $\mathscr{C}$, if $\alpha_1 \sim^{A,B} \alpha_2$ and $\beta_1 \sim^{A,B} \beta_2$, then $\alpha_1 + \beta_1 \sim^{A,B} \alpha_2 + \beta_2$. In this case, the quotient category $\mathscr{C}/\sim$ is also a (pre)additive category.

It turns out that the concept of additive congruence is equivalent to the concept of two-sided ideal of morphisms in (pre)additive categories:

Definition 2.43. Let $\mathscr{C}$ be a preadditive category. A **two sided ideal of morphisms** I in $\mathscr{C}$ is a set of morphisms in $\mathscr{C}$ such that

a. For a given pair of objects A, B in $\mathscr{C}$ the set $I_{A,B} := I \cap \mathrm{Hom}_{\mathscr{C}}(A, B)$ is a subgroup of $\mathrm{Hom}_{\mathscr{C}}(A, B)$.
b. For a given pair of morphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ in $\mathscr{C}$, if $\alpha \in I$ or $\beta \in I$ then $\alpha \cdot \beta \in I$.

Remark 2.44. Let $\mathscr{C}$ be a (pre)additive category.

(1) Let $\sim$ is an additive congruence in $\mathscr{C}$. Then the set of all morphisms α in $\mathscr{C}$ for which $[\alpha] = 0$ in $\mathscr{C}/\sim$ defines a two sided ideal of morphisms.
(2) Let I a two sided ideal of morphisms in $\mathscr{C}$. Then we can define the following additive congruence: For two morphisms α, β in $\mathscr{C}$, $\alpha \sim \beta$ if α and β have the same source and range and $\alpha - \beta \in I$. The (pre)additive category $\mathscr{C}/\sim$ will be called the **quotient category** of $\mathscr{C}$ by I and will usually be denoted by $\mathscr{C}/I$.
(3) By Remark 2.41, the category $\mathscr{C}/I$ has decidable equality of morphisms if and only if we have an algorithm which decides for a given morphism $\alpha\colon A \to B$ in $\mathscr{C}$ whether $\alpha \in I$.

Stable categories are special case of quotient categories:

Example 2.45. Let $\mathscr{C}$ be a (pre)additive category and $\mathcal{L}$ a class of objects in $\mathscr{C}$. We denote by I the two-sided ideal of morphisms that is generated by the identity morphisms of objects in $\mathcal{L}$. In this case, we call $\mathscr{C}/I$ the **stable category** of $\mathscr{C}$ modulo $\mathcal{L}$ (cf. Section 2.2.6). In particular, a morphism α in $\mathscr{C}$ becomes zero in $\mathscr{C}/I$ if and only if α factors through some object in $\mathcal{L}$.

2.2.5. Finitely Presented Categories Defined by Quivers with Relations. For a quiver $\mathfrak{q}$, we can construct the free category $\mathcal{F}_{\mathfrak{q}}$ (cf. Section 2.1.1). For a field k, we can construct the k-linear closure $k\mathcal{F}_{\mathfrak{q}}$ of $\mathcal{F}_{\mathfrak{q}}$ (cf. Section 2.2.1). Sometimes we want to enforce equality between certain paths in $\mathcal{F}_{\mathfrak{q}}$, or more generally, we want to consider one or more k-linear combinations of paths as zero morphisms in $k\mathcal{F}_{\mathfrak{q}}$. The right framework for achieving this goal is provided by finitely presented categories. Finitely presented categories have many applications in this thesis:

- Their functor categories provide models for categories of left and right modules over finite dimensional algebras (cf. Section 2.2.7).

- They can be used to visualize k-linear subcategories with a finite number of objects. See for example the abstraction k-algebroid of a strong exceptional sequences in Section 6.2.
- They can be used to test hypotheses which are based on a finite number of objects and morphisms with relations. In certain cases, we can provide generic constructive proofs of these hypotheses, particularly in the context of "equational proofs" that include checking complicated equalities or solving two-sided inhomogeneous linear equations. See for example Lemma 5.17. They can be even used to prove theorems in Abelian categories, see for example [**BK21**] and [**Pos22**].

In the following we state the definition of finitely presented categories defined by "categorical relations":

Definition 2.46. Let $\mathfrak{q}$ be a quiver and $\mathcal{F}_\mathfrak{q}$ the free category defined by $\mathfrak{q}$ (cf. Definition 2.2). If we want to identify two paths $p = r_1 r_2 \dots r_\ell$, $p' = r'_1 r'_2 \dots r'_t$ with $\mathfrak{s}(p) = \mathfrak{s}(p')$ and $\mathfrak{t}(p) = \mathfrak{t}(p')$, we might take the quotient category $\mathcal{F}_\mathfrak{q}/\sim$ where $\sim$ is the smallest congruence relation on $\mathcal{F}_\mathfrak{q}$ for which $p \sim p'$. The same procedure can be used to identify more paths. Such quotient categories are usually called **finitely presented categories**.

The following is the definition of k-linear finitely presented categories defined by k-linear relations:

Definition 2.47. Let k be a commutative ring and $k\mathcal{F}_\mathfrak{q}$ the k-linear closure of $\mathcal{F}_\mathfrak{q}$. For a finite set ρ of morphisms in $k\mathcal{F}_\mathfrak{q}$, we denote by $\langle\rho\rangle$ the two-sided ideal of morphisms generated by ρ. The quotient category $k\mathcal{F}_\mathfrak{q}/\langle\rho\rangle$ will be called the **k-linear finitely presented category** (or the **k-algebroid**) defined by $\mathfrak{q}$ subject to the set of k-relations ρ. The set ρ is called **admissible** if there exists $t \geq 2$ with $\mathfrak{m}^t \subseteq \langle\rho\rangle \subseteq \mathfrak{m}^2$ where $\mathfrak{m}$ is the two-sided ideal of $k\mathcal{F}_\mathfrak{q}$ generated by the arrows of $\mathfrak{q}$. In this case, we say $k\mathcal{F}_\mathfrak{q}/\langle\rho\rangle$ is admissible as well.

Remark 2.48. If $\mathfrak{q}$ is acyclic, then $k\mathcal{F}_\mathfrak{q}/\langle\rho\rangle$ is admissible if and only if every element in ρ is a formal k-linear combination of paths of length at least 2.

Remark 2.49. The `GAP` package `QPA2` [**Qt21**] provides, among other things, the basic interfaces to quivers, paths algebras and their quotient algebras. It has been used for implementing the finitely presented linear categories in `Algebroids` [**BHP+21**].

Example 2.50. The endomorphism algebra $\operatorname{End} k\mathcal{F}_\mathfrak{q}$ of $k\mathcal{F}_\mathfrak{q}$ is usually called the path k-algebra of $\mathfrak{q}$. If ρ is a set of relations in $k\mathcal{F}_\mathfrak{q}$ then

$$\operatorname{End}(k\mathcal{F}_\mathfrak{q}/\langle\rho\rangle) \cong (\operatorname{End} k\mathcal{F}_\mathfrak{q})/\langle\rho\rangle$$

where End(-) is defined in Definition A.29.

Remark 2.51. If the category $k\mathcal{F}_\mathfrak{q}/\langle\rho\rangle$ is admissible, then it is Hom-finite. In this case, as the endomorphism algebra $\operatorname{End}(k\mathcal{F}_\mathfrak{q}/\langle\rho\rangle)$ is finite dimensional, we can use the *theory of noncommutative Gröbner bases* to decide the equality of morphisms in $k\mathcal{F}_\mathfrak{q}/\langle\rho\rangle$ (see for example [**DMR99**, Section 2.4] and [**Gre99**]).

2.2.6. Stable Categories Defined by Classes of (Co)Lifting Objects. Let $\mathscr{C}$ be a preadditive category and let $\mathcal{L}$ be a class of objects in $\mathscr{C}$. We have already seen in Example 2.45 that the stable category $\mathscr{C}/\mathcal{L}$ is the quotient category of $\mathscr{C}$ modulo the two-sided ideal of all morphisms that factor through an object in $\mathcal{L}$. The class $\mathcal{L}$ might contain infinitely many objects, hence the problem of deciding the equality of morphisms in $\mathscr{C}/\mathcal{L}$ can not a priori be solved algorithmically. However, this changes if the class $\mathcal{L}$ is equipped with "extra properties": For example, if $\mathscr{C}$ is Abelian with enough projectives and $\mathcal{L}$ is the class of projective objects in $\mathscr{C}$, then a morphism $\varphi\colon A \to B$ in $\mathscr{C}$ factors through an object in $\mathcal{L}$ if and only if it is liftable along some epimorphism $\ell_A\colon P_A \twoheadrightarrow A$ for an object $P_A \in \mathcal{L}$. In this section we introduce the concept of classes of (co)lifting objects in preadditive categories as an abstraction of these "extra properties". Our approach provide the following two computational features:

- The decidability of the equality of morphisms in the stable category translates to a (co)lift problem in $\mathscr{C}$ (cf. Remark 2.56).
- Under relatively simple assumptions, we will be able to elevate a $\mathscr{D}$-homomorphism structure from $\mathscr{C}$ to the stable category (cf. Theorem 4.23).

In Theorem 3.29 and Corollary 2.65 we prove that homotopy categories and Freyd categories can be constructed as stable categories associated to certain classes of colifing objects. The concepts introduced in this chapter has been implemented in the `GAP` package `StableCategories` [**Sal21e**].

In the following we state the definition of classes of lifting objects:

Definition 2.52. Let $\mathscr{C}$ be an additive category. A **class of lifting objects** on $\mathscr{C}$ consists of the following data:

(1) A distinguished class of objects $\mathcal{L}$ in $\mathscr{C}$.
(2) Every object A in $\mathscr{C}$ is assigned a distinguished morphism $\ell_A\colon L_A \to A$ with $L_A \in \mathcal{L}$. Furthermore, if A belongs to $\mathcal{L}$, we require ℓ_A to be a split-epimorphism.
(3) For every morphism $\varphi\colon A \to B$, there exists a **lifting morphism** $L_\varphi\colon L_A \to L_B$ that renders the following diagram

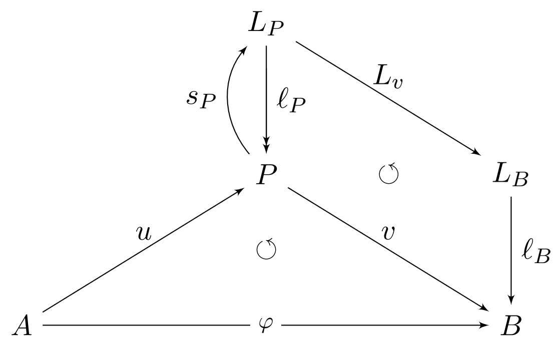
$$\begin{array}{ccc} L_A & \xrightarrow{\ell_A} & A \\ {\scriptstyle L_\varphi}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \varphi} \\ L_B & \xrightarrow[\ell_B]{} & B \end{array}$$

commutative.

The following lemma is the key property of classes of lifting objects:

Lemma 2.53. *Let $\mathscr{C}$ be an additive category equipped with a class of lifting objects $\mathcal{L}$. A morphism $\varphi\colon A \to B$ in $\mathscr{C}$ factors through an object in $\mathcal{L}$ if and only if it is liftable along $\ell_B\colon L_B \to B$.*

Proof. Let P be an object in $\mathcal{L}$ and $u\colon A \to P, v\colon P \to B$ be morphisms with $\varphi = u \cdot v$. We denote the section morphism of $\ell_P\colon L_P \to P$ by s_P, i.e., $s_P \cdot \ell_P = \mathrm{id}_P$. Hence, $\varphi = u \cdot v = u \cdot \mathrm{id}_P \cdot v = u \cdot s_P \cdot \ell_P \cdot v = u \cdot s_P \cdot L_v \cdot \ell_B$ and the claim follows. The converse is trivial.



□

Lemma 2.54. *Let $\mathscr{C}$ be an additive category. If $\mathcal{L}$ is class of lifting objects for $\mathscr{C}$, then*

$$\mathcal{I}_{\mathcal{L}} \coloneqq \{\varphi\colon A \to B \mid \varphi \textit{ is liftable along } \ell_B\}$$

is a two-sided ideal of morphisms in $\mathscr{C}$.

Proof. Suppose that $\varphi\colon A \to B$ is a morphism in $\mathcal{I}_{\mathcal{L}}$. We need to show that for any morphism $\varphi'\colon A \to B$ in $\mathcal{I}_{\mathcal{L}}$, the difference $\varphi - \varphi'$ also belongs to $\mathcal{I}_{\mathcal{L}}$; and that any composition of morphisms with φ from left or right is again a morphism in $\mathcal{I}_{\mathcal{L}}$. Since φ and φ' belong to $\mathcal{I}_{\mathcal{L}}$, there are two morphisms $\psi, \psi'\colon A \to L_B$ such that $\varphi = \psi \cdot \ell_B$ and $\varphi' = \psi' \cdot \ell_B$. Hence, $\varphi - \varphi' = (\psi - \psi') \cdot \ell_B$, i.e., $\varphi - \varphi'$ belongs to $\mathcal{I}_{\mathcal{L}}$. Let $f\colon A' \to A$ be any morphism with range equals to A. Then $f \cdot \varphi = f \cdot \psi \cdot \ell_B$, i.e., the morphism $f \cdot \varphi\colon A' \to B$ factors through ℓ_B, hence it belongs to $\mathcal{I}_{\mathcal{L}}$. Let $g\colon B \to B'$ be any morphism with source equals to B. Then $\varphi \cdot g = \psi \cdot \ell_B \cdot g = \psi \cdot L_g \cdot \ell_{B'}$, i.e., the morphism $\varphi \cdot g\colon A \to B'$ factors through $\ell_{B'}$, hence it belongs to $\mathcal{I}_{\mathcal{L}}$. Hence $\mathcal{I}_{\mathcal{L}}$ is a two-sided ideal of morphisms in $\mathscr{C}$. □

In the following we state the definition of the stable category associated to a class of lifting objects:

Definition 2.55. Let $\mathscr{C}$ be an additive category and let $\mathcal{L}$ be a class of lifting objects for $\mathscr{C}$. The quotient category $\mathscr{C}/\mathcal{L}$ will be called the **stable category**[26] of $\mathscr{C}$ w.r.t. the class $\mathcal{L}$. In particular

(1) The objects class is the same as that of $\mathscr{C}$.
(2) For two objects A, B in $\mathscr{C}$ we have

$$\mathrm{Hom}_{\mathscr{C}/\mathcal{L}}(A, B) \coloneqq \mathrm{Hom}_{\mathscr{C}}(A, B)/\mathcal{I}_{\mathcal{L}}(A, B)$$

where $\mathcal{I}_{\mathcal{L}}(A, B) \coloneqq \mathcal{I}_{\mathcal{L}} \cap \mathrm{Hom}_{\mathscr{C}}(A, B)$.

Remark 2.56. The stable category $\mathscr{C}/\mathcal{L}$ has decidable equality of morphisms if and only if we have an algorithm which decides for a given morphism $\varphi\colon A \to B$ in $\mathscr{C}$ whether φ lifts along ℓ_B. In particular, if $\mathscr{C}$ has decidable lifts, then $\mathscr{C}/\mathcal{L}$ has decidable equality of morphisms.

Remark 2.57. An object A in $\mathscr{C}/\mathcal{L}$ is zero if and only if the morphism $\ell_A\colon L_A \to A$ in $\mathscr{C}$ is split-epimorphism. Consequently, the class $\mathcal{L}$ collapses in $\mathscr{C}/\mathcal{L}$ to only one object, precisely, to the zero object.

[26]The associated category constructor is `StableCategoryByClassOfLiftingObjects` in [Sal21e].

Remark 2.58. It follows immediately from the definition that $\mathscr{C}/\mathcal{L}$ and the projection functor $[\,]\colon \mathscr{C} \to \mathscr{C}/\mathcal{L}$ are additive.

Example 2.59. Let $\mathscr{C}$ be an additive category.

(1) Let $\mathcal{L} := \mathrm{Obj}_{\mathscr{C}}$, and for every object A in $\mathscr{C}$, set $\ell_A := \mathrm{id}_A\colon A \to A$. In this case, $\mathscr{C}/\mathcal{L}$ is equivalent to the full subcategory of $\mathscr{C}$ generated by the zero object.
(2) Let $\mathcal{L} := \{0\}$, and for every object A in $\mathscr{C}$, set ℓ_A to be the universal morphism from the zero object to A. In this case, $\mathscr{C}/\mathcal{L} \simeq \mathscr{C}$.

Example 2.60. Let $\mathscr{C}$ be an additive category with enough projectives. That is, for any object A in $\mathscr{C}$, there exists an epimorphism $p_A\colon P_A \twoheadrightarrow A$ from some projective object P_A to A. The class $\mathcal{L}$ of all projective objects in $\mathscr{C}$ together with the morphisms $\ell_A := p_A\colon P_A \twoheadrightarrow A$ for all $A \in \mathscr{C}$ define a class of lifting objects in $\mathscr{C}$. Let $\mathfrak{P}$ be the set of all morphisms in $\mathscr{C}$ that factor through some projective object. A straight verification shows that $\mathcal{I}_{\mathcal{L}} = \mathfrak{P}$, hence the stable category $\mathscr{C}/\mathcal{L}$ coincides with the classical stable category of $\mathscr{C}$ by projectives $\underline{\mathscr{C}} := \mathscr{C}/\mathfrak{P}$.

In the following we define the dual concept of a class of lifting objects.

Definition 2.61. Let $\mathscr{C}$ be an additive category. A **class of colifting objects** on $\mathscr{C}$ consists of the following data:

(1) A distinguished class of objects $\mathcal{Q}$ in $\mathscr{C}$.
(2) Every object A in $\mathscr{C}$ is assigned a distinguished morphism $q_A\colon A \to Q_A$ with $Q_A \in \mathcal{Q}$. Furthermore, if A belongs to $\mathcal{Q}$, we require q_A to be a split-monomorphism.
(3) For every morphism $\varphi\colon A \to B$, there exists a **colifting morphism** $Q_\varphi\colon Q_A \to Q_B$ that renders the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\;q_A\;} & Q_A \\ {\scriptstyle\varphi}\downarrow & \circlearrowleft & \downarrow{\scriptstyle Q_\varphi} \\ B & \xrightarrow[\;q_B\;]{} & Q_B \end{array}$$

commutative.

Analogously to the categories with classs of lifting objects, we can prove that a morphism $\varphi\colon A \to B$ in $\mathscr{C}$ factors through some object U in $\mathcal{Q}$ if and only if φ is coliftable along q_A. Furthermore, the set

$$\mathcal{I}_{\mathcal{Q}} := \{\varphi\colon A \to B \mid \varphi \text{ is coliftable along } q_A\}$$

is a two-sided ideal of morphisms in $\mathscr{C}$. The quotient category $\mathscr{C}/\mathcal{Q}$ will be called the **stable category** of $\mathscr{C}$ associated to the class $\mathcal{Q}$.

Example 2.62. Let $\mathscr{C}$ be an additive category with enough injectives. That is, for any object A in $\mathscr{C}$, there exists a monomorphism $\iota_A\colon A \hookrightarrow I_A$ from A into some injective object I_A. The class $\mathcal{Q}$ of all injective objects in $\mathscr{C}$ together with the morphisms $q_A := \iota_A\colon A \hookrightarrow I_A$ for all $A \in \mathscr{C}$ define a class of colifting objects in $\mathscr{C}$. Let $\mathfrak{I}$ be the set of all morphisms in $\mathscr{C}$ that factor through some injective object. A straight verification shows that $\mathcal{I}_Q = \mathfrak{I}$, hence the stable category $\mathscr{C}/\mathcal{L}$ coincides with the classical stable category of $\mathscr{C}$ by injectives $\overline{\mathscr{C}} := \mathscr{C}/\mathfrak{I}$.

Example 2.63. Let $\mathscr{C}$ be an additive category and let $\operatorname{Arr}(\mathscr{C})$ be its arrow category. We denote by $\mathcal{Q}_{\operatorname{Arr}(\mathscr{C})}$ the class of all objects in $\operatorname{Arr}(\mathscr{C})$ that are represented by split-epimorphisms in $\mathscr{C}$. For an object $\vec{A} \coloneqq \left(A_1 \xrightarrow{\alpha} A_2\right)$ in $\operatorname{Arr}(\mathscr{C})$ set

$$Q_{\vec{A}} \coloneqq \left(A_1 \oplus A_2 \xrightarrow{\left(\begin{smallmatrix} \alpha \\ \mathrm{id}_{A_2} \end{smallmatrix}\right)} A_2 \right) \text{ and } q_{\vec{A}} \coloneqq \vec{A} \xrightarrow{\{(\,\mathrm{id}_{A_1}\ 0\,),\, \mathrm{id}_{A_2}\}} Q_{\vec{A}}.$$

The morphism $(\,0\ \mathrm{id}_{A_2}\,)\colon A_2 \to A_1 \oplus A_2$ is a section morphism for $\left(\begin{smallmatrix} \alpha \\ \mathrm{id}_{A_2} \end{smallmatrix}\right)\colon A_1 \oplus A_2 \to A_2$. Hence, the later is a split-epimorphism and $Q_{\vec{A}} \in \mathcal{Q}_{\operatorname{Arr}(\mathscr{C})}$.

Let us prove that if $\vec{A} \coloneqq \left(A_1 \xrightarrow{\alpha} A_2\right)$ in $\mathcal{Q}_{\operatorname{Arr}(\mathscr{C})}$, then $q_{\vec{A}}$ is a split-monomorphism. Since $A_1 \xrightarrow{\alpha} A_2$ is split-epimorphism in $\mathscr{C}$, α has a section morphism $A_2 \xrightarrow{\gamma} A_1$. The morphism

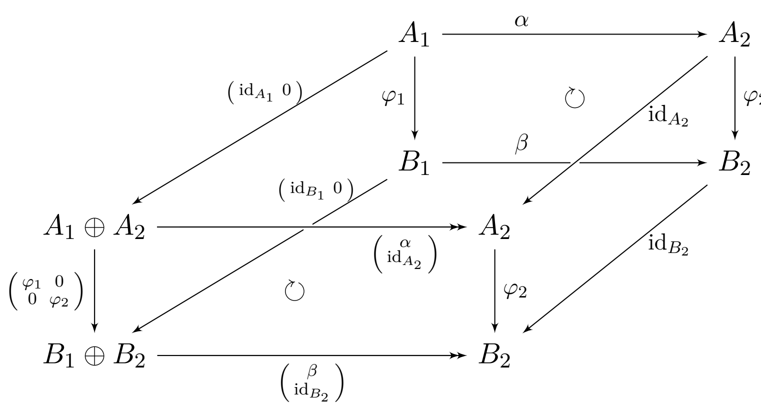
$$r_{\vec{A}} \coloneqq Q_{\vec{A}} \xrightarrow{\{\left(\begin{smallmatrix} \mathrm{id}_{A_1} \\ \gamma \end{smallmatrix}\right),\, \mathrm{id}_{A_2}\}} \vec{A}$$

in $\operatorname{Arr}(\mathscr{C})$ is well-defined and satisfies $q_{\vec{A}} \bullet r_{\vec{A}} = \mathrm{id}_{\vec{A}}$, i.e., $r_{\vec{A}}$ is a retraction of $q_{\vec{A}}$ and the claim follows.

For a morphism $\{\varphi_1, \varphi_2\}\colon \left(A_1 \xrightarrow{\alpha} A_2\right) \to \left(B_1 \xrightarrow{\beta} B_2\right)$ in $\operatorname{Arr}(\mathscr{C})$, we define

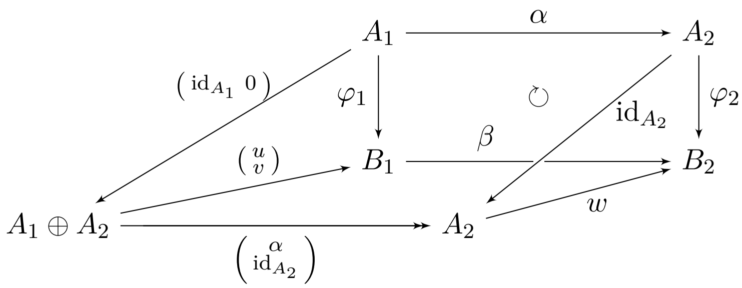
$$Q_{\{\varphi_1,\varphi_2\}} \coloneqq Q_{\vec{A}} \xrightarrow{\{\left(\begin{smallmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{smallmatrix}\right),\varphi_2\}} Q_{\vec{B}}.$$

The above data can be incorporated into the following commutative diagram:



Lemma 2.64. *Let A be an additive category and $\operatorname{Arr}(\mathscr{C})/\mathcal{Q}$ be the stable category of $\operatorname{Arr}(\mathscr{C})$ w.r.t. the above class of colifting objects. For any two objects $\vec{A} \coloneqq A_1 \xrightarrow{\alpha} A_2$ and $\vec{B} \coloneqq B_1 \xrightarrow{\beta} B_2$ in $\operatorname{Arr}(\mathscr{C})$, a morphism $\left[\vec{A} \xrightarrow{\{\varphi_1,\varphi_2\}} \vec{B}\right]$ in $\operatorname{Arr}(\mathscr{C})/\mathcal{Q}$ is zero if and only if φ_2 is liftable along β.*

PROOF. The morphism $\left[\vec{A} \xrightarrow{\{\varphi_1,\varphi_2\}} \vec{B}\right]$ is zero if and only if there exists a colift morphism $\{\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right), w\} \colon Q_{\vec{A}} \to \vec{B}$ of $\vec{A} \xrightarrow{\{\varphi_1,\varphi_2\}} \vec{B}$ along $q_{\vec{A}}$:



i.e., if and only if $u = \varphi_1, w = \varphi_2$ and $v \cdot \beta = w = \varphi_2$. □

Corollary 2.65. *Let $\mathscr{C}$ be an additive category and let $\mathrm{Arr}(\mathscr{C})/\mathcal{Q}$ be the stable category of $\mathrm{Arr}(\mathscr{C})$ w.r.t. the above class of colifting objects. Then*

$$\mathrm{Arr}(\mathscr{C})/\mathcal{Q} \cong \mathcal{A}(\mathscr{C})$$

where $\mathcal{A}(\mathscr{C})$ is the FREYD *category*[27] *of $\mathscr{C}$.*

2.2.7. Functor Categories and Quiver Representations. The natural generalization of rings are small[28] preadditive categories. For instance, a ring R corresponds to the preadditive category $\mathscr{C}(R)$ which has just one object, say $*$, and which has the morphism space $\mathrm{Hom}_{\mathscr{C}(R)}(*,*) := R$. Under this viewpoint, an R-module is nothing but an additive functor $\mathscr{C}(R) \to \mathbf{Ab}$. In other words, the natural generalization of modules are additive functors from small preadditive categories to $\mathbf{Ab}$.

Throughout this section, k is always a field.

Definition 2.66. Let $\mathscr{A}$ be a small k-linear category and $\mathscr{D}$ a k-linear category. The **category of k-linear functors**[29] from $\mathscr{A}$ to $\mathscr{D}$, denoted by $[\mathscr{A}, \mathscr{D}]$, is defined by the following data:

(1) $\mathrm{Obj}_{[\mathscr{A},\mathscr{D}]}$ is defined by the set of all k-linear functors from $\mathscr{A}$ to $\mathscr{D}$, i.e., every object in $[\mathscr{A}, \mathscr{D}]$ is determined by its values on the objects and the generating morphisms of $\mathscr{A}$.
(2) For two objects F and G in $[\mathscr{A}, \mathscr{D}]$ the morphisms from F to G are the natural transformations[30] from F to G, i.e., it is determined by its values on the objects of $\mathscr{A}$.
(3) The composition of two morphisms is given by their vertical composition as natural transformations.
(4) For a given object F in $[\mathscr{A}, \mathscr{D}]$ we define id_F by the identity natural transformation of F, i.e., it assigns to each object v in $\mathscr{A}$ the identity morphism of $F(v)$.

The category of functors inherits its fundamental properties from its range category $\mathscr{D}$:

Theorem 2.67. *Let $[\mathscr{A}, \mathscr{D}]$ be a category of k-linear functors as in Definition 2.66.*

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[27] See Definition 2.28.
[28] A category is called small both the collection of objects and morphisms are sets.
[29] The associated category constructor is `FunctorCategory` in [**BS21a**].
[30] See Definition A.16.

(1) If $\mathscr{D}$ is Abelian, then $[\mathscr{A}, \mathscr{D}]$ is Abelian.
(2) If $\mathscr{A}$ has finitely many objects and $\mathscr{D}$ has decidable equality of morphisms, then $[\mathscr{A}, \mathscr{D}]$ has decidable equality of morphisms.

Proof. It is well known that if $\mathscr{D}$ has certain type of limits or colimits, then the category of functors $[\mathscr{A}, \mathscr{D}]$ has those limits or colimits and they can be computed "object-wise" in $\mathscr{D}$. For more details we refer to [**Pre09**, Theorem 10.1.3], [**Rie16**, Section 3.3] or [**Fre64**, Theorem 5.11]. On the other hand, two morphisms in $[\mathscr{A}, \mathscr{D}]$ are equal if their values on the objects of $\mathscr{A}$ are equal. The assertion follows because $\mathscr{A}$ has finitely many objects. □

In the following we state the definition of right modules over a k-linear finitely presented category:

Definition 2.68. Let $\mathscr{A}$ be a small k-linear category. We call $[\mathscr{A}, k\text{-}\mathbf{mat}]$ the category of **k-finite dimensional right $\mathscr{A}$-modules** and we denote it by $\mathbf{mod}\text{-}\mathscr{A}$. We denote the category $\mathbf{mod}\text{-}\mathscr{A}^{\mathrm{op}}$ by $\mathscr{A}\text{-}\mathbf{mod}$ and call it the category of **k-finite dimensional (left) $\mathscr{A}$-modules**[31].

Notation 2.69. For a category $\mathscr{C}$ we denote by $\mathbf{proj}(\mathscr{C})$ (resp. $\mathbf{inj}(\mathscr{C})$) the full subcategory of $\mathscr{C}$ generated by all projective (resp. injective) objects in $\mathscr{C}$. If $\mathscr{A}$ is a small k-linear category, we will denote $\mathbf{proj}(\mathbf{mod}\text{-}\mathscr{A})$ by $\mathbf{proj}\text{-}\mathscr{A}$ and $\mathbf{inj}(\mathbf{mod}\text{-}\mathscr{A})$ by $\mathbf{inj}\text{-}\mathscr{A}$. Analogously, we will denote $\mathbf{proj}(\mathscr{A}\text{-}\mathbf{mod})$ by $\mathscr{A}\text{-}\mathbf{proj}$ and $\mathbf{inj}(\mathscr{A}\text{-}\mathbf{mod})$ by $\mathscr{A}\text{-}\mathbf{inj}$.

Let A be a k-algebra. The category of right A-modules will be denoted by $\mathbf{Mod}\text{-}A$. The full subcategory of $\mathbf{Mod}\text{-}A$ consisting of finitely generated right A-modules will be denoted by $\mathbf{mod}\text{-}A$. The full subcategory of $\mathbf{mod}\text{-}A$ consisting of finite dimensional right A-modules will be denoted by $\mathbf{fdmod}\text{-}A$. If A is a finite dimensional then $\mathbf{fdmod}\text{-}A = \mathbf{mod}\text{-}A$.

Theorem 2.70. *Let $\mathscr{A} := k\mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle$ be a k-linear finitely presented category defined by a quiver $\mathfrak{q}$ subject to a set of relations $\rho \subset k\mathcal{F}_{\mathfrak{q}}$. Then $\mathbf{mod}\text{-}\mathscr{A} \simeq \mathbf{fdmod}\text{-}\operatorname{End}\mathscr{A}$.*

Proof. For detailed proofs we refer to [**ARS97**, Proposition 1.7] or [**ASS06**, Theorem III.1.6]. In the following, we sketch the construction of the asserted equivalences:

$$\mathcal{G} \colon \mathbf{mod}\text{-}\mathscr{A} \xrightarrow{\sim} \mathbf{fdmod}\text{-}\operatorname{End}\mathscr{A} : \mathcal{F}.$$

For an object $F \colon \mathscr{A} \to k\text{-}\mathbf{mat}$ in $\mathbf{mod}\text{-}\mathscr{A}$, we define $\mathcal{G}(F) := \bigoplus_{v \in \mathscr{A}} F(v)$. The endomorphism algebra $\operatorname{End}\mathscr{A}$ is generated by the morphisms of $\mathscr{A}$ (cp. Definition A.29). For a morphism $a \in \mathscr{A}$ and a vector $x \in F(v), v \in \mathscr{A}$, we define

$$x \cdot a := \begin{cases} x \cdot F(a) & \text{if } v = \mathfrak{s}(a), \\ 0 & \text{otherwise.} \end{cases}$$

This operation can be linearly extended to an action $\mathcal{G}(F) \times \operatorname{End}\mathscr{A} \to \mathcal{G}(F)$ which equips $\mathcal{G}(F)$ with a structure of a k-finite dimensional right $\operatorname{End}\mathscr{A}$-module. For a morphism $\alpha \colon F \to G$ in $\mathbf{mod}\text{-}\mathscr{A}$, we define the morphism $\mathcal{G}_\alpha := \bigoplus_{v \in \mathscr{A}} \alpha(v) \colon \mathcal{G}(F) \to \mathcal{G}(G)$. It can be shown that $\mathcal{G}_\alpha$ is a morphism in $\mathbf{fdmod}\text{-}\operatorname{End}\mathscr{A}$.

For a k-finite dimensional right $\operatorname{End}\mathscr{A}$-module M, we define the object $\mathcal{F}(M) \colon \mathscr{A} \to k\text{-}\mathbf{mat}$ in $\mathbf{mod}\text{-}\mathscr{A}$ by mapping an object v in $\mathscr{A}$ to $\dim_k(M \cdot \mathrm{id}_v)$; and by mapping a morphism $a \colon u \to v$

[31] I.e., $\mathscr{A}$ can be embedded in $\mathscr{A}\text{-}\mathbf{mod}$ by the Yoneda Lemma 2.86.

to the matrix of the k-linear map

$$\begin{cases} M \cdot \mathrm{id}_u & \longrightarrow M \cdot \mathrm{id}_v, \\ x & \mapsto x \cdot a \end{cases}$$

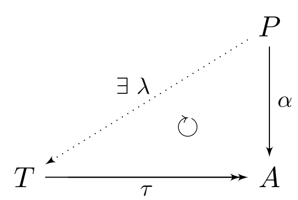
with respect to some throughout fixed bases $\mathcal{B}(M \cdot \mathrm{id}_u)$ and $\mathcal{B}(M \cdot \mathrm{id}_v)$.

For a morphism $\alpha \colon M \to N$ in **fdmod**-$\operatorname{End} \mathscr{A}$ we define $\mathcal{F}(\alpha) \colon \mathcal{F}(M) \to \mathcal{F}(N)$ by mapping an object v in $\mathscr{A}$ to the matrix of the k-linear map

$$\begin{cases} M \cdot \mathrm{id}_v & \longrightarrow N \cdot \mathrm{id}_v, \\ x & \mapsto \alpha(x) \end{cases}$$

with respect to some throughout fixed bases $\mathcal{B}(M \cdot \mathrm{id}_v)$ and $\mathcal{B}(N \cdot \mathrm{id}_v)$. □

Definition 2.71. Let $\mathscr{C}$ be an Abelian category. We say $\mathscr{C}$ **has enough projective objects** if we have an algorithm which for a given object A in $\mathscr{C}$ computes a projective object P_A and an epimorphism $\pi_A \colon P_A \twoheadrightarrow A$. Furthermore, we say $\mathscr{C}$ **has computable projective lifts** if we have an algorithm which for a given projective object P, a morphism $\alpha \colon P \to A$ and an epimorphism $\tau \colon T \twoheadrightarrow A$, computes a lift morphism of α along τ, i.e., a morphism $\lambda \colon P \to T$ that renders the following diagram commutative:



Remark 2.72. Let $\mathscr{C}$ be a category with enough projectives and computable projective lifts. We can derive an algorithm to decide whether an object A in $\mathscr{C}$ is projective. We compute an epimorphism $\pi_A \colon P_A \to A$ from some projective object P_A. An easy verification shows that A is projective if and only if there exists a lift morphism $\lambda \colon A \to P_A$ of id_A along π_A, i.e., with $\lambda \cdot \pi_A = \mathrm{id}_A$.

Definition 2.73. Let $\mathscr{C}$ be an Abelian category and A an object in $\mathscr{C}$. A projective object P together with an epimorphism $\pi \colon P \twoheadrightarrow A$ will be called a **projective cover** of A if π is a **superfluous epimorphism**[32], i.e., epimorphisms to A can be lifted along π only via epimorphisms. In particular, for any morphism $\ell \colon T \to P$, if $\ell \cdot \pi$ is an epimorphism, then ℓ is also an epimorphism.

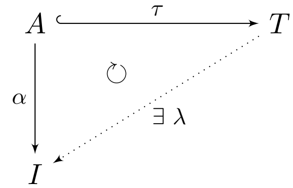
Lemma 2.74. *Let k be field and Λ a finite dimensional k-algebra. Let M be an object in* **mod**-Λ *and $\pi \colon P \to M$ be a projective cover of M. For any epimorphism $q \colon Q \to M$ where Q is a projective object, we have $\dim_k P \leq \dim_k Q$.*

Proof. There exists a lift morphism λ of q along π. Since π is superfluous, λ is an epimorphism, i.e., $\dim_k P \leq \dim_k Q$. □

Definition 2.75. Let $\mathscr{C}$ be an Abelian category. We say that $\mathscr{C}$ **has enough injective objects** if we have an algorithm which for a given object A in $\mathscr{C}$ computes an injective object I_A and a monomorphism $\iota_A \colon A \hookrightarrow I_A$. Furthermore, we say $\mathscr{C}$ **has computable injective**

[32]Some references call it **coessential epimorphism**.

colifts if we have an algorithm which for a given injective object I, a morphism $\alpha\colon A \to I$ and a monomorphism $\tau\colon A \hookrightarrow T$, computes a colift morphism of α along τ, i.e., a morphism $\lambda\colon T \to I$ that renders the following diagram commutative:



Remark 2.76. Similar to Remark 2.72, if $\mathscr{C}$ is a category with enough injectives and computable injective colifts, then we can decide whether an object A in $\mathscr{C}$ is injective.

Definition 2.77. Let $\mathscr{C}$ be an Abelian category and A an object in $\mathscr{C}$. An injective object I together with a monomorphism $\iota\colon A \hookrightarrow I$ will be called **injective envelope** for A if ι is an **essential monomorphism**, i.e., monomorphisms from A can be colifted along ι only via monomorphisms. In particular, for any morphism $\ell\colon I \to T$, if $\iota \cdot \ell$ is a monomorphism, then so is ℓ.

Definition 2.78. Let $\mathscr{C}$ be an additive category. A nonzero object A is said to be **indecomposable** if A has no nonzero direct summands. The full subcategory generated by all indecomposable objects in $\mathscr{C}$ will be denoted by $\mathbf{ind}(\mathscr{C})$. The skeleton category of $\mathbf{ind}(\mathscr{C})$ will be denoted by $\mathbf{ind}_0(\mathscr{C})$.

The following theorem enables us to classify the indecomposable projective and injective objects in $\mathbf{mod}\text{-}\mathscr{A}$ where $\mathscr{A}$ is a k-linear category defined by a quiver subject to an admissible set of relations. Details can be found in [**DW17**, Section 3.1] or [**ASS06**, Section III.2.].

Theorem 2.79. *Let $\mathscr{A} := k\mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle$ be a k-linear finitely presented category defined by a quiver $\mathfrak{q}$ subject to an admissible set of relations ρ. Then*

(1) *The indecomposable projective objects in $\mathbf{mod}\text{-}\operatorname{End}\mathscr{A} \simeq \mathbf{mod}\text{-}\mathscr{A}$ are, up to isomorphism, exactly the cyclic right $\operatorname{End}\mathscr{A}$-modules $P(v) := \mathrm{id}_v \cdot \operatorname{End}\mathscr{A}$, $v \in \mathscr{A}$.*
(2) *The indecomposable injective objects in $\mathbf{mod}\text{-}\operatorname{End}\mathscr{A}$ are, up to isomorphism, exactly the modules $I(v) := \operatorname{Hom}_k(\operatorname{End}\mathscr{A} \cdot \mathrm{id}_v, k)$, $v \in \mathscr{A}$ where the right action of $\operatorname{End}\mathscr{A}$ on $I(v)$ is given by*
$$\begin{cases} I(v) \times \operatorname{End}\mathscr{A} & \longrightarrow I(v) \\ (\varphi, a) & \longmapsto \varphi \cdot a\colon \begin{cases} \operatorname{End}\mathscr{A} \cdot \mathrm{id}_v & \to k, \\ x & \mapsto \varphi(a \cdot x). \end{cases} \end{cases}$$
(3) *The simple objects in $\mathbf{mod}\text{-}\operatorname{End}\mathscr{A}$ are, up to isomorphism, exactly $S(v) := k^1$, $v \in \mathscr{A}$ where the right action of $\operatorname{End}\mathscr{A}$ on k^1 is given by*
$$x \cdot a := \begin{cases} \lambda \cdot x & \text{if } a = \lambda \cdot \mathrm{id}_v \text{ for some } \lambda \in k, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.80. Let $P(v)$ be the indecomposable projective object in $\mathbf{mod}\text{-}\operatorname{End}\mathscr{A}$ associated to an object $v \in \mathscr{A}$. For every $u \in \mathscr{A}$, the k-vector space $\operatorname{Hom}_{\mathscr{A}}(v, u)$ equals the k-vector space $P(v) \cdot \mathrm{id}_u$. Thus, we set
$$\mathcal{B}(P(v) \cdot \mathrm{id}_u) := \mathcal{B}(\operatorname{Hom}_{\mathscr{A}}(v, u)).$$

By Theorem 2.70, $P(v)$ corresponds in $\mathbf{mod}\text{-}\mathscr{A}$ to the object

$$P_v := \mathcal{F}(P(v)) \colon \mathscr{A} \to k\text{-}\mathbf{mat}$$

which maps an object u in $\mathscr{A}$ to $\bigoplus_{b \in \mathcal{B}(\mathrm{Hom}_{\mathscr{A}}(v,u))} 1 = \dim_k \mathrm{Hom}_{\mathscr{A}}(v,u)$ and maps a morphism $a \colon u_1 \to u_2$ to the matrix of the k-linear map

$$\begin{cases} P(v) \cdot \mathrm{id}_{u_1} & \longrightarrow P(v) \cdot \mathrm{id}_{u_2}, \\ \quad x & \mapsto x \cdot a \end{cases}$$

with respect to the bases $\mathcal{B}(P(v) \cdot \mathrm{id}_{u_1})$ and $\mathcal{B}(P(v) \cdot \mathrm{id}_{u_2})$. That is, the full subcategory of $\mathbf{mod}\text{-}\mathscr{A}$ that is generated by the objects $P_v, v \in \mathscr{A}$ is a model for $\mathbf{ind}_0(\mathbf{proj}\text{-}\mathscr{A})$.

Remark 2.81. Let $P(u)$ and $P(v)$ be the indecomposable projective objects in $\mathbf{mod}\text{-}\operatorname{End} \mathscr{A}$ associated to objects u and v in $\mathscr{A}$. Every element $a \in \operatorname{End} \mathscr{A}$ defines a morphism

$$p(a) \colon \begin{cases} P(v) & \to P(u), \\ \quad x & \mapsto a \cdot x. \end{cases}$$

in $\mathbf{mod}\text{-}\operatorname{End} \mathscr{A}$. By Theorem 2.70, the morphism $p(a)$ corresponds in $\mathbf{mod}\text{-}\mathscr{A}$ to the morphism $p_a := \mathcal{F}(p(a)) \colon P_v \to P_u$ defined at object w in $\mathscr{A}$ by the matrix of the k-linear map

$$\begin{cases} P(v) \cdot \mathrm{id}_w & \longrightarrow P(u) \cdot \mathrm{id}_w, \\ \quad x & \mapsto a \cdot x \end{cases}$$

with respect to the bases $\mathcal{B}(P(v) \cdot \mathrm{id}_w)$ and $\mathcal{B}(P(u) \cdot \mathrm{id}_w)$.

Remark 2.82. Let $S(v)$ be the simple object in $\mathbf{mod}\text{-}\operatorname{End} \mathscr{A}$ associated to a vertex $v \in \mathscr{A}$. Then $S(v)$ correspondence in $\mathbf{mod}\text{-}\mathscr{A}$ to the object $S_v := \mathcal{F}(S(v)) \colon \mathscr{A} \to k\text{-}\mathbf{mat}$ which maps the object v in $\mathscr{A}$ to 1 and all other objects to 0; and maps all the generating morphisms of $\mathscr{A}$ to the corresponding zero morphisms.

Lemma 2.83. *Let $\mathscr{A} := k\mathcal{F}_{\mathfrak{q}}/\langle \rho \rangle$ be a k-linear finitely presented category defined by a quiver $\mathfrak{q}$ subject to an admissible set of relations ρ. Then for every object $F \colon \mathscr{A} \to k\text{-}\mathbf{mat}$ in $\mathbf{mod}\text{-}\mathscr{A}$ and every $v \in \mathfrak{q}_0$ we have*

$$\mathrm{Hom}_{\mathbf{mod}\text{-}\mathscr{A}}(P_v, F) \cong \mathrm{Hom}_{k\text{-}\mathbf{mat}}(1, F(v)) \cong F(v).$$

Proof. By Remark 2.80, $P_v(v) = \bigoplus_{b \in \mathcal{B}(\mathrm{Hom}_{\mathscr{A}}(v,v))} 1$. Let $\xi_v \colon 1 \to P_v(v)$ be the natural injection of the direct summand that is indexed by the morphism $\mathrm{id}_v \colon v \to v \in \mathcal{B}(\mathrm{Hom}_{\mathscr{A}}(v,v))$.

A straightforward verification shows that the k-linear map

$$\varphi \colon \begin{cases} \mathrm{Hom}_{\mathbf{mod}\text{-}\mathscr{A}}(P_v, F) & \to \mathrm{Hom}_{k\text{-}\mathbf{mat}}(1, F(v)), \\ \qquad \alpha & \mapsto \xi_v \cdot \alpha(v) \end{cases}$$

is an isomorphism and its inverse is given by

$$\begin{cases} \mathrm{Hom}_{k\text{-}\mathbf{mat}}(1, F(v)) & \to \mathrm{Hom}_{\mathbf{mod}\text{-}\mathscr{A}}(P_v, F), \\ \qquad \ell & \mapsto \begin{cases} P_v & \to F, \\ u & \mapsto P_v(u) = \bigoplus_{b \in \mathcal{B}(\mathrm{Hom}_{\mathscr{A}}(v,u))} 1 \xrightarrow{(\ell \cdot F(b))_{b,1}} F(u). \end{cases} \end{cases}$$

∎

Remark 2.84. The above proof is the categorical formulation of the fact that $P(v)$ is the cyclic right $\operatorname{End}\mathscr{A}$-module generated by $\mathrm{id}_v \in \operatorname{End}\mathscr{A}$; which means that every morphism from $P(v)$ is uniquely determined by its value on id_v.

Remark 2.85. With the same assumptions and notations of Lemma 2.83, let $\alpha\colon P_v \to G$ and $\tau\colon F \twoheadrightarrow G$ be two morphisms in $\mathbf{mod}\text{-}\mathscr{A}$. Since P_v is a projective object, there exists a lift morphism $\lambda\colon P_v \to F$ of α along τ. Lemma 2.83 provides an algorithm to compute such a λ. Let $s\colon G(v) \to F(v)$ be a section morphism for $\tau(v)\colon F(v) \to G(v)$, i.e., $s \cdot \tau(v) = \mathrm{id}_{G(v)}$. Then $\lambda \coloneqq \varphi^{-1}(\xi_v \cdot \alpha(v) \cdot s)\colon P_v \to F$ is a lift morphism of α along τ.

Let $m_v,\ v \in \mathfrak{q}_0$ be a list of nonnegative integers, then the above trick, together with the universal property of the direct sum object, can be used to compute a lift morphism of any morphism $\bigoplus_{v\in\mathfrak{q}_0} P_v^{m_v} \to G$ along $\tau\colon F \twoheadrightarrow G$.

The main Yoneda Lemma applies to local small categories and functors to **Set**. The following lemma is the additive version of Yoneda Lemma which applies on a preadditive category $\mathscr{C}$ and the category of additive functors from $\mathscr{C}$ to **Ab**. The Lemma still applies for R-linear categories and the category of R-linear functors to R-**Mod**. For more details, see e.g., [**Bor94b**, Theorem 6.3.5].

Lemma 2.86 (Additive Yoneda lemma)**.** *Let $\mathscr{C}$ be a preadditive category and let A be an object in $\mathscr{C}$. Then for any additive functor $F\colon \mathscr{C} \to \mathbf{Ab}$, there is an isomorphism of Abelian groups*

$$\Phi\colon \begin{cases} \mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(V,-),F) & \to F(V), \\ \alpha & \mapsto \alpha_V(\mathrm{id}_V). \end{cases}$$

which is natural in both V and F.

Proof. It can be shown that the inverse of Φ is given by

$$\Phi^{-1}\colon \begin{cases} F(V) & \to \mathrm{Nat}(\mathrm{Hom}_{\mathscr{C}}(V,-),F), \\ x & \mapsto \alpha_x\colon \begin{cases} \mathrm{Hom}_{\mathscr{C}}(V,-) & \to F, \\ U & \mapsto \alpha_{x,U}\colon \begin{cases} \mathrm{Hom}_{\mathscr{C}}(V,U) & \to F(U), \\ f & \mapsto F(f)(x). \end{cases} \end{cases} \end{cases}$$

□

By substituting F in the previous lemma with the functor $\mathrm{Hom}_{\mathscr{C}}(U,-)\colon \mathscr{C} \to \mathbf{Ab}$ we get the following corollary:

Corollary 2.87. *Let $\mathscr{C}$ be a preadditive category. Then the functor*

$$\mathcal{Y}_{\mathscr{C}}\colon \begin{cases} \mathscr{C}^{\mathrm{op}} & \to [\mathscr{C},\mathbf{Ab}], \\ V & \mapsto \mathrm{Hom}_{\mathscr{C}}(V,-), \\ \alpha^{\mathrm{op}}\colon V \to U & \mapsto \mathcal{Y}_{\mathscr{C}}(\alpha^{\mathrm{op}})\colon \begin{cases} \mathcal{Y}_{\mathscr{C}}(V) & \to \mathcal{Y}_{\mathscr{C}}(U), \\ W & \mapsto \mathrm{Hom}_{\mathscr{C}}(\alpha,W) = \alpha\cdot - \end{cases} \end{cases}$$

is a fully faithful embedding.

Remark 2.88 (Yoneda Embedding)**.** We call the functor $\mathcal{Y}_{\mathscr{C}}$ in Corollary 2.87 the "contravariant" Yoneda embedding of $\mathscr{C}$. The "covariant" Yoneda embedding of $\mathscr{C}$ is defined by

$$\mathcal{Z}_{\mathscr{C}} \coloneqq \mathcal{Y}_{\mathscr{C}^{\mathrm{op}}}\colon \mathscr{C} \to [\mathscr{C}^{\mathrm{op}},\mathbf{Ab}].$$

Corollary 2.89. *With the same assumptions and notations in Corollary 2.87, a morphism* α *in* $\mathscr{C}$ *is an isomorphism if and only if* $\mathcal{Y}_{\mathscr{C}}(\alpha^{\mathrm{op}})$ *is a natural isomorphism if and only if* $\mathcal{Z}_{\mathscr{C}}(\alpha)$ *is a natural isomorphism.*

We are mainly interested in constructing the following instance of the Yoneda embedding:

Corollary 2.90. *Let* $\mathscr{A} = k\mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle$ *be* k*-linear finitely presented category defined by a quiver* $\mathfrak{q}$ *subject to an admissible set of relations* ρ*. Then the* Yoneda *embedding*

$$\mathcal{Y}_{\mathscr{A}}\colon \begin{cases} \mathscr{A}^{\mathrm{op}} & \to \mathbf{mod}\text{-}\mathscr{A}, \\ v & \mapsto \mathrm{Hom}_{\mathscr{A}}(v,-), \\ a^{\mathrm{op}}\colon v\to u & \mapsto \mathcal{Y}_{\mathscr{A}}(a^{\mathrm{op}})\colon \begin{cases} \mathcal{Y}_{\mathscr{A}}(v) & \to \mathcal{Y}_{\mathscr{A}}(u), \\ w & \mapsto \mathrm{Hom}_{\mathscr{A}}(a,w) = a\cdot - \end{cases} \end{cases}$$

induces an isomorphism $\mathscr{A}^{\mathrm{op}} \cong \mathbf{ind}_0(\mathbf{proj}\text{-}\mathscr{A})$*. In particular,* $\mathscr{A}^{\mathrm{op}}$ *and* $\mathbf{ind}(\mathbf{proj}\text{-}\mathscr{A})$ *are equivalent.*

Proof. It follows from Remark 2.80 that $\mathcal{Y}_{\mathscr{A}}(v)$ and P_v are equal for every object $v \in \mathscr{A}$. The assertion follows from Theorem 2.79.(1) and Corollary 2.87. □

Definition 2.91. Let k be a field, A a k-algebra and M a right A-module. The radical of M, denoted by $\mathbf{rad}(M)$, is defined by the intersection of all maximal submodules in M. The top of M, denoted by $\mathbf{top}(M)$, is defined by $M/\mathbf{rad}(M)$.

Example 2.92. Let $P(v)$ be the projective indecomposable right $\mathrm{End}\,\mathscr{A}$-module associated to the object $v \in \mathscr{A}$ as introduced in Theorem 2.79. According to [**ASS06**, Lemma III.2.4], $\mathbf{rad}(P(v))$ is the right $\mathrm{End}\,\mathscr{A}$-module generated by the set of all morphisms in $\mathscr{A}$ that are represented by paths p in $k\mathcal{F}_{\mathfrak{q}}$ with $\mathrm{Source}(p) = v$ and $p \neq \mathrm{id}_v$.

The following lemma is a reformulation of [**ASS06**, Lemma III.2.2.c] in categorical language:

Lemma 2.93. *Let* $\mathscr{A} = k\mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle$ *be* k*-linear finitely presented category defined by a quiver* $\mathfrak{q}$ *subject to an admissible set of relations* ρ*. Then*

(1) *For an object* M *in* $\mathbf{mod}\text{-}\mathrm{End}\,\mathscr{A}$*, the radical* $\mathbf{rad}(M)$ *is given by* $M\cdot\mathbf{rad}(\mathrm{End}\,\mathscr{A})$ *where* $\mathbf{rad}(\mathrm{End}\,\mathscr{A})$ *is the Jacobson radical of* $\mathrm{End}\,\mathscr{A}$*.*

(2) *For an object* $F\colon \mathscr{A}\to k\text{-}\mathbf{mat}$ *in* $\mathbf{mod}\text{-}\mathscr{A}$*, the radical* $\mathbf{rad}(F)$ *is given by the object*

$$\mathbf{rad}(F)\colon \begin{cases} \mathscr{A} & \to k\text{-}\mathbf{mat}, \\ v & \mapsto \mathrm{im}(\mu_v), \\ a\colon u\to v & \mapsto \text{the unique lift of } \iota_u\cdot F(a) \text{ along } \iota_v \end{cases}$$

where

$$\mu_v := \bigoplus_{s\in Q_1,s\colon w\to v} F(w) \xrightarrow{(F([s]))_{s,1}} F(v)$$

and $\iota_v\colon \mathrm{im}(\mu_v)\hookrightarrow F(v)$ *is the image embedding of* μ_v*. Moreover, the collection* $(\iota_v)_{v\in\mathfrak{q}_0}$ *defines a monomorphism*

$$\iota_F\colon \begin{cases} \mathbf{rad}(F) & \to F, \\ v & \mapsto \iota_v \end{cases}$$

in $\mathbf{mod}\text{-}\mathscr{A}$*. The morphism* i_F *will be called the* ***radical embedding*** *of* F*. Hence,* $\mathbf{top}(F)$ *is given by the cokernel object* $\mathrm{coker}(\iota_F)$*.*

The following remark follows from [**ASS06**, Corollaries I.5.9 and I.5.17]:

Remark 2.94. Let P_v be the indecomposable projective object in $\mathbf{mod}\text{-}\mathscr{A}$ associated to an object $v \in \mathscr{A}$ and $\iota_{P_v} \colon \mathbf{rad}(P_v) \hookrightarrow P_v$ the radical embedding of P_v. Then $\mathbf{top}(P_v) = S_v$ and the cokernel projection $s_v \colon P_v \twoheadrightarrow S_v$ of ι_{P_v} is a projective cover of S_v.

The following theorem states that the category $\mathbf{mod}\text{-}\mathscr{A} = k\mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle$ for an admissible set of relations $\rho \subset k\mathcal{F}_{\mathfrak{q}}$ admits projective covers.

Theorem 2.95. *Let $\mathscr{A} = k\mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle$ be k-linear finitely presented category defined by a quiver $\mathfrak{q}$ subject to an admissible set of relations ρ. For each object F in $\mathbf{mod}\text{-}\mathscr{A}$ there exists, up to the order of elements, a unique list of nonnegative integers m_v, $v \in \mathscr{A}$ and a projective cover*

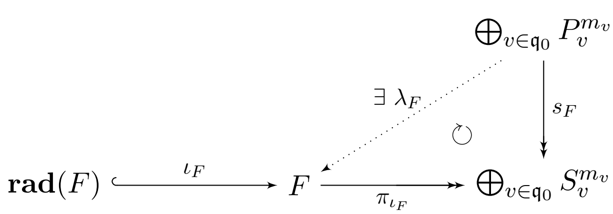
$$\lambda_F \colon \bigoplus_{v \in \mathscr{A}} P_v^{m_v} \twoheadrightarrow F$$

for F.

Proof. Let $\iota_F \colon \mathbf{rad}(F) \hookrightarrow F$ be the radical embedding of F and $\pi_{\iota_F} \colon F \twoheadrightarrow \operatorname{coker}(\iota_F)$ its cokernel projection. Then $\mathbf{top}(F) \coloneqq \operatorname{coker}(\iota_F)$ is semisimple (cf. [**ASS06**, Theorem I.5.8] or [**Zim14**, Proposition 1.9.6]), i.e., it can be decomposed as a direct sum of simple objects in $\mathbf{mod}\text{-}\mathscr{A}$. For a vertex $v \in \mathscr{A}$, let m_v be the multiplicity of S_v in such a decomposition. Hence, $\mathbf{top}(F) = \bigoplus_{v \in \mathfrak{q}_0} S_v^{m_v}$ and the following direct sum of the projective covers $s_v \colon P_v \to S_v$, $v \in \mathscr{A}$ introduced in Remark 2.94

$$s_F \coloneqq \bigoplus_{v \in \mathscr{A}} s_v^{m_v} \colon \bigoplus_{v \in \mathfrak{q}_0} P_v^{m_v} \twoheadrightarrow \bigoplus_{v \in \mathfrak{q}_0} S_v^{m_v}$$

defines a projective cover of $\mathbf{top}(F)$. It can be shown that any lift morphism, say λ_F, of s_F along π_{ι_F}:



is a projective cover of F, see e.g., [**ASS06**, Theorem I.5.8] or [**Zim14**, Proposition 1.9.6]. Note that Remark 2.85 can be used to compute λ_F. □

The following corollary is essential for performing homological algebra computations over the Abelian category $\mathbf{mod}\text{-}\mathscr{A}$.

Corollary 2.96. *Let $\mathscr{A} = k\mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle$ be k-linear finitely presented category defined by a quiver $\mathfrak{q}$ subject to an admissible set of relations $\rho \subset k\mathcal{F}_{\mathfrak{q}}$. Then the Abelian category $\mathbf{mod}\text{-}\mathscr{A}$ has enough projective and injective objects and has computable projective lifts and injective colifts. Moreover, if $\mathfrak{q}$ is acyclic*[33] *then $\mathbf{gldim}\,\mathbf{mod}\text{-}\mathscr{A} \le |\mathfrak{q}_0|$.*

Proof. The assertion of having enough projectives and computable projective lifts follows from Remark 2.85, Theorem 2.95 and [**DW17**, Proposition 3.1.7]. The assertion of having enough

[33] If $\mathfrak{q}$ is not acyclic, then we can use its *Kupisch series* to decide whether the global dimension is finite. For details we refer to [**Ful68**].

injective objects and computable injective colifts follows by a dual argument (See e.g., [**DW17**, Chapters 2 and 3]). □

Corollary 2.97. *Let $\mathscr{A} = k\mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle$ be k-linear finitely presented category defined by a quiver $\mathfrak{q}$ subject to an admissible set of relations $\rho \subset k\mathcal{F}_{\mathfrak{q}}$. Then the additive category $\mathscr{A}^{\mathrm{op},\oplus}$ has weak kernels. Consequently, the associated* Freyd *category $\mathcal{A}(\mathscr{A}^{\mathrm{op},\oplus})$ is Abelian. Furthermore, we have*

$$\mathcal{A}(\mathscr{A}^{\mathrm{op},\oplus}) \cong \mathbf{mod}\text{-}\mathscr{A} \cong \mathbf{mod}\text{-}\operatorname{End}\mathscr{A}.$$

Proof. Let $\mathbf{proj}\text{-}\mathscr{A}$ be the full subcategory of $\mathbf{mod}\text{-}\mathscr{A}$ generated by the projective objects. By Corollary 2.90, $\mathscr{A}^{\mathrm{op},\oplus} \cong \mathbf{proj}\text{-}\mathscr{A}$, therefore, it is sufficient to prove that $\mathbf{proj}\text{-}\mathscr{A}$ has weak kernels.

Let $\varphi\colon P \to Q$ be a morphism in $\mathbf{proj}\text{-}\mathscr{A}$ and let $\iota_\varphi\colon K \hookrightarrow P$ in $\mathbf{mod}\text{-}\mathscr{A}$ be the kernel embedding of φ. By Corollary 2.96, the category $\mathbf{mod}\text{-}\mathscr{A}$ has enough projectives, i.e., there exists an P_K in $\mathbf{proj}\text{-}\mathscr{A}$ and an epimorphism $\pi_K\colon P_K \twoheadrightarrow K$.

We claim that P_K together with $\pi_K \cdot \iota_\varphi\colon P_K \to P$ in $\mathbf{proj}\text{-}\mathscr{A}$ defines a weak kernel for φ. Let $\tau\colon T \to P$ be a morphism in $\mathbf{proj}\text{-}\mathscr{A}$ such that $\tau \cdot \varphi = 0$. By the universal property of the kernel object K, there exists a lift morphism $\lambda\colon T \to K$ of τ along ι_φ. Since T is a projective object, there exists a lift morphism $\mu\colon T \to P_K$ of λ along the epimorphism π_K. Hence, $\tau = \lambda \cdot \iota_\varphi = \mu \cdot \pi_K \cdot \iota_\varphi$ and by Theorem 2.31, the category $\mathcal{A}(\mathscr{A}^{\mathrm{op},\oplus})$ is Abelian. The equivalence of categories follows from [**Pos21a**, Theorem 4.1] and Theorem 2.70. □

Remark 2.98. By replacing $\mathscr{A}$ with $\mathscr{A}^{\mathrm{op}}$ we get

$$\mathcal{A}(\mathscr{A}^{\oplus}) \cong \mathscr{A}\text{-}\mathbf{mod} \cong \operatorname{End}\mathscr{A}\text{-}\mathbf{mod}.$$

CHAPTER 3

Category Constructors in Homological Algebra

This chapter provides an algorithmic approach to the basic homological computations on the additive and Abelian categories already introduced in the Chapter 2. In Sections 3.1 and 3.2, we content ourselves with the basic concepts and constructions related to complexes and homotopy categories. We show that the class of contractible objects in the category of complexes $\mathcal{C}^b(\mathscr{C})$ forms a class of colifting objects (cf. Remark 3.28) and use this class to construct the homotopy category $\mathcal{K}^b(\mathscr{C})$ as a stable category (Theorem 3.29). This enables us to reduce deciding the equality of morphisms (and in the affirmative case, computing a chain homotopy witnessing the equality) in $\mathcal{K}^b(\mathscr{C})$ to a colift problem in $\mathcal{C}^b(\mathscr{C})$ (cf. Corollary 3.26), which in turn reduces to solving a system of inhomogeneous two-sided linear equations in $\mathscr{C}$ (cf. Corollary 3.26). As a result, it will be vital to equip the category $\mathscr{C}$ with an appropriate homomorphism structure, because, as we shall observe in Chapter 4, homomorphism structures provide the ideal categorical framework for solving systems of inhomogeneous two-sided linear equations in categories.

In Section 3.3, we provide an explicit computation of projective and injective resolutions of objects and morphisms in the bounded homotopy categories of Abelian categories with enough projective resp. injective objects, and review their relations to quasi-isomorphisms and localization functors.

In Section 3.4, we quickly review the definition of the bounded derived category $\mathcal{D}^b(\mathscr{C})$ of an Abelian category $\mathscr{C}$. The tricky definition of morphisms in the bounded derived categories makes it very hard to algorithmically decide the equality of morphisms in $\mathcal{D}^b(\mathscr{C})$. However, we are interested only in the case the provided Abelian category $\mathscr{C}$ has enough projective or injective objects and finite global dimension, in which case we can employ the projective or injective resolutions to model the bounded derived category $\mathcal{D}^b(\mathscr{C})$ in terms of a bounded homotopy category $\mathcal{K}^b(\mathbf{proj}(\mathscr{C}))$ or $\mathcal{K}^b(\mathbf{inj}(\mathscr{C}))$, which then enables us to decide the equality of morphisms in $\mathcal{D}^b(\mathscr{C})$. We review basic homological constructions such as the extension groups $\mathrm{Ext}^n(-,-)$ and the computation of the left and right derived functors.

All the constructions presented in this chapter are implemented within the `GAP` meta-package `HigherHomologicalAlgebra` [**Sal21a**]. The constructive approach to these categories provided in this chapter will be extended further in Chapter 4, where we elevate homomorphism structures from a category $\mathscr{C}$ to $\mathcal{C}^b(\mathscr{C})$, $\mathcal{K}^b(\mathscr{C})$ and $\mathcal{D}^b(\mathscr{C})$; and in Chapter 5, where we discuss the computability of $\mathcal{K}^b(\mathscr{C})$ as a triangulated category.

The primary instances for $\mathscr{C}$ we have in mind are the following:

- $\mathscr{C}$ is the Abelian category $\mathbf{mod}\text{-}\mathscr{A}$ (resp. $\mathscr{A}\text{-}\mathbf{mod}$) of functors from an (admissible) k-linear finitely presented category $\mathscr{A}$ (resp. $\mathscr{A}^{\mathrm{op}}$) to $k\text{-}\mathbf{mat}$. In particular, if $\mathscr{A}$ is the abstraction k-algebroid $\mathbf{A}_{\mathscr{E}}$ of a strong exceptional sequence $\mathscr{E}$ in a triangulated category $\mathfrak{T}$, then $\mathbf{A}_{\mathscr{E}}\text{-}\mathbf{mod} \simeq \operatorname{End} T_{\mathscr{E}}\text{-}\mathbf{mod}$ where $T_{\mathscr{E}}$ is the tilting object of $\mathscr{E}$. Furthermore, the categories $\mathbf{A}_{\mathscr{E}}\text{-}\mathbf{mod}$, $\mathcal{C}^b(\mathbf{A}_{\mathscr{E}}\text{-}\mathbf{mod})$, $\mathcal{K}^b(\mathbf{A}_{\mathscr{E}}\text{-}\mathbf{mod})$ and $\mathcal{D}^b(\mathbf{A}_{\mathscr{E}}\text{-}\mathbf{mod})$ have decidable equality of morphisms and are equipped with a ($k\text{-}\mathbf{mat}$)-homomorphism structure.

- $\mathscr{C}$ is the additive closure category $\mathscr{A}^{\oplus}$ of a k-linear finitely presented category $\mathscr{A}$, or $\mathscr{E}^{\oplus}$ of a strong exceptional sequence in a triangulated category $\mathfrak{T}$ (cf. Chapter 1).
- $\mathscr{C}$ is the Abelian Freyd category $\mathcal{A}(R\text{-}\mathbf{rows}) \simeq R\text{-}\mathbf{fpmod}$ for a computable commutative ring R. The categories $\mathcal{A}(R\text{-}\mathbf{rows}), \mathcal{C}^b(\mathcal{A}(R\text{-}\mathbf{rows}))$ and $\mathcal{K}^b(\mathcal{A}(R\text{-}\mathbf{rows}))$ have decidable equality of morphisms and are equipped with a $\mathcal{A}(R\text{-}\mathbf{rows})$-homomorphism structure. If, furthermore, R has finite global dimension, then $\mathcal{D}^b(R\text{-}\mathbf{rows})$ has decidable equality of morphisms and is equipped with a $\mathcal{A}(R\text{-}\mathbf{rows})$-homomorphism structure (cf. Appendix C).

3.1. Complex Categories

Let $\mathscr{C}$ be an additive category. The **cochain complex category**[1] $\mathcal{C}(\mathscr{C})$ is defined by the following data:

(1) An object $A := (A^i, \partial_A^i)_{i\in\mathbb{Z}}$ in $\mathcal{C}(\mathscr{C})$ is a sequence of objects and morphisms in $\mathscr{C}$

$$\cdots \longrightarrow A^{i-1} \xrightarrow{\partial_A^{i-1}} A^i \xrightarrow{\partial_A^i} A^{i+1} \longrightarrow \cdots$$

such that $\partial_A^i \cdot \partial_A^{i+1} = 0$ for all $i \in \mathbb{Z}$.

(2) A morphism $\varphi\colon A \to B$ in $\mathcal{C}(\mathscr{C})$ is a family of morphisms $(\varphi^i\colon A^i \to B^i)_{i\in\mathbb{Z}}$

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A^{i-1} & \xrightarrow{\partial_A^{i-1}} & A^i & \xrightarrow{\partial_A^i} & A^{i+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \varphi^{i-1}} & \circlearrowleft & \downarrow{\scriptstyle \varphi^{i}} & \circlearrowleft & \downarrow{\scriptstyle \varphi^{i+1}} & & \\
\cdots & \longrightarrow & B^{i-1} & \xrightarrow[\partial_B^{i-1}]{} & B^i & \xrightarrow[\partial_B^i]{} & B^{i+1} & \longrightarrow & \cdots
\end{array}$$

such that $\partial_A^i \cdot \varphi^{i+1} = \varphi^i \cdot \partial_B^i$ for all $i \in \mathbb{Z}$. Composition of morphisms is defined by degree-wise composition of morphisms and the identity morphism of an object $A := (A^i, \partial_A^i)_{i\in\mathbb{Z}}$ is defined by the family $(\mathrm{id}_{A^i})_{i\in\mathbb{Z}}$.

Definition 3.1. Let $\mathscr{C}$ be an additive category and $\mathcal{C}(\mathscr{C})$ its complex category and let A be an object in $\mathcal{C}(\mathscr{C})$.

(1) A is said to be **bounded below** if there exists $\ell_A \in \mathbb{Z}$ with $A^i = 0$ for all $i < \ell_A$.
(2) A is said to be **bounded above** if there exists $u_A \in \mathbb{Z}$ with $A^i = 0$ for all $i > u_A$.
(3) A is said to be **bounded** if its is bounded below and bounded above.

We define $\mathcal{C}^+(\mathscr{C})$, $\mathcal{C}^-(\mathscr{C})$ and $\mathcal{C}^b(\mathscr{C})$ by the full subcategories of $\mathcal{C}(\mathscr{C})$ generated by the objects which are bounded below, bounded above, resp. bounded.

Definition 3.2. Let $\mathscr{C}$ be an additive category and $\mathcal{C}(\mathscr{C})$ its complex category.

(1) Objects of $\mathcal{C}(\mathscr{C})$ will be called **cochain complexes** over $\mathscr{C}$.
(2) Morphisms of $\mathcal{C}(\mathscr{C})$ will be called **cochain morphisms** over $\mathscr{C}$.
(3) The **support** of a cochain complex A over $\mathscr{C}$ is defined by

$$\mathrm{Supp}_A := \{i \in \mathbb{Z} \mid A^i \neq 0\}.$$

[1]The associated category constructor is `ComplexCategoryByCohains`(-) [**Sal21b**].

(4) An n^{th}**-stalk cochain complex** over $\mathscr{C}$ is an object A in $\mathcal{C}(\mathscr{C})$ with $A^i = 0$ for all $i \neq n$. In this case, we denote A by $\lceil A^n \rfloor_n$.
(5) An n^{th}**-stalk cochain morphism** over $\mathscr{C}$ is a morphism φ in $\mathcal{C}(\mathscr{C})$ with $\varphi^i = 0$ for all $i \neq n$. If furthermore the source and range are also n-stalk complexes, then φ is denoted by $\lceil \varphi^n \rfloor_n$. We define the n**-stalk functor** by

$$\lceil - \rfloor_n \colon \begin{cases} \mathscr{C} & \to \mathcal{C}(\mathscr{C}), \\ C & \mapsto \lceil C \rfloor_n, \\ \ell & \mapsto \lceil \ell \rfloor_n. \end{cases}$$

Definition 3.3. Let $\mathscr{C}$ be an Abelian category and $\mathcal{C}(\mathscr{C})$ its complex category. We can define the following functors:

(1) For every $i \in \mathbb{Z}$, we define the i^{th}**-cycles functor**[2] by

$$\mathrm{Z}^i \colon \begin{cases} \mathcal{C}(\mathscr{C}) & \to \mathscr{C}, \\ A & \mapsto \ker(\partial_A^i), \\ A \xrightarrow{\varphi} B & \mapsto \textit{the} \text{ lift morphism of } \iota_{\partial_A^i} \cdot \varphi^i \text{ along } \iota_{\partial_B^i}, \end{cases}$$

where $\iota_{\partial_A^i} \colon \ker(\partial_A^i) \hookrightarrow A^i$ and $\iota_{\partial_B^i} \colon \ker(\partial_B^i) \hookrightarrow B^i$ are the kernel embeddings of ∂_A^i resp. ∂_B^i.
(2) For every $i \in \mathbb{Z}$, we define the i^{th}**-boundaries functor**[3] by

$$\mathrm{B}^i \colon \begin{cases} \mathcal{C}(\mathscr{C}) & \to \mathscr{C}, \\ A & \mapsto \mathrm{im}\left(\partial_A^{i-1}\right), \\ A \xrightarrow{\varphi} B & \mapsto \textit{the} \text{ lift morphism of } \epsilon_{\partial_A^{i-1}} \cdot \varphi^i \text{ along } \epsilon_{\partial_B^{i-1}}, \end{cases}$$

where $\epsilon_{\partial_A^{i-1}} \colon \mathrm{im}(\partial_A^i) \hookrightarrow A^i$ and $\epsilon_{\partial_B^{i-1}} \colon \mathrm{im}(\partial_B^i) \hookrightarrow B^i$ are the image embeddings of ∂_A^{i-1} resp. ∂_B^{i-1}.
(3) Let A be an object in $\mathscr{C}$. Since $\mathscr{C}$ is Abelian, there exists for every $i \in \mathbb{Z}$ a lift epimorphism $\mu_A^i \colon A^{i-1} \to \mathrm{im}\left(\partial_A^{i-1}\right)$ of ∂_A^{i-1} along $\epsilon_{\partial_A^{i-1}}$. Hence

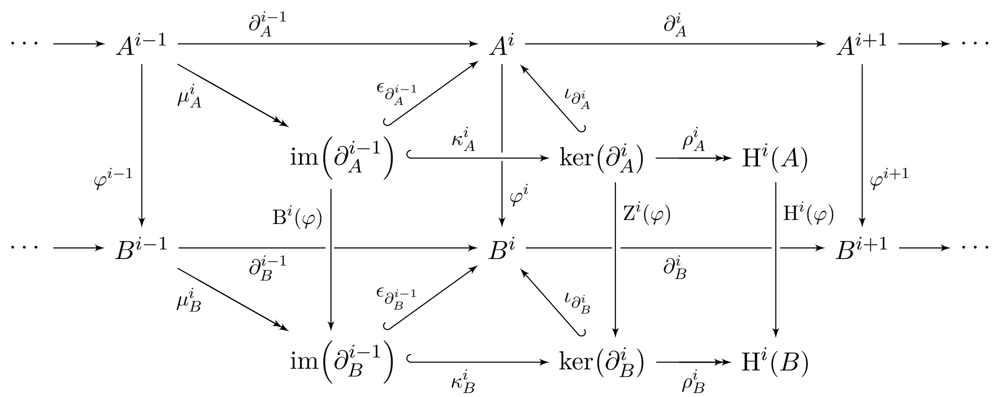
$$\mu_A^i \cdot \left(\epsilon_{\partial_A^{i-1}} \cdot \partial_A^i\right) = \partial_A^{i-1} \cdot \partial_A^i = 0,$$

i.e., $\epsilon_{\partial_A^{i-1}} \cdot \partial_A^i = 0$, hence there exists a unique lift morphism $\kappa_A^i \colon \mathrm{im}\left(\partial_A^{i-1}\right) \to \ker(\partial_A^i)$ of $\epsilon_{\partial_A^{i-1}}$ along $\iota_{\partial_A^i}$. An easy verification shows that κ_A^i is a monomorphism. We define the i^{th}**-cohomology functor**[4] on $\mathcal{C}(\mathscr{C})$ by

$$\mathrm{H}^i \colon \begin{cases} \mathcal{C}(\mathscr{C}) & \to \mathscr{C}, \\ A & \mapsto \mathrm{coker}(\kappa_A^i), \\ \varphi \colon A \to B & \mapsto \textit{the} \text{ colift morphism of } \mathrm{Z}^i(\varphi) \cdot \rho_B^i \text{ along } \rho_A^i, \end{cases}$$

where $\rho_A^i \colon \ker(\partial_A^i) \twoheadrightarrow \mathrm{H}^i(A)$ and $\rho_B^i \colon \ker(\partial_B^i) \twoheadrightarrow \mathrm{H}^i(B)$ are the cokernel projections of κ_A^i and κ_B^i.

[2] See `CyclesAt(-)` [Sal21b].
[3] See `BoundariesAt(-)` [Sal21b].
[4] See `CohomologyAt(-)` in [Sal21b].



(4) For every $i \in \mathbb{Z}$ we define the i^{th}-**boundaries-to-cycles** natural transformation by

$$\kappa^i \colon \begin{cases} \mathrm{B}^i & \to \mathrm{Z}^i, \\ A & \mapsto \kappa_A^i \colon \mathrm{B}^i(A) \hookrightarrow \mathrm{Z}^i(A) \end{cases}$$

and the i^{th}-**cycles-to-cohomology** natural transformation by

$$\rho^i \colon \begin{cases} \mathrm{Z}^i & \to \mathrm{H}^i, \\ A & \mapsto \rho_A^i \colon \mathrm{Z}^i(A) \twoheadrightarrow \mathrm{H}^i(A). \end{cases}$$

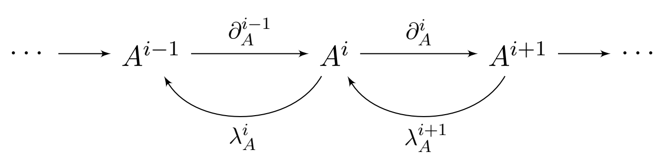
Definition 3.4. Let $\mathscr{C}$ be an Abelian category. A morphism $\varphi \colon A \to B$ in $\mathcal{C}(\mathscr{C})$ is called **quasi-isomorphism** if $\mathrm{H}^i(\varphi) \colon \mathrm{H}^i(A) \to \mathrm{H}^i(B)$ is an isomorphism for all $i \in \mathbb{Z}$.

Remark 3.5. We observe that in any expression $\tau = \varphi \cdot \psi$ if two morphisms are quasi-isomorphisms, then so is the third.

Definition 3.6. Let $\mathscr{C}$ be an Abelian category. An object A in $\mathcal{C}(\mathscr{C})$ is called **exact** (or **acyclic**) if $\kappa_A^i \colon B^i(A) \hookrightarrow Z^i(A)$ is an isomorphism for all $i \in \mathbb{Z}$; or equivalently, if $\mathrm{H}^i(A) = 0$ for all $i \in \mathbb{Z}$.

Remark 3.7. An immediate observation from the above definition is that A is exact if and only if ∂_A^i lifts along the kernel embedding of ∂_A^{i+1} via an epimorphism for all $i \in \mathbb{Z}$.

Definition 3.8. Let $\mathscr{C}$ be an additive category. An object A in $\mathcal{C}(\mathscr{C})$ is called **contractible** if there exists a family of morphisms $\left(\lambda_A^i \colon A^i \to A^{i-1}\right)_{i \in \mathbb{Z}}$



such that

$$\partial_A^i \cdot \lambda_A^{i+1} + \lambda_A^i \cdot \partial_A^{i-1} = \mathrm{id}_{A^i}$$

for all $i \in \mathbb{Z}$.

Lemma 3.9. *Let $\mathscr{C}$ be an Abelian category and A an object in $\mathcal{C}(\mathscr{C})$. If A is contractible, then A is exact.*

Proof. We use the same notations as in Definition 3.3. For all $i \in \mathbb{Z}$, we have

$$\begin{aligned}
\iota_{\partial_A^i} &= \iota_{\partial_A^i} \bullet \mathrm{id}_{A^i} \\
&= \iota_{\partial_A^i} \bullet \left(\partial_A^i \bullet \lambda_A^{i+1} + \lambda_A^i \bullet \partial_A^{i-1}\right) \\
&= \iota_{\partial_A^i} \bullet \lambda_A^i \bullet \partial_A^{i-1} \\
&= \iota_{\partial_A^i} \bullet \lambda_A^i \bullet \mu_A^i \bullet \epsilon_{\partial_A^{i-1}} \\
&= \iota_{\partial_A^i} \bullet \lambda_A^i \bullet \mu_A^i \bullet \kappa_A^i \bullet \iota_{\partial_A^i};
\end{aligned}$$

and since $\iota_{\partial_A^i}$ is a monomorphism, we get $\mathrm{id}_{\ker\left(\partial_A^i\right)} = \iota_{\partial_A^i} \bullet \lambda_A^i \bullet \mu_A^i \bullet \kappa_A^i$, i.e., κ_A^i is a split-epimorphism, thus an epimorphism. This means κ_A^i is an isomorphism for all $i \in \mathbb{Z}$ and A is exact. ☐

Corollary 3.10. *Let $\mathscr{C}$ be an Abelian category. If a morphism $\varphi\colon A \to B$ in $\mathcal{C}(\mathscr{C})$ factors through some contractible object, then $\mathrm{H}^i(\varphi) = 0$ for all $i \in \mathbb{Z}$.*

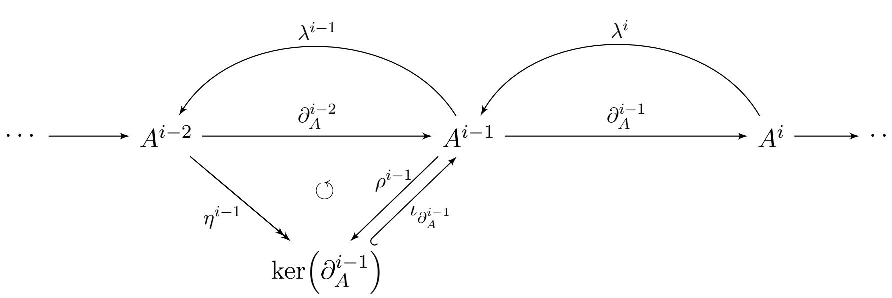
Lemma 3.11. *Let $\mathscr{C}$ be an Abelian category and A an object in $\mathcal{C}^-(\mathscr{C})$. If A is exact and A^i is a projective object for all $i \in \mathbb{Z}$, then A is contractible.*

Proof. We will iteratively construct a family of morphisms $\left(\lambda^i\colon A^i \to A^{i-1}\right)_{i\in\mathbb{Z}}$ such that $\partial_A^i \bullet \lambda^{i+1} + \lambda^i \bullet \partial_A^{i-1} = \mathrm{id}_{A^i}$ for all $i \in \mathbb{Z}$. Let u_A be an upper bound for A. For each $i > u_A$, we define λ^i by $A^i \xrightarrow{0} A^{i-1}$. Suppose we have already computed λ^{i+1} and λ^i and let us compute λ^{i-1}.

It follows from the assumption that

$$\begin{aligned}
\left(\mathrm{id}_{A^{i-1}} - \partial_A^{i-1} \bullet \lambda^i\right) \bullet \partial_A^{i-1} &= \partial_A^{i-1} - \partial_A^{i-1} \bullet \lambda^i \bullet \partial_A^{i-1} \\
&= \partial_A^{i-1} - \partial_A^{i-1} \bullet \left(\mathrm{id}_{A^i} - \partial_A^i \bullet \lambda^{i+1}\right) \\
&= \partial_A^{i-1} - \partial_A^{i-1} \\
&= 0.
\end{aligned}$$

Let ρ^{i-1} be the kernel lift of $\mathrm{id}_{A^{i-1}} - \partial_A^{i-1} \bullet \lambda^i$ along $\iota_{\partial_A^{i-1}}$; and let λ^{i-1} be a projective lift of ρ^{i-1} along η^{i-1}.



It follows

$$\begin{aligned}\partial_A^{i-1} \cdot \lambda^i + \lambda^{i-1} \cdot \partial_A^{i-2} &= \partial_A^{i-1} \cdot \lambda^i + \lambda^{i-1} \cdot \eta^{i-1} \cdot \iota_{\partial_A^{i-1}} \\ &= \partial_A^{i-1} \cdot \lambda^i + \rho^{i-1} \cdot \iota_{\partial_A^{i-1}} \\ &= \partial_A^{i-1} \cdot \lambda^i + \mathrm{id}_{A^{i-1}} - \partial_A^{i-1} \cdot \lambda^i \\ &= \mathrm{id}_{A^{i-1}}\end{aligned}$$

as desired. Hence A is indeed contractible. □

Theorem 3.12. *Let $\mathscr{C}$ be an additive category.*

(1) If $\mathscr{C}$ is additive (Abelian), then $\mathcal{C}^b(\mathscr{C})$ is also additive (Abelian).
(2) If $\mathscr{C}$ has decidable equality of morphisms, then so does $\mathcal{C}^b(\mathscr{C})$.

Proof. This first assertion is evident. Basically, the required computations can be performed in $\mathcal{C}^b(\mathscr{C})$ by computing them "index-wise" in $\mathscr{C}$. For two morphism $\varphi, \psi \colon A \to B$ in $\mathcal{C}^b(\mathscr{C})$, $\varphi = \psi$ if and only if $\varphi^i = \psi^i$ for $i \in \mathrm{Supp}_A \cap \mathrm{Supp}_B$. The assertion follows because $\mathrm{Supp}_A \cap \mathrm{Supp}_B$ is a finite set. □

The following constructions will be used later to construct homotopy categories as stable categories. They are also essential in proving that homotopy categories are triangulated.

Definition 3.13. Let $\mathscr{C}$ be an additive category and $\mathcal{C}(\mathscr{C})$ its complex category. The **mapping cone** of a morphism $\alpha \colon A \to B$ in $\mathcal{C}(\mathscr{C})$, denoted by $\mathrm{Cone}(\alpha)$, is defined by the object in $\mathcal{C}(\mathscr{C})$ whose differential at index $i \in \mathbb{Z}$ is given by

$$\partial^i_{\mathrm{Cone}(\alpha)} := \quad A^{i+1} \oplus B^i \xrightarrow{\begin{pmatrix} -\partial_A^{i+1} & \alpha^{i+1} \\ 0 & \partial_B^i \end{pmatrix}} A^{i+2} \oplus B^{i+1}$$

The mapping cone is well-defined because

$$\begin{aligned}\partial^i_{\mathrm{Cone}(\alpha)} \cdot \partial^{i+1}_{\mathrm{Cone}(\alpha)} &= \begin{pmatrix} -\partial_A^{i+1} & \alpha^{i+1} \\ 0 & \partial_B^i \end{pmatrix} \cdot \begin{pmatrix} -\partial_A^{i+2} & \alpha^{i+2} \\ 0 & \partial_B^{i+1} \end{pmatrix} \\ &= \begin{pmatrix} \partial_A^{i+1} \cdot \partial_A^{i+2} & -\partial_A^{i+1} \cdot \alpha^{i+2} + \alpha^{i+1} \cdot \partial_B^{i+1} \\ 0 & \partial_B^i \cdot \partial_B^{i+1} \end{pmatrix} \\ &= 0.\end{aligned}$$

Definition 3.14. Let $\mathscr{C}$ be an additive category and let $\alpha \colon A \to B$ be a morphism in $\mathcal{C}(\mathscr{C})$.

- The **natural injection in the mapping cone** of α is the morphism

  $$\iota(\alpha) \colon B \to \mathrm{Cone}(\alpha)$$

  whose component at index $i \in \mathbb{Z}$ is given by the matrix $\begin{pmatrix} 0 & \mathrm{id}_B^i \end{pmatrix}$.
- The **natural projection from the mapping cone** of α is the morphism

  $$\pi(\alpha) \colon \mathrm{Cone}(\alpha) \to \Sigma(A)$$

  whose component at index $i \in \mathbb{Z}$ is given by the matrix $\begin{pmatrix} \mathrm{id}_A^{i+1} & 0 \end{pmatrix}^{\mathrm{tr}}$, where $\Sigma(A)$ is the object whose differential at index $i \in \mathbb{Z}$ is $\partial^i_{\Sigma(A)} := -\partial_A^{i+1}$; or equivalently, is equal to $\mathrm{Cone}(A \to 0)$.

Remark 3.15. The above constructions Cone$(-)$,$\iota(-)$ and $\pi(-)$ are functorial in α, that is, we have three functors:

- $\mathrm{Cone}(-)\colon \mathrm{Arr}(\mathcal{C}(\mathscr{C})) \to \mathcal{C}(\mathscr{C})$,
- $\iota\colon \mathrm{Arr}(\mathcal{C}(\mathscr{C})) \to \mathrm{Arr}(\mathcal{C}(\mathscr{C}))$,
- $\pi\colon \mathrm{Arr}(\mathcal{C}(\mathscr{C})) \to \mathrm{Arr}(\mathcal{C}(\mathscr{C}))$.

The above functors can be illustrated in the following commutative diagram

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\iota(\alpha_1)} & \mathrm{Cone}(\alpha_1) & \xrightarrow{\pi(\alpha_1)} & \Sigma(A_1) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} & & \vdots\ {\scriptstyle \mathrm{Cone}_{\alpha_1,\alpha_2}(u,v)} & & \downarrow{\scriptstyle \Sigma(u)} \\ A_2 & \xrightarrow[\alpha_2]{} & B_2 & \xrightarrow[\iota(\alpha_2)]{} & \mathrm{Cone}(\alpha_2) & \xrightarrow[\pi(\alpha_2)]{} & \Sigma(A_2), \end{array}$$

whose component at index $i \in \mathbb{Z}$ is given by the commutative diagram

$$\begin{array}{ccccccc} A_1^i & \xrightarrow{\alpha_1^i} & B_1^i & \xrightarrow{\left(\begin{smallmatrix} 0 & \mathrm{id}^i_{B_1} \end{smallmatrix}\right)} & A_1^{i+1} \oplus B_1^i & \xrightarrow{\left(\begin{smallmatrix} \mathrm{id}^{i+1}_{A_1} \\ 0 \end{smallmatrix}\right)} & A_1^{i+1} \\ {\scriptstyle u^n}\downarrow & & \downarrow{\scriptstyle v^n} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} u^{i+1} & 0 \\ 0 & v^n \end{smallmatrix}\right)} & & \downarrow{\scriptstyle u^{n+1}} \\ A_2^i & \xrightarrow[\alpha_2^i]{} & B_2^i & \xrightarrow[\left(\begin{smallmatrix} 0 & \mathrm{id}^i_{B_2} \end{smallmatrix}\right)]{} & A_2^{i+1} \oplus B_2^i & \xrightarrow[\left(\begin{smallmatrix} \mathrm{id}^{i+1}_{A_2} \\ 0 \end{smallmatrix}\right)]{} & A_2^{i+1} \end{array}$$

Lemma 3.16. *Let $\mathscr{C}$ be an Abelian category. A morphism $\varphi\colon A \to B$ in $\mathcal{C}(\mathscr{C})$ is quasi-isomorphism if and only if* Cone(φ) *is exact.*

Proof. The assertion follows by inspecting the long exact sequence of cohomology associated to the short exact sequence $0 \to B \xrightarrow{\iota(\varphi)} \mathrm{Cone}(\varphi) \xrightarrow{\pi(\varphi)} \Sigma(A) \to 0$. □

Definition 3.17. Let $\mathscr{C}$ be an additive category and $\alpha\colon A \to B$ be a morphism in $\mathcal{C}(\mathscr{C})$.

(1) α is called **null-homotopic** if there exists a family of morphisms $\left(h^i\colon A^i \to B^{i-1}\right)_{i\in\mathbb{Z}}$ such that $\partial_A^i \cdot h^{i+1} + h^i \cdot \partial_B^{i-1} = \alpha^i$ for all $i \in \mathbb{Z}$. The family $\left(h^i\right)_{i\in\mathbb{Z}}$ will be called a **chain homotopy** of α. This can be depicted as

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A^{i-1} & \xrightarrow{\partial_A^{i-1}} & A^i & \xrightarrow{\partial_A^i} & A^{i+1} & \longrightarrow & \cdots \\ & & {\scriptstyle \alpha^{i-1}}\downarrow & {\scriptstyle h^i}\swarrow & \downarrow{\scriptstyle \alpha^i} & {\scriptstyle h^{i+1}}\swarrow & \downarrow{\scriptstyle \alpha^{i+1}} & & \\ \cdots & \longrightarrow & B^{i-1} & \xrightarrow[\partial_B^{i-1}]{} & B^i & \xrightarrow[\partial_B^i]{} & B^{i+1} & \longrightarrow & \cdots. \end{array}$$

(2) α is called **homotopy-equivalence** if there exists a morphism $\beta\colon B \to A$ such that $\alpha \cdot \beta - \mathrm{id}_A$ and $\beta \cdot \alpha - \mathrm{id}_B$ are both null-homotopic. In such a case, β is called a **homotopy-inverse** of α, and we say A and B are **homotopy-equivalent**.

The following lemma allows us to translate the problem of deciding whether a morphism $\mathcal{C}(\mathscr{C})$ is null-homotopic to a colift problem in $\mathcal{C}(\mathscr{C})$.

Lemma 3.18. *Let $\mathscr{C}$ be an additive category. A morphism $\alpha\colon A \to B$ in $\mathcal{C}(\mathscr{C})$ is null-homotopic if and only if there exists a colift morphism $\lambda\colon \mathrm{Cone}(\mathrm{id}_A) \to B$ of α along $\iota(\mathrm{id}_A)\colon A \to \mathrm{Cone}(\mathrm{id}_A)$:*

$$\begin{array}{ccc} A & \xrightarrow{\ \alpha\ } & B \\ {\scriptstyle \iota(\mathrm{id}_A)}\big\downarrow & \circlearrowleft & \nearrow{\scriptstyle \lambda} \\ \mathrm{Cone}(\mathrm{id}_A) & & \end{array}$$

Proof. Let λ be a colift morphism of α along $\iota(\mathrm{id}_A)$ and its component at $i \in \mathbb{Z}$ is defined by

$$\lambda^i : A^{i+1} \oplus A^i \xrightarrow{\begin{pmatrix} u^i \\ v^i \end{pmatrix}} B^i.$$

Since $\iota(\mathrm{id}_A) \cdot \lambda = \alpha$, it follows that $v^i = \alpha^i$ for all $i \in \mathbb{Z}$. Since λ is a morphism in $\mathcal{C}(\mathscr{C})$, we have $\partial^i_{\mathrm{Cone}(\mathrm{id}_A)} \cdot \lambda^{i+1} = \lambda^i \cdot \partial^i_B$, i.e.,

$$\begin{pmatrix} -\partial_A^{i+1} & \mathrm{id}_A^i \\ 0 & \partial_A^i \end{pmatrix} \cdot \begin{pmatrix} u^{i+1} \\ \alpha^{i+1} \end{pmatrix} = \begin{pmatrix} u^i \\ \alpha^i \end{pmatrix} \cdot \begin{pmatrix} \partial_B^i \end{pmatrix}$$

for all $i \in \mathbb{Z}$. Hence, for all $i \in \mathbb{Z}$, we have $\partial_A^{i+1} \cdot u^{i+1} + u^i \cdot \partial_B^i = \alpha^{i+1}$ which implies that α is null-homotopic and the collection $\left(h^i := u^{i-1}\colon A^i \to B^{i-1}\right)_{i\in\mathbb{Z}}$ defines a chain homotopy associated to α.

Suppose that α is null-homotopic and let $\left(h^i\colon A^i \to B^{i-1}\right)_{i\in\mathbb{Z}}$ be a chain homotopy associated to it. Then $\lambda\colon \mathrm{Cone}(\mathrm{id}_A) \to B$ whose component at $i \in \mathbb{Z}$ is

$$A^{i+1} \oplus A^i \xrightarrow{\begin{pmatrix} h^{i+1} \\ \alpha^i \end{pmatrix}} B^i;$$

is a colift morphism of α along $\iota(\mathrm{id}_A)$. □

Remark 3.19. Let $\mathscr{C}$ be an additive category. A morphism $\alpha\colon A \to B$ in $\mathcal{C}^b(\mathscr{C})$ is null-homotopic if and only if there exists a family of morphisms $\left(h^i\colon A^i \to B^{i-1}\right)_{i\in\mathbb{Z}}$ with

$$\partial_A^i \cdot h^{i+1} + h^i \cdot \partial_B^{i-1} = \alpha^i$$

for all $i \in \mathbb{Z}$. Since A is a bounded complex, the question boils down to verifying the solvability of a *system of two-sided inhomogeneous linear equations* in $\mathscr{C}$:

$$\{\partial_A^i \cdot \chi^{i+1} + \chi^i \cdot \partial_B^{i-1} = \alpha^i \mid i \in \mathrm{Supp}_A\}.$$

Our approach to solve such systems is based on the concept of *homomorphism structures* (cf. Theorem 4.17). In particular, if we equip $\mathscr{C}$ with a $\mathscr{D}$-homomorphism structure, then we can translate solving the above system into a lift problem in $\mathscr{D}$, which is usually much easier to perform.

Remark 3.20. Let $\mathscr{C}$ be an additive category and let $\alpha\colon A \to B$ be an isomorphism in $\mathcal{C}(\mathscr{C})$. Then $\mathrm{Cone}(\alpha)$ is contractible.

Proof. Let $\beta\colon B \to A$ denote the inverse of α. For each $i \in \mathbb{Z}$, define $h^i\colon A^{i+1} \oplus B^i \to A^i \oplus B^{i-1}$ by the matrix $\left(\begin{smallmatrix} 0 & 0 \\ \beta^i & 0 \end{smallmatrix}\right)$. We have then $h^i \cdot \partial^{i-1}_{\mathrm{Cone}(\alpha)} + \partial^i_{\mathrm{Cone}(\alpha)} \cdot h^{i+1} = \mathrm{id}^i_{\mathrm{Cone}(\alpha)}$ for each $i \in \mathbb{Z}$, thus $\mathrm{Cone}(\alpha)$ is contractible. □

3.2. Homotopy Categories

Definition 3.21. Let $\mathscr{C}$ be an additive category and let $* \in \{+,-,b,\ \}$. The set of all null-homotopic morphisms in $\mathcal{C}^*(\mathscr{C})$ defines a two-sided ideal. The additive quotient category of $\mathcal{C}^*(\mathscr{C})$ by this ideal will be called the **homotopy category** of $\mathscr{C}$ and will be denoted by $\mathcal{K}^*(\mathscr{C})$.

The associated additive quotient functor to $\mathcal{K}^*(\mathscr{C})$ will be denoted by

$$[\,]\colon \mathcal{C}^*(\mathscr{C}) \to \mathcal{K}^*(\mathscr{C}).$$

Remark 3.22. By Definition 3.8, an object A in $\mathcal{C}^*(\mathscr{C})$ is contractible if and only if id_A is null-homotopic. In other words, the object A is contractible if and only if $[A]$ in $\mathcal{K}^*(\mathscr{C})$ is zero. This means if a morphism $\alpha\colon A \to B$ in $\mathcal{C}^*(\mathscr{C})$ factors through any contractible object, then $[\alpha] = 0$ and α is null-homotopic.

Remark 3.23. A morphism $[\alpha]$ in $\mathcal{K}^*(\mathscr{C})$ is an isomorphism if and only if $[\mathrm{Cone}(\alpha)]$ in $\mathcal{K}^*(\mathscr{C})$ is zero (cf. Lemma B.22). In other words, α is a homotopy-equivalence if and only if $\mathrm{Cone}(\alpha)$ is contractible. If $\mathscr{C}$ is Abelian, then by Lemma 3.9, every contractible object is exact, hence the mapping cone of any homotopy-equivalence is exact. Thus, by Lemma 3.16, every homotopy-equivalence is a quasi-isomorphism.

Remark 3.24. Let $\mathscr{C}$ be an Abelian category and A, B are objects in $\mathcal{C}^-(\mathscr{C})$ where A^i, B^i are projective objects for all $i \in \mathbb{Z}$. Suppose $\alpha\colon A \to B$ is a quasi-isomorphism. By Lemma 3.16, $\mathrm{Cone}(\alpha)$ is exact, and by Lemma 3.11, $\mathrm{Cone}(\alpha)$ is contractible. Hence, by the previous Remark, α is a homotopy-equivalence.

Remark 3.25. Let $\mathscr{C}$ be an Abelian category and let $\mathrm{H}^i\colon \mathcal{C}(\mathscr{C}) \to \mathscr{C}$ be the i^{th}-cohomology functor. By Lemma 3.18, Remark 3.20 and Corollary 3.10, the functor H^i factors uniquely along the quotient functor $[\,]\colon \mathcal{C}(\mathscr{C}) \to \mathcal{K}(\mathscr{C})$. The colift functor of H^i along $[\,]$ will as well be denoted by H^i. Obviously, H^i is defined by

$$\mathrm{H}^i\colon \begin{cases} \mathcal{K}(\mathscr{C}) & \to \mathscr{C}, \\ [A] & \mapsto \mathrm{H}^i(A), \\ [\alpha]\colon [A] \to [B] & \mapsto \mathrm{H}^i(\alpha)\colon\ \mathrm{H}^i(A) \to \mathrm{H}^i(B). \end{cases}$$

Two morphisms $[\alpha], [\beta]\colon A \to B$ in $\mathcal{K}^b(\mathscr{C})$ are equal if $\alpha - \beta$ is null-homotopic. Hence, Lemma 3.18, Remark 3.19 and Theorem 4.17 provide an algorithmic description[5] for the equality of morphisms in bounded homotopy categories:

Corollary 3.26. *Let $\mathscr{C}$ be an additive category. If any of the following hold:*

(1) The category $\mathcal{C}^b(\mathscr{C})$ has decidable colifts;
(2) The category $\mathcal{C}^b(\mathscr{C})$ is equipped with a $\mathscr{D}$-homomorphism structure where $\mathscr{D}$ has decidable lifts;
(3) The category $\mathscr{C}$ is equipped with a $\mathscr{D}$-homomorphism structure where $\mathscr{D}$ has decidable lifts;

then $\mathcal{K}^b(\mathscr{C})$ has decidable equality of morphisms.

Example 3.27. Let R be a commutative left computable ring. Since R is left computable, it follows that R-**rows** has decidable lifts; and by Example 4.6, the category R-**rows** can be equipped with an (R-**rows**)-homomorphism structure. On the other hand, by [**Pos21a**, Corollary 6.17], the

[5]See `IsNullHomotopic(-)` and `HomotopyMorphisms(-)` operations in [**Sal21b**].

category $\mathcal{A}(R\text{-}\mathbf{rows})$ has decidable lifts and can be equipped with an $\mathcal{A}(R\text{-}\mathbf{rows})$-homomorphism structure. By Corollary 3.26, both $\mathcal{K}^b(R\text{-}\mathbf{rows})$ and $\mathcal{K}^b(\mathcal{A}(R\text{-}\mathbf{rows}))$ have decidable equality of morphisms.

The following remark enables us to construct homotopy categories as stable categories associated to a class of colifting objects:

Remark 3.28. Let $\mathscr{C}$ be an additive category and $\mathcal{C}^*(\mathscr{C})$ be its complex category where $* \in \{+,-,b,\ \}$. We denote by $\mathcal{Q}_{\mathcal{C}^*(\mathscr{C})}$ the set of all contractible objects in $\mathcal{C}^*(\mathscr{C})$. We claim that $\mathcal{Q}_{\mathcal{C}^*(\mathscr{C})}$ defines a class of colifting objects in $\mathcal{C}^*(\mathscr{C})$. For an object A in $\mathcal{C}^*(\mathscr{C})$ we define Q_A by $\mathrm{Cone}(\mathrm{id}_A)$ and $q_A\colon A \to Q_A$ by the natural injection in the mapping cone (cf. Definition 3.14). It follows by Remark 3.20 that Q_A is contractible. For a morphism $\varphi\colon A \to B$ in $\mathcal{C}^*(\mathscr{C})$, we define $Q_\varphi\colon Q_A \to Q_B$ by $\mathrm{Cone}_{\mathrm{id}_A,\mathrm{id}_B}(\varphi,\varphi)$ as introduced in Remark 3.15. It remains to show that if A is contractible, then q_A is a split-monomorphism. Since A is contractible, there exists a family of morphisms $\left(\lambda^i_A\colon A^i \to A^{i-1}\right)_{i\in\mathbb{Z}}$ such that $\partial^i_A \cdot \lambda^{i+1}_A + \lambda^i_A \cdot \partial^{i-1}_A = \mathrm{id}_{A^i}$ for all $i \in \mathbb{Z}$. The morphism $r_A\colon Q_A \to A$ which is defined at $i \in \mathbb{Z}$ by

$$b^n_A \coloneqq A^{i+1} \oplus A^i \xrightarrow{\begin{pmatrix}\lambda^{i+1}_A \\ \mathrm{id}_{A^i}\end{pmatrix}} A^i.$$

is a retraction morphism of q_A, hence q_A is indeed a split-monomorphism.

Theorem 3.29. *Let $\mathscr{C}$ be an additive category. Then there is an isomorphism*

$$\mathcal{K}^*(\mathscr{C}) \cong \mathcal{C}^*(\mathscr{C})/\mathcal{Q}$$

where $\mathcal{C}^*(\mathscr{C})/\mathcal{Q}$ is the stable category of $\mathcal{C}^*(\mathscr{C})$ w.r.t. the above system of colifting objects.

PROOF. The assertion follows by Lemma 3.18. □

3.3. Computing Projective and Injective Resolutions of Complexes

In this section we provide an algorithmic description for the following very useful constructions in homological algebra:

- Let $\mathscr{C}$ be Abelian with enough projective objects and $\mathbf{proj}(\mathscr{C})$ the full subcategory of $\mathscr{C}$ generated by the projective objects of $\mathscr{C}$. Construct the adjunction:

  $$\iota\colon \mathcal{K}^-(\mathbf{proj}(\mathscr{C})) \rightleftarrows \mathcal{K}^-(\mathscr{C}) :\mathcal{P}$$

  where ι is the inclusion functor and $\mathcal{P}$ is the *projective resolution functor* which maps objects and morphisms of $\mathcal{K}^-(\mathscr{C})$ to their projective resolutions in $\mathcal{K}^-(\mathbf{proj}(\mathscr{C}))$.
- Analogously, let $\mathscr{C}$ be Abelian with enough injective objects and $\mathbf{inj}(\mathscr{C})$ the full subcategory of $\mathscr{C}$ generated by the injective objects of $\mathscr{C}$. Construct the adjunction:

  $$\mathcal{I}\colon \mathcal{K}^+(\mathscr{C}) \rightleftarrows \mathcal{K}^+(\mathbf{inj}(\mathscr{C})) :\iota$$

  where ι is the inclusion functor and $\mathcal{I}$ is the *injective resolution functor* which maps objects and morphisms of $\mathcal{K}^+(\mathscr{C})$ to their injective resolutions in $\mathcal{K}^+(\mathbf{inj}(\mathscr{C}))$.

In Section 3.4 we employ these adjunctions to perform the following tasks:

(1) If $\mathscr{C}$ is Abelian with enough projectives, we use them to compute left derived functors of right exact functors from $\mathscr{C}$ to another Abelian category $\mathscr{E}$.

(2) If $\mathscr{C}$ is Abelian with enough projectives, they induce an exact equivalences
$$\mathcal{K}^-(\mathbf{proj}(\mathscr{C})) \simeq \mathcal{D}^-(\mathscr{C})$$
which restrict in case $\mathscr{C}$ has finite global dimension to
$$\mathcal{K}^b(\mathbf{proj}(\mathscr{C})) \simeq \mathcal{D}^b(\mathscr{C})$$
allowing us to translate computations from $\mathcal{D}^b(\mathscr{C})$ to $\mathcal{K}^b(\mathbf{proj}(\mathscr{C}))$.

(3) If $\mathscr{C}$ is Abelian with enough injectives, we use them to compute the right derived functors of left exact functors from $\mathscr{C}$ to another Abelian category $\mathscr{E}$.

(4) If $\mathscr{C}$ is Abelian with enough injectives and finite global dimension, then these functors induce an exact equivalence
$$\mathcal{D}^+(\mathscr{C}) \simeq \mathcal{K}^+(\mathbf{inj}(\mathscr{C}))$$
which restrict in case $\mathscr{C}$ has finite global dimension to
$$\mathcal{D}^b(\mathscr{C}) \simeq \mathcal{K}^b(\mathbf{inj}(\mathscr{C}))$$
allowing us to translate computations from $\mathcal{D}^b(\mathscr{C})$ to $\mathcal{K}^b(\mathbf{inj}(\mathscr{C}))$.

As already mentioned in the introduction of this chapter, our primary instances for the Abelian category $\mathscr{C}$ are

- The Abelian category $\mathbf{mod}$-$\mathscr{A}$ (resp. $\mathscr{A}$-$\mathbf{mod}$) of functors from an admissible k-linear finitely presented category $\mathscr{A}$ (resp. $\mathscr{A}^{\mathrm{op}}$) to k-$\mathbf{mat}$. We use the techniques of this section to render the Happel theorem constructive (cf. Corollary 6.7).
- The Abelian Freyd category $\mathcal{A}(R\text{-}\mathbf{rows}) \simeq R\text{-}\mathbf{fpmod}$ for a computable commutative ring R with finite global dimension, e.g., the polynomial ring $R = k[x_0, \dots, x_n]$. Appendix C provides a software demonstration for computing the extension groups $\mathrm{Ext}^n_{\mathscr{C}}(-,-)$ as
$$\mathrm{Ext}^n_{\mathscr{C}}(A,B) \coloneqq \mathrm{Hom}_{\mathcal{D}^b(\mathscr{C})}(A, \Sigma^n(B))$$
for $\mathscr{C} = \mathcal{A}(\mathbb{Q}[x,y]\text{-}\mathbf{rows}) \simeq \mathbb{Q}[x,y]\text{-}\mathbf{fpmod}$.

All the constructions presented in this section are implemented in `ComplexesCategories` [**Sal21b**].

We start by defining projective resolutions of complexes.

Definition 3.30. Let $\mathscr{C}$ be an Abelian category and $\mathcal{C}(\mathscr{C})$ be its complex category. A **projective resolution** for an object A in $\mathcal{C}(\mathscr{C})$ consists of the following data:

- An object $\mathcal{P}_A$ such that $\mathcal{P}_A^i$ is projective for all $i \in \mathbb{Z}$.
- A quasi-isomorphism $q_A \colon \mathcal{P}_A \to A$.

A *projective resolution* of an object in $\mathscr{C}$ is a projective resolution of the 0-stalk complex[6] in $\mathcal{C}(\mathscr{C})$ defined by that object.

The following theorem enables us to construct projective resolutions of bounded-above complexes over Abelian categories with enough projective objects.

Theorem 3.31. *Let $\mathscr{C}$ be an Abelian category with enough projectives and let $\mathcal{C}^-(\mathscr{C})$ be its bounded-above complex category. Then each A in $\mathcal{C}^-(\mathscr{C})$ admits a projective resolution $q_A \colon \mathcal{P}_A \to A$ where $\mathcal{P}_A$ belongs to $\mathcal{C}^-(\mathscr{C})$. Furthermore, if A is contractible, then so is $\mathcal{P}_A$.*

[6]For the definition of a stalk complex, see Definition 3.2.

Proof. Let u_A be the upper bound of A. For all $i > u_A$, define $\partial^i_{\mathcal{P}_A}$ by $0 \xrightarrow{0} 0$ and q^i_A by $0 \xrightarrow{0} A^i$. Suppose $\left(\partial^i_{\mathcal{P}_A}, q^i_A\right)$ has been computed and let us compute $\left(\partial^{i-1}_{\mathcal{P}_A}, q^{i-1}_A\right)$. Define the morphism

$$\tau^{i-1}_A := \begin{pmatrix} -\partial^i_{\mathcal{P}_A} & q^i_A \\ 0 & \partial^{i-1}_A \end{pmatrix} \colon \mathcal{P}^i_A \oplus A^{i-1} \to \mathcal{P}^{i+1}_A \oplus A^i$$

and let $\kappa^{i-1}_A = \left(\epsilon^{i-1}_A \ \delta^{i-1}_A \right) \colon K^{i-1}_A \hookrightarrow \mathcal{P}^i_A \oplus A^{i-1}$ be its kernel embedding. Since $\mathscr{C}$ has enough projectives, there exists an epimorphism $\lambda^{i-1}_A \colon \mathcal{P}^{i-1}_A \twoheadrightarrow K^{i-1}_A$ where $\mathcal{P}^{i-1}_A$ is a projective object. We set

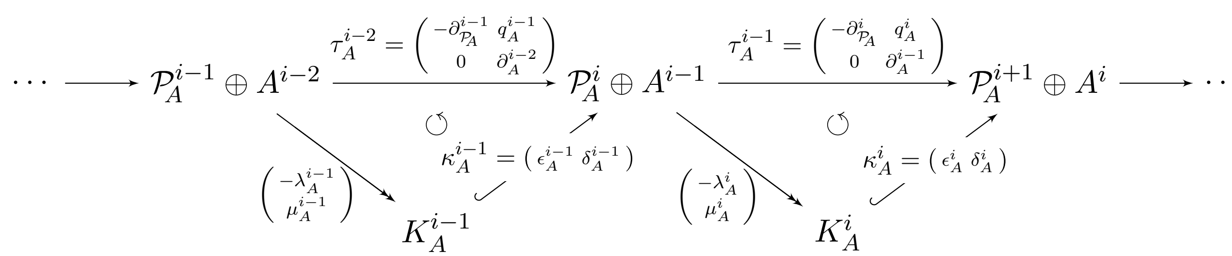
$$\partial^{i-1}_{\mathcal{P}_A} := \lambda^{i-1}_A \cdot \epsilon^{i-1}_A \colon \mathcal{P}^{i-1}_A \to \mathcal{P}^i_A$$

and

$$q^{i-1}_A := -\lambda^{i-1}_A \cdot \delta^{i-1}_A \colon \mathcal{P}^{i-1}_A \to A^{i-1}.$$

An straight verification shows that for every $i \in \mathbb{Z}$, the morphism $-\lambda^{i-1}_A \colon \mathcal{P}^{i-1}_A \twoheadrightarrow K^{i-1}_A$ is the kernel lift morphism of $\left(-\partial^{i-1}_{\mathcal{P}_A} \ q^{i-1}_A \right) \colon \mathcal{P}^{i-1}_A \to \mathcal{P}^i_A \oplus A^{i-1}$ along $\kappa^{i-1}_A \colon K^{i-1}_A \hookrightarrow \mathcal{P}^i_A \oplus A^{i-1}$. For each $i \in \mathbb{Z}$, denote by $\mu^{i-1}_A \colon A^{i-2} \twoheadrightarrow K^{i-1}_A$ the kernel lift morphism of $\left(0 \ \partial^{i-2}_A \right) \colon A^{i-2} \to \mathcal{P}^i_A \oplus A^{i-1}$ along $\kappa^{i-1}_A \colon K^{i-1}_A \hookrightarrow \mathcal{P}^i_A \oplus A^{i-1}$.

The above data incorporates in the following diagram:



Since λ^{i-1}_A is an epimorphism, the kernel lift of τ^{i-2}_A along κ^{i-1}_A is an epimorphism for all $i \le u_A$. Hence, the natural embedding $\mathrm{im}\left(\tau^{i-2}_A\right) \hookrightarrow \ker\left(\tau^{i-1}_A\right)$ is an epimorphism as well, consequently an isomorphism for all $i \le u_A$. Thus, by Remark 3.7, $\mathrm{Cone}(q_A)$ is exact, and by Lemma 3.16, q_A is a quasi-isomorphism as desired.

If A is contractible, then by Lemma 3.9, A is exact. Since q_A is a quasi-isomorphism, we have $\mathrm{H}^i(\mathcal{P}_A) \cong \mathrm{H}^i(A) \cong 0$ for all $i \in \mathbb{Z}$, i.e., $\mathcal{P}_A$ is exact as well. It follows by Lemma 3.11 that $\mathcal{P}_A$ is contractible as asserted. □

Also morphisms between bounded-above complexes lift to morphisms between the corresponding projective resolutions. Any two such morphisms coincide in the homotopy category.

Theorem 3.32. *Let $\mathscr{C}$ be an Abelian category with enough projectives and let $\mathcal{C}^-(\mathscr{C})$ be its bounded-above complex category. Let A and B be objects in $\mathcal{C}^-(\mathscr{C})$ and q_A and q_B their projective resolutions as constructed in Theorem 3.31. Then any morphism $\varphi \colon A \to B$ lifts, uniquely up to homotopy, to a morphism $\mathcal{P}_\varphi \colon \mathcal{P}_A \to \mathcal{P}_B$. Moreover, if φ is a quasi-isomorphism, then $\mathcal{P}_\varphi$ is a homotopy-equivalence.*

Proof. Let u_φ be a common upper bound for A and B. For all $i > u_\varphi$ define $\mathcal{P}^i_\varphi$ by $0 \xrightarrow{0} 0$. Suppose $\mathcal{P}^i_\varphi$ and $\mathcal{P}^{i+1}_\varphi$ has been computed and let us compute $\mathcal{P}^{i-1}_\varphi$. We have the following commutative diagram:

$$\begin{array}{ccccccc}
\mathcal{P}_A^{i-1} & \xrightarrow{\lambda_A^{i-1}} & K_A^{i-1} & \xrightarrow{\kappa_A^{i-1} = \left(\epsilon_A^{i-1}\ \delta_A^{i-1}\right)} & \mathcal{P}_A^{i} \oplus A^{i-1} & \xrightarrow{\tau_A^{i-1} = \begin{pmatrix} -\partial_{\mathcal{P}_A}^{i} & q_A^{i} \\ 0 & \partial_A^{i-1} \end{pmatrix}} & \mathcal{P}_A^{i+1} \oplus A^{i} \\
\Big\downarrow \mathcal{P}_\varphi^{i-1} & \circlearrowleft & \Big\downarrow \ell_\varphi^{i-1} & \circlearrowleft & \Big\downarrow \tau_\varphi^{i-1} := \begin{pmatrix} \mathcal{P}_\varphi^{i} & 0 \\ 0 & \varphi^{i-1} \end{pmatrix} & \circlearrowleft & \Big\downarrow \tau_\varphi^{i} := \begin{pmatrix} \mathcal{P}_\varphi^{i+1} & 0 \\ 0 & \varphi^{i} \end{pmatrix} \\
\mathcal{P}_B^{i-1} & \xrightarrow[\lambda_B^{i-1}]{} & K_B^{i-1} & \xrightarrow[\kappa_B^{i-1} = \left(\epsilon_B^{i-1}\ \delta_B^{i-1}\right)]{} & \mathcal{P}_B^{i} \oplus B^{i-1} & \xrightarrow[\tau_B^{i-1} = \begin{pmatrix} -\partial_{\mathcal{P}_B}^{i} & q_B^{i} \\ 0 & \partial_B^{i-1} \end{pmatrix}]{} & \mathcal{P}_B^{i+1} \oplus B^{i}
\end{array}$$

where ℓ_φ^{i-1} is the kernel lift of $\kappa_A^{i-1} \cdot \tau_\varphi^{i-1}$ along κ_B^{i-1}, and $\mathcal{P}_\varphi^{i-1}$ is a projective lift of $\lambda_A^{i-1} \cdot \ell_\varphi^{i-1}$ along λ_B^{i-1}.

The commutativity of the above diagram implies

$$\lambda_A^{i-1} \cdot \begin{pmatrix} \epsilon_A^{i-1} & \delta_A^{i-1} \end{pmatrix} \cdot \begin{pmatrix} \mathcal{P}_\varphi^{i} & 0 \\ 0 & \varphi^{i-1} \end{pmatrix} = \mathcal{P}_\varphi^{i-1} \cdot \lambda_B^{i-1} \cdot \begin{pmatrix} \epsilon_B^{i-1} & \delta_B^{i-1} \end{pmatrix}$$

i.e.,

$$\begin{pmatrix} \partial_{\mathcal{P}_A}^{i-1} \cdot \mathcal{P}_\varphi^{i} & -q_A^{i-1} \cdot \varphi^{i-1} \end{pmatrix} = \begin{pmatrix} \mathcal{P}_\varphi^{i-1} \cdot \partial_{\mathcal{P}_B}^{i-1} & -\mathcal{P}_\varphi^{i-1} \cdot q_B^{i-1} \end{pmatrix}$$

for all $i \in \mathbb{Z}$. Hence, $\mathcal{P}_\varphi \colon \mathcal{P}_A \to \mathcal{P}_B$ is well-defined and $q_A \cdot \varphi = \mathcal{P}_\varphi \cdot q_B$ as desired.

Let $\rho, \zeta \colon \mathcal{P}_A \to \mathcal{P}_B$ be two morphisms with $\rho \cdot q_B = \zeta \cdot q_B = q_A \cdot \varphi$. We want to prove that $\rho - \zeta$ is null-homotopic. Using the notation of Remark 3.15, we define the morphism

$$\psi := \mathrm{Cone}_{q_A, q_B}(\rho, \varphi) - \mathrm{Cone}_{q_A, q_B}(\zeta, \varphi) \colon \mathrm{Cone}(q_A) \to \mathrm{Cone}(q_B).$$

The component of ψ at index $i \in \mathbb{Z}$ is

$$\begin{pmatrix} \rho^{i+1} - \zeta^{i+1} & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{P}_A^{i+1} \oplus A^{i} \to \mathcal{P}_B^{i+1} \oplus B^{i}.$$

We will prove that ψ is null-homotopic and then use the corresponding chain-homotopy to construct a chain-homotopy for $\rho - \zeta$. We need to construct a family $\left(\mathcal{P}_A^{i+1} \oplus A^{i} \xrightarrow{\ell^{i}} \mathcal{P}_B^{i} \oplus B^{i-1}\right)_{i \in \mathbb{Z}}$ such that $\psi^{i} = \tau_A^{i} \cdot \ell^{i+1} + \ell^{i} \cdot \tau_B^{i-1}$ for all $i \in \mathbb{Z}$.

For all $i > u_\varphi$ we set

$$\ell^{i} := \begin{pmatrix} h^{i+1} & 0 \\ 0 & 0 \end{pmatrix} := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{P}_A^{i+1} \oplus A^{i} \to \mathcal{P}_B^{i} \oplus B^{i-1}.$$

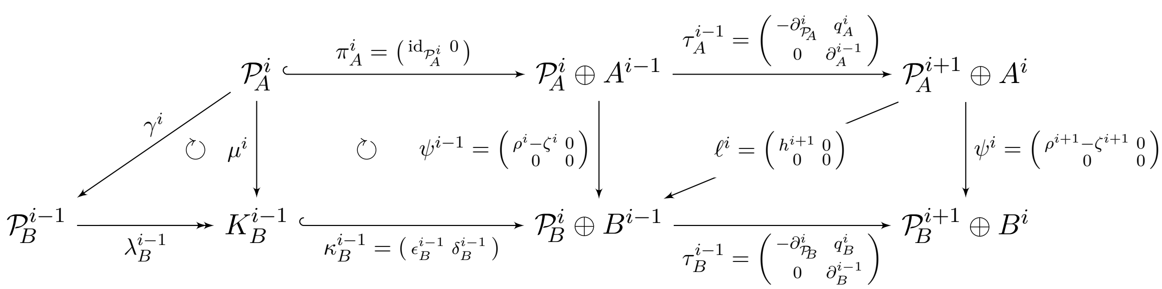
Suppose we have already computed ℓ^{i} and ℓ^{i+1} and let us compute ℓ^{i-1}. The equality $\psi^{i} = \tau_A^{i} \cdot \ell^{i+1} + \ell^{i} \cdot \tau_B^{i-1}$ translates to

$$\begin{pmatrix} \rho^{i+1} - \zeta^{i+1} + \partial_{\mathcal{P}_A}^{i+1} \cdot h^{i+2} + h^{i+1} \cdot \partial_{\mathcal{P}_B}^{i} & -h^{i+1} \cdot q_B^{i} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\pi_A^i := \begin{pmatrix}\mathrm{id}_{\mathcal{P}_A^i} & 0\end{pmatrix}\colon \mathcal{P}_A^i \to \mathcal{P}_A^i \oplus A^{i-1}$ be the natural injection of $\mathcal{P}_A^i$ in $\mathcal{P}_A^i \oplus A^{i-1}$. We get the following equality:

$$\begin{aligned}
\pi_A^i \cdot (\psi^{i-1} - \tau_A^{i-1} \cdot \ell^i) \cdot \tau_B^{i-1} &= \left(\begin{pmatrix}\rho^i - \zeta^i & 0\end{pmatrix} - \begin{pmatrix}-\partial_{\mathcal{P}_A}^i & q_A^i\end{pmatrix} \cdot \begin{pmatrix} h^{i+1} & 0 \\ 0 & 0 \end{pmatrix} \right) \cdot \begin{pmatrix} -\partial_{\mathcal{P}_B}^i & q_B^i \\ 0 & \partial_B^{i-1} \end{pmatrix} \\
&= \begin{pmatrix}\rho^i - \zeta^i + \partial_{\mathcal{P}_A}^i \cdot h^{i+1} & 0\end{pmatrix} \cdot \begin{pmatrix} -\partial_{\mathcal{P}_B}^i & q_B^i \\ 0 & \partial_B^{i-1} \end{pmatrix} \\
&= \begin{pmatrix}-\rho^i \cdot \partial_{\mathcal{P}_B}^i + \zeta^i \cdot \partial_{\mathcal{P}_B}^i - \partial_{\mathcal{P}_A}^i \cdot h^{i+1} \cdot \partial_{\mathcal{P}_B}^i & \rho^i \cdot q_B^i - \zeta^i \cdot q_B^i + \partial_{\mathcal{P}_A}^i \cdot h^{i+1} \cdot q_B^i\end{pmatrix} \\
&= \begin{pmatrix}-\rho^i \cdot \partial_{\mathcal{P}_B}^i + \zeta^i \cdot \partial_{\mathcal{P}_B}^i - \partial_{\mathcal{P}_A}^i \cdot h^{i+1} \cdot \partial_{\mathcal{P}_B}^i & 0\end{pmatrix} \\
&= \begin{pmatrix}-\rho^i \cdot \partial_{\mathcal{P}_B}^i + \zeta^i \cdot \partial_{\mathcal{P}_B}^i - \partial_{\mathcal{P}_A}^i \cdot \left(-\rho^{i+1} + \zeta^{i+1} - \partial_{\mathcal{P}_A}^{i+1} \cdot h^{i+2}\right) & 0\end{pmatrix} \\
&= \begin{pmatrix}-\rho^i \cdot \partial_{\mathcal{P}_B}^i + \partial_{\mathcal{P}_A}^i \cdot \rho^{i+1} + \zeta^i \cdot \partial_{\mathcal{P}_B}^i - \partial_{\mathcal{P}_A}^i \cdot \zeta^{i+1} & 0\end{pmatrix} \\
&= \begin{pmatrix}0 & 0\end{pmatrix}.
\end{aligned}$$

Let $\mu^i\colon \mathcal{P}_A^i \to K_B^{i-1}$ be the kernel lift morphism of $\pi_A^i \cdot \left(\psi^{i-1} - \tau_A^{i-1} \cdot \ell^i\right)$ along κ_B^{i-1} and $\gamma^i\colon \mathcal{P}_A^i \to \mathcal{P}_B^{i-1}$ a projective lift of μ^i along λ_B^{i-1}.



Define h^i by $-\gamma^i$, then the equality $\gamma^i \cdot \lambda_B^{i-1} \cdot \kappa_B^{i-1} = \pi_A^i \cdot \left(\psi^{i-1} - \tau_A^{i-1} \cdot \ell^i\right)$ translates to

$$\begin{pmatrix}-h^i \cdot \partial_{\mathcal{P}_B}^{i-1} & h^i \cdot q_B^{i-1}\end{pmatrix} = \begin{pmatrix}\rho^i - \zeta^i + \partial_{\mathcal{P}_A}^i \cdot h^{i+1} & 0\end{pmatrix},$$

i.e.,

$$\begin{pmatrix}\rho^i - \zeta^i + \partial_{\mathcal{P}_A}^i \cdot h^{i+1} + h^i \cdot \partial_{\mathcal{P}_B}^{i-1} & h^i \cdot q_B^{i-1}\end{pmatrix} = \begin{pmatrix}0 & 0\end{pmatrix};$$

hence defining ℓ^i by $\begin{pmatrix} h^i & 0 \\ 0 & 0 \end{pmatrix}$ implies $\psi^{i-1} = \tau_A^{i-1} \cdot \ell^i + \ell^{i-1} \cdot \tau_B^{i-2}$ as desired. Hence, ψ is indeed null-homotopic. The family $\left(h^i\colon \mathcal{P}_A^i \to \mathcal{P}_B^{i-1}\right)_{i \in \mathbb{Z}}$ defines a chain homotopy for $\zeta - \rho$.

Suppose φ is a quasi-isomorphism. It follows from the equality $\mathcal{P}_\varphi \cdot q_B = q_A \cdot \varphi$ and Remark 3.5 that $\mathcal{P}_\varphi$ is a quasi-isomorphism. Hence, by Remark 3.24, $\mathcal{P}_\varphi$ is a homotopy-equivalence as desired. □

Given a category $\mathscr{C}$ and a class W of morphisms in $\mathscr{C}$. A localization of $\mathscr{C}$ by W is a category $\mathscr{C}[W^{-1}]$ and a "localization by W" functor $Q\colon \mathscr{C} \to \mathscr{C}[W^{-1}]$ with the properties: $Q(f)$ is an isomorphism for all $f \in W$ and any other such functor $\mathscr{C} \to \mathscr{D}$ factors uniquely along Q. Some approaches for the construction of localization categories can be found in [**GM03**, Chapter 3] or [**DS95**, Chapter 6]. The following theorem provides an example of a localization functor of the the above-bounded complex category of an Abelian category with enough projectives where W is the class of quasi-isomorphisms:

Theorem 3.33. *Let $\mathscr{C}$ be an Abelian category with enough projectives. Then*

$$\mathcal{P}\colon \begin{cases} \mathcal{C}^-(\mathscr{C}) & \to \mathcal{K}^-(\mathbf{proj}(\mathscr{C})), \\ A & \mapsto [\mathcal{P}_A], \\ \varphi\colon A \to B & \mapsto [\mathcal{P}_\varphi]\colon [\mathcal{P}_A] \to [\mathcal{P}_B] \end{cases}$$

is a functor. Moreover, $\mathcal{P}$ factors along the quotient functor $[\,]\colon \mathcal{C}^-(\mathscr{C}) \to \mathcal{K}^-(\mathscr{C})$ via

$$\mathcal{P}\colon \begin{cases} \mathcal{K}^-(\mathscr{C}) & \to \mathcal{K}^-(\mathbf{proj}(\mathscr{C})), \\ [A] & \mapsto [\mathcal{P}_A], \\ [\varphi]\colon [A] \to [B] & \mapsto [\mathcal{P}_\varphi]\colon [\mathcal{P}_A] \to [\mathcal{P}_B]. \end{cases}$$

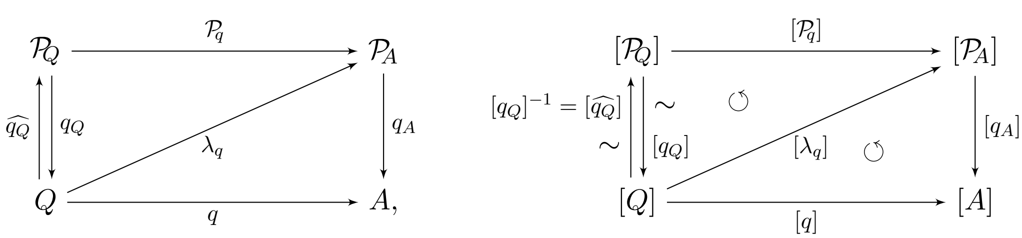
Proof. The functoriality follows by Theorems 3.31 and 3.32. By Theorem 3.31, if A is contractible, then so is $\mathcal{P}_A$, i.e., $[\mathcal{P}_A] = 0$. Since every null-homotopic morphism factors along a contractible object, the functor $\mathcal{P}$ maps null-homotopic morphisms to zero, and consequently $\mathcal{P}$ factors along the quotient functor as asserted. □

Lemma 3.34. *Let $\mathscr{C}$ be an Abelian category with enough projectives. Let A, Q be objects in $\mathcal{C}^-(\mathscr{C})$ with Q^i projective for all $i \in \mathbb{Z}$; and let $q\colon Q \to A$ be some morphism in $\mathcal{C}^-(\mathscr{C})$, then there exists a unique lift morphism $[\lambda_q]\colon [Q] \to [P_A]$ of $[q]$ along $[q_A]$. Furthermore, if q is a quasi-isomorphism, then $[\lambda_q]$ is an isomorphism, and its inverse is the unique lift morphism of $[q_A]$ along $[q]$.*

Proof. Let $q_A\colon \mathcal{P}_A \to A, q_Q\colon \mathcal{P}_Q \to Q$ and $\mathcal{P}_q\colon \mathcal{P}_Q \to \mathcal{P}_A$ be the morphisms asserted by Theorems 3.31 and 3.32, It follows by Remark 3.24 that q_Q is a homotopy-equivalence, i.e., $[q_Q]$ is an isomorphism. Let $\widehat{q_Q}$ be representative of $[q_Q]^{-1}$ and define λ_q by $\widehat{q_Q} \bullet \mathcal{P}_q$, then

$$\begin{aligned} [\lambda_q] \bullet [q_A] &= [\lambda_q \bullet q_A] \\ &= [\widehat{q_Q} \bullet \mathcal{P}_q \bullet q_A] \\ &= [\widehat{q_Q} \bullet q_Q \bullet q] \\ &= [\widehat{q_Q}] \bullet [q_Q] \bullet [q] \\ &= [q_Q]^{-1} \bullet [q_Q] \bullet [q] \\ &= [\mathrm{id}_Q] \bullet [q] \\ &= [q], \end{aligned}$$

i.e., $[\lambda_q]$ is a lift morphism of $[q]$ along $[q_A]$.



Suppose $[\mu]\colon [Q] \to [\mathcal{P}_A]$ is a lift morphism of q along q_A, i.e., with $[\mu] \cdot [q_A] = [q]$, then $\mu \cdot q_A - q$ is null-homotopic. It follows by Theorems 3.32 and 3.33 and the following computation

$$\begin{aligned}(q_Q \cdot \mu - \mathcal{P}_q) \cdot q_A &= q_Q \cdot \mu \cdot q_A - \mathcal{P}_q \cdot q_A\\ &= q_Q \cdot \mu \cdot q_A - q_Q \cdot q\\ &= q_Q \cdot (\mu \cdot q_A - q),\end{aligned}$$

that $q_Q \cdot \mu - \mathcal{P}_q$ is also null-homotopic, hence $[q_Q] \cdot [\mu] = [\mathcal{P}_q]$ and $[\mu] = [q_Q]^{-1} \cdot [\mathcal{P}_q] = [\widehat{q_Q}] \cdot [\mathcal{P}_q] = [\widehat{q_Q} \cdot \mathcal{P}_q] = [\lambda_q]$ as asserted.

Suppose now that q is a quasi-isomorphism. It follows by Theorem 3.32 that $\mathcal{P}_q$ is a homotopy-equivalence, i.e., $[\mathcal{P}_q]$ is an isomorphism. Consequently, $[\lambda_q]$ is an isomorphism as well. It is obvious that $[\lambda_q]^{-1}$ is a lift morphism of $[q_A]$ along $[q]$. Suppose $[\ell]\colon [P_A] \to [Q]$ is a another lift morphism of $[q_A]$ along $[q]$, i.e., with $[\ell] \cdot [q] = [q_A]$. Since q and q_A are quasi-isomorphisms, it follows that ℓ is a quasi-isomorphism as well. Hence, by Remark 3.24, $[\ell]$ is an isomorphism, thus $[q] = [\ell]^{-1} \cdot [q_A]$. It follows by the first assertion of the Lemma that $[\ell]^{-1} = [\lambda_q]$, i.e., $[\ell] = [\lambda_q]^{-1}$; and $[\lambda_q]^{-1}$ is then, as desired, the unique lift morphism of $[q_A]$ along $[q]$. □

An immediate consequence from Theorem 3.32 and Lemma 3.34 are the following two lemmas:

Lemma 3.35. *Let $\mathscr{C}$ be an Abelian category with enough projectives. Let A be an object in $\mathcal{C}^-(\mathscr{C})$. For any two projective resolutions $p\colon P \to A$, $q\colon Q \to A$ in $\mathcal{C}^-(\mathscr{C})$ of A, the morphisms $[p]$ and $[q]$ lift uniquely along each other via an isomorphism and its inverse.*

Lemma 3.36. *Let $\mathscr{C}$ be an Abelian category with enough projectives. Let $\varphi\colon A \to B$ be a morphism in $\mathcal{C}^-(\mathscr{C})$ and let $r_A\colon Q_A \to A$ and $r_B\colon Q_B \to B$ in $\mathcal{C}^-(\mathscr{C})$ be projective resolutions of A resp. B. Then there exists a unique morphism $[Q_\varphi]\colon [Q_A] \to [Q_B]$ with $[Q_\varphi] \cdot [r_B] = [r_A] \cdot [\varphi]$.*

The following corollary enables us to detect equalities in homotopy categories, in a similar way to monomorphisms and epimorphisms in general categories.

Corollary 3.37. *Let $\mathscr{C}$ be an Abelian category with enough projectives. Let Q be an object in $\mathcal{C}^-(\mathscr{C})$ with Q^i projective for all $i \in \mathbb{Z}$, and let $p\colon P \to A$ in $\mathcal{C}^-(\mathscr{C})$ be some projective resolution. Then for any morphism $s\colon Q \to P$, $[s] \cdot [p] = 0$ if and only if $[s] = 0$.*

Proof. $\mathrm{id}_Q\colon Q \to Q$ is a projective resolution for Q and $[s] \cdot [p] = [\mathrm{id}_Q] \cdot 0$, hence, by Lemma 3.36, $[s] = 0$. The converse is trivial. □

Corollary 3.38. *Let $\mathscr{C}$ be an Abelian category with enough projectives and $\mathbf{proj}(\mathscr{C})$ the full subcategory generated by all projective objects in $\mathscr{C}$. Then the projective resolution functor $\mathcal{P}$ is a right adjoint to the inclusion functor:*

$$\iota\colon \mathcal{K}^-(\mathbf{proj}(\mathscr{C})) \rightleftarrows \mathcal{K}^-(\mathscr{C}) :\mathcal{P}\,.$$

Proof. For a given pair of objects $[A]$ in $\mathcal{K}^-(\mathscr{C})$ and $[Q]$ in $\mathcal{K}^-(\mathbf{proj}(\mathscr{C}))$, we define the map

$$\Phi_{[Q],[A]}\colon \begin{cases}\mathrm{Hom}_{\mathcal{K}^-(\mathbf{proj}(\mathscr{C}))}([Q], \mathcal{P}([A])) & \to \mathrm{Hom}_{\mathcal{K}^-(\mathscr{C})}([Q],[A]),\\ \qquad [\lambda] & \mapsto [\lambda] \cdot q_{[A]}.\end{cases}$$

where $q_{[A]}\colon \mathcal{P}([A]) \to [A]$ is a projective resolution of $[A]$ asserted in Theorem 3.31.

By Lemma 3.34, $\Phi_{[Q],[A]}$ is bijective. Let $[\psi]\colon [Q'] \to [Q]$ in $\mathcal{K}^-(\mathbf{proj}(\mathscr{C}))$ and $[\varphi]\colon [A] \to [A']$ in $\mathcal{K}^-(\mathscr{C})$ be two morphisms, then for any $[\lambda]\colon [Q] \to \mathcal{P}([A])$ we have

$$[\psi] \cdot [\lambda] \cdot q_{[A]} \cdot [\varphi] = [\psi] \cdot [\lambda] \cdot \mathcal{P}([\varphi]) \cdot q_{[A']},$$

which translates to the commutativity of the following diagram:

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathcal{K}^-(\mathbf{proj}(\mathscr{C}))}([Q],\mathcal{P}([A])) & \xrightarrow{-\bullet q_{[A]}} & \operatorname{Hom}_{\mathcal{K}^-(\mathscr{C})}([Q],[A]) \\
\big\downarrow{\scriptstyle [\psi]\bullet - \bullet \mathcal{P}([\varphi])} & \circlearrowleft & \big\downarrow{\scriptstyle [\psi]\bullet - \bullet [\varphi]} \\
\operatorname{Hom}_{\mathcal{K}^-(\mathbf{proj}(\mathscr{C}))}([Q'],\mathcal{P}([A'])) & \xrightarrow[-\bullet q_{[A']}]{} & \operatorname{Hom}_{\mathcal{K}^-(\mathscr{C})}([Q'],[A']),
\end{array}$$

i.e., the map Φ is natural. □

In fact, the components of aforementioned adjunction are both *exact functors* (cf. Definition B.24). It is obvious that the injection

$$\iota\colon \mathcal{K}^-(\mathbf{proj}(\mathscr{C})) \to \mathcal{K}^-(\mathscr{C})$$

is exact. The exactness of $\mathcal{P}$ follows by the following Lemma:

Lemma 3.39. *Let $\mathscr{C}$ be Abelian category with enough projectives. The projective resolution functor*

$$\mathcal{P}\colon \mathcal{K}^-(\mathscr{C}) \to \mathcal{K}^-(\mathbf{proj}(\mathscr{C}))$$

is exact.

Proof. We start by showing that $\mathcal{P}$ commutes with the shift functors up to a natural isomorphism $\mu\colon \Sigma\bullet\mathcal{P} \to \mathcal{P}\bullet\Sigma$, then we show that $\mathcal{P}$ is exact with respect to μ.

Let $q_{\Sigma(A)}$ and $q_{\Sigma(B)}$ be the projective resolutions of $\Sigma(A)$ resp. $\Sigma(B)$ as computed in Theorem 3.31. Similarly, let q_A and q_B be projective resolutions of A resp. B. Applying Σ on a morphism only shifts the induced morphisms on cohomology, hence $\Sigma(q_A)$ and $\Sigma(q_B)$ are both quasi-isomorphisms, hence projective resolutions of $\Sigma(A)$ resp. $\Sigma(B)$. It follows from Lemma 3.34 that $q_{\Sigma(A)}$ and $\Sigma(q_A)$ lift uniquely along each other via an isomorphism and its inverse. The same holds for $q_{\Sigma(B)}$ and $\Sigma(q_B)$.

Let μ_A be the unique lift *isomorphism* of $q_{\Sigma(A)}$ along $\Sigma(q_A)$. Analogously, we define μ_B.

$$\begin{array}{ccccc}
 & & \overset{\mu_A}{\sim}\ (\mathcal{P}(\Sigma(A)) \to \Sigma(\mathcal{P}(A)))\quad \circlearrowleft & & \\
\mathcal{P}(\Sigma(A)) & \xrightarrow{q_{\Sigma(A)}} & \Sigma(A) & \xleftarrow{\Sigma(q_A)} & \Sigma(\mathcal{P}(A)) \\
\big\downarrow{\scriptstyle \mathcal{P}(\Sigma(\varphi))} & \circlearrowleft & \big\downarrow{\scriptstyle \Sigma(\varphi)} & \circlearrowleft & \big\downarrow{\scriptstyle \Sigma(\mathcal{P}(\varphi))} \\
\mathcal{P}(\Sigma(B)) & \xrightarrow[q_{\Sigma(B)}]{} & \Sigma(B) & \xleftarrow[\Sigma(q_B)]{} & \Sigma(\mathcal{P}(B)) \\
 & & \circlearrowleft\quad \underset{\mu_B}{\sim}\ (\mathcal{P}(\Sigma(B)) \to \Sigma(\mathcal{P}(B))) & &
\end{array}$$

A simple diagram chase shows that

$$(\mu_A\bullet\Sigma(\mathcal{P}(\varphi)) - \mathcal{P}(\Sigma(\varphi))\bullet\mu_B)\bullet\Sigma(q_B) = 0,$$

hence, by Corollary 3.37,

$$\mu_A \bullet \Sigma(\mathcal{P}(\varphi)) = \mathcal{P}(\Sigma(\varphi)) \bullet \mu_B,$$

i.e., the assignment

$$\mu\colon \begin{cases} \Sigma \bullet \mathcal{P} & \to \mathcal{P} \bullet \Sigma, \\ A & \mapsto \mu_A \end{cases}$$

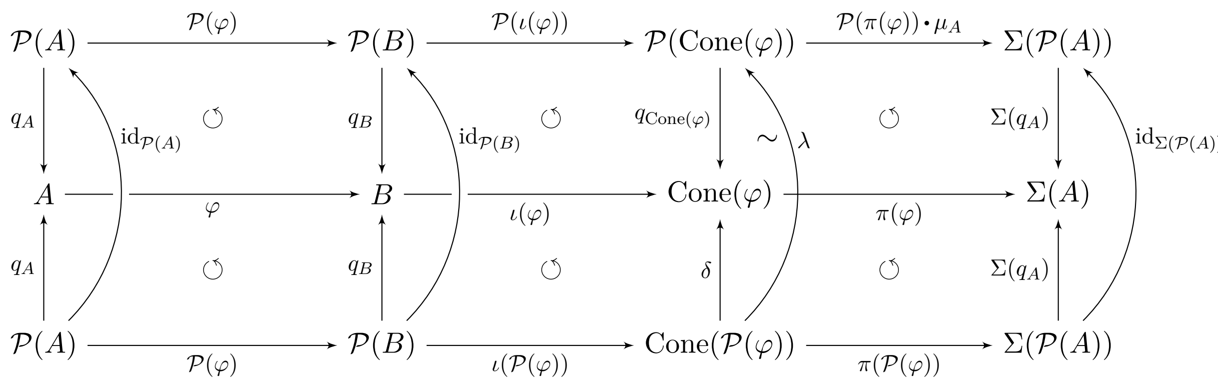
defines a natural isomorphism.

We still need to show that for any morphism $\varphi\colon A \to B$ in $\mathcal{K}^-(\mathscr{C})$, the triangle

$$\mathcal{P}(A) \xrightarrow{\mathcal{P}(\varphi)} \mathcal{P}(B) \xrightarrow{\mathcal{P}(\iota(\varphi))} \mathcal{P}(\mathrm{Cone}(\varphi)) \xrightarrow{\mathcal{P}(\pi(\varphi)) \bullet \mu_A} \Sigma(\mathcal{P}(A))$$

is exact. Let q_A and q_B be projective resolutions of A resp. B, then $\mathcal{P}(\varphi) \bullet q_B = q_A \bullet \varphi$. Let $\delta\colon \mathrm{Cone}(\mathcal{P}(\varphi)) \to \mathrm{Cone}(\varphi)$ and $\epsilon\colon \mathrm{Cone}(q_A) \to \mathrm{Cone}(q_B)$ be the morphisms resulted by the axiom **TR 4**. Since q_A and q_B are quasi-isomorphisms, $\mathrm{Cone}(q_A)$ and $\mathrm{Cone}(q_B)$ are, by Lemma 3.16, both exact; hence ϵ is a quasi-isomorphism. It follows by the same Lemma that $\mathrm{Cone}(\epsilon)$ is exact as well. By the $3{\times}3$ -Lemma (see e.g., [**May01**, Lemma 2.6]), $\mathrm{Cone}(\epsilon) \cong \mathrm{Cone}(\delta)$, thus $\mathrm{Cone}(\delta)$ is exact and δ is then a quasi-isomorphism, i.e., δ is a projective resolution for $\mathrm{Cone}(\varphi)$. Let $q_{\mathrm{Cone}(\varphi)}\colon \mathcal{P}(\mathrm{Cone}(\varphi)) \to \mathrm{Cone}(\varphi)$ be the projective resolution of $\mathrm{Cone}(\varphi)$ asserted by Theorem 3.32. By Lemma 3.34, δ lifts uniquely along $q_{\mathrm{Cone}(\varphi)}$ via an isomorphism, say λ.

We depict the above data by the following diagram whose upper and lower parts are commutative:



By a diagram chase we get the following two equalities:

$$(\mathcal{P}(\iota(\varphi)) - \iota(\mathcal{P}(\varphi)) \bullet \lambda) \bullet q_{\mathrm{Cone}(\varphi)} = 0$$

and

$$(\lambda \bullet \mathcal{P}(\pi(\varphi)) \bullet \mu_A - \pi(\mathcal{P}(\varphi))) \bullet \Sigma(q_A) = 0,$$

hence, by Corollary 3.37, we have

$$\mathcal{P}(\iota(\varphi)) - \iota(\mathcal{P}(\varphi)) \bullet \lambda = 0$$

and

$$\lambda \bullet \mathcal{P}(\pi(\varphi)) \bullet \mu_A - \pi(\mathcal{P}(\varphi)) = 0.$$

Hence, the top and bottom triangles are isomorphic; and since the bottom triangle is exact, so is then the top. □

Remark 3.40. All statements about projective resolutions can be dualized to a similar statements for the existence of injective resolutions. An **injective resolution** for an object A in $\mathcal{C}(\mathscr{C})$ is a quasi-isomorphism $q_A\colon A \to \mathcal{I}_A$ such that $\mathcal{I}_A^i$ is injective for all $i \in \mathbb{Z}$. If $\mathscr{C}$ is an Abelian category with enough injectives, then each object A in $\mathcal{C}^+(\mathscr{C})$ admits an injective resolution, and every morphism $\varphi\colon A \to B$ can, up to homotopy, uniquely be lifted to a morphism $\mathcal{I}_\varphi\colon \mathcal{I}_A \to \mathcal{I}_B$ with $\varphi \cdot q_B = q_A \cdot \mathcal{I}_\varphi$. We get a functor

$$\mathcal{I}\colon \begin{cases} \mathcal{K}^+(\mathscr{C}) & \to \mathcal{K}^+(\mathbf{inj}(\mathscr{C})), \\ \quad [A] & \mapsto [\mathcal{I}_A], \\ \quad [\varphi] & \mapsto [\mathcal{I}_\varphi]. \end{cases}$$

which maps quasi-isomorphisms to isomorphisms. Furthermore, the functor $\mathcal{I}$ is a left adjoint to the inclusion functor.

$$\mathcal{I}\colon \mathcal{K}^+(\mathscr{C}) \rightleftarrows \mathcal{K}^+(\mathbf{inj}(\mathscr{C})) : \iota$$

and the bijection associated to a pair of objects $[I]$ in $\mathcal{K}^+(\mathbf{inj}(\mathscr{C}))$ and $[A]$ in $\mathcal{K}^+(\mathscr{C})$ is given by

$$\Psi_{[A],[I]}\colon \begin{cases} \mathrm{Hom}_{\mathcal{K}^-(\mathbf{inj}(\mathscr{C}))}(\mathcal{I}([A]),[I]) & \to \mathrm{Hom}_{\mathcal{K}^-(\mathscr{C})}([A],[I]), \\ \qquad\qquad [\lambda] & \mapsto q_{[A]} \cdot [\lambda] \end{cases}$$

where $q_{[A]} \coloneqq [q_A]\colon [A] \to [\mathcal{I}_A]$.

3.4. Derived Categories and Derived Functors

This chapter provides an overview on derived categories and some of their associated concepts like extension groups $\mathrm{Ext}^n(-,-)$ and the derived functors. Deciding the equality of morphisms in the derived category directly is a priori extremely difficult. As a result, one seeks a more friendly category that is equivalent to the derived category in question. For example, if $\mathscr{C}$ has enough projective objects then we can use the equivalence

$$\mathcal{D}^-(\mathscr{C}) \cong \mathcal{K}^-(\mathbf{proj}(\mathscr{C})),$$

where $\mathbf{proj}(\mathscr{C})$ is the full subcategory of $\mathscr{C}$ generated by all projective objects in $\mathscr{C}$. Similarly, if $\mathscr{C}$ has enough injective objects, then can use the equivalence

$$\mathcal{D}^+(\mathscr{C}) \cong \mathcal{K}^+(\mathbf{inj}(\mathscr{C}))$$

where $\mathbf{inj}(\mathscr{C})$ is the full subcategory of $\mathscr{C}$ generated by all injective objects in $\mathscr{C}$.

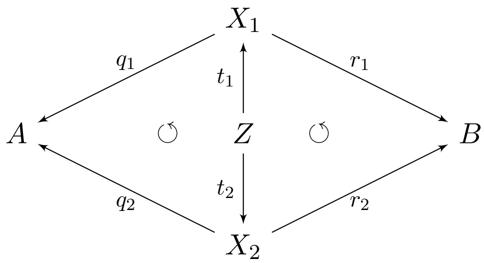
With these techniques we implement versions of *derived equivalences* on computer. Namely, the Happel theorem where the tilting module is the direct some of objects of a complete strong exceptional sequence (cf. Corollary 6.7 and Appendix E).

Definition 3.41. Let $\mathscr{C}$ be an Abelian category and $* \in \{\text{“ ”}, +, -, b\}$. The **derived category** $\mathcal{D}^*(\mathscr{C})$ is defined by the following data:

(1) $\mathrm{Obj}(\mathcal{D}^*(\mathscr{C})) \coloneqq \mathrm{Obj}(\mathcal{K}^*(\mathscr{C}))$.
(2) For a given pair of objects A and B in $\mathcal{D}^*(\mathscr{C})$, we define the $\mathrm{Hom}_{\mathcal{D}^*(\mathscr{C})}(A,B)$ by the set of all equivalence classes of *roofs* of the form $A \xleftarrow{q} X \xrightarrow{r} B$ where q, r live in $\mathcal{K}^*(\mathscr{C})$ and q is a quasi-isomorphism; where two such roofs

$$A \xleftarrow{q_1} X_1 \xrightarrow{r_1} B \text{ and } A \xleftarrow{q_2} X_2 \xrightarrow{r_2} B$$

are equivalent if there exists an object Z in $\mathcal{K}^*(\mathscr{C})$ and two quasi-isomorphisms $t_1\colon Z \to X_1$ and $t_2\colon Z \to X_2$ rendering the following diagram in $\mathcal{K}^*(\mathscr{C})$



commutative. A morphism that is represented by a roof $A \xleftarrow{q} X \xrightarrow{r} B$ is usually denoted by r/q.

(3) The composition of two morphisms represented by the roofs

$$A \xleftarrow{q_1} X \xrightarrow{r_1} B \text{ and } B \xleftarrow{q_2} Y \xrightarrow{r_2} C$$

is the morphism represented by the roof $A \xleftarrow{q} Z \xrightarrow{r} C$ where Z is the object in $\mathcal{K}^*(\mathscr{C})$ whose differential at $i \in \mathbb{Z}$ is

$$\partial_Z^i \coloneqq X^i \oplus Y^i \oplus B^{i-1} \xrightarrow{\begin{pmatrix} \partial_X^i & 0 & -r_1^i \\ 0 & \partial_Y^i & -q_2^i \\ 0 & 0 & -\partial_B^{i-1} \end{pmatrix}} X^{i+1} \oplus Y^{i+1} \oplus B^i,$$

and q, r are the morphisms whose components at $i \in \mathbb{Z}$ are

$$q^i \coloneqq X^i \oplus Y^i \oplus B^{i-1} \xrightarrow{\begin{pmatrix} -q_1^i \\ 0 \\ 0 \end{pmatrix}} A^i$$

resp.

$$r^i \coloneqq X^i \oplus Y^i \oplus B^{i-1} \xrightarrow{\begin{pmatrix} 0 \\ r_2^i \\ 0 \end{pmatrix}} C^i.$$

(4) The identity morphism of an object A is given by $\mathrm{id}_A/\mathrm{id}_A$.

The categories $\mathcal{D}^-(\mathscr{C}), \mathcal{D}^+(\mathscr{C})$ and $\mathcal{D}^b(\mathscr{C})$ are called the bounded above, bounded below resp. bounded derived categories of $\mathscr{C}$.

Remark 3.42. Let $\mathscr{C}$ be an Abelian category and let $\mathcal{D}^*(\mathscr{C})$ be its derived category. We have the following facts:

(1) If $r/q\colon A \to B$ in $\mathcal{D}^*(\mathscr{C})$ is represented by the roof $A \xleftarrow{q} X \xrightarrow{r} B$, then

$$r/q = \mathrm{id}_X/q \bullet r/\mathrm{id}_X = (q/\mathrm{id}_X)^{-1} \bullet r/\mathrm{id}_X.$$

(2) There is a natural functor

$$Q\colon \begin{cases} \mathcal{K}^*(\mathscr{C}) & \to \mathcal{D}^*(\mathscr{C}), \\ A & \mapsto A, \\ r\colon A \to B & \mapsto r/\mathrm{id}_A\colon A \to B, \end{cases}$$

which maps quasi-isomorphisms to isomorphisms. In particular, if r is a quasi-isomorphism, then $Q(r)^{-1} = \mathrm{id}_A/r$. Furthermore, Q is universal with this property, i.e., if $U\colon \mathcal{K}^*(\mathscr{C}) \to E$ is a functor which maps quasi-isomorphisms to isomorphisms, then there exists, up to a natural isomorphism, a unique functor $\widetilde{U}\colon \mathcal{D}^*(\mathscr{C}) \to E$ such that $U \cong Q \bullet \widetilde{U}$. Since each morphism r/q can be written as $Q(q)^{-1} \bullet Q(r)$, $\widetilde{U}$ is given by

$$\widetilde{U}\colon \begin{cases} \mathcal{D}^*(\mathscr{C}) & \to E, \\ \quad A & \mapsto U(A), \\ \quad r/q & \mapsto U(q)^{-1} \bullet U(r). \end{cases}$$

(3) The cohomology functors $\mathrm{H}^i\colon \mathcal{K}^*(\mathscr{C}) \to \mathscr{C},\ i \in \mathbb{Z}$ map quasi-isomorphisms to isomorphisms, hence they can be regarded as functors from $\mathcal{D}^*(\mathscr{C})$. In particular, we define the i^{th}-cohomology functor by

$$\mathrm{H}^i\colon \begin{cases} \mathcal{D}^*(\mathscr{C}) & \to \mathscr{C}, \\ \quad A & \mapsto \mathrm{H}^i(A), \\ \quad r/q & \mapsto \mathrm{H}^i(q)^{-1} \bullet \mathrm{H}^i(r). \end{cases}$$

(4) A morphism $r/q\colon A \to B$ in $\mathcal{D}^*(\mathscr{C})$ is an isomorphism if and only if $\mathrm{H}^i(r/q)$ is an isomorphism for all $i \in \mathbb{Z}$ if and only if r is a quasi-isomorphism.

Remark 3.43. The category $\mathcal{D}^*(\mathscr{C})$ is additive.

(1) An object A in $\mathcal{D}^*(\mathscr{C})$ is zero if and only if $\mathrm{H}^i(A) = 0$ for all $i \in \mathbb{Z}$, i.e., if and only if A is exact.

(2) The product and coproduct can be inherited from $\mathcal{K}^*(\mathscr{C})$, for example if A and B are two objects in $\mathcal{D}^*(\mathscr{C})$ then the natural injection of A into $A \oplus B$ is represented by the roof

$$A \xleftarrow{\mathrm{id}_A} A \xrightarrow{(\mathrm{id}_A\ 0)} A \oplus B$$

and the natural projection from $A \oplus B$ onto A is represented by the roof

$$A \oplus B \xleftarrow{\begin{pmatrix} \mathrm{id}_A & 0 \\ 0 & \mathrm{id}_B \end{pmatrix}} A \oplus B \xrightarrow{\begin{pmatrix} \mathrm{id}_A \\ 0 \end{pmatrix}} A.$$

(3) For a given pair of objects A, B in $\mathcal{D}^*(\mathscr{C})$, the zero morphism from A to B is given by $0/\mathrm{id}_A$; and the addition of morphisms $r_1/q_1, r_2/q_2\colon A \to B$ is given, as can be done in any additive category, by the composition of the triple

$$A \xrightarrow{(\mathrm{id}_A\ \mathrm{id}_A)} A \oplus A \xrightarrow{\begin{pmatrix} r_1/q_1 & 0 \\ 0 & r_2/q_2 \end{pmatrix}} B \oplus B \xrightarrow{\begin{pmatrix} \mathrm{id}_B \\ \mathrm{id}_B \end{pmatrix}} B.$$

Definition 3.44. The shift automorphism $\widetilde{\Sigma}$ on $\mathcal{D}^*(\mathscr{C})$ is the functor determined by the relation $\Sigma \bullet Q = Q \bullet \widetilde{\Sigma}$ where Q is the natural functor $Q\colon \mathcal{K}^*(\mathscr{C}) \to \mathcal{D}^*(\mathscr{C})$ and Σ is the shift automorphism on $\mathcal{K}^*(\mathscr{C})$. A triangle

$$A \xrightarrow{r} B \xrightarrow{\iota} C \xrightarrow{\pi} \widetilde{\Sigma}(A)$$

in $\mathcal{D}^*(\mathscr{C})$ will be called exact if it is isomorphic to the image under Q of some exact triangle in $\mathcal{K}^*(\mathscr{C})$, i.e., to some triangle of the form

$$Q(X) \xrightarrow{Q(f)} Q(Y) \xrightarrow{Q(\iota(f))} Q(\mathrm{Cone}(f)) \xrightarrow{Q(\pi(f))} Q(\Sigma(X)) = \widetilde{\Sigma}(Q(X)).$$

It can be shown this class of exact triangles turns $\mathcal{D}^*(\mathscr{C})$ into a triangulated category. Moreover, the natural functor $Q\colon \mathcal{K}^*(\mathscr{C}) \to \mathcal{D}^*(\mathscr{C})$ is exact (cf. [**GM03**]).

For $* \in \{+, -, b\}$, we have natural embeddings $\mathcal{D}^*(\mathscr{C}) \hookrightarrow \mathcal{D}(\mathscr{C})$ defined by forgetting the boundedness conditions.

Proposition 3.45. *The natural embeddings $\mathcal{D}^*(\mathscr{C}) \hookrightarrow \mathcal{D}(\mathscr{C})$ for $* = +, -$ or b, define equivalences of $\mathcal{D}^*(\mathscr{C})$ with the full triangulated subcategories of $\mathcal{D}(\mathscr{C})$ generated by all objects A with $\mathrm{H}^i(A) = 0$ for $i \ll 0$, $i \gg 0$ resp. $|i| \gg 0$.*

The concept of $\mathcal{K}$-projectives and $\mathcal{K}$-injectives allows us to identify derived categories with homotopy categories. For extensive treatment we refer to [**Spa88**], [**Yek12**] and [**Yek20**].

Definition 3.46. Let $\mathscr{C}$ be an Abelian category and let $* \in \{+, -, b, \text{“ ”}\}$.

(1) An object P in $\mathcal{K}^*(\mathscr{C})$ is called $\mathcal{K}$-**projective** if for every acyclic object U in $\mathcal{K}^*(\mathscr{C})$, $\mathrm{Hom}_{\mathcal{K}^*(\mathscr{C})}(P, U) = 0$.
(2) A $\mathcal{K}$-**projective resolution** of an object A in $\mathcal{K}^*(\mathscr{C})$ is a quasi-isomorphism $P \to A$ from some $\mathcal{K}$-projective object P in $\mathcal{K}^*(\mathscr{C})$.
(3) We say $\mathcal{K}^*(\mathscr{C})$ has enough $\mathcal{K}$-projectives if every A in $\mathcal{K}^*(\mathscr{C})$ has a $\mathcal{K}$-projective resolution.
(4) The full subcategory of $\mathcal{K}^*(\mathscr{C})$ generated by $\mathcal{K}$-projective objects will be denoted by $\mathcal{K}^*_{\mathrm{proj}}(\mathscr{C})$. It can be shown that $\mathcal{K}^*_{\mathrm{proj}}(\mathscr{C})$ is a triangulated subcategory of $\mathcal{K}^*(\mathscr{C})$.

Example 3.47. Let $\mathscr{C}$ be an Abelian category and P an object in $\mathcal{K}^-(\mathscr{C})$ where P^i is a projective object for all $i \in \mathbb{Z}$, then P is $\mathcal{K}$-projective.

Lemma 3.48. *Let $\mathscr{C}$ be an Abelian category and let P be an object in $\mathcal{K}^*(\mathscr{C})$. The following statements are equivalent*

(1) P is $\mathcal{K}$-projective,
(2) For every quasi-isomorphism $q\colon A \to B$ in $\mathcal{K}^*(\mathscr{C})$, the map

$$\mathrm{Hom}_{\mathcal{K}^*(\mathscr{C})}(P, A) \xrightarrow{- \bullet q} \mathrm{Hom}_{\mathcal{K}^*(\mathscr{C})}(P, B)$$

is bijective,
(3) Every quasi-isomorphism $q\colon A \to P$ in $\mathcal{K}^*(\mathscr{C})$ is a split-epimorphism,
(4) For every B in $\mathcal{K}^*(\mathscr{C})$ the map

$$Q_{P,B}\colon \begin{cases} \mathrm{Hom}_{\mathcal{K}^*(\mathscr{C})}(P, B) & \to \mathrm{Hom}_{\mathcal{D}^*(\mathscr{C})}(P, B), \\ \qquad \psi & \mapsto \psi / \mathrm{id}_P \end{cases}$$

is an isomorphism.

Corollary 3.49. *Any quasi-isomorphism in $\mathcal{K}^*(\mathscr{C})$ between two $\mathcal{K}$-projective objects is an isomorphism.*

Theorem 3.50. *Let $\mathscr{C}$ be an Abelian category. Then the natural functor*

$$\zeta\colon \mathcal{K}^*_{\mathrm{proj}}(\mathscr{C}) \to \mathcal{D}^*(\mathscr{C})$$

is fully faithful. Moreover, if $\mathcal{K}^*(\mathscr{C})$ has enough $\mathcal{K}$-projectives, then ζ defines an exact equivalence.

Proof. ζ is exact because it is defined by the composition of the exact functors

$$\mathcal{K}^*_{\mathrm{proj}}(\mathscr{C}) \hookrightarrow \mathcal{K}^*(\mathscr{C}) \xrightarrow{Q} \mathcal{D}^*(\mathscr{C}).$$

Since Q maps quasi-isomorphisms to isomorphisms and $\mathcal{K}^*(\mathscr{C})$ has enough $\mathcal{K}$-projectives, ζ is essentially surjective. □

Corollary 3.51. *Let $\mathscr{C}$ be an Abelian category with enough projectives and* $\mathbf{proj}(\mathscr{C})$ *the full subcategory generated by projective objects. The natural functor*

$$\mathcal{K}^-(\mathbf{proj}(\mathscr{C})) \to \mathcal{D}^-(\mathscr{C})$$

defines an exact equivalence.

Proof. By Theorem 3.31, $\mathcal{K}^b(\mathscr{C})$ has enough $\mathcal{K}$-projectives. By Corollary 3.49, the exact natural embedding $\mathcal{K}^-(\mathbf{proj}(\mathscr{C})) \hookrightarrow \mathcal{K}^-_{\mathrm{proj}}(\mathscr{C})$ is essentially surjective, hence an equivalence. Hence, the assertion follows by Theorem 3.50. □

Definition 3.52. Let $\mathscr{C}$ be an Abelian category. For two objects A and B in $\mathscr{C}$, we define the i^{th}-**extension group** of A and B by

$$\operatorname{Ext}^i_{\mathscr{C}}(A,B) \coloneqq \operatorname{Hom}_{\mathcal{D}^b(\mathscr{C})}\Big(A, \Sigma^i(B)\Big)$$

where A and B are considered as objects in $\mathcal{D}^b(\mathscr{C})$.

Remark 3.53. Since Σ is an autoequivalence, we can identify the extension group $\operatorname{Ext}^i_{\mathscr{C}}(A,B)$ with $\operatorname{Hom}_{\mathcal{D}^b(\mathscr{C})}\Big(\Sigma^k(A), \Sigma^{k+i}(B)\Big)$ for all $i,k \in \mathbb{Z}$. Hence, we can define a composition law of extensions:

$$*\colon \begin{cases} \operatorname{Ext}^i_{\mathscr{C}}(A,B) \times \operatorname{Ext}^j_{\mathscr{C}}(B,C) & \to \operatorname{Ext}^{i+j}_{\mathscr{C}}(A,C), \\ (r,\psi) & \mapsto r \bullet \Sigma^i(\psi). \end{cases}$$

Remark 3.54. $\operatorname{Ext}^i_{\mathscr{C}}(A,B) = 0$ for all $i < 0$ (cf. [**GM03**, III.5]).

Definition 3.55. Let A be an object in $\mathscr{C}$. We define the homological **projective dimension** and **injective dimension** of A by

$$\operatorname{prodim}(A) \coloneqq \sup\{n \mid \exists B \in \mathscr{C}, \operatorname{Ext}^n_{\mathscr{C}}(A,B) \neq 0\}$$

resp.

$$\operatorname{injdim}(A) \coloneqq \sup\{n \mid \exists B \in \mathscr{C}, \operatorname{Ext}^n_{\mathscr{C}}(B,A) \neq 0\}.$$

The **homological dimension** of the category $\mathscr{C}$ is the maximum n such that there exists two objects A, B in $\mathscr{C}$ with $\operatorname{Ext}^n_{\mathscr{C}}(A,B) \neq 0$ (or ∞ if no such d exists).

Lemma 3.56. *The following properties of an object A in $\mathscr{C}$ are equivalent:*

(1) $\operatorname{prodim}(A) = 0$*;*
(2) $\operatorname{Ext}^1_{\mathscr{C}}(A,B) = 0$ *for all B in $\mathscr{C}$;*
(3) A is a projective object.

Similarly, the following properties are equivalent:

(1) $\operatorname{injdim}(A) = 0$*;*
(2) $\operatorname{Ext}^1_{\mathscr{C}}(B,A) = 0$ *for all B in $\mathscr{C}$;*
(3) A is an injective object.

Proof. See [**GM03**, Lemma III.9.10]. □

Lemma 3.57. *Let*

$$0 \to B \to P^{-(k-1)} \to \cdots \to P^0 \to A \to 0$$

be an acyclic object in $\mathcal{C}^b(\mathscr{C})$ with all P^i projective, then

$$\operatorname{prodim}(B) = \max\{0, \operatorname{prodim}(A) - k\}.$$

Similarly, if

$$0 \to A \to I^0 \to \cdots \to I^{k-1} \to B \to 0$$

is an acyclic object in $\mathcal{C}^b(\mathscr{C})$ with all I^i injective, then

$$\operatorname{injdim}(B) = \max\{0, \operatorname{injdim}(A) - k\}.$$

Proof. See [**GM03**, Lemma III.9.11]. □

Corollary 3.58. *Let $\mathscr{C}$ be an Abelian category with enough projectives and A an object in $\mathscr{C}$. The following statements are equivalent:*

(1) $\operatorname{prodim}(A) \leq k$*;*
(2) if the complex

$$0 \to B \to P^{-(k-1)} \to \cdots \to P^0 \to A \to 0$$

is acyclic and every P^i is projective, then B is also projective.
(3) there exists an acyclic complex

$$0 \to P^{-k} \to P^{-(k-1)} \to \cdots \to P^0 \to A \to 0$$

in which every P^i projective.

Proof. (1) → (2) follows from Lemmas 3.56 and 3.57. Now we show (2) → (3): Since $\mathscr{C}$ has enough projectves, we can compute a projective resolution, say P_A, for A. Let $\iota \colon K \hookrightarrow P_A^{-(k-1)}$ be the kernel embedding of $\partial_{P_A}^{-(k-1)}$, then

$$0 \to K \overset{\iota}{\hookrightarrow} P^{-(k-1)} \to \cdots \to P^0 \to A \to 0$$

is acyclic, hence, by assumption, K is projective and the above complex is a projective resolution of length $\leq k$ of A. Now we show (3) → (1): By Lemma 3.56, $\operatorname{prodim}\left(P^{-k}\right) = 0$ and by Lemma 3.57, $0 = \operatorname{prodim}(B) = \max\{0, \operatorname{prodim}(A) - k\}$, i.e., $\operatorname{prodim}(A) \leq k$ as desired. □

Remark 3.59. Let $\mathscr{C}$ be an Abelian category with enough projectives and a finite homological dimension $d \geq 0$. Then we have an algorithm which constructs for a given object A in $\mathscr{C}$ a finite projective resolution of length at most d. We start by constructing some projective resolution P of A, then we let $P \twoheadrightarrow \tau^{\geq -d}(P)$ be the natural projection[7] of P on the smart d-bellow truncation of P, i.e., the morphism

$$\begin{array}{ccccccccccccccc}
\cdots & \xrightarrow{\partial_P^{-(d+2)}} & P^{-(d+1)} & \xrightarrow{\partial_P^{-(d+1)}} & P^{-d} & \xrightarrow{\partial_P^{-d}} & P^{-(d-1)} & \xrightarrow{\partial_P^{-(d-1)}} & \cdots & \xrightarrow{\partial_P^{-1}} & P^0 & \twoheadrightarrow & A & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \pi} & & \Vert & & & & & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & \operatorname{coker}\left(\partial_P^{-(d+1)}\right) & \xrightarrow[\iota]{} & P^{-(d-1)} & \xrightarrow[\partial_P^{-(d-1)}]{} & \cdots & \xrightarrow[\partial_P^{-1}]{} & P^0 & \twoheadrightarrow & A & \longrightarrow & 0
\end{array}$$

[7] It is a quasi-isomorphism (cf. [**KS06**, Definition 8.3.8]).

where π is the cokerel projection of $\partial_P^{-(d+1)}$ and ι is the unique colift of ∂_P^{-d} along π. Since the top row is acyclic and the truncation is smart, the bottom row is also acyclic. By Corollary 3.58, $\operatorname{coker}\left(\partial_P^{-(d+1)}\right)$ is projective. This means $\tau^{\geq -d}(P)$ defines a finite projective resolution for A of length at most d. Of course P and $\tau^{\geq -d}(P)$ are homotopy-equivalent.

Lemma 3.60. *Let $\mathscr{C}$ be an Abelian category with enough projectives and a finite homological dimension $d \geq 0$. Let P in $\mathcal{K}^-(\mathscr{C})$ be a complex of projectives such that $\mathrm{H}^i(P) = 0$ for $i \ll 0$, then P is isomorphic to a bounded complex of projectives.*

Proof. Let $\ell \in \mathbb{Z}$ be a lower homological bound for P, i.e., $\mathrm{H}^i(P) = 0$ for all $i < \ell$. Let π be the cokernel projection of $\partial_P^{\ell-(d+1)}$ and ι the unique colift of $\partial_P^{\ell-d}$ along π. Then the natural projection of P on the smart $(\ell-d)$-below truncation $\tau^{\geq \ell-d}(P)$

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{\partial_P^{\ell-(d+2)}} & P^{\ell-(d+1)} & \xrightarrow{\partial_P^{\ell-(d+1)}} & P^{\ell-d} & \xrightarrow{\partial_P^{\ell-d}} & P^{\ell-(d-1)} & \xrightarrow{\partial_P^{\ell-(d-1)}} & \cdots & \xrightarrow{\partial_P^{\ell-1}} & P^{\ell} & \longrightarrow P^{\ell+1} \longrightarrow & \cdots \\
 & & & & \big\downarrow{\scriptstyle \pi} & & \Vert & & & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & \operatorname{coker}\left(\partial_P^{\ell-(d+1)}\right) & \xrightarrow[\iota]{} & P^{\ell-(d-1)} & \xrightarrow[\partial_P^{\ell-(d-1)}]{} & \cdots & \xrightarrow[\partial_P^{\ell-1}]{} & P^{\ell} & \longrightarrow P^{\ell+1} \longrightarrow & \cdots
\end{array}$$

is a quasi-isomorphism. This means $\mathrm{H}^i\left(\tau^{\geq \ell-d}(P)\right) = 0$ for all $i < \ell$. Hence, the complex

$$0 \to \operatorname{coker}\left(\partial_P^{\ell-(d+1)}\right) \xrightarrow{\iota} P^{\ell-(d-1)} \to \cdots \to P^{\ell} \twoheadrightarrow \operatorname{coker}\left(\partial_A^{\ell-1}\right) \to 0$$

is acyclic, and by Corollary 3.58, $\operatorname{coker}\left(\partial_P^{\ell-(d+1)}\right)$ is a projective object. If we define Q by $\tau^{\geq \ell-d}(P)$, then the assertion follows by Remark 3.24. □

Theorem 3.61. *Let $\mathscr{C}$ be an Abelian category with enough projectives and finite homological dimension d, then the natural functor*

$$\mathcal{K}^b(\mathbf{proj}(\mathscr{C})) \to \mathcal{D}^b(\mathscr{C})$$

defines an exact equivalence.

Proof. The functor is fully faithful and exact due to Corollary 3.51. By Theorem 3.31 and Lemma 3.60 the functor $\mathcal{K}^b(\mathbf{proj}(\mathscr{C})) \to \mathcal{D}^b(\mathscr{C})$ is essentially surjective. □

Theorem 3.62. *Let $\mathscr{C}$ be an Abelian category with enough injectives and finite homological dimension $d \geq 0$, then the natural functor*

$$\mathcal{K}^b(\mathbf{inj}(\mathscr{C})) \to \mathcal{D}^b(\mathscr{C})$$

defines an exact equivalence.

Theorem 3.63. *Let $\mathscr{C}$ be a Abelian category with enough projectives and finite homological dimension. If $\mathcal{K}^b(\mathscr{C})$ has decidable equality of morphisms, then so does $\mathcal{D}^b(\mathscr{C})$.*

Definition 3.64. Let $\mathscr{C}$ be an Abelian category and $\mathfrak{T}$ a triangulated category. Suppose $* \in \{+, -, b, \text{“ ”}\}$ and $Q\colon \mathcal{K}^*(\mathscr{C}) \to \mathcal{D}^*(\mathscr{C})$ is the natural localization functor. Let $F\colon \mathcal{K}^*(\mathscr{C}) \to \mathfrak{T}$ be an exact functor. The **left derived functor** of F is an exact functor

$$\mathbf{L}F\colon \mathcal{D}^*(\mathscr{C}) \to \mathfrak{T},$$

together with a natural transformation

$$\eta\colon Q\bullet \mathbf{L}F \to F$$

which is universal in the sense that if $\mathbb{G}\colon \mathcal{D}^*(\mathscr{C}) \to \mathfrak{T}$ is another exact functor equipped with a natural transformation $\zeta\colon Q\bullet\mathbb{G} \to F$, then there exists a unique natural transformation $\lambda\colon \mathbb{G} \to \mathbf{L}F$ such $\zeta_A = \lambda_{Q(A)}\bullet\eta_A$ for all A in $\mathcal{K}^*(A)$.

With the same assumptions as above, we have the following very useful lemma:

Lemma 3.65. *If $\mathcal{K}^*(\mathscr{C})$ has enough $\mathcal{K}$-projectives, then F has a left derived functor $(\mathbf{L}F, \eta)$ given by*

$$\mathbf{L}F\colon \begin{cases} \mathcal{D}^*(\mathscr{C}) & \to \mathfrak{T}, \\ A & \mapsto F(\mathcal{P}_A), \\ r/q\colon A \to B & \mapsto F(\mathcal{P}_q)^{-1}\bullet F(\mathcal{P}_r)\colon F(\mathcal{P}_A) \to F(\mathcal{P}_B) \end{cases}$$

and the natural transformation

$$\eta\colon \begin{cases} Q\bullet\mathbf{L}F & \to F, \\ A & \mapsto F(\pi_A). \end{cases}$$

Definition 3.66. Let $F\colon \mathscr{C} \to \mathscr{E}$ be any functor between two Abelian categories, then F can naturally be lifted to an exact functor $\mathcal{K}^-(\mathscr{C}) \to \mathcal{K}^-(\mathscr{E})$ whose composition with the localization functor of $\mathcal{K}^-(\mathscr{E})$ gives another exact functor $\widetilde{F}\colon \mathcal{K}^-(\mathscr{C}) \to \mathcal{D}^-(\mathscr{E})$. If $\widetilde{F}$ has a left derived functor $\left(\mathbf{L}\widetilde{F}, \eta\right)$, then we say F has a left derived functor $\mathbb{L}F \coloneqq \mathbf{L}\widetilde{F}$; and we define the i^{th}-**left derived functor** of F by

$$\mathbb{L}^i F \coloneqq \iota_C\bullet\mathbb{L}F\bullet\mathrm{H}^i\colon \mathscr{C} \to \mathscr{E}$$

where $\iota_{\mathscr{C}}$ is the natural embedding of $\mathscr{C}$ in $\mathcal{D}^-(\mathscr{C})$ and H^i is the i^{th}-cohomology functor. The natural transformation η induces a natural transformation $\mathbb{L}^0F \to F$.

Example 3.67. Let $\mathscr{C}$ be an Abelian category with enough projectives. Then $\mathcal{K}^-(\mathscr{C})$ has enough $\mathcal{K}$-projectves (cf. Section 3.3). Hence, any exact functor $\mathcal{K}^-(\mathscr{C}) \to \mathfrak{T}$ has a left derived functor.

Let $F\colon \mathscr{C} \to \mathscr{E}$ be a functor to an Abelian category $\mathscr{E}$, then F has as well a left derived functor $\mathbb{L}F\colon \mathcal{D}^-(\mathscr{C}) \to \mathcal{D}^-(\mathscr{E})$. Furthermore, the natural transformation $\mathbb{L}^0F \to F$ is an isomorphism if and only if F is left exact.

Definition 3.68. Let $\mathscr{C}$ be an Abelian category and $\mathfrak{T}$ a triangulated category. Suppose $* \in \{+, -, b, \text{“ ”}\}$ and $Q\colon \mathcal{K}^*(\mathscr{C}) \to \mathcal{D}^*(\mathscr{C})$ is the natural localization functor. Let $F\colon \mathcal{K}^*(\mathscr{C}) \to \mathfrak{T}$ be an exact functor. The **right derived functor** of F is an exact functor

$$\mathbf{R}F\colon \mathcal{D}^*(\mathscr{C}) \to \mathfrak{T},$$

together with a natural transformation

$$\eta\colon F \to Q\bullet\mathbf{R}F$$

which is universal in the sense that if $\mathbb{G}\colon \mathcal{D}^*(\mathscr{C}) \to \mathfrak{T}$ is another exact functor equipped with a natural transformation $\zeta\colon F \to Q\bullet\mathbb{G}$, then there exists a unique natural transformation $\lambda\colon \mathbf{R}F \to \mathbb{G}$ such $\zeta_A = \eta_A\bullet\lambda_{Q(A)}$ for all A in $\mathcal{K}^*(A)$.

Definition 3.69. Let $\mathscr{C}$ be an Abelian category and $\mathfrak{T}$ a triangulated category. Suppose $* \in \{+, -, b, \text{“ ”}\}$ and $Q\colon \mathcal{K}^*(\mathscr{C}) \to \mathcal{D}^*(\mathscr{C})$ is the natural localization functor. Let $F\colon \mathcal{K}^*(\mathscr{C}) \to \mathfrak{T}$

be an exact functor. If $\mathcal{K}^*(\mathscr{C})$ has enough $\mathcal{K}$-injectives, we define the **right derived functor** of F by the pair $(\mathbf{R}F, \eta)$ consisting of the functor

$$\mathbf{R}F\colon \begin{cases} \mathcal{D}^*(\mathscr{C}) & \to \mathfrak{T}, \\ A & \mapsto F(\mathcal{I}_A), \\ r/q\colon A \to B & \mapsto F(\mathcal{I}_q)^{-1} \bullet F(\mathcal{I}_r)\colon F(\mathcal{I}_A) \to F(\mathcal{I}_B) \end{cases}$$

and the natural transformation

$$\eta\colon \begin{cases} F & \to Q \bullet \mathbf{R}F, \\ A & \mapsto F(\iota_A). \end{cases}$$

Definition 3.70. Let $F\colon \mathscr{C} \to \mathscr{E}$ be any functor between two Abelian categories, then F can naturally be lifted to an exact functor $\mathcal{K}^+(\mathscr{C}) \to \mathcal{K}^+(\mathscr{E})$, whose composition with the localization functor of $\mathcal{K}^+(\mathscr{E})$ gives another exact functor $\widetilde{F}\colon \mathcal{K}^+(\mathscr{C}) \to \mathcal{D}^+(\mathscr{E})$. If $\widetilde{F}$ has a right derived functor $\left(\mathbf{R}\widetilde{F}, \eta\right)$, then we say F has a right derived functor $\mathbb{R}F \coloneqq \mathbf{R}\widetilde{F}$; and we define the i^{th}**-right derived functor** of F by

$$\mathbb{R}^i F \coloneqq \iota_C \bullet \mathbb{R}F \bullet \mathrm{H}^i\colon \mathscr{C} \to \mathscr{E}$$

where ι_C is the natural embedding of $\mathscr{C}$ in $\mathcal{D}^+(\mathscr{C})$ and H^i is the i^{th}-cohomology functor. The natural transformation η induces a natural transformation $F \to \mathbb{R}^0 F$.

Example 3.71. Let $\mathscr{C}$ be an Abelian category with enough injectives. Then $\mathcal{K}^+(\mathscr{C})$ has enough $\mathcal{K}$-injectives (cf. Section 3.3). Hence, any exact functor $\mathcal{K}^+(\mathscr{C}) \to \mathfrak{T}$ has a right derived functor.

Let $F\colon \mathscr{C} \to \mathscr{E}$ be a functor to an Abelian category $\mathscr{E}$, then F has as well a right derived functor $\mathbb{R}F\colon \mathcal{D}^+(\mathscr{C}) \to \mathcal{D}^+(\mathscr{E})$. Furthermore, the induced natural transformation $F \to \mathbb{R}^0 F$ is an isomorphism if and only if F is right exact.

CHAPTER 4

Homomorphism Structures

We have already seen in Corollary 3.26 that solving two-sided inhomogeneous linear system of equations in a category $\mathscr{C}$ is necessary to equip its bounded homotopy category $\mathcal{K}^b(\mathscr{C})$ with decidable equality of morphisms: Precisely, verifying the equality of two morphisms $\alpha, \beta\colon A \to B$ in $\mathcal{K}^b(\mathscr{C})$ amounts to verifying the solvability of the system

$$\{\partial_A^i \cdot \chi^{i+1} + \chi^i \cdot \partial_B^{i-1} = \alpha^i - \beta^i \mid i \in \mathrm{Supp}_A\},$$

for the given differentials $\partial_A^i\colon A^i \to A^{i+1}, \partial_B^i\colon B^i \to B^{i+1}$ and unknown morphisms $\chi^i\colon A^i \to B^{i-1}$ for $i \in \mathrm{Supp}_A$. A particular solution $(\chi^i)_{i\in\mathrm{Supp}_A}$ gives us a *chain homotopy* witnessing the equality "$\alpha = \beta$" in $\mathcal{K}^b(\mathscr{C})$.

We will see later in Definition 5.3 that two-sided inhomogeneous linear system of equations are necessary to render a triangulated category $(\mathfrak{T}, \Sigma, \triangle)$ computable: Precisely, verifying the exactness of a triangle

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$$

over $\mathfrak{T}$ amounts to (1) finding a particular solution of the system

$$\iota \cdot \chi = \iota(\alpha), \qquad \chi \cdot \pi(\alpha) = \pi,$$

where $\iota(\alpha)$ and $\pi(\alpha)$ are taken from *the* standard exact triangle associated to α:

$$A \xrightarrow{\alpha} B \xrightarrow{\iota(\alpha)} \mathrm{Cone}(\alpha) \xrightarrow{\pi(\alpha)} \Sigma(A)$$

(cf. Definition 5.1) and $\chi\colon C \to \mathrm{Cone}(\alpha)$ is an unknown morphism; and then (2) verifying that this particular solution χ is an isomorphism in $\mathfrak{T}$. Checking whether the particular solution χ is an isomorphism *also* amounts to verifying the solvability of the system

$$\chi \cdot \xi = \mathrm{id}_C, \qquad \xi \cdot \chi = \mathrm{id}_{\mathrm{Cone}(\alpha)}$$

for an unknown morphism $\xi\colon \mathrm{Cone}(\alpha) \to C$.

Solving two-sided linear systems is very usefull in functor categories: Let k be a commutative ring and $\mathscr{A}$ a k-linear finitely presented category defined by a quiver $\mathfrak{q}$ subject to a set of k-relations ρ. Let $[\mathscr{A}, \mathscr{E}]$ be the category of k-linear functors from $\mathscr{A}$ into a category $\mathscr{E}$ and consider two objects F and G in $[\mathscr{A}, \mathscr{E}]$. Then computing the external $\mathrm{Hom}_{[\mathscr{A},\mathscr{E}]}(F, G)$ amounts to finding the solution set of a system of two-sided inhomogeneous linear equations in $\mathscr{E}$ (a linear equation for each arrow in the quiver $\mathfrak{q}$). If $\mathscr{E}$ is the category R-**rows** for some commutative ring R, then we can easily use the classical Kronecker product trick in solving matrix equations [**LT85**]. However, to cover as many cases as possible, we will have to use a categorical approach that is context-independent.

Which categorical constructions can help? For a locally small category $\mathscr{C}$, the *external* Hom bifunctor is defined as follows:

$$\mathrm{Hom}_{\mathscr{C}}(-,-)\colon \begin{cases} \mathscr{C}^{\mathrm{op}} \times \mathscr{C} & \to \mathbf{Set}, \\ (A,D) & \mapsto \mathrm{Hom}_{\mathscr{C}}(A,D), \\ (\alpha^{\mathrm{op}},\beta)\colon (B,C) \to (A,D) & \mapsto \mathrm{Hom}_{\mathscr{C}}(\alpha,\beta)\colon \begin{cases} \mathrm{Hom}_{\mathscr{C}}(B,C) & \to \mathrm{Hom}_{\mathscr{C}}(A,D), \\ \chi & \mapsto \alpha \cdot \chi \cdot \beta. \end{cases} \end{cases}$$

Choosing an element in the set $\mathrm{Hom}_{\mathscr{C}}(A,D)$ is similar to choosing a map in **Set** from some singleton set, say $\{*\}$, to $\mathrm{Hom}_{\mathscr{C}}(A,D)$. This simple idea allows us to define a natural isomorphism

$$\nu\colon \begin{cases} \mathrm{Hom}_{\mathscr{C}}(-,-) & \to \mathrm{Hom}_{\mathbf{Set}}(\{*\},\mathrm{Hom}_{\mathscr{C}}(-,-)), \\ (A,D) & \mapsto \nu_{A,D}\colon \begin{cases} \mathrm{Hom}_{\mathscr{C}}(A,D) & \to \mathrm{Hom}_{\mathbf{Set}}(\{*\},\mathrm{Hom}_{\mathscr{C}}(A,D)), \\ \gamma & \mapsto \nu_{A,D}(\gamma)\colon \begin{cases} \{*\} & \to \mathrm{Hom}_{\mathscr{C}}(A,D), \\ * & \mapsto \gamma \end{cases} \end{cases} \end{cases}$$

where the naturality of ν translates to the equality[1]

$$\nu_{A,D}(\alpha \cdot \chi \cdot \beta) = \nu_{B,C}(\chi) \cdot \mathrm{Hom}_{\mathscr{C}}(\alpha,\beta).$$

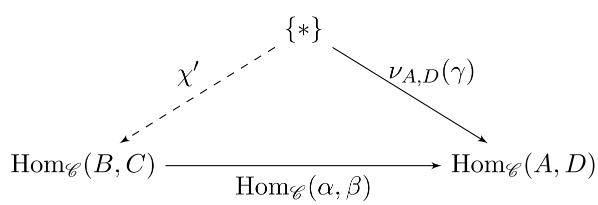
for all triples $A \xrightarrow{\alpha} B \xrightarrow{\chi} C \xrightarrow{\beta} D$ of morphisms in $\mathscr{C}$. This equality enables us to translate any two-sided equation in $\mathscr{C}$

$$\alpha \cdot \chi \cdot \beta = \gamma$$

for given morphisms $\alpha\colon A \to B$, $\beta\colon C \to D$ and $\gamma\colon A \to D$ and an unknown morphism $\chi\colon B \to C$ into a left-sided equation in **Set**

$$\chi' \cdot \mathrm{Hom}_{\mathscr{C}}(\alpha,\beta) = \nu_{A,D}(\gamma)$$

where a solution χ can be recovered from χ' as $\chi = \nu_{B,C}^{-1}(\chi')$.

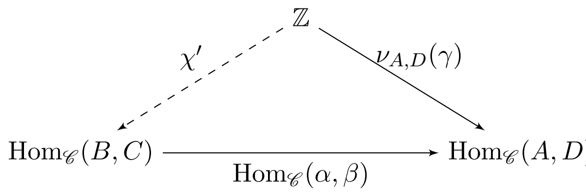


If $\mathscr{C}$ is preadditive, the range of the external Hom bifunctor can be taken to be the category **Ab** of abelian groups. In this case, we get a new natural isomorphism

$$\nu\colon \begin{cases} \mathrm{Hom}_{\mathscr{C}}(-,-) & \to \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z},\mathrm{Hom}_{\mathscr{C}}(-,-)), \\ (A,D) & \mapsto \nu_{A,D}\colon \begin{cases} \mathrm{Hom}_{\mathscr{C}}(A,D) & \to \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z},\mathrm{Hom}_{\mathscr{C}}(A,D)), \\ \gamma & \mapsto \nu_{A,D}(\gamma)\colon \begin{cases} \mathbb{Z} & \to \mathrm{Hom}_{\mathscr{C}}(A,D), \\ 1 & \mapsto \gamma \end{cases} \end{cases} \end{cases}$$

[1]This equality holds since $\nu_{A,D}(\alpha \cdot \chi \cdot \beta)(*) = \alpha \cdot \chi \cdot \beta = \mathrm{Hom}_{\mathscr{C}}(\alpha,\beta)(\chi) = \mathrm{Hom}_{\mathscr{C}}(\alpha,\beta)(\nu_{B,C}(\chi)(*))$.

which also enables us to translate any two-sided equation $\alpha \cdot \chi \cdot \beta = \gamma$ in $\mathscr{C}$ into a left-sided equation $\chi' \cdot \mathrm{Hom}_{\mathscr{C}}(\alpha, \beta) = \nu_{A,D}(\gamma)$ in **Ab**.



Suppose now that $\mathscr{C}$ is a closed symmetric monoidal category whose tensor unit is 1, tensor bifunctor is $\otimes \colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and internal Hom bifunctor is $\underline{\mathrm{Hom}} \colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ (cf. [**Pos17**, Section 3.2]). In this case, we can also construct a natural isomorphism

$$\nu \colon \begin{cases} \mathrm{Hom}_{\mathscr{C}}(-,-) & \to \mathrm{Hom}_{\mathscr{C}}(1, \underline{\mathrm{Hom}}(-,-)), \\ (A, D) & \mapsto \nu_{A,D} \colon \begin{cases} \mathrm{Hom}_{\mathscr{C}}(A, D) & \to \mathrm{Hom}_{\mathscr{C}}(1, \underline{\mathrm{Hom}}(A, D)), \\ \gamma & \mapsto \delta_{A,D}(\lambda_A \cdot \gamma) \end{cases} \end{cases}$$

where

$$\lambda_A \colon 1 \otimes A \xrightarrow{\sim} A$$

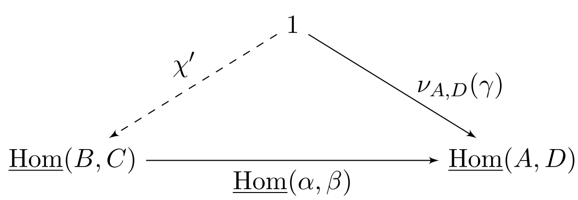
is the left unitor of A and

$$\delta_{A,D} \colon \mathrm{Hom}_{\mathscr{C}}(1 \otimes A, D) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(1, \underline{\mathrm{Hom}}(A, D))$$

is the isomorphism induced by the adjunction $(-) \otimes A \dashv \underline{\mathrm{Hom}}(A, -)$. Again, the naturality of ν translates to the equality

$$\nu_{A,D}(\alpha \cdot \chi \cdot \beta) = \nu_{B,C}(\chi) \cdot \underline{\mathrm{Hom}}(\alpha, \beta)$$

for all triples $A \xrightarrow{\alpha} B \xrightarrow{\chi} C \xrightarrow{\beta} D$ in $\mathscr{C}$. This means that a closed symmetric monoidal structure enables us to translate two-sided inhomogeneous equation $\alpha \cdot \chi \cdot \beta = \gamma$ in $\mathscr{C}$ into a left-sided equation $\chi' \cdot \underline{\mathrm{Hom}}(\alpha, \beta) = \nu_{A,D}(\gamma)$ in $\mathscr{C}$.



From a computer algebra viewpoint, this says that if the axioms of closed symmetric monoidal categories are realized in $\mathscr{C}$ by algorithms and $\mathscr{C}$ has decidable lifts (cf. Definition A.8), then we can automatically derive an algorithm to solve two-sided inhomogeneous equations $\alpha \cdot \chi \cdot \beta = \gamma$ in $\mathscr{C}$.

The concept of a $\mathscr{D}$-homomorphism structure on a category $\mathscr{C}$ was first formulated by Posur in his constructive approach to Freyd categories [**Pos21a**]. This concept requires far less prerequisites than those discussed previously while retaining the ability to transform two-sided equations in a category $\mathscr{C}$ to left-sided equations in $\mathscr{D}$.

A $\mathscr{D}$-homomorphism structure $(\mathbb{1}, H(-,-), \nu)$ on a category $\mathscr{C}$ consists of an object $\mathbb{1}$ in $\mathscr{D}$, a bifunctor $H \colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{D}$ and a natural isomorphism $\nu \colon \mathrm{Hom}_{\mathscr{C}}(-,-) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, H(-,-))$ (cf. Definition 4.2).

The main advantage of the homomorphism structure is that it allows us to convert a two-sided equation

$$\alpha \cdot \chi \cdot \beta = \gamma : A \to D$$

for with given morphisms α, β, γ and an unknown morphism $\chi : B \to C$ in $\mathscr{C}$ into a *left-sided* equation or a lifting problem

$$\chi' \cdot H(\alpha, \beta) = \nu_{A,D}(\gamma)$$

in $\mathscr{D}$, where χ can be recovered as $\chi = \nu_{B,C}^{-1}(\chi')$.

This can be extended in the additive case to solve two-sided inhomogeneous linear *systems* (cf. Theorem 4.17). Solving two-sided inhomogeneous linear systems is indispensable for almost all constructive approaches of thesis, for instance:

- In order to decided the exactness of a given triangle in a triangulated category $\mathfrak{T}$ (and in the affirmative case to compute an isomorphism witnessing the exactness), we need to solve a two-sided inhomogeneous linear system of equations in $\mathfrak{T}$. So once $\mathfrak{T}$ is equipped with a $\mathscr{D}$-homomorphism structure we can reduce this two-sided linear system to a left-sided equation in $\mathscr{D}$ (cf. Lemma 5.4).
- Let $\mathscr{P}$ be an additive category with weak kernels and let $\mathcal{A}(\mathscr{P})$ be the Abelian Freyd category of $\mathscr{P}$. The class $\mathcal{L}$ of all projective objects in $\mathcal{A}(\mathscr{P})$ defines a class of lifting objects in $\mathcal{A}(\mathscr{P})$. Deciding the equality of morphisms in the stable category $\mathcal{A}(\mathscr{P})/\mathcal{L}$ (and the affirmative case compute the so-called lift morphism witnessing the equality) requires the ability to compute lifts in $\mathcal{A}(\mathscr{P})$. It is shown in [**Pos21a**, Section 6] that a lift (i.e., a left-sided equation) in the Freyd category $\mathcal{A}(\mathscr{P})$ in turn requires solving a two-sided inhomogeneous linear system in the underlying category $\mathscr{P}$. So once $\mathscr{P}$ is equipped with a $\mathscr{D}$-homomorphism structure we can again reduce this two-sided inhomogeneous linear system to a left-sided equation in $\mathscr{D}$ and hence compute the desired lift in $\mathcal{A}(\mathscr{P})$, and finally decide the equality of morphisms in the stable category $\mathcal{A}(\mathscr{P})/\mathcal{L}$. (See Remark 2.56 and Examples 2.60 and 5.37).
- In order to decide the equality of morphisms in the bounded homotopy category $\mathcal{K}^b(\mathscr{C})$ (and in the affirmative case to compute a chain homotopy witnessing the equality), we need to be able to solve two-sided inhomogeneous linear systems in the underlying additive category $\mathscr{C}$. So once $\mathscr{C}$ is equipped with a $\mathscr{D}$-homomorphism structure, we can again reduce this two-sided inhomogeneous linear system to a left-sided equation in $\mathscr{D}$ and hence compute the desired chain homotopy witness (cf. Corollary 3.26). Furthermore, the computation of chain-homotopies witnessing the equality of morphisms in $\mathcal{K}^b(\mathscr{C})$ is essential for
  - turning a bounded homotopy category into a computable triangulated category (cf. Section 5.2),
  - computing Postnikov systems and their associated convolutions and finally the *convolution functor* (cf. Algorithms 4 and 5). The convolution functor is left adjoint functor in the adjoint pair of exact equivalences induced by strong exceptional sequences in bounded homotopy categories.

Remark 4.1. For some applications we need the homomorphism structure to be *equivalent* to the external Hom bifunctor. For example to

- compute the Yoneda embedding of some finitely presented k-algebroid $\mathscr{A}$ into the functor category $\mathscr{A}\text{-}\mathbf{mod} := [\mathscr{A}^{\mathrm{op}}, k\text{-}\mathbf{mat}]$ (cf. Corollary 2.90).

- compute the Ext-groups in the context of bounded derived categories, which is defined by
$$\mathrm{Ext}^n_{\mathscr{C}}(A, B) \coloneqq \mathrm{Hom}_{\mathcal{D}^b(\mathscr{C})}(A, \Sigma^n(B))$$
(cf. Appendix C).
- compute the abstraction k-algebroid of a strong exceptional sequences $\mathscr{E}$ in a triangulated category $\mathfrak{T}$ (cf. Section 6.2).
- compute the functor
$$\mathrm{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, -) \colon \mathfrak{T} \to \mathbf{A}_{\mathscr{E}}\text{-}\mathbf{mod}$$
where $\mathfrak{T}$ is a k-linear triangulated category, $\mathscr{E}$ is a strong exceptional sequence in $\mathfrak{T}$, $T_{\mathscr{E}}$ is the tilting object associated to $\mathscr{E}$, $\mathbf{A}_{\mathscr{E}}$ is the abstraction k-algebroid of $\mathscr{E}$ and $\mathbf{A}_{\mathscr{E}}$-**mod** is the category of k-linear functors from $\mathbf{A}_{\mathscr{E}}^{\mathrm{op}}$ to the category k-**mat** of matrices over k (Remark 6.36). This functor is essential for computing the *replacement functor*, the latter being the right adjoint in adjoint pair of exact equivalences induced by strong exceptional sequences in bounded homotopy categories (cf. Section 6.4).

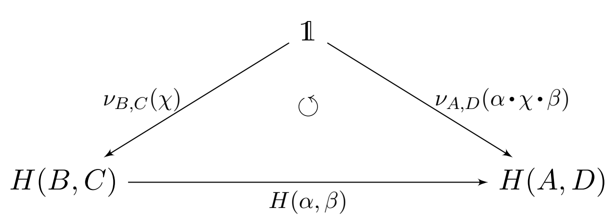
It was, therefore, of fundamental importance to investigate ways to enhance the category constructors so that they automatically lift the homomorphism structures from the input categories to the output category. In the first section, we summarize the key characteristics of homomorphism structures and demonstrate them with examples. The original treatment can be found in [**Pos21a**] or [**Pos21b**]. The second section is devoted to the construction of new homomorphism structures from existing ones.

4.1. Basics

The following is the formal definition of a $\mathscr{D}$-homomorphism structure of a category $\mathscr{C}$.

Definition 4.2. Let $\mathscr{C}$ and $\mathscr{D}$. A $\mathscr{D}$-**homomorphism structure** for $\mathscr{C}$ consists of the following data:

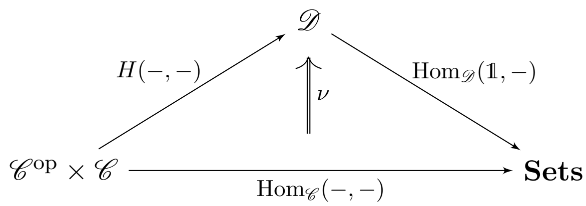
(1) An object $\mathbb{1} \in \mathscr{D}$ called the **distinguished object**.
(2) A bifunctor $H(-,-)\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{D}$. If $\mathscr{C}$ is an Ab-category then we require $\mathscr{D}$ to be an Ab-category as well and H to be bilinear i.e., it acts linearly on morphisms in each component.
(3) An isomorphism $\nu_{B,C}\colon \mathrm{Hom}_{\mathscr{C}}(B, C) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, H(B, C))$ for each pair of objects $B, C \in \mathscr{C}$ satisfying
$$\nu_{A,D}(\alpha \cdot \chi \cdot \beta) = \nu_{B,C}(\chi) \cdot H(\alpha, \beta)$$
for all composable triples of morphisms $A \xrightarrow{\alpha} B \xrightarrow{\varphi} C \xrightarrow{\beta} D$. In other words, the following diagram commutes:



In the preadditive case we require $\nu_{B,C}$ to be an isomorphism of Abelian groups.

Remark 4.3. The third axiom is equivalent to the existence of a natural isomorphism

$$\nu\colon \operatorname{Hom}_{\mathscr{C}}(-,-) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{D}}(\mathbb{1}, H(-,-)).$$



In the preadditive case we replace **Sets** by **Ab**. The existence of the natural isomorphism ν means that for any two morphisms $\alpha\colon A \to B$ and $\beta\colon C \to D$ we have the following commutative diagram:

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathscr{C}}(B,C) & \xrightarrow[\sim]{\nu_{B,C}} & \operatorname{Hom}_{\mathscr{D}}(\mathbb{1}, H(B,C)) \\
\Big\downarrow{\scriptstyle \operatorname{Hom}_{\mathscr{C}}(\alpha,\beta)} & \circlearrowleft & \Big\downarrow{\scriptstyle \operatorname{Hom}_{\mathscr{D}}(\mathbb{1}, H(\alpha,\beta))} \\
\operatorname{Hom}_{\mathscr{C}}(A,D) & \xrightarrow[\nu_{A,D}]{\sim} & \operatorname{Hom}_{\mathscr{D}}(\mathbb{1}, H(A,D))
\end{array}$$

which translates for any morphism $\chi\colon B \to C$ to the equality

$$\begin{aligned}
\nu_{A,D}(\alpha \cdot \chi \cdot \beta) &= \nu_{A,D}(\operatorname{Hom}_{\mathscr{C}}(\alpha,\beta)(\chi)) \\
&= (\operatorname{Hom}_{\mathscr{C}}(\alpha,\beta) \cdot \nu_{A,D})(\chi) \\
&= (\nu_{B,C} \cdot \operatorname{Hom}_{\mathscr{D}}(\mathbb{1}, H(\alpha,\beta)))(\chi) \\
&= \operatorname{Hom}_{\mathscr{D}}(\mathbb{1}, H(\alpha,\beta))(\nu_{B,C}(\chi)) \\
&= \nu_{B,C}(\chi) \cdot H(\alpha,\beta);
\end{aligned}$$

i.e., to the third axiom.

Having a homomorphism structure enables us to reduce verifying the equality of morphisms in $\mathscr{C}$ to verifying equality of morphisms in $\mathscr{D}$.

Corollary 4.4. *Suppose $\mathscr{C}$ is an Ab-category equipped with a $\mathscr{D}$-homomorphism structure. If $\mathscr{D}$ has decidable equality of morphisms, then so does $\mathscr{C}$.*

Proof. Two morphisms $\varphi, \psi\colon B \to C$ in $\mathscr{C}$ are equal if and only if $\nu_{B,C}(\varphi) = \nu_{B,C}(\psi)$. □

Sometimes we may want to switch the range category of a $\mathscr{D}$-homomorphism structure to another category, say $\mathscr{E}$. For example, when $\mathscr{E}$ provides more computational features than $\mathscr{D}$. The existence of a fully faithful functor $F\colon \mathscr{D} \to \mathscr{E}$ simplifies such transition. For instance, $\mathscr{E}$ could be the Freyd category $\mathcal{A}(\mathscr{D})$.

Lemma 4.5. *Let $\mathscr{C}$ be a preadditive category equipped with a $\mathscr{D}$-homomorphism structure $(\mathbb{1}, H(-,-), \nu)$. If $F\colon \mathscr{D} \to \mathscr{E}$ is a fully faithful functor, then $\mathscr{C}$ can be equipped with an $\mathscr{E}$-homomorphism structure $\left(F(\mathbb{1}), \widetilde{H}(-,-), \widetilde{\nu}\right)$ where $\widetilde{H} := H \cdot F$ and $\widetilde{\nu}$ is the vertical composition of the natural transformations*

$$\operatorname{Hom}_{\mathscr{C}}(-,-) \xrightarrow{\nu} H(-,-) \cdot \operatorname{Hom}_{\mathscr{D}}(\mathbb{1},-) \xrightarrow{\epsilon} \widetilde{H}(-,-) \cdot \operatorname{Hom}_{\mathscr{E}}(F(\mathbb{1}),-)$$

where ϵ maps an object (B,C) in $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ to the morphism

$$\epsilon_{B,C} \colon \begin{cases} \mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, H(B,C)) & \to \mathrm{Hom}_{\mathscr{E}}\Big(F(\mathbb{1}), \widetilde{H}(B,C)\Big), \\ \ell & \mapsto F(\ell). \end{cases}$$

Proof. It is sufficient to prove that ϵ is a natural isomorphism. Let $(\alpha^{\mathrm{op}}, \beta) \colon (B,C) \to (A,D)$ be a morphism in $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$. The assignment ϵ defines a natural transformation because for every $\ell \colon \mathbb{1} \to H(B,C)$, we have

$$\epsilon_{B,C}(\ell) \cdot \widetilde{H}(\alpha,\beta) = F(\ell) \cdot F(H(\alpha,\beta)) = F(\ell \cdot H(\alpha,\beta)) = \epsilon_{A,D}(\ell \cdot H(\alpha,\beta)).$$

The morphism $\epsilon_{B,C}$ is an isomorphism because F is fully faithful; hence ϵ is indeed a natural isomorphism, and consequently so is the vertical composition $\widetilde{\nu} := \nu \cdot \epsilon$. ☐

The following can be found in [**Pos21a**, Example 6.5].

Example 4.6. Let R be a commutative ring. Then R-**rows** is equipped with an R-**rows**-homomorphism structure. The associated R-**rows**-homomorphism structure $(R^{1\times 1}, H(-,-), \nu)$ is given by

$$H(-,-) \colon \begin{cases} R\text{-}\mathbf{rows}^{\mathrm{op}} \times R\text{-}\mathbf{rows} & \to R\text{-}\mathbf{rows}, \\ \Big(R^{1\times b}, R^{1\times c}\Big) & \mapsto R^{1\times bc}, \\ (\alpha^{\mathrm{op}}, \beta) \colon \Big(R^{1\times b}, R^{1\times c}\Big) \to \Big(R^{1\times a}, R^{1\times d}\Big) & \mapsto \alpha^{\mathrm{tr}} \otimes \beta \colon R^{1\times bc} \xrightarrow{(\alpha_{j,i}\cdot\beta)_{i,j}} R^{1\times ad} \end{cases}$$

and

$$\nu \colon \begin{cases} \mathrm{Hom}_{R\text{-}\mathbf{rows}}(-,-) & \to \mathrm{Hom}_{R\text{-}\mathbf{rows}}(R^{1\times 1}, H(-,-)), \\ \Big(R^{1\times b}, R^{1\times c}\Big) & \mapsto \nu_{R^{1\times b}, R^{1\times c}} \end{cases}$$

where $\nu_{R^{1\times b}, R^{1\times c}}$ is the assignment

$$\nu_{R^{1\times b}, R^{1\times c}} \colon \begin{cases} \mathrm{Hom}_{R\text{-}\mathbf{rows}}\Big(R^{1\times b}, R^{1\times c}\Big) & \to \mathrm{Hom}_{R\text{-}\mathbf{rows}}\Big(R^{1\times 1}, R^{1\times bc}\Big), \\ \varphi & \mapsto \mathbf{vec}(\varphi) \end{cases}$$

and $\mathbf{vec}(\varphi)$ is the *vectorization* of φ, i.e., the row defined by the concatenation of all rows of φ. The induced equality

$$\mathbf{vec}(\alpha \cdot \varphi \cdot \beta) = \mathbf{vec}(\varphi) \cdot \Big(\alpha^{\mathrm{tr}} \otimes \beta\Big)$$

is the Kronecker product trick to solve matrix equations (cf. [**LT85**]).

The following three examples can be found in [**Pos21a**, Example 6.7], [**Pos21b**, Construction 1.27] and [**Pos21a**, Theorem 6.14].

Example 4.7. Let $\mathscr{C}$ be an additive closed symmetric monoidal category. Then the tensor unit $1 \in \mathscr{C}$ and the internal Hom-functor define a $\mathscr{C}$-homomorphism structure for $\mathscr{C}$. If the axioms of a closed symmetric monoidal category are realized in $\mathscr{C}$ by algorithms, then $\mathscr{C}$ is equipped with a $\mathscr{C}$-homomorphism structure.

Example 4.8. Let $\mathscr{C}$ be a preadditive category and $\mathscr{D}$ an additive category. Then any $\mathscr{D}$-homomorphism structure of $\mathscr{C}$ can be lifted to a $\mathscr{D}$-homomorphism structure of $\mathscr{C}^{\oplus}$. In particular, if $\mathscr{C}$ is equipped with a $\mathscr{D}$-homomorphism structure, then so is $\mathscr{C}^{\oplus}$.

Example 4.9. Let $\mathscr{C}$ be an additive category equipped with a $\mathscr{D}$-homomorphism structure $(\mathbb{1}, H(-,-), \nu)$. If $\mathscr{D}$ is Abelian and if $\mathbb{1}$ is a projective object, then $\mathcal{A}(\mathscr{C})$ can be equipped with a $\mathscr{D}$-homomorphism structure.

Definition 4.10. Let $\mathscr{C}$ be equipped with a $\mathscr{D}$-homomorphism structure $(\mathbb{1}, H(-,-), \nu)$. We say, the $\mathscr{D}$-homomorphism structure of $\mathscr{C}$ is **equivalent** to the external Hom functor $\mathrm{Hom}_{\mathscr{C}}(-,-)$ if the functor $\mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, -)$ is faithful and preserves all finite limits and colimits.

The following lemma enables us to derive a (k-**mat**)-homomorphism structure for Hom-finite k-linear categories over a field k.

Lemma 4.11. *Let k be a field, $\mathscr{C}$ a k-linear category and k-**mat** the category of matrices over k introduced in Example 2.16. Suppose we have*

(1) an algorithm which for a given pair of objects B, C in $\mathscr{C}$, computes an ordered basis $\mathcal{B}(\mathrm{Hom}_{\mathscr{C}}(B,C))$ of $\mathrm{Hom}_{\mathscr{C}}(B,C)$,

(2) an algorithm which for a given morphism $\varphi\colon B \to C$, computes its k-linear coefficients with respect to $\mathcal{B}(\mathrm{Hom}_{\mathscr{C}}(B,C))$, i.e., the *row $\lambda_\varphi \in k^{1\times \dim_k \mathrm{Hom}_{\mathscr{C}}(B,C)}$ with*

$$\lambda_\varphi \cdot \mathcal{B}(\mathrm{Hom}_{\mathscr{C}}(B,C)) = \varphi.$$

*Then $\mathscr{C}$ can be equipped with a (k-**mat**)-homomorphism structure (which is equivalent to the external Hom).*

Proof. Define $H\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to k\text{-}\mathbf{mat}$ by mapping an object (B, C) to $\dim_k \mathrm{Hom}_{\mathscr{C}}(B,C)$ and a given morphism $(\alpha^{\mathrm{op}}, \beta)\colon (B,C) \to (A,D)$ to the matrix of the k-linear map

$$\mathrm{Hom}_{\mathscr{C}}(\alpha,\beta)\colon \begin{cases} \mathrm{Hom}_{\mathscr{C}}(B,C) & \to \mathrm{Hom}_{\mathscr{C}}(A,D), \\ \quad\varphi & \mapsto \alpha \bullet \varphi \bullet \beta \end{cases}$$

with respect to the bases $\mathcal{B}(\mathrm{Hom}_{\mathscr{C}}(B,C))$ and $\mathcal{B}(\mathrm{Hom}_{\mathscr{C}}(A,D))$. In other words,

$$H(\alpha,\beta) = \begin{pmatrix} \vdots \\ \lambda_{\alpha\bullet b\bullet\beta} \\ \vdots \end{pmatrix}_{b\in\mathcal{B}(B,C)} \in k^{H(B,C)\times H(A,D)}.$$

For each object (B, C) in $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$, we define the bijection

$$\nu_{B,C}\colon \begin{cases} \mathrm{Hom}_{\mathscr{C}}(B,C) & \to \mathrm{Hom}_{k\text{-}\mathbf{mat}}(1, H(B,C)), \\ \quad\varphi & \mapsto \lambda_\varphi. \end{cases}$$

For any triple $A \xrightarrow{\alpha} B \xrightarrow{\varphi} C \xrightarrow{\beta} D$, we have

$$\begin{aligned} \nu_{A,D}(\alpha\bullet\varphi\bullet\beta)\cdot\mathcal{B}(\mathrm{Hom}_{\mathscr{C}}(A,D)) &= \lambda_{\alpha\bullet\varphi\bullet\beta}\cdot\mathcal{B}(\mathrm{Hom}_{\mathscr{C}}(A,D)) \\ &= \alpha\bullet\varphi\bullet\beta \\ &= \mathrm{Hom}_{\mathscr{C}}(\alpha,\beta)(\varphi) \\ &= \lambda_\varphi\cdot H(\alpha,\beta)\cdot\mathcal{B}(\mathrm{Hom}_{\mathscr{C}}(A,D)) \\ &= \nu_{B,C}(\varphi)\cdot H(\alpha,\beta)\cdot\mathcal{B}(\mathrm{Hom}_{\mathscr{C}}(A,D)). \end{aligned}$$

Since $\mathcal{B}(\mathrm{Hom}_{\mathscr{C}}(A,D))$ is a basis, it follows that $\nu_{A,D}(\alpha\bullet\varphi\bullet\beta) = \nu_{B,C}(\varphi)\cdot H(\alpha,\beta)$. Consequently, the assignment $(B,C) \mapsto \nu_{B,C}$ is a natural isomorphism and the triple $(1, H(-,-), \nu)$ defines a (k-**mat**)-homomorphism structure for $\mathscr{C}$. ∎

Remark 4.12. Let k be any field and $\mathscr{C}$ a k-linear category equipped with some (k-**mat**)-homomorphism structure $(1, H(-,-), \nu)$. For a pair of objects B, C in $\mathscr{C}$, applying the isomorphism $\nu_{B,C}^{-1}$ on the elements of the canonical basis of $\mathrm{Hom}_{\mathscr{D}}(1, H(B,C))$ gives a basis of $\mathrm{Hom}_{\mathscr{C}}(B,C)$. In this case the k-linear coefficients of any $\varphi\colon B \to C$ with respect to the this basis are the entries of the row $\nu(\varphi)$.

Example 4.13. Let k be a field and $\mathscr{A} = \mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle$ a finitely presented category defined by a quiver $\mathfrak{q}$ subject to an admissible set of relations ρ. Then $\mathscr{A}$ can be equipped with a (k-**mat**)-homomorphism structure.

Remark 4.14. Let G be an additively written finitely presented Abelian group, $R = \bigoplus_{g\in G} R_g$ a G-graded ring and let $\mathcal{C}\Big(\bigoplus_{g\in G} R_g\Big)$ be the category associated to R introduced in Definition 2.7. If R_0 is a field and for each $g \in G$ the R_0-vector space R_g is finite dimensional, then $\mathcal{C}\Big(\bigoplus_{g\in G} R_g\Big)$ can be equipped with an (R_0-**mat**)-homomorphism structure. The fact that R-**grrows** $\cong \mathcal{C}\Big(\bigoplus_{g\in G} R_g\Big)^{\oplus}$ implies that any homomorphism structure on $\mathcal{C}\Big(\bigoplus_{g\in G} R_g\Big)$ can be lifted to R-**grrows**.

Example 4.15. Let k be a commutative ring and $R = k[x_1, \dots, x_n]$ be a $\mathbb{Z}^t$-graded polynomial ring with $\deg x_1 = m_1, \dots, \deg x_n = m_n$. Then $R = \bigoplus_{m\in\mathbb{Z}^t} R_m$ where $k \subseteq R_0$ and each R_m is a free k-module generated by the monomials of degree m. The monomials $x_1^{s_1} \dots x_n^{s_n}$ of degree m corresponds to the integral solutions[2] of the equation $m_1y_1 + \cdots + m_ny_n = m$ which can be rephrased as $My = m$ with $M \in \mathbb{Z}^{t\times n}$ and $y \in \mathbb{Z}^n$.

The set of real solutions of $My = m$ forms a polyhedron. By the the theory of convex geometry, such a polyhedron can be written as *Minkowski sum* of a polytope and a cone where the cone consists of the solutions of the equation $My = 0$ (see e.g., [**Zie95**] or [**BG09**]). That is, if $My = 0$ has just the trivial solution $0 \in \mathbb{Z}^n$, then $My = m$ has a finite number of integral solutions for every $m \in \mathbb{Z}^t$. In such a case, $R_0 = k$ and R_m is a finite dimensional k-vector space for all $m \in \mathbb{Z}^t$. The same holds for the $\mathbb{Z}^t$ graded exterior algebra $\Lambda = k[e_0, \dots, e_n]$.

Definition 4.16. Let $\mathscr{C}$ be an additive category. A **linear system** $\Big((\alpha_{ij})_{ij}, (\beta_{ij})_{ij}, (\gamma_i)_i\Big)$ in $\mathscr{C}$ with $m \in \mathbb{N}$ equations and $n \in \mathbb{N}$ indeterminates is defined by the following data:

(1) Objects $(A_i)_i, (D_i)_i$ and $(B_j)_j, (C_j)_j$ in $\mathscr{C}$ for $i = 1, \dots, m,\ j = 1, \dots, n$.
(2) Morphisms $(\alpha_{ij}\colon A_i \to B_j)_{ij}$ and $(\beta_{ij}\colon C_j \to D_i)_{ij}$ in $\mathscr{C}$ for $i = 1, \dots, m,\ j = 1, \dots, n$.
(3) Morphisms $(\gamma_i\colon A_i \to D_i)_i$ in $\mathscr{C}$ for $i = 1, \dots, m$.

$$\begin{array}{cccccccc}
A_1 \xrightarrow{\alpha_{11}} B_1 & C_1 \xrightarrow{\beta_{11}} D_1 & \cdots & A_1 \xrightarrow{\alpha_{1n}} B_n & C_n \xrightarrow{\beta_{1n}} D_1 & A_1 \xrightarrow{\gamma_1} D_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A_m \xrightarrow{\alpha_{m1}} B_1 & C_1 \xrightarrow{\beta_{m1}} D_m & \cdots & A_m \xrightarrow{\alpha_{mn}} B_n & C_n \xrightarrow{\beta_{mn}} D_m & A_m \xrightarrow{\gamma_m} D_m.
\end{array}$$

[2] See `MonomialsWithGivenDegree` in [**BGK+21b**].

A **solution** for the system is given by morphisms $(X_j\colon B_j \to C_j)$ for $j = 1, \ldots, n$, such that the equations

$$\begin{array}{ccccc} \alpha_{11} \cdot X_1 \cdot \beta_{11} & + \cdots + & \alpha_{1n} \cdot X_n \cdot \beta_{1n} & = & \gamma_1 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{m1} \cdot X_1 \cdot \beta_{m1} & + \cdots + & \alpha_{mn} \cdot X_n \cdot \beta_{mn} & = & \gamma_m \end{array}$$

hold. We say $\mathscr{C}$ has **decidable linear systems**[3] if we have an algorithm that constructs for a given linear system a solution or disproves its existence.

The following theorem illustrates the use of homomorphism structure in solving linear systems. The following is a replication of [**Pos21a**, Theorem 6.10].

Theorem 4.17. *Let $\mathscr{C}$ be an additive category equipped with $\mathscr{D}$-homomorphism structure $(\mathbb{1}, H(-,-), \nu)$. Given a linear system $\left((\alpha_{ij})_{ij}, (\beta_{ij})_{ij}, (\gamma_i)_i\right)$ with m equations and n indeterminates in $\mathscr{C}$, then there exists a solution for the linear system if and only if there exists a solution in $\mathscr{D}$ to the lift problem:*

$$\begin{array}{ccc} & & \mathbb{1} \\ & \overset{\exists ?}{\swarrow} \quad \circlearrowleft & \big\downarrow {\scriptstyle (\nu_{A_i,D_i}(\gamma_i))_{1i}} \\ \bigoplus_{j=1}^n H(B_j, C_j) & \xrightarrow[(H(\alpha_{ij},\beta_{ij}))_{ji}]{} & \bigoplus_{i=1}^m H(A_i, D_i) \end{array}$$

Proof. Suppose that $(X_j)_j$ is a solution for the linear system, hence

$$\sum_{j=1}^{n} \alpha_{ij} \cdot X_j \cdot \beta_{ij} = \gamma_i \text{ for } i = 1, \ldots, m$$

It follows from the following computation

$$\begin{aligned} \left(\nu_{B_j,C_j}(X_j)\right)_{1j} \cdot (H(\alpha_{ij}, \beta_{ij}))_{ji} &= \left(\sum_{j=1}^{n} \nu_{B_j,C_j}(X_j) \cdot H(\alpha_{ij}, \beta_{ij})\right)_{1i} \\ &= \left(\sum_{j=1}^{n} \nu_{A_i,D_i}(\alpha_{ij} \cdot X_j \cdot \beta_{ij})\right)_{1i} \\ &= \left(\nu_{A_i,D_i}\left(\sum_{j=1}^{n} \alpha_{ij} \cdot X_j \cdot \beta_{ij}\right)\right)_{1i} \\ &= (\nu_{A_i,D_i}(\gamma_i))_{1i}, \end{aligned}$$

that $\ell := \left(\nu_{B_j,C_j}(X_j)\right)_{1j}\colon \mathbb{1} \to \bigoplus_{j=1}^n H(B_j, C_j)$ is a solution to the above lift problem; which proves the "*only if*" part of the theorem.

Suppose now that we are given a lift $\ell = (\ell_j)_{1j}\colon \mathbb{1} \to \bigoplus_{j=1}^n H(B_j, C_j)$ for the above diagram. Define $X_j\colon B_j \to C_j$ by $\nu^{-1}_{B_j,C_j}(\ell_j)$ for $j = 1, \ldots, n$.

[3] See the operation `SolveLinearSystemInAbCategory` in [**GSP22**].

Since all ν_{A_i,D_i} for $i = 1, \dots, n$ are isomorphisms, we conclude by the following computation

$$
\begin{aligned}
(\nu_{A_i,D_i}(\gamma_i))_{1i} &= (\ell_j)_{1j} \bullet (H(\alpha_{ij}, \beta_{ij}))_{ji} \\
&= \left(\sum_{j=1}^{n} \ell_j \bullet H(\alpha_{ij}, \beta_{ij}) \right)_{1i} \\
&= \left(\sum_{j=1}^{n} \nu_{B_j,C_j}(X_j) \bullet H(\alpha_{ij}, \beta_{ij}) \right)_{1i} \\
&= \left(\sum_{j=1}^{n} \nu_{A_i,D_i}(\alpha_{ij} \bullet X_j \bullet \beta_{ij}) \right)_{1i} \\
&= \left(\nu_{A_i,D_i} \left(\sum_{j=1}^{n} \alpha_{ij} \bullet X_j \bullet \beta_{ij} \right) \right)_{1i},
\end{aligned}
$$

that $\gamma_i = \sum_{j=1}^{n} \alpha_{ij} \bullet X_j \bullet \beta_{ij}$ for $i = 1, \dots, m$; which proves the "*if*" part of the theorem. □

4.2. Homomorphism Structure on Functor Categories

Let $\mathfrak{q}$ be a quiver and $\mathscr{A} = k\mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle$ be a the k-linear finitely presented category defined by $\mathfrak{q}$ subject to the set $\rho \subset k\mathcal{F}_{\mathfrak{q}}$. By Theorem 2.67, the category of k-linear functors **mod**-$\mathscr{A}$ is Abelian. Equipping **mod**-$\mathscr{A}$ with a (k-**mat**)-homomorphism structure enables us to solve systems of two-sided inhomogeneous linear equations, which are vital for the decidability of equality of morphisms in $\mathcal{K}^b(\mathbf{mod}\text{-}\mathscr{A})$, as well as for computing chain homotopy of null-homotopic morphisms (cf. Corollary 3.26).

When $\mathfrak{q}$ is acyclic and ρ is admissible, then **mod**-$\mathscr{A}$ is Abelian with enough projective and injective objects and a finite global dimension (cf. Corollary 2.96). In this case, we obtain the equivalences

$$\mathcal{D}^b(\mathbf{mod}\text{-}\mathscr{A}) \cong \mathcal{K}^b(\mathbf{proj}\text{-}\mathscr{A}) \cong \mathcal{K}^b(\mathbf{inj}\text{-}\mathscr{A})$$

which turns $\mathcal{D}^b(\mathbf{mod}\text{-}\mathscr{A})$ into a category with decidable equality of morphisms (cf. Section 3.4).

Theorem 4.18. *Let $\mathfrak{q}$ be a quiver and $\mathscr{A} = k\mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle$ be the k-linear finitely presented category defined by $\mathfrak{q}$ subject to a set of relations ρ. For any category $[\mathscr{A}, \mathscr{E}]$ of k-linear functors, if $\mathscr{E}$ is equipped with a $\mathscr{D}$-homomorphism structure and $\mathscr{D}$ is Abelian, then $[\mathscr{A}, \mathscr{E}]$ can be equipped with a $\mathscr{D}$-homomorphism structure. In particular, the category* **mod**-$\mathscr{A} := [\mathscr{A}, k\text{-}\mathbf{mat}]$ *is equipped with a* (k-**mat**)*-homomorphism structure.*

Proof. We denote the data of the $\mathscr{D}$-homomorphism structure of $\mathscr{E}$ by $(\mathbb{1}, H(-,-), \nu)$. For a pair of objects F and G in $[\mathscr{A}, \mathscr{E}]$ we define the morphism

$$\Psi_{F,G} \colon \bigoplus_{v \in \mathfrak{q}_0} H(F(v), G(v)) \to \bigoplus_{\sigma \in \mathfrak{q}_1} H(F(\mathfrak{s}_\sigma), G(\mathfrak{t}_\sigma))$$

by the matrix

$$\begin{array}{c} \\ \vdots \\ H(F(\mathfrak{s}_\sigma),G(\mathfrak{s}_\sigma)) \\ \vdots \\ H(F(\mathfrak{r}_\sigma),G(\mathfrak{r}_\sigma)) \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccc} \dots & H(F(\mathfrak{s}_\sigma),G(\mathfrak{r}_\sigma)) & \dots \end{array} \\ \begin{pmatrix} * & 0 & * \\ * & -H(F(\mathfrak{s}_\sigma),G(\sigma)) & * \\ * & 0 & * \\ * & H(F(\sigma),G(\mathfrak{r}_\sigma)) & * \\ * & 0 & * \end{pmatrix} \end{array}.$$

For a given pair of morphisms $\eta\colon X \to F$ and $\zeta\colon G \to Y$ we define the morphisms

$$\Theta_{\eta,\zeta} \coloneqq \bigoplus_{v\in\mathfrak{q}_0} H(\eta(v),\zeta(v))\colon \bigoplus_{v\in\mathfrak{q}_0} H(F(v),G(v)) \to \bigoplus_{v\in\mathfrak{q}_0} H(X(v),Y(v))$$

$$\Delta_{\eta,\zeta} \coloneqq \bigoplus_{\sigma\in\mathfrak{q}_1} H(\eta(\mathfrak{s}_\sigma),\zeta(\mathfrak{r}_\sigma))\colon \bigoplus_{\sigma\in\mathfrak{q}_1} H(F(\mathfrak{s}_\sigma),G(\mathfrak{r}_\sigma)) \to \bigoplus_{\sigma\in\mathfrak{q}_1} H(X(\mathfrak{s}_\sigma),Y(\mathfrak{r}_\sigma)).$$

For all $\sigma \in \mathfrak{q}_1$ we have the following two equalities

$$\begin{aligned} H(F(\mathfrak{s}_\sigma),G(\sigma)) \cdot H(\eta(\mathfrak{s}_\sigma),\zeta(\mathfrak{r}_\sigma)) &= H(\eta(\mathfrak{s}_\sigma),G(\sigma)\cdot\zeta(\mathfrak{r}_\sigma)) \\ &= H(\eta(\mathfrak{s}_\sigma),\zeta(\mathfrak{s}_\sigma)\cdot Y(\sigma)) \\ &= H(\eta(\mathfrak{s}_\sigma),\zeta(\mathfrak{s}_\sigma)) \cdot H(X(\mathfrak{s}_\sigma),Y(\sigma)) \end{aligned}$$

and

$$\begin{aligned} H(F(\sigma),G(\mathfrak{r}_\sigma)) \cdot H(\eta(\mathfrak{s}_\sigma),\zeta(\mathfrak{r}_\sigma)) &= H(\eta(\mathfrak{s}_\sigma)\cdot F(\sigma),\zeta(\mathfrak{r}_\sigma)) \\ &= H(X(\sigma)\cdot\eta(\mathfrak{r}_\sigma),\zeta(\mathfrak{r}_\sigma)) \\ &= H(\eta(\mathfrak{r}_\sigma),\zeta(\mathfrak{r}_\sigma)) \cdot H(X(\sigma),Y(\mathfrak{r}_\sigma)); \end{aligned}$$

hence $\Psi_{F,G} \cdot \Delta_{\eta,\zeta} = \Theta_{\eta,\zeta} \cdot \Psi_{X,Y}$. This gives a bifunctor

$$\widehat{H}(-,-)\colon \begin{cases} [\mathscr{A},\mathscr{E}]^{\mathrm{op}} \times [\mathscr{A},\mathscr{E}] & \to \mathscr{D}, \\ (F,G) & \mapsto \ker(\Psi_{F,G}), \\ (\eta^{\mathrm{op}},\zeta)\colon (F,G) \to (X,Y) & \mapsto \textit{the } \text{kernel lift of } \iota_{F,G}\cdot\Theta_{\eta,\zeta} \text{ along } \iota_{X,Y} \end{cases}$$

where $\iota_{F,G}$ and $\iota_{X,Y}$ are the kernel embeddings of $\Psi_{F,G}$ resp. $\Psi_{X,Y}$.

Next, we construct the natural isomorphism

$$\mathrm{Hom}_{[\mathscr{A},\mathscr{E}]}(F,G) \simeq \mathrm{Hom}_{\mathscr{D}}(\mathbb{1},H(F,G)).$$

For a morphism $\varphi\colon F \to G$ we define $\lambda_\varphi\colon \mathbb{1} \to \bigoplus_{v\in\mathfrak{q}_0} H(F(v),G(v))$ by the matrix

$$\begin{array}{c} \\ \mathbb{1} \end{array} \begin{array}{c} \begin{array}{ccc} \dots & H(F(v),G(v)) & \dots \end{array} \\ \begin{pmatrix} \dots & \nu_{F(v),G(v)}(\varphi(v)) & \dots \end{pmatrix} \end{array}.$$

For all $\sigma \in \mathfrak{q}_1$ we have

$$\begin{aligned}
-\nu_{F(\mathfrak{s}_\sigma),G(\mathfrak{s}_\sigma)}(\varphi(\mathfrak{s}_\sigma)) \cdot H(F(\mathfrak{s}_\sigma), G(\sigma)) + \nu_{F(\mathfrak{r}_\sigma),G(\mathfrak{r}_\sigma)}(\varphi(\mathfrak{r}_\sigma)) \cdot H(F(\sigma), G(\mathfrak{r}_\sigma)) \\
&= -\nu_{F(\mathfrak{s}_\sigma),G(\mathfrak{r}_\sigma)}(\varphi(\mathfrak{s}_\sigma) \cdot G(\sigma)) + \nu_{F(\mathfrak{s}_\sigma),G(\mathfrak{r}_\sigma)}(F(\sigma) \cdot \varphi(\mathfrak{r}_\sigma)) \\
&= \nu_{F(\mathfrak{s}_\sigma),G(\mathfrak{r}_\sigma)}(-\varphi(\mathfrak{s}_\sigma) \cdot G(\sigma) + F(\sigma) \cdot \varphi(\mathfrak{r}_\sigma)) \\
&= \nu_{F(\mathfrak{s}_\sigma),G(\mathfrak{r}_\sigma)}(0) \\
&= 0;
\end{aligned}$$

hence $\lambda_\varphi \cdot \Psi_{F,G} = 0$. This means for each object (F, G) in $[\mathscr{A}, \mathscr{E}]^{\mathrm{op}} \times [\mathscr{A}, \mathscr{E}]$ we can define the map

$$\widehat{\nu}_{F,G} \colon \begin{cases} \mathrm{Hom}_{[\mathscr{A},\mathscr{E}]}(F, G) & \to \mathrm{Hom}_{\mathscr{D}}\Big(\mathbb{1}, \widehat{H}(F, G)\Big), \\ \qquad \varphi & \mapsto \textit{the} \text{ kernel lift of } \lambda_\varphi \text{ along } \iota_{F,G}, \end{cases}$$

i.e., $\widehat{\nu}_{F,G}(\varphi)$ is the unique morphism for which $\widehat{\nu}_{F,G}(\varphi) \cdot \iota_{F,G} = \lambda_\varphi$. The uniqueness of $\widehat{\nu}_{F,G}(\varphi)$ is justified by the universal property of kernels, and it implies that $\widehat{\nu}_{F,G}$ is injective.

We still need to show that $\widehat{\nu}_{F,G}$ is surjective. For a given morphism $\ell \colon \mathbb{1} \to \widehat{H}(F, G)$, the composition $\ell \cdot \iota_{F,G}$ is uniquely determined by a matrix

$$\begin{array}{cccc} & \dots & H(F(v), G(v)) & \dots \\ \mathbb{1} & (\dots & \ell_v & \dots) \end{array}.$$

For every $v \in \mathscr{A}$, we set $\varphi_{\ell,v} := \nu^{-1}_{F(v),G(v)}(\ell_v) \colon F(v) \to G(v)$. Since $\iota_{F,G} \cdot \Theta_{F,G} = 0$, we have the equality

$$-\ell_{\mathfrak{s}_\sigma} \cdot H(F(\mathfrak{s}_\sigma), G(\sigma)) + \ell_{\mathfrak{r}_\sigma} \cdot H(F(\sigma), G(\mathfrak{r}_\sigma)) = 0$$

for every $\sigma \in \mathfrak{q}_1$, i.e.,

$$-\nu_{F(\mathfrak{s}_\sigma),G(\mathfrak{s}_\sigma)}(\varphi_{\ell,\mathfrak{s}_\sigma}) \cdot H(F(\mathfrak{s}_\sigma), G(\sigma)) + \nu_{F(\mathfrak{r}_\sigma),G(\mathfrak{r}_\sigma)}(\varphi_{\ell,\mathfrak{r}_\sigma}) \cdot H(F(\sigma), G(\mathfrak{r}_\sigma)) = 0$$

which, due to the naturality of ν, is equivalent to

$$\nu_{F(\mathfrak{s}_\sigma),G(\mathfrak{r}_\sigma)}(-\varphi_{\ell,\mathfrak{s}_\sigma} \cdot G(\sigma) + F(\sigma) \cdot \varphi_{\ell,\mathfrak{r}_\sigma}) = 0.$$

Since ν is a natural isomorphism, we have $-\varphi_{\ell,\mathfrak{s}_\sigma} \cdot G(\sigma) + F(\sigma) \cdot \varphi_{\ell,\mathfrak{r}_\sigma} = 0$. In particular, the assignment

$$\varphi_\ell \colon \begin{cases} F & \to G, \\ v & \mapsto \varphi_{\ell,v} \end{cases}$$

defines a morphism in $[\mathscr{A}, \mathscr{E}]$. By construction that $\lambda_{\varphi_\ell} = \ell \cdot \iota_{F,G}$, hence $\widehat{\nu}_{F,G}(\varphi_\ell) = \ell$ and $\widehat{\nu}_{F,G}$ is indeed surjective.

It remains to show that the assignment

$$\widehat{\nu} \colon \begin{cases} \mathrm{Hom}_{[\mathscr{A},\mathscr{E}]}(-, -) & \to H(-, -) \cdot \mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, -), \\ \qquad (F, G) & \mapsto \widehat{\nu}_{F,G} \end{cases}$$

defines a natural isomorphism. The data associated to the triple η, ζ and φ incorporate into the following commutative diagram

$$\begin{array}{ccccc}
 & & \mathbb{1} & & \\
 & \swarrow^{\widehat{\nu}_{F,G}(\varphi)} \quad \circlearrowleft & \downarrow{\scriptstyle \lambda_\varphi} & & \\
\widehat{H}(F,G) & \xhookrightarrow{\iota_{F,G}} & \bigoplus_{v\in\mathfrak{q}_0} H(F(v),G(v)) & \xrightarrow{\Psi_{F,G}} & \bigoplus_{\sigma\in\mathfrak{q}_1} H(F(\mathfrak{s}_\sigma),G(\mathfrak{t}_\sigma)) \\
{\scriptstyle \widehat{H}(\eta,\zeta)}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \Theta_{\eta,\zeta}} & \circlearrowleft & \downarrow{\scriptstyle \Delta_{\eta,\zeta}} \\
\widehat{H}(X,Y) & \xhookrightarrow[\iota_{X,Y}]{} & \bigoplus_{v\in\mathfrak{q}_0} H(X(v),Y(v)) & \xrightarrow[\Psi_{X,Y}]{} & \bigoplus_{\sigma\in\mathfrak{q}_1} H(X(\mathfrak{s}_\sigma),Y(\mathfrak{t}_\sigma))
\end{array}$$

For every $v \in \mathfrak{q}_0$, we have

$$\nu_{X(v),Y(v)}((\eta\cdot\varphi\cdot\zeta)(v)) = \nu_{X(v),Y(v)}(\eta(v)\cdot\varphi(v)\cdot\zeta(v)) = \nu_{F(v),G(v)}(\varphi(v))\cdot H(\eta(v),\zeta(v)),$$

hence $\lambda_{\eta\cdot\varphi\cdot\zeta} = \lambda_\varphi\cdot\Theta_{\eta,\zeta}$. On the other hand, $\lambda_\varphi\cdot\Theta_{\eta,\zeta} = \left(\widehat{\nu}_{F,G}(\varphi)\cdot\widehat{H}(\eta,\zeta)\right)\cdot\iota_{X,Y}$, consequently $\lambda_{\eta\cdot\varphi\cdot\zeta} = \left(\widehat{\nu}_{F,G}(\varphi)\cdot\widehat{H}(\eta,\zeta)\right)\cdot\iota_{X,Y}$. By the definition of $\widehat{\nu}$, we get $\widehat{\nu}_{X,Y}(\eta\cdot\varphi\cdot\zeta) = \widehat{\nu}_{F,G}(\varphi)\cdot\widehat{H}(\eta,\zeta)$. In particular, the following diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{[\mathscr{A},\mathscr{E}]}(F,G) & \xrightarrow{\widehat{\nu}_{F,G}} & \mathrm{Hom}_{\mathscr{D}}\left(\mathbb{1},\widehat{H}(F,G)\right) \\
{\scriptstyle \eta\cdot - \cdot\zeta}\downarrow & \circlearrowleft & \downarrow{\scriptstyle -\cdot\widehat{H}(\eta,\zeta)} \\
\mathrm{Hom}_{[\mathscr{A},\mathscr{E}]}(X,Y) & \xrightarrow[\widehat{\nu}_{X,Y}]{} & \mathrm{Hom}_{\mathscr{D}}\left(\mathbb{1},\widehat{H}(X,Y)\right)
\end{array}$$

is commutative and $\widehat{\nu}$ is indeed a natural transformation. □

Remark 4.19. For an implementation of the above theorem we refer to the `GAP` package `FunctorCategories` [**BS21a**]. A software-demo of the theorem can be found in Appendix E.

4.3. Homomorphism Structure on Stable Categories

Our aim in this section is to equip stable categories defined by classes of lifting or colifting objects (cf. Definition 2.55) with homomorphism structures.

We start by the following construction:

Construction 4.20. Let $\mathscr{C}$ be an additive category equipped with a class of lifting objects $\mathcal{L}$. Suppose $\mathscr{C}$ is equipped with a $\mathscr{D}$-homomorphism structure $(\mathbb{1}, H(-,-), \nu)$ for some Abelian category $\mathscr{D}$. We define the bifunctor

$$H_{\mathcal{L}}(-,-)\colon \begin{cases} \mathscr{C}^{\mathrm{op}}\times\mathscr{C} & \to \mathscr{D}, \\ (B,C) & \mapsto \mathrm{coker}(H(B,\ell_C)), \\ (\alpha^{\mathrm{op}},\beta)\colon (B,C)\to(A,D) & \mapsto \textit{the}\text{ cokernel colift of } \pi_{B,C} \text{ along } H(\alpha,\beta)\cdot\pi_{A,D} \end{cases}$$

where $\pi_{B,C}$ and $\pi_{A,D}$ are the cokernel projections of $H(B,\ell_C)$ resp. $H(A,\ell_D)$:

$$\begin{array}{ccccc} H(B,L_C) & \xrightarrow{H(B,\ell_C)} & H(B,C) & \xrightarrow{\pi_{B,C}} & H_{\mathcal{L}}(B,C) \\ {\scriptstyle H(\alpha,L_\beta)}\downarrow & \circlearrowright & \downarrow{\scriptstyle H(\alpha,\beta)} & \circlearrowright & \downarrow{\scriptstyle H_{\mathcal{L}}(\alpha,\beta)} \\ H(A,L_D) & \xrightarrow[H(A,\ell_D)]{} & H(A,D) & \xrightarrow[\pi_{A,D}]{} & H_{\mathcal{L}}(A,D) \end{array}$$

Lemma 4.21. *Let $\mathscr{C}$ be an additive category equipped with a class of lifting objects $\mathcal{L}$ and a $\mathscr{D}$-homomorphism structure $(\mathbb{1}, H(-,-), \nu)$ such that $\mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, -)$ is a faithful functor*[4]*. Then $H(Q, \ell_A)$ is an epimorphism for all $Q \in \mathcal{L}$ and $A \in \mathscr{C}$.*

Proof. Since $Q \in \mathcal{L}$, $\ell_Q \colon L_Q \to Q$ is a split-epimorphism. Let $\delta_Q \colon Q \to L_Q$ be a section morphism for ℓ_Q. For any morphism $\varphi \colon Q \to A$, we have

$$\varphi = \mathrm{id}_Q \cdot \varphi = \delta_Q \cdot \ell_Q \cdot \varphi = \delta_Q \cdot L_\varphi \cdot \ell_A,$$

i.e., the homomorphism of Abelian groups $\mathrm{Hom}_{\mathscr{C}}(Q, \ell_A)$ is surjective, hence an epimorphism.

By applying the functor $\mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, -)$ on $H(Q, \ell_A)$ and using the naturality of ν, we get the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}}(Q, L_A) & \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(Q,\ell_A)} & \mathrm{Hom}_{\mathscr{C}}(Q, A) \\ {\scriptstyle \nu_{Q,L_A}}\downarrow{\scriptstyle \sim} & \circlearrowright & {\scriptstyle \sim}\downarrow{\scriptstyle \nu_{Q,A}} \\ \mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, H(Q, L_A)) & \xrightarrow[\mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, H(Q,\ell_A))]{} & \mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, H(Q, A)) \end{array}$$

It follows that $\mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, H(Q, \ell_A))$ is an epimorphism as well. The assertion follows from the fact that faithful functors reflect epimorphisms (cf. Lemma A.14). □

Lemma 4.22. *Let $\mathscr{C}$ be an additive category equipped with a class of lifting objects $\mathcal{L}$ and with a $\mathscr{D}$-homomorphism structure $(\mathbb{1}, H(-,-), \nu)$ with*

(1) $\mathscr{D}$ is Abelian,

(2) $H(Q, \ell_C)$ is an epimorphism for all $Q \in \mathcal{L}$ and $C \in \mathscr{C}$.

Then, for any pair of objects B, C in $\mathscr{C}$, if either B or C lives in $\mathcal{L}$, then $H_{\mathcal{L}}(B, C) = 0$. Consequently, $H_{\mathcal{L}}(-,-)$ factors through $\mathscr{C}/\mathcal{L}^{\mathrm{op}} \times \mathscr{C}/\mathcal{L}$.

Proof. By construction $H_{\mathcal{L}}(B, C) \coloneqq \mathrm{coker}(H(B, \ell_C))$, hence $H_{\mathcal{L}}(B, C) = 0$ if and only if $H(B, \ell_C)$ is an epimorphism.

If $B \in \mathcal{L}$ then, by the assumption, we have $H(B, \ell_C)$ is as desired an epimorphism. If $C \in \mathcal{L}$, then ℓ_C is a split-epimorphism, i.e., there exists a morphism δ_C with $\delta_C \cdot \ell_C = \mathrm{id}_C$. This means

$$\mathrm{id}_{H(B,C)} = H(B, \mathrm{id}_C) = H(B, \delta_C \cdot \ell_C) = H(B, \delta_C) \cdot H(B, \ell_C),$$

i.e., $H(B, \ell_C)$ is a split-epimorphism, hence an epimorphism.

Let $(\alpha^{\mathrm{op}}, \beta) \colon (B, C) \to (A, D)$ be a morphism in $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$. If α belongs to $\mathcal{I}_{\mathcal{L}}$, then there exists a lift morphism, say $\tau_\alpha \colon A \to L_B$, of α along ℓ_B. In this case, $H_{\mathcal{L}}(\alpha, \beta)$ can be written as $H_{\mathcal{L}}(\ell_B, \beta) \cdot H_{\mathcal{L}}(\tau_\alpha, \mathrm{id}_D)$, i.e., $H_{\mathcal{L}}(\alpha, \beta)$ factors through $H_{\mathcal{L}}(L_B, D) = 0$, i.e., $H_{\mathcal{L}}(\alpha, \beta) = 0$. On

[4]This assumption is fulfilled for all examples of this thesis.

the other hand, if β belongs to $\mathcal{I}_{\mathcal{L}}$, then there exists a lift, say $\tau_\beta\colon C \to L_D$, of β along ℓ_D. In this case, $H_{\mathcal{L}}(\alpha,\beta)$ can be written as $H_{\mathcal{L}}(\mathrm{id}_B,\tau_\beta)\cdot H_{\mathcal{L}}(\alpha,\ell_D)$, i.e., $H_{\mathcal{L}}(\alpha,\beta)$ factors through $H_{\mathcal{L}}(B,L_D)=0$, i.e., $H_{\mathcal{L}}(\alpha,\beta)=0$.

Hence, we can define the bifunctor

$$H_{\mathcal{L}}(-,-)\colon \begin{cases} \mathscr{C}/\mathcal{L}^{\mathrm{op}} \times \mathscr{C}/\mathcal{L} & \to \mathscr{D}, \\ ([B],[C]) & \mapsto H_{\mathcal{L}}(B,C), \\ ([\alpha]^{\mathrm{op}},[\beta])\colon ([B],[C]) \to ([A],[D]) & \mapsto H(\alpha,\beta)\colon H(B,C)\to H(A,D), \end{cases}$$

which is a colift of $H_{\mathcal{L}}(-,-)\colon \mathscr{C}^{\mathrm{op}}\times\mathscr{C}\to\mathscr{D}$ along the functor

$$[\,](-,-)\colon \begin{cases} \mathscr{C}^{\mathrm{op}} \times \mathscr{C} & \to \mathscr{C}/\mathcal{L}^{\mathrm{op}} \times \mathscr{C}/\mathcal{L}, \\ (B,C) & \mapsto ([B],[C]), \\ (\alpha^{\mathrm{op}},\beta)\colon (B,C) \to (A,D) & \mapsto ([\alpha]^{\mathrm{op}},[\beta])\colon ([B],[C]) \to ([A],[D]). \end{cases}$$

□

The following is the main theorem in this section:

Theorem 4.23. *Let $\mathscr{C}$ be an additive category equipped with a class of lifting objects $\mathcal{L}$ and with a $\mathscr{D}$-homomorphism structure $(\mathbb{1},H(-,-),\nu)$ such that*

(1) $\mathscr{D}$ is Abelian,
(2) $\mathbb{1}$ is a projective object,
(3) $\mathrm{Hom}_{\mathscr{D}}(\mathbb{1},-)$ is a faithful functor[5].

Then $\mathscr{C}/\mathcal{L}$ can be equipped with a $\mathscr{D}$-homomorphism structure.

Proof. Since $\mathrm{Hom}_{\mathscr{D}}(\mathbb{1},-)$ is faithful, $H(Q,\ell_C)$ is an epimorphism for all $Q\in\mathcal{L}$ and $C\in\mathscr{C}$ (cf. Lemma 4.21). Let $H_{\mathcal{L}}(-,-)$ be the bifunctor asserted in Lemma 4.22. We claim that

$$\mathrm{Hom}_{\mathscr{C}/\mathcal{L}}(-,-)\cong \mathrm{Hom}_{\mathscr{D}}(\mathbb{1},H_{\mathcal{L}}(-,-)).$$

Since $\mathbb{1}$ is a projective object in $\mathscr{D}$, the functor $\mathrm{Hom}_{\mathscr{D}}(\mathbb{1},-)$ is exact. Hence, applying it on the exact sequence

$$H(B,L_C) \xrightarrow{\;H(B,\ell_C)\;} H(B,C) \xrightarrow{\;\pi_{B,C}\;} H_{\mathcal{L}}([B],[C]),$$

yields another exact sequence

$$\mathrm{Hom}_{\mathscr{D}}(\mathbb{1},H(B,L_C)) \xrightarrow{\mathrm{Hom}_{\mathscr{D}}(\mathbb{1},H(B,\ell_C))} \mathrm{Hom}_{\mathscr{D}}(\mathbb{1},H(B,C)) \xrightarrow{\mathrm{Hom}_{\mathscr{D}}(\mathbb{1},\pi_{B,C})} \mathrm{Hom}_{\mathscr{D}}(\mathbb{1},H_{\mathcal{L}}([B],[C]))$$

By the naturality of ν we can create the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathscr{C}}(B,L_C) & \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(B,\ell_C)} & \mathrm{Hom}_{\mathscr{C}}(B,C) & \xrightarrow{[\,]_{B,C}} & \mathrm{Hom}_{\mathscr{C}/\mathcal{L}}([B],[C]) \\ \sim\big\downarrow \nu_{B,L_C} & \circlearrowleft & \sim\big\downarrow \nu_{B,C} & \circlearrowleft & \big\downarrow \zeta_{[B],[C]} \\ \mathrm{Hom}_{\mathscr{D}}(\mathbb{1},H(B,L_C)) & \xrightarrow[\mathrm{Hom}_{\mathscr{D}}(\mathbb{1},H(B,\ell_C))]{} & \mathrm{Hom}_{\mathscr{D}}(\mathbb{1},H(B,C)) & \xrightarrow[\mathrm{Hom}_{\mathscr{D}}(\mathbb{1},\pi_{B,C})]{} & \mathrm{Hom}_{\mathscr{D}}(\mathbb{1},H_{\mathcal{L}}([B],[C])), \end{array}$$

---

[5]If $\mathscr{D}$ is a module category R-**mod** then $\mathbb{1}$ is called a generator (see e.g., [**Jac89**, Theorem 3.21]).

where $\zeta_{[B],[C]}$ is the cokernel colift of $[\ ]_{B,C}$ along $\nu_{B,C} \bullet \mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, \pi_{B,C})$. That is, for any morphism $[\varphi]\colon [B] \to [C]$, we have $\zeta_{[B],[C]}([\varphi]) = \nu_{B,C}(\varphi) \bullet \pi_{B,C}$. It follows from the 5-Lemma [**Wei94**, Ex. 1.3.3] that $\zeta_{[B],[C]}$ is an isomorphism. Its inverse is given by

$$\zeta_{B,C}^{-1}\colon \begin{cases} \mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, H_{\mathcal{L}}([B],[C])) & \to \mathrm{Hom}_{\mathscr{C}/\mathcal{L}}([B],[C]), \\ \qquad \ell & \mapsto \left[\nu_{B,C}^{-1}(\lambda_\ell)\right] \end{cases}$$

where $\lambda_\ell\colon \mathbb{1} \to H(B,C)$ is a projective-lift of ℓ along $\pi_{B,C}$.

The following computation

$$\begin{aligned} \Big(\zeta_{[B],[C]} \bullet \mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, H_{\mathcal{L}}([\alpha],[\beta]))\Big)([\varphi]) &= \mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, H_{\mathcal{L}}([\alpha],[\beta]))\Big(\zeta_{[B],[C]}([\varphi])\Big) \\ &= \nu_{B,C}(\varphi) \bullet \pi_{B,C} \bullet H_{\mathcal{L}}([\alpha],[\beta]) \\ &= \nu_{B,C}(\varphi) \bullet H(\alpha,\beta) \bullet \pi_{A,D} \\ &= \nu_{A,D}(\alpha \bullet \varphi \bullet \beta) \bullet \pi_{A,D} \\ &= \zeta_{[A],[D]}([\alpha \bullet \varphi \bullet \beta]) \\ &= \zeta_{[A],[D]}\Big(\mathrm{Hom}_{\mathscr{C}/\mathcal{L}}([\alpha],[\beta])([\varphi])\Big) \\ &= \Big(\mathrm{Hom}_{\mathscr{C}/\mathcal{L}}([\alpha],[\beta]) \bullet \zeta_{[A],[D]}\Big)([\varphi]) \end{aligned}$$

translates to the commutativity of the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}/\mathcal{L}}([B],[C]) & \xrightarrow{\ \zeta_{[B],[C]}\ } & \mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, H_{\mathcal{L}}([B],[C])) \\ {\scriptstyle \mathrm{Hom}_{\mathscr{C}/\mathcal{L}}([\alpha],[\beta])}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, H_{\mathcal{L}}([\alpha],[\beta]))} \\ \mathrm{Hom}_{\mathscr{C}/\mathcal{L}}([A],[D]) & \xrightarrow[\ \zeta_{[A],[D]}\ ]{} & \mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, H_{\mathcal{L}}([A],[D])) \end{array}$$

Hence, the assignment

$$\zeta\colon \begin{cases} \mathrm{Hom}_{\mathscr{C}/\mathcal{L}}(-,-) & \to \mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, H_{\mathcal{L}}(-,-)), \\ \quad ([B],[C]) & \mapsto \zeta_{[B],[C]}\colon \begin{cases} \mathrm{Hom}_{\mathscr{C}/\mathcal{L}}([B],[C]) & \to \mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, H_{\mathcal{L}}([B],[C])), \\ \qquad [\varphi] & \mapsto \nu_{B,C}(\varphi) \bullet \pi_{B,C} \end{cases} \end{cases}$$

is a natural isomorphism. That is, $(\mathbb{1}, H_{\mathcal{L}}(-,-), \zeta)$ is a $\mathscr{D}$-homomorphism structure of $\mathscr{C}/\mathcal{L}$. □

The same statement holds for stable categories defined by classes of colifting objects:

Corollary 4.24. *Let $\mathscr{C}$ be an additive category equipped with a class of colifting objects $\mathcal{Q}$ and with a $\mathscr{D}$-homomorphism structure $(\mathbb{1}, H(-,-), \nu)$ such that*

(1) $\mathscr{D}$ is Abelian,
(2) $\mathbb{1}$ is a projective object,
(3) $\mathrm{Hom}_{\mathscr{D}}(\mathbb{1}, -)$ is a faithful functor.

Then $\mathscr{C}/\mathcal{Q}$ can be equipped with a $\mathscr{D}$-homomorphism structure.

Proof. If a category is equipped with a $\mathscr{D}$-homomorphism structure, then so is its opposite category. On the other hand, the class of colifting objects in $\mathscr{C}$ defines a class of lifting objects in $\mathscr{C}^{\mathrm{op}}$. Thus, the assertion follows by Theorem 4.23. ☐

Remark 4.25. A software-demo for the homomorphism structure can be found in Appendix D or in the manual of the `GAP` package `StableCategories` [**Sal21e**].

4.4. Homomorphism Structure on Categories of Bounded Complexes

In this section we discuss how to elevate a $\mathscr{D}$-homomorphism structures on an additive or Abelian category $\mathscr{C}$ to the category of bounded complexes $\mathcal{C}^b(\mathscr{C})$.

Theorem 4.26. *Let $\mathscr{C}$ be an additive category equipped with a $\mathscr{D}$-homomorphism structure where $\mathscr{D}$ is also additive. Then $\mathcal{C}^b(\mathscr{C})$ can be equipped with a $\mathcal{C}^b(\mathscr{D})$-homomorphism structure.*

Proof. Let

$$\underline{H}\colon \mathcal{C}^b(\mathscr{C})^{\mathrm{op}} \times \mathcal{C}^b(\mathscr{C}) \to \mathcal{C}^b(\mathscr{D})$$

be the bifunctor defined as follows:

(1) An object (B, C) in $\mathcal{C}^b(\mathscr{C})^{\mathrm{op}} \times \mathcal{C}^b(\mathscr{C})$ is mapped to the totalisation of the double complex

$$\begin{array}{cccccccc}
 & & & \vdots & & \vdots & & \\
 & & & \uparrow & & \uparrow & & \\
j+1 & \cdots & \longrightarrow & H(B^{-i}, C^{j+1}) & \xrightarrow{(-1)^{i+j+2} H\left(\partial_B^{-i-1}, C^{j+1}\right)} & H(B^{-i-1}, C^{j+1}) & \longrightarrow & \cdots \\
 & & & \Big\uparrow{\scriptstyle H\left(B^{-i}, \partial_C^j\right)} & & \Big\uparrow{\scriptstyle H\left(B^{-i-1}, \partial_C^j\right)} & & \\
j & \cdots & \longrightarrow & H(B^{-i}, C^{j}) & \xrightarrow[(-1)^{i+j+1} H\left(\partial_B^{-i-1}, C^{j}\right)]{} & H(B^{-i-1}, C^{j}) & \longrightarrow & \cdots \\
 & & & \uparrow & & \uparrow & & \\
 & & & \vdots & & \vdots & & \\
 & & & i & & i+1, & &
\end{array}$$

i.e., to the complex $\underline{H}(B, C)$ in $\mathcal{C}^b(\mathscr{D})$ whose object at index $n \in \mathbb{Z}$ is

$$\underline{H}(B, C)^n := \bigoplus_{j \in \mathbb{Z}} H\left(B^{j-n}, C^j\right)$$

and whose differential $\partial^n_{\underline{H}(B,C)}$ at $n \in \mathbb{Z}$ is given by the matrix

$$\begin{array}{c} \\ \vdots \\ H(B^{j-n}, C^j) \\ H(B^{j+1-n}, C^{j+1}) \\ \vdots \end{array}
\begin{array}{c} \begin{array}{ccccc} \cdots & H(B^{j-1-n}, C^j) & H(B^{j-n}, C^{j+1}) & H(B^{j+1-n}, C^{j+2}) & \cdots \end{array} \\
\begin{pmatrix}
* & * & 0 & 0 & 0 \\
0 & (-1)^{n+1} H\left(\partial_B^{j-1-n}, C^j\right) & H\left(B^{j-n}, \partial_C^j\right) & 0 & 0 \\
0 & 0 & (-1)^{n+1} H\left(\partial_B^{j-n}, C^{j+1}\right) & H\left(B^{j+1-n}, \partial_C^{j+1}\right) & 0 \\
0 & 0 & 0 & * & *
\end{pmatrix} \end{array}.$$

In particular, the differential $\partial^0_{\underline{H}(B,C)}$ is given by the matrix

$$\begin{array}{c} \\ \vdots \\ H(B^j,C^j) \\ H(B^{j+1},C^{j+1}) \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} \dots & H(B^{j-1},C^j) & H(B^j,C^{j+1}) & H(B^{j+1},C^{j+2}) & \dots \end{array} \\ \begin{pmatrix} * & * & 0 & 0 & 0 \\ 0 & -H\left(\partial_B^{j-1},C^j\right) & H\left(B^j,\partial_C^j\right) & 0 & 0 \\ 0 & 0 & -H\left(\partial_B^j,C^{j+1}\right) & H\left(B^{j+1},\partial_C^{j+1}\right) & 0 \\ 0 & 0 & 0 & * & * \end{pmatrix} \end{array}.$$

(2) A morphism $(\alpha^{\mathrm{op}},\beta)\colon (B,C)\to(A,D)$ in $\mathcal{C}^b(\mathscr{C})^{\mathrm{op}}\times\mathcal{C}^b(\mathscr{C})$ is mapped to the morphism $\underline{H}(\alpha,\beta)\colon \underline{H}(B,C)\to\underline{H}(A,D)$ defined by the totalisation of the morphism of double complexes whose component at index $(i,j)\in\mathbb{Z}^2$ is given by $H(\alpha^{-i},\beta^j)\colon H(B^{-i},C^j)\to H(A^{-i},D^j)$. This means the component of $\underline{H}(\alpha,\beta)$ at index $n\in\mathbb{Z}$ is given by the matrix

$$\begin{array}{c} \\ \vdots \\ H(B^{j-n},C^j) \\ H(B^{j+1-n},C^{j+1}) \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} \dots & H(A^{j-n},D^j) & H(A^{j+1-n},D^{j+1}) & \dots \end{array} \\ \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & H(\alpha^{j-n},\beta^j) & 0 & 0 \\ 0 & 0 & H(\alpha^{j+1-n},\beta^{j+1}) & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \end{array}.$$

Let $\lceil\mathbb{1}\rfloor_0$ denote the 0-stalk complex in $\mathcal{C}^b(\mathscr{D})$ defined by $\mathbb{1}$. We define the natural transformation

$$\underline{\nu}\colon \begin{cases} \mathrm{Hom}_{\mathcal{C}^b(\mathscr{C})}(-,-) & \to \mathrm{Hom}_{\mathcal{C}^b(\mathscr{D})}(\lceil\mathbb{1}\rfloor_0,\underline{H}(-,-)), \\ (B,C) & \mapsto \underline{\nu}_{B,C}\colon \mathrm{Hom}_{\mathcal{C}^b(\mathscr{C})}(B,C)\to\mathrm{Hom}_{\mathcal{C}^b(\mathscr{D})}(\lceil\mathbb{1}\rfloor_0,\underline{H}(B,C)) \end{cases}$$

where $\underline{\nu}_{B,C}$ is defined by mapping a morphism $\varphi\colon B\to C$ to the 0-stalk morphism

$$\underline{\nu}_{B,C}(\varphi)\colon \lceil\mathbb{1}\rfloor_0\to\underline{H}(B,C)$$

defined by the morphism $\mathbb{1}\to\bigoplus_{j\in\mathbb{Z}}H(B^j,C^j)$ whose matrix is

$$\begin{array}{c} \\ \mathbb{1} \end{array} \begin{array}{ccccc} & \dots & H\left(B^j,C^j\right) & H\left(B^{j+1},C^{j+1}\right) & \dots \\ (& \dots & \nu_{B^j,C^j}\left(\varphi^j\right) & \nu_{B^{j+1},C^{j+1}}\left(\varphi^{j+1}\right) & \dots\;) \end{array}.$$

For all $j\in\mathbb{Z}$, the column of $\underline{\nu}_{B,C}(\varphi)^0\bullet\partial^0_{\underline{H}(B,C)}$ that is indexed by $H(B^j,C^{j+1})$ is given by

$$\begin{aligned} \nu_{B^j,C^j}\left(\varphi^j\right)\bullet H\left(B^j,\partial_C^j\right) - \nu_{B^{j+1},C^{j+1}}\left(\varphi^{j+1}\right)\bullet H\left(\partial_B^j,C^{j+1}\right) \\ &= \nu_{B^j,C^{j+1}}\left(\varphi^j\bullet\partial_C^j\right) - \nu_{B^j,C^{j+1}}\left(\partial_B^j\bullet\varphi^{j+1}\right) \\ &= \nu_{B^j,C^{j+1}}\left(\varphi^j\bullet\partial_C^j - \partial_B^j\bullet\varphi^{j+1}\right) \\ &= \nu_{B^j,C^{j+1}}(0) \\ &= 0, \end{aligned}$$

i.e., $\underline{\nu}_{B,C}(\varphi)^0\bullet\partial^0_{\underline{H}(B,C)}=0$, and consequently $\underline{\nu}_{B,C}(\varphi)$ is indeed a complex morphism in $\mathcal{C}^b(\mathscr{D})$ and $\underline{\nu}_{B,C}$ is well-defined. Moreover, $\nu_{B,C}$ is an isomorphism and its inverse

$$\underline{\nu}_{B,C}^{-1}\colon \mathrm{Hom}_{\mathcal{C}^b(\mathscr{D})}(\lceil\mathbb{1}\rfloor_0,\underline{H}(B,C))\to\mathrm{Hom}_{\mathcal{C}^b(\mathscr{C})}(B,C)$$

is defined by mapping the 0-stalk morphism $\ell\colon \lceil \mathbb{1} \rfloor_0 \to \underline{H}(B,C)$ defined by $\ell^0\colon \mathbb{1} \to \bigoplus_{j\in\mathbb{Z}} H(B^j, C^j)$ with matrix

$$\begin{array}{cccccc} & \dots & H(B^j,C^j) & H(B^{j+1},C^{j+1}) & \dots \\ \mathbb{1} & (\dots & \ell^0_j & \ell^0_{j+1} & \dots) \end{array}$$

to the morphism $\underline{\nu}^{-1}_{B,C}(\ell)\colon B \to C$ whose component at index $j \in \mathbb{Z}$ is given by $\nu^{-1}_{B^j,C^j}\left(\ell^0_j\right)\colon B^j \to C^j$. For all $j \in \mathbb{Z}$, we have

$$\begin{aligned} \nu_{B^j,C^{j+1}}&\left(\nu^{-1}_{B^j,C^j}\left(\ell^0_j\right)\cdot\partial^j_C - \partial^j_B\cdot\nu^{-1}_{B^{j+1},C^{j+1}}\left(\ell^0_{j+1}\right)\right) \\ &= \ell^0_j\cdot H\left(B^j,\partial^j_C\right) - \ell^0_{j+1}\cdot H\left(\partial^j_B, C^{j+1}\right) \\ &= 0, \end{aligned}$$

i.e., $\nu^{-1}_{B^j,C^j}\left(\ell^0_j\right)\cdot\partial^j_C - \partial^j_B\cdot\nu^{-1}_{B^{j+1},C^{j+1}}\left(\ell^0_{j+1}\right) = 0$ because $\nu_{B^j,B^{j+1}}$ is an isomorphism. Hence, $\underline{\nu}^{-1}_{B,C}(\ell)\colon B \to C$ is indeed a morphism in $\mathcal{C}^b(\mathscr{C})$, i.e., $\underline{\nu}^{-1}_{B,C}$ is well-defined. The naturality of $\underline{\nu}$ follows from the naturality of ν. □

Corollary 4.27. *Let $\mathscr{C}$ be an additive category equipped with a $\mathscr{D}$-homomorphism structure where $\mathscr{D}$ is Abelian. Then $\mathcal{C}^b(\mathscr{C})$ can be equipped with a $\mathscr{D}$-homomorphism structure.*

PROOF. The category $\mathcal{C}^b(\mathscr{C})$ has a $\mathscr{D}$-homomorphism structure $\left(\mathbb{1}, \widetilde{H}(-,-), \widetilde{\nu}\right)$, where $\widetilde{H}(-,-)$ is defined by the composition[6]

$$\mathcal{C}^b(\mathscr{C})^{\text{op}} \times \mathcal{C}^b(\mathscr{C}) \xrightarrow{\underline{H}(-,-)} \mathcal{C}^b(\mathscr{D}) \xrightarrow{\mathrm{Z}^0} \mathscr{D},$$

and $\widetilde{\nu}$ is defined by the vertical composition

$$\mathrm{Hom}_{\mathcal{C}^b(\mathscr{D})}(-,-) \xrightarrow{\underline{\nu}} \mathrm{Hom}_{\mathcal{C}^b(\mathscr{D})}(\lceil \mathbb{1} \rfloor_0, \underline{H}(-,-)) \xrightarrow{\zeta} \mathrm{Hom}_{\mathscr{D}}\left(\mathbb{1}, \widetilde{H}(-,-)\right),$$

where $\zeta_{B,C}$ is defined by

$$\zeta_{B,C}\colon \begin{cases} \mathrm{Hom}_{\mathcal{C}^b(\mathscr{D})}(\lceil \mathbb{1} \rfloor_0, \underline{H}(B,C)) & \to \mathrm{Hom}_{\mathscr{D}}\left(\mathbb{1}, \widetilde{H}(B,C)\right), \\ \qquad\qquad \ell & \mapsto \mathrm{Z}^0(\ell). \end{cases}$$

For an object (B,C) in $\mathcal{C}^b(\mathscr{C})^{\text{op}} \times \mathcal{C}^b(\mathscr{C})$, let $\iota_{B,C}\colon \widetilde{H}(B,C) \hookrightarrow \underline{H}(B,C)^0$ be the kernel embedding of $\partial^0_{\underline{H}(B,C)}$. For any morphism $\ell\colon \lceil \mathbb{1} \rfloor_0 \to \underline{H}(B,C)$ the morphism $\zeta_{B,C}(\ell)$ is the lift of $\ell^0\colon \mathbb{1} \to \underline{H}(B,C)^0$ along $\iota_{B,C}$.

[6]The functor Z^0 is introduced in Definition 3.3.

It is sufficient to prove that ζ is indeed a natural isomorphism. For any morphism $(\alpha^{\mathrm{op}}, \beta)\colon (B, C) \to (A, D)$ and any morphism $\ell\colon \lceil \mathbb{1} \rfloor_0 \to \underline{H}(B, C)$, we can create the following commutative diagram:

$$\begin{array}{ccccc}
 & & \mathbb{1} & & \\
 & \swarrow^{\zeta_{B,C}(\ell)} \quad \circlearrowleft & \downarrow{\scriptstyle \ell^0} & & \\
\widetilde{H}(B, C) & \xhookrightarrow{\iota_{B,C}} & \underline{H}(B, C)^0 & \xrightarrow{\partial^0_{\underline{H}(B,C)}} & \underline{H}(B, C)^1 \\
\downarrow{\scriptstyle \widetilde{H}(\alpha,\beta)} & \circlearrowleft & \downarrow{\scriptstyle \underline{H}(\alpha,\beta)^0} & \circlearrowleft & \downarrow{\scriptstyle \underline{H}(\alpha,\beta)^1} \\
\widetilde{H}(A, D) & \xhookrightarrow[\iota_{A,D}]{} & \underline{H}(A, D)^0 & \xrightarrow[\partial^0_{\underline{H}(A,D)}]{} & \underline{H}(A, D)^1
\end{array}$$

from which we conclude the equality $\zeta_{B,C}(\ell) \cdot \widetilde{H}(\alpha, \beta) \cdot \iota_{A,D} = \ell^0 \cdot \underline{H}(\alpha, \beta)^0$. Hence, by the definition of ζ, we have $\zeta_{A,D}(\ell \cdot \underline{H}(\alpha, \beta)) = \zeta_{B,C}(\ell) \cdot \widetilde{H}(\alpha, \beta)$; which translates into the commutativity of the following diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}^b(\mathscr{D})}(\lceil \mathbb{1} \rfloor_0, \underline{H}(B, C)) & \xrightarrow{\zeta_{B,C}} & \mathrm{Hom}_{\mathscr{D}}\Big(\mathbb{1}, \widetilde{H}(B, C)\Big) \\
{\scriptstyle - \cdot \underline{H}(\alpha,\beta)}\downarrow & \circlearrowleft & \downarrow{\scriptstyle - \cdot \widetilde{H}(\alpha,\beta)} \\
\mathrm{Hom}_{\mathcal{C}^b(\mathscr{D})}(\lceil \mathbb{1} \rfloor_0, \underline{H}(A, D)) & \xrightarrow[\zeta_{A,D}]{} & \mathrm{Hom}_{\mathscr{D}}\Big(\mathbb{1}, \widetilde{H}(A, D)\Big),
\end{array}$$

i.e., ζ is indeed a natural transformation. Moreover, the component $\zeta_{B,C}$ is an isomorphism and its inverse $\zeta^{-1}_{B,C}$ maps a morphism $\tau\colon \mathbb{1} \to \widetilde{H}(B, C)$ to the 0-stalk complex morphism $\ell_\tau\colon \lceil \mathbb{1} \rfloor_0 \to \underline{H}(B, C)$ defined by $\tau \cdot \iota_{B,C}$. This means ζ is a natural isomorphism, hence, so is the vertical composition $\widetilde{\nu} := \underline{\nu} \cdot \delta$. □

Corollary 4.28. *Let $\mathscr{C}$ be an additive category equipped with a $\mathscr{D}$-homomorphism structure $(\mathbb{1}, H(-, -), \nu)$ where $\mathscr{D}$ is an additive category with weak kernels. Then the category $\mathcal{C}^b(\mathscr{C})$ has an $\mathcal{A}(\mathscr{D})$-homomorphism structure, where $\mathcal{A}(\mathscr{D})$ is the* Freyd *category of $\mathscr{D}$.*

Proof. By Theorem 2.31 the category $\mathscr{D}$ has weak kernels if and only if its Freyd category $\mathcal{A}(\mathscr{D})$ is Abelian. Moreover, the natural embedding $\mathscr{D} \xrightarrow{\iota} \mathcal{A}(\mathscr{D})$ is always fully faithful. Hence, by Lemma 4.5, $\mathscr{C}$ can be equipped by an $\mathcal{A}(\mathscr{D})$-homomorphism structure; consequently by Corollary 4.27, $\mathcal{C}^b(\mathscr{C})$ can also be equipped by an $\mathcal{A}(\mathscr{D})$-homomorphism structure. □

Example 4.29. Let R be a commutative left coherent ring. In Example 4.6, we found that R-**rows** is equipped with a (R-**rows**)-homomorphism structure. By Section 2.1.1, $\mathcal{A}(R\text{-}\mathbf{rows})$ is an Abelian category. Hence, by Corollaries 4.27 and 4.28, the category $\mathcal{C}^b(R\text{-}\mathbf{rows})$ can be equipped with an $\mathcal{A}(R\text{-}\mathbf{rows})$-homomorphism structure.

Example 4.30. Let R be a commutative left coherent ring. Then $\mathcal{A}(R\text{-}\mathbf{rows})$ can be equipped with an $\mathcal{A}(R\text{-}\mathbf{rows})$-homomorphism structure [**Pos21a**]. Hence, $\mathcal{C}^b(\mathcal{A}(R\text{-}\mathbf{rows}))$ can be equipped with an $\mathcal{A}(R\text{-}\mathbf{rows})$-homomorphism structure.

Example 4.31. Let $\mathfrak{q}$ be a quiver and $\mathscr{A} = k\mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle$ be the k-linear finitely presented category defined by $\mathfrak{q}$ subject to a set of relations ρ. According to Theorem 4.18, the category **mod**-$\mathscr{A}$ can be equipped with a (k-**mat**)-homomorphism structure. Hence, $\mathcal{C}^b(\mathbf{mod}\text{-}\mathscr{A})$ can be equipped with a (k-**mat**)-homomorphism structure.

4.5. Homomorphism Structure on Bounded Homotopy and Derived Categories

In this section we discuss how to elevate a $\mathscr{D}$-homomorphism structures on an additive or Abelian category $\mathscr{C}$ to the bounded homotopy category $\mathcal{K}^b(\mathscr{C})$.

Corollary 4.32. *With the same assumptions and notations as in Theorem 4.26, if B, C are objects in $\mathcal{C}^b(\mathscr{C})$ and either of which is contractible, then $\underline{H}(B,C)$ is also contractible.*

Proof. In the case where B is contractible, let $\left(\lambda_B^n\colon B^n \to B^{n-1}\right)_{n\in\mathbb{Z}}$ be a family of morphisms satisfying $\partial_B^n \cdot \lambda_B^{n+1} + \lambda_B^n \cdot \partial_B^{n-1} = \mathrm{id}_{B^n}$ for all $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$, we define the morphism

$$\lambda^n_{\underline{H}(B,C)}\colon \underline{H}(B,C)^n \to \underline{H}(B,C)^{n-1}$$

by the matrix

$$\begin{array}{c} \\ \vdots \\ H(B^{j-n},C^j) \\ H(B^{j+1-n},C^{j+1}) \\ \vdots \end{array}
\begin{array}{c} \begin{array}{cccc} \dots & H(B^{j+1-n},C^j) & H(B^{j+2-n},C^{j+1}) & \dots \end{array} \\
\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & (-1)^n H\left(\lambda_B^{j+1-n},C^j\right) & 0 & 0 \\ 0 & 0 & (-1)^n H\left(\lambda_B^{j+2-n},C^{j+1}\right) & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \end{array}.$$

A direct computation shows that $\partial^n_{\underline{H}(B,C)} \cdot \lambda^{n+1}_{\underline{H}(B,C)} + \lambda^n_{\underline{H}(B,C)} \cdot \partial^{n-1}_{\underline{H}(B,C)}$ is given by the matrix

$$\begin{array}{c} \\ \vdots \\ H(B^{j-n},C^j) \\ \vdots \end{array}
\begin{array}{c} \begin{array}{ccc} \dots & H(B^{j-n},C^j) & \dots \end{array} \\
\begin{pmatrix} * & 0 & 0 \\ 0 & H\left(\partial_B^{j-n} \cdot \lambda_B^{j+1-n},C^j\right) + H\left(\lambda_B^{j-n} \cdot \partial_B^{j-1-n},C^j\right) & 0 \\ 0 & 0 & * \end{pmatrix} \end{array},$$

which, by the functoriality of $H(-,-)$, is equal to $\mathrm{id}_{\underline{H}(B,C)^n}$.

In the case where C is contractible, let $\left(\lambda_C^n\colon C^n \to C^{n-1}\right)_{n\in\mathbb{Z}}$ be a family of morphisms statisfying $\partial_C^n \cdot \lambda_C^{n+1} + \lambda_C^n \cdot \partial_C^{n-1} = \mathrm{id}_{C^n}$ for all $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$, we define the morphism

$$\lambda^n_{\underline{H}(B,C)}\colon \underline{H}(B,C)^n \to \underline{H}(B,C)^{n-1}$$

by the matrix

$$\begin{array}{c} \\ \vdots \\ H(B^{j-n},C^j) \\ H(B^{j+1-n},C^{j+1}) \\ \vdots \end{array}
\begin{array}{c} \begin{array}{cccc} \dots & H(B^{j-n},C^{j-1}) & H(B^{j+1-n},C^j) & \dots \end{array} \\
\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & H\left(B^{j-n},\lambda_C^j\right) & 0 & 0 \\ 0 & 0 & H\left(B^{j+1-n},\lambda_C^{j+1}\right) & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \end{array}.$$

A direct computation shows that $\partial^n_{\underline{H}(B,C)} \bullet \lambda^{n+1}_{\underline{H}(B,C)} + \lambda^n_{\underline{H}(B,C)} \bullet \partial^{n-1}_{\underline{H}(B,C)}$ is given by the matrix

$$\begin{array}{c} \\ \vdots \\ H(B^{j-n},C^j) \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccc} \dots & H(B^{j-n},C^j) & \dots \end{array} \\ \begin{pmatrix} * & 0 & 0 \\ 0 & H\Big(B^{j-n}, \partial^j_C \bullet \lambda^{j+1}_C\Big) + H\Big(B^{j-n}, \lambda^j_C \bullet \partial^{j-1}_C\Big) & 0 \\ 0 & 0 & * \end{pmatrix} \end{array},$$

which, by the functoriality of $H(-,-)$, is equal to $\mathrm{id}_{\underline{H}(B,C)^n}$. This means, in either case, the object $\underline{H}(B,C)$ is contractible. □

Theorem 4.33. *Let $\mathscr{C}$ be an additive category equipped with $\mathscr{D}$-homomorphism structure $(\mathbb{1}, H(-,-), \nu)$. Then $\mathcal{K}^b(\mathscr{C})$ can be equipped with a $\mathcal{K}^b(\mathscr{D})$-homomorphism structure.*

Proof. By Theorem 4.26, $\mathcal{C}^b(\mathscr{C})$ can be equipped with a $\mathcal{C}^b(\mathscr{D})$-homomorphism structure $(\lceil \mathbb{1} \rfloor_0, \underline{H}(-,-), \underline{\nu})$. Let $(\alpha^{\mathrm{op}}, \beta)\colon (B,C) \to (A,D)$ be a morphism in $\mathcal{C}^b(\mathscr{C})^{\mathrm{op}} \times \mathcal{C}^b(\mathscr{C})$. If α is null-homotopic, then α factors through $\mathrm{Cone}(\mathrm{id}_A)$, hence $\underline{H}(\alpha,\beta)$ factors through the object $\underline{H}(\mathrm{Cone}(\mathrm{id}_A), C)$. Analogously, if β is null-homotopic, then $\underline{H}(\alpha,\beta)$ factors through the object $\underline{H}(A, \mathrm{Cone}(\mathrm{id}_C))$. By Corollary 4.32, if either α or β is null-homotopic then $\underline{H}(\alpha,\beta)$ factors through a contractible object, i.e., by Remark 3.22, $\underline{H}(\alpha,\beta)$ is then null-homotopic.

The component of the natural isomorphism $\underline{\nu}$ at some object (B,C) in $\mathcal{C}^b(\mathscr{C})^{\mathrm{op}} \times \mathcal{C}^b(\mathscr{C})$ is given by an isomorphism

$$\underline{\nu}_{B,C} : \mathrm{Hom}_{\mathcal{C}^b(\mathscr{C})}(B,C) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}^b(\mathscr{D})}(\lceil \mathbb{1} \rfloor_0, \underline{H}(B,C)).$$

We claim that a morphism $\alpha\colon A \to B$ is null homotopic if and only if $\underline{\nu}_{B,C}(\varphi)$ is null-homotopic: Suppose that φ is null-homotopic and let $\big(h^i\colon A^i \to B^{i-1}\big)_{i\in\mathbb{Z}}$ is a chain homotopy of φ. Let $j^0\colon \mathbb{1}_D \to \underline{H}(B,C)^{-1}$ be the morphism defined by the matrix

$$\begin{array}{cccccc} & \dots & H(B^j, C^{j-1}) & H(B^{j+1}, C^j) & \dots \\ \mathbb{1} & (\dots & \nu_{B^j,C^{j-1}}(h^j) & \nu_{B^{j+1},C^j}(h^{j+1}) & \dots). \end{array}$$

whose composition with $\partial^{-1}_{\underline{H}(B,C)}$ is given by the matrix

$$\begin{array}{ccccc} & \dots & H(B^j, C^j) & \dots \\ \mathbb{1} & \Big(\dots & \nu_{B^j,C^{j-1}}(h^j) \bullet H\Big(B^j, \partial^{j-1}_C\Big) + \nu_{B^{j+1},C^j}(h^{j+1}) \bullet H\Big(\partial^j_B, C^j\Big) & \dots\Big) \end{array}$$

which, by the naturality of ν, can be simplified to

$$\begin{array}{cccc} & \dots & H(B^j, C^j) & \dots \\ \mathbb{1} & \Big(\dots & \nu_{B^j,C^j}\Big(h^j \bullet \partial^{j-1}_C + \partial^j_B \bullet h^{j+1}\Big) & \dots\Big) \end{array} = \begin{array}{cccc} & \dots & H(B^j, C^j) & \dots \\ \mathbb{1} & (\dots & \nu_{B^j,C^j}(\varphi^j) & \dots); \end{array}$$

hence $j^0 \bullet \partial^{-1}_{\underline{H}(B,C)} = \underline{\nu}_{B,C}(\varphi)^0$ and $\underline{\nu}_{B,C}(\varphi)$ is then null-homotopic. Conversely, suppose $\underline{\nu}_{B,C}(\varphi)$ is null-homotopic and let $\ell^0\colon \mathbb{1} \to \underline{H}(B,C)^{-1}$ be a morphism such that $\ell^0 \bullet \partial^{-1}_{\underline{H}(B,C)} = \underline{\nu}_{B,C}(\varphi)^0$. Thus, if ℓ^0 is defined by the matrix

$$\begin{array}{cccccc} & \dots & H(B^j, C^{j-1}) & H(B^{j+1}, C^j) & \dots \\ \mathbb{1} & (\dots & \ell^0_j & \ell^0_{j+1} & \dots), \end{array}$$

then $\ell^0_j \bullet H\left(B^j, \partial_C^{j-1}\right) + \ell^0_{j+1} \bullet H\left(\partial_B^j, C^j\right) = \nu_{B^j,C^j}(\varphi^j)$ for all $j \in \mathbb{Z}$. We define the family $\left(h^j := \nu^{-1}_{B^j,C^{j-1}}\left(\ell^0_j\right) \colon B^j \to C^{j-1}\right)_{j\in\mathbb{Z}}$, then

$$\nu_{B^j,C^{j-1}}\left(h^j\right) \bullet H\left(B^j, \partial_C^{j-1}\right) + \nu_{B^{j+1},C^j}\left(h^{j+1}\right) \bullet H\left(\partial_B^j, C^j\right) = \nu_{B^j,C^j}\left(\varphi^j\right)$$

for all $j \in \mathbb{Z}$. Because of the naturality of ν, we have

$$\nu_{B^j,C^j}\left(h^j \bullet \partial_C^{j-1} + \partial_B^j \bullet h^{j+1}\right) = \nu_{B^j,C^j}\left(\varphi^j\right)$$

for all $j \in \mathbb{Z}$. Since ν is natural isomorphism, we have the equalities

$$h^j \bullet \partial_C^{j-1} + \partial_B^j \bullet h^{j+1} = \varphi^j$$

for all $j \in \mathbb{Z}$, hence φ is null-homotopic.

Hence, we can define the functor

$$\widehat{H}(-,-)\colon \begin{cases} \mathcal{K}^b(\mathscr{C})^{\,\mathrm{op}} \times \mathcal{K}^b(\mathscr{C}) & \to \mathcal{K}^b(\mathscr{D}),\\ ([B],[C]) & \mapsto [\underline{H}(B,C)],\\ ([\alpha]^{\mathrm{op}},[\beta])\colon ([B],[C]) \to ([A],[D]) & \mapsto [\underline{H}(\alpha,\beta)]; \end{cases}$$

and the natural isomorphism

$$\widehat{\nu}\colon \begin{cases} \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(-,-) & \to \mathrm{Hom}_{\mathcal{K}^b(\mathscr{D})}\left([\lceil \mathbb{1} \rfloor_0], \widehat{H}(-,-)\right),\\ ([B],[C]) & \mapsto \widehat{\nu}_{[B],[C]} \end{cases}$$

where $\widehat{\nu}_{B,C}$ is defined by

$$\widehat{\nu}_{[B],[C]}\colon \begin{cases} \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}([B],[C]) & \to \mathrm{Hom}_{\mathcal{K}^b(\mathscr{D})}\left([\lceil \mathbb{1} \rfloor_0], \widehat{H}([B],[C])\right),\\ [\varphi] & \mapsto \left[\underline{\nu}_{B,C}(\varphi)\right]. \end{cases}$$

□

Theorem 4.34. *Let $\mathscr{C}$ be an additive category equipped with $\mathscr{D}$-homomorphism structure $(\mathbb{1}, H(-,-), \nu)$. If $\mathscr{D}$ is Abelian and $\mathbb{1}$ is a projective object, then $\mathcal{K}^b(\mathscr{C})$ can be equipped with a $\mathscr{D}$-homomorphism structure.*

Proof. In the previous theorem, we found that $\mathcal{K}^b(\mathscr{C})$ can be equipped with a $\mathcal{K}^b(\mathscr{D})$-homomorphism structure $\left([\lceil \mathbb{1} \rfloor_0], \widehat{H}(-,-), \widehat{\nu}\right)$.

Define the bifunctor $\widetilde{H}(-,-)$ by the composition

$$\mathcal{K}^b(\mathscr{C})^{\,\mathrm{op}} \times \mathcal{K}^b(\mathscr{C}) \xrightarrow{\widehat{H}(-,-)} \mathcal{K}^b(\mathscr{D}) \xrightarrow{\mathrm{H}^0} \mathscr{D},$$

and the natural transformation $\widetilde{\nu}$ by vertical composition

$$\mathrm{Hom}_{\mathcal{K}^b(\mathscr{D})}(-,-) \xrightarrow{\widehat{\nu}} \mathrm{Hom}_{\mathcal{K}^b(\mathscr{D})}\left([\lceil \mathbb{1} \rfloor_0], \widehat{H}(-,-)\right) \xrightarrow{\zeta} \mathrm{Hom}_{\mathscr{D}}\left(\mathbb{1}, \widetilde{H}(-,-)\right),$$

where ζ is defined by

$$\zeta\colon \begin{cases} \mathrm{Hom}_{\mathcal{K}^b(\mathscr{D})}\left([\lceil \mathbb{1} \rfloor_0], \widehat{H}(-,-)\right) & \to \mathrm{Hom}_{\mathscr{D}}\left(\mathbb{1}, \widetilde{H}(-,-)\right),\\ ([B],[C]) & \mapsto \zeta_{[B],[C]} \end{cases}$$

and $\zeta_{[B],[C]}$ is the map

$$\zeta_{[B],[C]}\colon \begin{cases} \mathrm{Hom}_{\mathcal{K}^b(\mathscr{D})}\Big([\lceil \mathbb{1} \rfloor_0], \widehat{H}([B],[C])\Big) & \to \mathrm{Hom}_{\mathscr{D}}\Big(\mathbb{1}, \widetilde{H}([B],[C])\Big), \\ [\ell] & \mapsto \mathrm{H}^0([\ell]). \end{cases}$$

It is sufficient to prove that ζ is indeed a natural isomorphism. In the following we show how ζ can be computed, then use the procedure in proving the naturality of ζ.

For an object $([B],[C])$ in $\mathcal{K}^b(\mathscr{C})^{\mathrm{op}} \times \mathcal{K}^b(\mathscr{C})$, we define the morphisms:

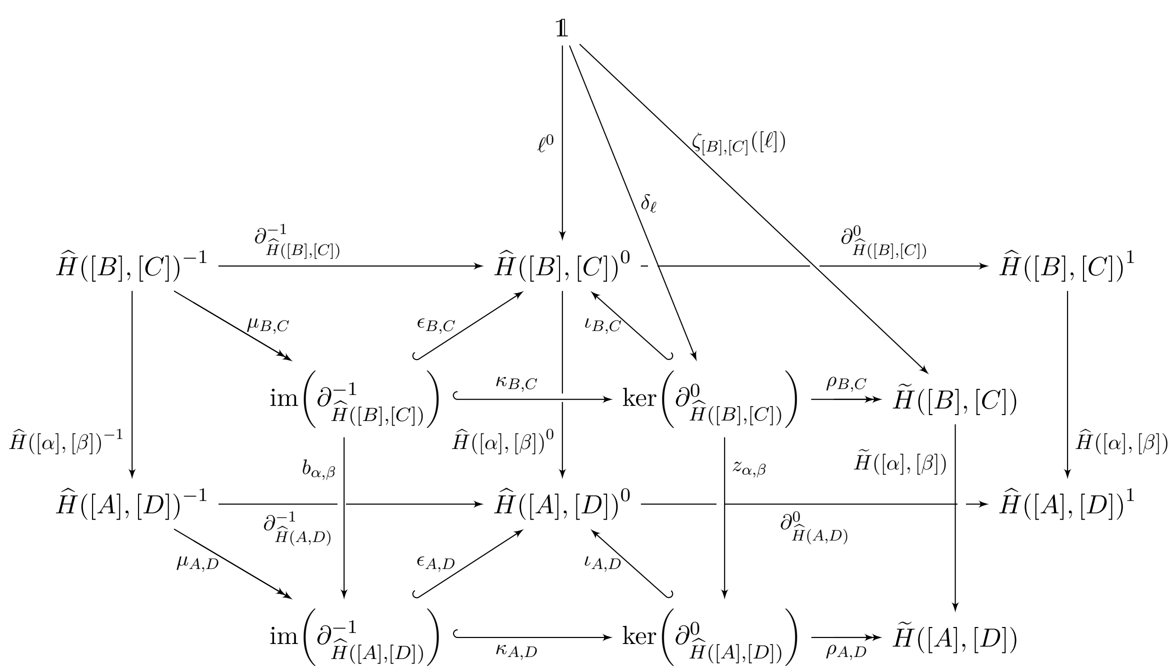
- $\iota_{B,C}$ by the kernel embedding of $\partial^0_{\widehat{H}([B],[C])}$,
- $\epsilon_{B,C}$ by the image embedding of $\partial^{-1}_{\widehat{H}([B],[C])}$,
- $\kappa_{B,C}$ by the lift of $\epsilon_{B,C}$ along $\iota_{B,C}$,
- $\rho_{B,C}$ by the cokernel projection of $\kappa_{B,C}$ and
- $\mu_{B,C}$ by the lift of $\partial^{-1}_{\widehat{H}([B],[C])}$ along $\epsilon_{B,C}$.

For every morphism $[\ell]\colon [\lceil \mathbb{1} \rfloor_0] \to \widehat{H}([B],[C])$, there exists a unique lift, say δ_ℓ, of ℓ^0 along $\iota_{B,C}$, and $\zeta_{[B],[C]}([\ell]) = \delta_\ell \cdot \rho_{B,C}$.

For a morphism $([\alpha]^{\mathrm{op}}, [\beta])\colon ([B],[C]) \to ([A],[D])$ in $\mathcal{K}^b(\mathscr{C})^{\mathrm{op}} \times \mathcal{K}^b(\mathscr{C})$, we define the morphisms:

- $z_{\alpha,\beta}$ by the lift of $\iota_{B,C} \cdot \widehat{H}([\alpha],[\beta])^0$ along $\iota_{A,C}$ and
- $b_{\alpha,\beta}$ by the lift of $\epsilon_{B,C} \cdot \widehat{H}([\alpha],[\beta])^0$ along $\epsilon_{A,C}$.

For given morphisms $([\alpha]^{\mathrm{op}}, [\beta])\colon ([B],[C]) \to ([A],[D])$ and $[\ell]\colon [\lceil \mathbb{1} \rfloor_0] \to \widehat{H}(B,C)$, we get the following commutative diagram:



from which we conclude the equality $\delta_\ell \bullet z_{\alpha,\beta} \bullet \iota_{A,D} = \ell^0 \bullet \widehat{H}([\alpha],[\beta])^0$. Hence, $\delta_{(\ell \bullet \underline{H}(\alpha,\beta))} = \delta_\ell \bullet z_{\alpha,\beta}$. The following computation

$$\begin{aligned} \zeta_{[A],[D]}\Big([\ell] \bullet \widehat{H}([\alpha],[\beta])\Big) &= \zeta_{[A],[D]}([\ell \bullet \underline{H}(\alpha,\beta)]) \\ &= \delta_{(\ell \bullet \underline{H}(\alpha,\beta))} \bullet \rho_{A,D} \\ &= \delta_\ell \bullet z_{\alpha,\beta} \bullet \rho_{A,D} \\ &= \delta_\ell \bullet \rho_{B,C} \bullet \widetilde{H}([\alpha],[\beta]) \\ &= \zeta_{[B],[C]}([\ell]) \bullet \widetilde{H}([\alpha],[\beta]) \end{aligned}$$

translates into the commutativity of the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{K}^b(\mathscr{D})}\Big([\lceil \mathbb{1} \rfloor_0], \widehat{H}([B],[C])\Big) & \xrightarrow{\quad \zeta_{[B],[C]} \quad} & \mathrm{Hom}_{\mathscr{D}}\Big(\mathbb{1}, \widetilde{H}([B],[C])\Big) \\ {\scriptstyle - \bullet \widehat{H}([\alpha],[\beta])} \Big\downarrow & \circlearrowleft & \Big\downarrow {\scriptstyle - \bullet \widetilde{H}([\alpha],[\beta])} \\ \mathrm{Hom}_{\mathcal{K}^b(\mathscr{D})}\Big([\lceil \mathbb{1} \rfloor_0], \widehat{H}([A],[D])\Big) & \xrightarrow[\quad \zeta_{[A],[D]} \quad]{} & \mathrm{Hom}_{\mathscr{D}}\Big(\mathbb{1}, \widetilde{H}([A],[D])\Big), \end{array}$$

i.e., ζ is indeed a natural transformation.

ζ is a natural isomorphism if for every object $([B],[C])$ in $\mathcal{K}^b(\mathscr{C})^{\mathrm{op}} \times \mathcal{K}^b(\mathscr{C})$, the component $\zeta_{[B],[C]}$ is an isomorphism.

We start by showing the $\zeta_{[B],[C]}$ is injective. Let $[\ell]\colon [\mathbb{1}] \to \widehat{H}([B],[C])$ be a morphism with $\zeta_{[B],[C]}([\ell]) = 0$. This means $\delta_\ell \bullet \rho_{B,C} = 0$. Since, $\mathscr{D}$ is Abelian, every monomorphism is a kernel embedding of its cokernel projection, i.e., $\kappa_{B,C}$ is a kernel embedding for $\rho_{B,C}$, hence, there exists a lift, say λ_ℓ, of δ_ℓ along $\kappa_{B,C}$. Since $\mathbb{1}$ is projective and $\mu_{B,C}$ is an epimorphism, there exists a lift $h_{B,C}$ of $\lambda_{B,C}$ along $\mu_{B,C}$. Hence

$$\begin{aligned} h_{B,C} \bullet \partial^{-1}_{\widehat{H}([B],[C])} &= h_{B,C} \bullet \mu_{B,C} \bullet \epsilon_{B,C} \\ &= \lambda_{B,C} \bullet \kappa_{B,C} \bullet \iota_{B,C} \\ &= \delta_\ell \bullet \iota_{B,C} \\ &= \ell^0, \end{aligned}$$

which, since ℓ is concentrated in degree 0, implies $[\ell] = 0$. Consequently, $\zeta_{[B],[C]}$ is injective.

It remains to show that $\zeta_{[B],[C]}$ is surjective. Let $\tau\colon \mathbb{1} \to \widetilde{H}([B],[C])$ be a morphism. Since $\mathbb{1}$ is projective and $\rho_{B,C}$ is an epimorphism, there exists a lift $d_\tau\colon \mathbb{1} \to \ker\Big(\partial^0_{\widehat{H}([B],[C])}\Big)$ of τ along $\rho_{B,C}$. We define $[\ell_\tau]\colon [\lceil \mathbb{1} \rfloor_0] \to \widehat{H}([B],[C])$ by the 0-stalk morphism defined by $d_\tau \bullet \iota_{B,C}$. It follows $\delta_{\ell_\tau} = d_\tau$ and $\zeta_{[B],[C]}([\ell_\tau]) = d_\tau \bullet \rho_{B,C} = \tau$. Consequently, $\zeta_{[B],[C]}$ is surjective.

This means ζ is a natural isomorphism and so is then the vertical composition $\widetilde{\nu} = \widehat{\nu} \bullet \zeta$ as desired. □

Corollary 4.35. *Let $\mathscr{C}$ be a Abelian category with enough projectives and finite global dimension. If $\mathscr{C}$ is equipped with a $\mathscr{D}$-homomorphism structure $(\mathbb{1}, H(-,-), \nu)$ where $\mathscr{D}$ is an Abelian category and $\mathbb{1}$ is a projective object, then $\mathcal{D}^b(\mathscr{C})$ can be equipped with a $\mathscr{D}$-homomorphism structure.*

PROOF. It follows from Theorems 3.61 and 4.33. ☐

Corollary 4.36. *Let $\mathscr{C}$ be an additive category equipped with a $\mathscr{D}$-homomorphism structure $(\mathbb{1}, H(-,-), \nu)$ where $\mathscr{D}$ is an additive category with weak kernels. Then the category $\mathcal{K}^b(\mathscr{C})$ can be equipped with an $\mathcal{A}(\mathscr{D})$-homomorphism structure, where $\mathcal{A}(\mathscr{D})$ is the* FREYD *category of $\mathscr{D}$.*

PROOF. By Theorem 2.31 the category $\mathscr{D}$ has weak kernels if and only if its FREYD category $\mathcal{A}(\mathscr{D})$ is Abelian. Moreover, the natural embedding $\mathscr{D} \xrightarrow{\iota} \mathcal{A}(\mathscr{D})$ is always fully faithful. Hence, by Lemma 4.5, $\mathscr{C}$ can be equipped by an $\mathcal{A}(\mathscr{D})$-homomorphism structure; consequently by Theorem 4.34, $\mathcal{K}^b(\mathscr{C})$ can also be equipped by an $\mathcal{A}(\mathscr{D})$-homomorphism structure. ☐

Example 4.37. Let R be a commutative left coherent ring. In Example 4.6, we found that R-**rows** is equipped with a (R-**rows**)-homomorphism structure. By Section 2.1.1, $\mathcal{A}(R\text{-}\mathbf{rows})$ is an Abelian category. Hence, by Theorem 4.34 and Corollary 4.28, the category $\mathcal{K}^b(R\text{-}\mathbf{rows})$ can be equipped with an $\mathcal{A}(R\text{-}\mathbf{rows})$-homomorphism structure.

Example 4.38. Let R be a commutative left coherent ring. Then $\mathcal{A}(R\text{-}\mathbf{rows})$ can be equipped with an $\mathcal{A}(R\text{-}\mathbf{rows})$-homomorphism structure [**Pos21a**]. Hence, $\mathcal{K}^b(\mathcal{A}(R\text{-}\mathbf{rows}))$ can be equipped with an $\mathcal{A}(R\text{-}\mathbf{rows})$-homomorphism structure.

Example 4.39. Let $\mathfrak{q}$ be a quiver and $\mathscr{A} = k\mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle$ be the k-linear finitely presented category defined by $\mathfrak{q}$ subject to a set of relations ρ. According to Theorem 4.18, the category $\mathbf{mod}\text{-}\mathscr{A}$ can be equipped with a (k-**mat**)-homomorphism structure. Hence, $\mathcal{K}^b(\mathbf{mod}\text{-}\mathscr{A})$ can be equipped with a (k-**mat**)-homomorphism structure.

CHAPTER 5

Computable Triangulated Categories

The Appendix B provides a brief review of the basic definitions and facts in triangulated categories which need in the next chapters. In this section we provide the constructive interpretation of the axioms in the definition of triangulated categories. We start the section by introducing the notion of a (pre)computable triangulated categories, and afterwards, we provide two examples: The bounded homotopy category of an additive category (cf. Section 5.2) and the stable category of a FROBENIUS category[1] (cf. Section 5.3).

5.1. Computable Triangulated Categories

In the following we state the definition of precomputable triangulated categories (cf. Definition B.1):

Definition 5.1. A **precomputable triangulated** category is a computable additive category together with an autoequivalence Σ and a class $\triangle$ of exact triangles subject to the following axioms:

TR′ 0. The functors Σ and Σ^{-1} and the associated natural isomorphisms[2] are realized by algorithms.

TR′ 1. The following requirements are satisfied:

(a) There is an algorithm which for a given morphism $\alpha\colon A \to B$ in $\mathfrak{T}$ constructs an object $\operatorname{Cone}(\alpha)$ and two morphisms $\iota(\alpha)$ and $\pi(\alpha)$ such that

$$A \xrightarrow{\alpha} B \xrightarrow{\iota(\alpha)} \operatorname{Cone}(\alpha) \xrightarrow{\pi(\alpha)} \Sigma(A)$$

belongs to the class $\triangle$.

(b) For any object A in $\mathfrak{T}$, we have $\operatorname{Cone}(\mathrm{id}_A) \cong 0$.

(c) A triangle

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$$

[1] In fact, homotopy categories can be constructed as stable categories of Frobenius categories; however, for performance reasons, we describe their triangulated structure directly.

[2] Depending on the use case, it might be desirable to choose the unit $\eta\colon \mathrm{id}_{\mathfrak{T}} \Rightarrow \Sigma^{-1}\bullet\Sigma$ and counit $\epsilon\colon \Sigma\bullet\Sigma^{-1} \Rightarrow \mathrm{id}_{\mathfrak{T}}$ of the adjunction $\Sigma^{-1} \dashv \Sigma$.

is exact if and only if there exists an isomorphism $\lambda\colon C \to \mathrm{Cone}(\alpha)$ with $\iota \cdot \lambda = \iota(\alpha)$ and $\lambda \cdot \pi(\alpha) = \pi$:

$$\begin{array}{ccccccc}
A & \xrightarrow{\alpha} & B & \xrightarrow{\iota} & C & \xrightarrow{\pi} & \Sigma(A) \\
 & & & \searrow^{\iota(\alpha)} \circlearrowleft & \lambda \downarrow \wr & \circlearrowleft \nearrow_{\pi(\alpha)} & \\
 & & & & \mathrm{Cone}(\alpha). & &
\end{array}$$

TR′ 2. We have an algorithm which for a given morphism $\alpha\colon A \to B$ in $\mathfrak{T}$ computes an isomorphism $\lambda\colon \Sigma(A) \xrightarrow{\sim} \mathrm{Cone}(\iota(\alpha))$ such that $\pi(\alpha) \cdot \lambda = \iota(\iota(\alpha))$ and $\lambda \cdot \pi(\iota(\alpha)) = -\Sigma(\alpha)$. In other words, λ induces an isomorphism of triangles

$$\begin{array}{ccccccc}
B & \xrightarrow{\iota(\alpha)} & \mathrm{Cone}(\alpha) & \xrightarrow{\pi(\alpha)} & \Sigma(A) & \xrightarrow{-\Sigma(\alpha)} & \Sigma(B) \\
\| & \circlearrowleft & \| & \circlearrowleft & \vdots\, \exists\lambda & \circlearrowleft & \| \\
B & \xrightarrow[\iota(\alpha)]{} & \mathrm{Cone}(\alpha) & \xrightarrow[\iota(\iota(\alpha))]{} & \mathrm{Cone}(\iota(\alpha)) & \xrightarrow[\pi(\iota(\alpha))]{} & \Sigma(B)
\end{array}$$

TR′ 3. We have an algorithm which for a given quadruple of morphisms α_1, u, v and α_2 with $\alpha_1 \cdot v = u \cdot \alpha_2$, computes a morphism $w\colon \mathrm{Cone}(\alpha_1) \to \mathrm{Cone}(\alpha_2)$ (not necessarily unique) that renders the following diagram

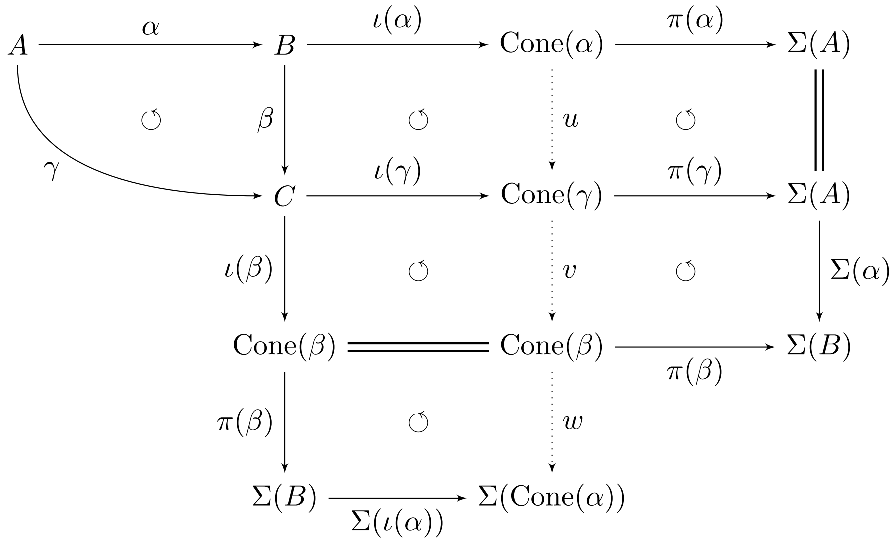
$$\begin{array}{ccccccc}
A_1 & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\iota(\alpha_1)} & \mathrm{Cone}(\alpha_1) & \xrightarrow{\pi(\alpha_1)} & \Sigma(A_1) \\
u \downarrow & \circlearrowleft & v \downarrow & \circlearrowleft & \vdots\, \exists w & \circlearrowleft & \downarrow \Sigma(u) \\
A_2 & \xrightarrow[\alpha_2]{} & B_2 & \xrightarrow[\iota(\alpha_2)]{} & \mathrm{Cone}(\alpha_2) & \xrightarrow[\pi(\alpha_2)]{} & \Sigma(A_2)
\end{array}$$

commutative.

TR′ 4. We have an algorithm which for a given triple of morphisms α, β and γ with $\gamma = \alpha \cdot \beta$ computes another triple of morphisms

$$\mathrm{Cone}(\alpha) \xrightarrow{u} \mathrm{Cone}(\gamma) \xrightarrow{v} \mathrm{Cone}(\beta) \xrightarrow{w} \Sigma(\mathrm{Cone}(\alpha))$$

which renders the following diagram



commutative; and computes an isomorphism $\lambda\colon \mathrm{Cone}(\beta) \xrightarrow{\sim} \mathrm{Cone}(u)$ with $v \cdot \lambda = \iota(u)$ and $\lambda \cdot \pi(u) = w$.

Remark 5.2. Let $\mathfrak{T}$ be a precomputable triangulated category. Then $\mathfrak{T}$ satisfies **TR 1**. Since every exact triangle in $\mathfrak{T}$ is isomorphic to a standard exact triangle, $\mathfrak{T}$ satisfies **TR 2**, **TR 3** and **TR 4**. In particular, every precomputable triangulated category is triangulated in the sense of Definition B.1. By Remark B.20, the converse is true if all existential quantifiers in the Definition B.1 are realized by algorithms.

Definition 5.3. A precomputable triangulated category $\mathfrak{T}$ will be called **computable triangulated** if there is an algorithm which computes the isomorphism in **TR′ 1.c** or disproves its existence. In other words, there is an algorithm which decides whether a given triangle is exact.

Lemma 5.4. *Let $\mathfrak{T}$ be a precomputable triangulated category. If $\mathfrak{T}$ is equipped with a $\mathscr{D}$-homomorphism structure and $\mathscr{D}$ has decidable lifts, then $\mathfrak{T}$ is computable triangulated.*

Proof. Suppose, we are given a triangle

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A).$$

By Theorem 4.17, $\mathfrak{T}$ has decidable linear systems. We check the solvability of the two-sided linear system

$$\iota \cdot \chi = \iota(\alpha), \qquad \chi \cdot \pi(\alpha) = \pi,$$

and in the affirmative case, we compute a solution χ and check whether it is an isomorphism[3].

If the system is solvable and χ is an isomorphism, then the triangle is exact by **TR′ 1.c**. Otherwise, by Lemma B.11, the triangle is not exact. □

Lemma 5.5. *Let $\mathfrak{T}$ be an additive category which satisfies* **TR′ 0, 1.a, 1.b, 2, 3, 4**. *Then, the following two axioms are equivalent*

[3]The morphism χ is an isomorphism if and only if the two-sided linear system $\chi \cdot \chi' = \mathrm{id}_C, \chi' \cdot \chi = \mathrm{id}_{\mathrm{Cone}(\alpha)}$ is solvable.

- **TR′ 1.c**.
- **TR″ 1.c** : *A triangle*
$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$$
is exact if and only if it is isomorphic to some standard exact triangle.

Proof. The direct implication is obvious. For the converse, let (u, v, w) be an isomorphism of triangles:

$$\begin{array}{ccccccc}
A & \xrightarrow{\alpha} & B & \xrightarrow{\iota} & C & \xrightarrow{\pi} & \Sigma(A) \\
u \downarrow \wr & \circlearrowleft & v \downarrow \wr & \circlearrowleft & w \downarrow \wr & \circlearrowleft & \wr \downarrow \Sigma(u) \\
X & \xrightarrow[f]{} & Y & \xrightarrow[\iota(f)]{} & \mathrm{Cone}(f) & \xrightarrow[\pi(f)]{} & \Sigma(X)
\end{array}$$

By **TR′ 3**, there exists a morphism $\mu\colon \mathrm{Cone}(f) \to \mathrm{Cone}(\alpha)$ which induces a morphism of exact triangles

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{\iota(f)} & \mathrm{Cone}(f) & \xrightarrow{\pi(f)} & \Sigma(X) \\
u^{-1} \downarrow \wr & \circlearrowleft & v^{-1} \downarrow \wr & \circlearrowleft & \mu \downarrow & \circlearrowleft & \wr \downarrow \Sigma(u^{-1}) \\
A & \xrightarrow[\alpha]{} & B & \xrightarrow[\iota(\alpha)]{} & \mathrm{Cone}(\alpha) & \xrightarrow[\pi(\alpha)]{} & \Sigma(A)
\end{array}$$

Analogously to the proof of Lemma B.11, μ is an isomorphism. Hence the isomorphism $\lambda := w \cdot \mu\colon C \xrightarrow{\sim} \mathrm{Cone}(\alpha)$ satisfies $\iota \cdot \lambda = \iota(\alpha)$ and $\lambda \cdot \pi(\alpha) = \pi$. □

5.2. Homotopy Categories are Triangulated

It is a well-known fact that homotopy categories are triangulated. However, due to the algorithmic requirements, they can be computable only if they are bounded. Hence, we consider in this section only the bounded homotopy categories of additive categories.

We start by specifying the shift automorphism:

Definition 5.6. Let $\mathscr{C}$ be an additive category and $\mathcal{K}^b(\mathscr{C})$ its bounded homotopy category. The **shift automorphism** on $\mathcal{K}^b(\mathscr{C})$ is defined by

$$\Sigma\colon \begin{cases} \mathcal{K}^b(\mathscr{C}) & \to \mathcal{K}^b(\mathscr{C}), \\ A = (\partial_A^i)_{i\in\mathbb{Z}} & \mapsto \Sigma(A) := \left(-\partial_A^{i+1}\right)_{i\in\mathbb{Z}}, \\ \varphi = (\varphi^i)_{i\in\mathbb{Z}}\colon A \to B & \mapsto \Sigma(\varphi) = (\varphi^{i+1})_{i\in\mathbb{Z}}\colon \Sigma(A) \to \Sigma(B) \end{cases}$$

and we denote its inverse by Σ^{-1}. It is obvious that $\Sigma \cdot \Sigma^{-1} = \mathrm{id}_{\mathcal{K}^b(\mathscr{C})} = \Sigma \cdot \Sigma^{-1}$ “on the nose”.

Definition 5.7. For a morphism $\alpha\colon A \to B$ in $\mathcal{K}^b(\mathscr{C})$, we define

(1) the **mapping cone** $\operatorname{Cone}(\alpha)$ by the object in $\mathcal{K}^b(\mathscr{C})$ whose differential at $i \in \mathbb{Z}$ is given by

$$\partial^i_{\operatorname{Cone}(\alpha)} \coloneqq A^{i+1} \oplus B^i \xrightarrow{\begin{pmatrix} -\partial_A^{i+1} & \alpha^{i+1} \\ 0 & \partial_B^i \end{pmatrix}} A^{i+2} \oplus B^{i+1}.$$

(2) the **natural injection to the mapping cone** $\iota(\alpha)$ by the morphism

$$\iota(\alpha)\colon B \to \operatorname{Cone}(\alpha)$$

whose component at $i \in \mathbb{Z}$ is

$$B^i \xrightarrow{(0 \ \mathrm{id}_{B^i})} A^{i+1} \oplus B^i.$$

(3) the **natural projection from the mapping cone** $\pi(\alpha)$ by the morphism

$$\pi(\alpha)\colon \operatorname{Cone}(\alpha) \to \Sigma(A)$$

whose component at $i \in \mathbb{Z}$ is given by

$$A^{i+1} \oplus B^i \xrightarrow{\begin{pmatrix} \mathrm{id}_{A^{i+1}} \\ 0 \end{pmatrix}} A^{i+1}.$$

(4) the **mapping cone triangle** $\mathrm{Tr}^{st}(\alpha)$ by the triangle

$$A \xrightarrow{\alpha} B \xrightarrow{\iota(\alpha)} \operatorname{Cone}(\alpha) \xrightarrow{\pi(\alpha)} \Sigma(A).$$

Definition 5.8. A triangle $A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$ in $\mathcal{K}^b(\mathscr{C})$ will be called **exact** if it is isomorphic to some mapping cone triangle. The class of all exact triangles in $\mathscr{C}$ will be denoted by $\triangle$.

Theorem 5.9. *Let $\mathscr{C}$ be a computable additive category. Suppose $\mathscr{C}$ is equipped with a $\mathscr{D}$-homomorphism structure where $\mathscr{D}$ is Abelian and has decidable lifts, then $\left(\mathcal{K}^b(\mathscr{C}), \Sigma, \triangle\right)$ is a computable triangulated category.*

Proof. By Theorem 4.34, $\mathcal{K}^b(\mathscr{C})$ can be equipped with a $\mathscr{D}$-homomorphism structure, hence has decidable linear systems by Theorem 4.17. According to Lemma 5.4, it is sufficient to prove that $\mathcal{K}^b(\mathscr{C})$ is a precomputable triangulated category. In the following we show that the axioms of Definition 5.1 are satisfied:

TR′ 0 The shift functor Σ, its inverse Σ^{-1} and the associated natural transformations are already introduced in Definition 5.6.

TR′ 1 (a) For a given morphism $\alpha\colon A \to B$ in $\mathcal{K}^b(\mathscr{C})$, we can compute $\operatorname{Cone}(\alpha)$, $\iota(\alpha)$ and $\pi(\alpha)$ as introduced in Definition 5.7. In particular, the standard exact triangles are the mapping cone triangles.

(b) Let A be an object in $\mathcal{K}^b(\mathscr{C})$. By Remark 3.20, the mapping cone $\operatorname{Cone}(\mathrm{id}_A)$ is contractible, hence is isomorphic to the zero object by Remark 3.22.

(c) Any exact triangle

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$$

in $\mathcal{K}^b(\mathscr{C})$ is by definition isomorphic to a mapping cone triangle. By Lemma 5.5, **TR′ 1.c** follows.

TR′ 2 For a given morphism $\alpha\colon A \to B$, we define $\lambda\colon \Sigma(A) \to \mathrm{Cone}(\iota(\alpha))$ by the morphism whose component at $i \in \mathbb{Z}$ is

$$\lambda^i := A^{i+1} \xrightarrow{\left(\alpha^{i+1}\ \mathrm{id}_{A^{i+1}}\ 0\right)} B^{i+1} \oplus A^{i+1} \oplus B^i.$$

A direct verification shows that $\pi(\alpha)\cdot\lambda = \iota(\iota(\alpha))$, $-\Sigma(\alpha) = \lambda\cdot\pi(\iota(\alpha))$. Furthermore, λ is an isomorphism and its inverse $\mu\colon \mathrm{Cone}(\iota(\alpha)) \to \Sigma(A)$ is given at $i \in \mathbb{Z}$ by

$$A^{i+1} \xrightarrow{\begin{pmatrix} 0 \\ \mathrm{id}_{A^{i+1}} \\ 0 \end{pmatrix}} B^{i+1} \oplus A^{i+1} \oplus B^i.$$

TR′ 3 We should prove that any commutative square

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha_1} & B_1 \\ {\scriptstyle u}\downarrow & \circlearrowleft & \downarrow{\scriptstyle v} \\ A_2 & \xrightarrow[\alpha_2]{} & B_2 \end{array}$$

can be completed into a morphism between the standard exact triangles associated to α_1 and α_2. We start by computing a chain homotopy $\left(h^i\colon A_1^i \to B_2^{i-1}\right)_{i\in\mathbb{Z}}$ associated to $\alpha_1\cdot v - u\cdot\alpha_2$. Then $w\colon \mathrm{Cone}(\alpha_1) \to \mathrm{Cone}(\alpha_2)$ whose component at $i \in \mathbb{Z}$ is

$$w^i := A_1^{i+1} \oplus B_1^i \xrightarrow{\begin{pmatrix} u^{i+1} & h^{i+1} \\ 0 & v^i \end{pmatrix}} A_2^{i+1} \oplus B_2^i$$

renders the diagram

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\iota(\alpha_1)} & \mathrm{Cone}(\alpha_1) & \xrightarrow{\pi(\alpha_1)} & \Sigma(A_1) \\ {\scriptstyle u}\downarrow & \circlearrowleft & {\scriptstyle v}\downarrow & \circlearrowleft & {\scriptstyle w}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \Sigma(u)} \\ A_2 & \xrightarrow[\alpha_2]{} & B_2 & \xrightarrow[\iota(\alpha_2)]{} & \mathrm{Cone}(\alpha_2) & \xrightarrow[\pi(\alpha_2)]{} & \Sigma(A_2) \end{array}$$

commutative. The set of all morphisms of this form will be called **standard morphisms between the standard cone objects** and will be denoted by $\mathrm{ConeMors}^{st}_{\alpha_1,\alpha_2}(u, v)$.

TR′ 4 We should prove that any triple of morphisms $\alpha\colon A \to B$, $\beta\colon B \to C$ and $\gamma = \alpha \bullet \beta$ can be completed to the following diagram

$$
\begin{array}{ccccccc}
A & \xrightarrow{\ \alpha\ } & B & \xrightarrow{\ \iota(\alpha)\ } & \mathrm{Cone}(\alpha) & \xrightarrow{\ \pi(\alpha)\ } & \Sigma(A) \\
 & \circlearrowleft & \downarrow{\scriptstyle \beta} & \circlearrowleft & \downarrow{\scriptstyle u} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}_{\Sigma(A)}} \\
{\scriptstyle \gamma}\ \searrow & & C & \xrightarrow{\ \iota(\gamma)\ } & \mathrm{Cone}(\gamma) & \xrightarrow{\ \pi(\gamma)\ } & \Sigma(A) \\
 & & \downarrow{\scriptstyle \iota(\beta)} & \circlearrowleft & \downarrow{\scriptstyle v} & \circlearrowleft & \downarrow{\scriptstyle \Sigma(\alpha)} \\
 & & \mathrm{Cone}(\beta) & \xrightarrow[\ \mathrm{id}_{\mathrm{Cone}(\beta)}\ ]{} & \mathrm{Cone}(\beta) & \xrightarrow[\ \pi(\beta)\ ]{} & \Sigma(B) \\
 & & \downarrow{\scriptstyle \pi(\beta)} & \circlearrowleft & \downarrow{\scriptstyle w} & & \\
 & & \Sigma(B) & \xrightarrow[\ \Sigma(\iota(\alpha))\ ]{} & \Sigma(\mathrm{Cone}(\alpha)), & &
\end{array}
$$

where the middle column is an exact triangle. Let $\left(h^i\colon A^i \to C^{i-1}\right)_{i\in\mathbb{Z}}$ be a chain homotopy associated to $\alpha\bullet\beta - \gamma$. A straightforward verification shows that the morphisms:

- $u\colon \mathrm{Cone}(\alpha) \to \mathrm{Cone}(\gamma)$ whose component at $i \in \mathbb{Z}$ is
$$u^i \coloneqq A^{i+1} \oplus B^i \xrightarrow{\begin{pmatrix} \mathrm{id}_{A^{i+1}} & h^{i+1} \\ 0 & \beta^i \end{pmatrix}} A^{i+1} \oplus C^i,$$
- $v\colon \mathrm{Cone}(\gamma) \to \mathrm{Cone}(\beta)$ whose component at $i \in \mathbb{Z}$ is
$$v^i \coloneqq A^{i+1} \oplus C^i \xrightarrow{\begin{pmatrix} \alpha^{i+1} & -h^{i+1} \\ 0 & \mathrm{id}_{C^i} \end{pmatrix}} B^{i+1} \oplus C^i,$$
- $w\colon \mathrm{Cone}(\beta) \to \Sigma(\mathrm{Cone}(\alpha))$ whose component at $i \in \mathbb{Z}$ is
$$w^i \coloneqq B^{i+1} \oplus C^i \xrightarrow{\begin{pmatrix} 0 & \mathrm{id}_{B^{i+1}} \\ 0 & 0 \end{pmatrix}} A^{i+2} \oplus B^{i+1}$$

render the above diagram commutative. Moreover, the triangle

$$\mathrm{Cone}(\alpha) \xrightarrow{u} \mathrm{Cone}(\gamma) \xrightarrow{v} \mathrm{Cone}(\beta) \xrightarrow{w} \Sigma(\mathrm{Cone}(\alpha))$$

is isomorphic to the standard cone triangle $\mathrm{Tr}^{st}(u)$ via the isomorphism

$$p\colon \mathrm{Cone}(\beta) \to \mathrm{Cone}(u)$$

defined at $i \in \mathbb{Z}$ by

$$p^i \coloneqq B^{i+1} \oplus C^i \xrightarrow{\begin{pmatrix} 0 & \mathrm{id}_{B^{i+1}} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{C^i} \end{pmatrix}} A^{i+2} \oplus B^{i+1} \oplus A^{i+1} \oplus C^i;$$

whose inverse $q\colon \mathrm{Cone}(u) \to \mathrm{Cone}(\beta)$ is given at $i \in \mathbb{Z}$ by

$$q^i := B^{i+1} \oplus C^i \xrightarrow{\begin{pmatrix} 0 & 0 \\ \mathrm{id}_{B^{i+1}} & 0 \\ \alpha^{i+1} & -h^{i+1} \\ 0 & \mathrm{id}_{C^i} \end{pmatrix}} A^{i+2} \oplus B^{i+1} \oplus A^{i+1} \oplus C^i.$$

□

Remark 5.10. Let $\alpha\colon A \to B$ be a morphism in $\mathcal{K}^b(\mathscr{C})$ and

$$A \xrightarrow{\alpha} B \xrightarrow{\iota(\alpha)} \mathrm{Cone}(\alpha) \xrightarrow{\pi(\alpha)} \Sigma(A)$$

the associated standard exact triangle. The inverse rotation[4]

$$\Sigma^{-1}(\mathrm{Cone}(\alpha)) \xrightarrow{-\Sigma^{-1}(\pi(\alpha))} A \xrightarrow{\alpha} B \xrightarrow{\iota(\alpha)} \mathrm{Cone}(\alpha)$$

is isomorphic to the standard exact triangle associated to $-\Sigma^{-1}(\pi(\alpha))$ via the isomorphism $\lambda\colon B \to \mathrm{Cone}(-\Sigma^{-1}(\pi(\alpha)))$ given at $i \in \mathbb{Z}$ by

$$\lambda^i := B^i \xrightarrow{(0\ \mathrm{id}_{B^i}\ 0)} A^{i+1} \oplus B^i \oplus A^i;$$

and whose inverse $\mu := \lambda^{-1}\colon \mathrm{Cone}(-\Sigma^{-1}(\pi(\alpha))) \to B$ is given at $i \in \mathbb{Z}$ by

$$\mu^i := A^{i+1} \oplus B^i \oplus A^i \xrightarrow{\begin{pmatrix} 0 \\ \mathrm{id}_{B^i} \\ \alpha^i \end{pmatrix}} B^i.$$

The object $\Sigma^{-1}(\mathrm{Cone}(\alpha))$ will be called the **standard cocone object** of α, and will be denoted by $\mathrm{Cocone}(\alpha)$.

Remark 5.11. By the previous Remark and Lemma B.5, every morphism $\alpha\colon A \to B$ in $\mathcal{K}^b(\mathscr{C})$ can be completed to an exact triangle:

$$\mathrm{Cocone}(\alpha) \xrightarrow{\Sigma^{-1}(\pi(\alpha))} A \xrightarrow{\alpha} B \xrightarrow{-\iota(\alpha)} \mathrm{Cone}(\alpha).$$

Suppose α_1, u, v, α_2 are morphisms $\mathcal{K}^b(\mathscr{C})$ as in **TR′ 3**, then each morphism $w \in \mathrm{ConeMors}^{st}_{\alpha_1,\alpha_2}(u,v)$ gives rise to a morphism of exact triangles

$$\begin{array}{ccccccc}
\mathrm{Cocone}(\alpha_1) & \xrightarrow{\Sigma^{-1}(\pi(\alpha_1))} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{-\iota(\alpha_1)} & \mathrm{Cone}(\alpha_1) \\
\Big\downarrow{\scriptstyle \Sigma^{-1}(w)} & \circlearrowleft & \Big\downarrow{\scriptstyle u} & \circlearrowleft & \Big\downarrow{\scriptstyle v} & \circlearrowleft & \Big\downarrow{\scriptstyle w} \\
\mathrm{Cocone}(\alpha_2) & \xrightarrow[\Sigma^{-1}(\pi(\alpha_2))]{} & B_1 & \xrightarrow[\alpha_2]{} & B_2 & \xrightarrow[-\iota(\alpha_2)]{} & \mathrm{Cone}(\alpha_2)
\end{array}$$

The set $\{\Sigma^{-1}(w), w \in \mathrm{ConeMors}^{st}_{\alpha_1,\alpha_2}(u,v)\}$ will be denoted by $\mathrm{CoconeMors}^{st}_{\alpha_1,\alpha_2}(u,v)$. We will refer to the elements of $\mathrm{CoconeMors}^{st}_{\alpha_1,\alpha_2}(u,v)$ as the **standard morphisms between the standard cocone objects**.

[4] See Corollary B.15.

Lemma 5.12. *Let $\mathscr{C}$ be an additive category and $\mathcal{K}^b(\mathscr{C})$ be its homotopy category. Then for any commutative diagram*

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha_1} & B_1 \\ {\scriptstyle u_1}\downarrow & \circlearrowleft & \downarrow{\scriptstyle v_1} \\ A_2 & \xrightarrow[\alpha_2]{} & B_2 \\ {\scriptstyle u_2}\downarrow & \circlearrowleft & \downarrow{\scriptstyle v_2} \\ A_3 & \xrightarrow[\alpha_3]{} & B_3; \end{array}$$

if $w_1 \in \operatorname{ConeMors}^{st}_{\alpha_1,\alpha_2}(u_1, v_1)$ and $w_2 \in \operatorname{ConeMors}^{st}_{\alpha_2,\alpha_3}(u_2, v_2)$, then

$$w_1 \bullet w_2 \in \operatorname{ConeMors}^{st}_{\alpha_1,\alpha_3}(u_1 \bullet u_2, v_1 \bullet v_2).$$

Proof. Suppose w_1 and w_2 have been constructed by using chain-homotopies $\left(h_1^i\colon A_1^i \to B_2^{i-1}\right)_{i\in\mathbb{Z}}$ resp. $\left(h_2^i\colon A_2^i \to B_3^{i-1}\right)_{i\in\mathbb{Z}}$. In other words, we have

$$\alpha_1^i \bullet v_1^i - u_1^i \bullet \alpha_2^i = \partial^i_{A_1} \bullet h_1^{i+1} + h_1^i \bullet \partial^{i-1}_{B_2} \quad \text{and}$$
$$\alpha_2^i \bullet v_2^i - u_2^i \bullet \alpha_3^i = \partial^i_{A_2} \bullet h_2^{i+1} + h_2^i \bullet \partial^{i-1}_{B_3}$$

for all $i \in \mathbb{Z}$. Then, by the following computation

$$\begin{aligned} \alpha_1^i \bullet \left(v_1^i \bullet v_2^i\right) - \left(u_1^i \bullet u_2^i\right) \bullet \alpha_3^i =& \left(\alpha_1^i \bullet v_1^i\right) \bullet v_2^i - u_1^i \bullet \left(u_2^i \bullet \alpha_3^i\right) \\ =& \left(u_1^i \bullet \alpha_2^i + \partial^i_{A_1} \bullet h_1^{i+1} + h_1^i \bullet \partial^{i-1}_{B_2}\right) \bullet v_2^i \\ & - u_1^i \bullet \left(\alpha_2^i \bullet v_2^i - \partial^i_{A_2} \bullet h_2^{i+1} - h_2^i \bullet \partial^{i-1}_{B_3}\right) \\ =& u_1^i \bullet \alpha_2^i \bullet v_2^i + \partial^i_{A_1} \bullet h_1^{i+1} \bullet v_2^i + h_1^i \bullet \partial^{i-1}_{B_2} \bullet v_2^i \\ & - u_1^i \bullet \alpha_2^i \bullet v_2^i + u_1^i \bullet \partial^i_{A_2} \bullet h_2^{i+1} + u_1^i \bullet h_2^i \bullet \partial^{i-1}_{B_3} \\ =& \partial^i_{A_1} \bullet h_1^{i+1} \bullet v_2^i + h_1^i \bullet v_2^{i-1} \bullet \partial^{i-1}_{B_3} + \partial^i_{A_1} \bullet u_1^{i+1} \bullet h_2^{i+1} + u_1^i \bullet h_2^i \bullet \partial^{i-1}_{B_3} \\ =& \partial^i_{A_1} \bullet \left(h_1^{i+1} \bullet v_2^i + u_1^{i+1} \bullet h_2^{i+1}\right) + \left(h_1^i \bullet v_2^{i-1} + u_1^i \bullet h_2^i\right) \bullet \partial^{i-1}_{B_3}; \end{aligned}$$

the family $\left(h_1^i \bullet v_2^{i-1} + u_1^i \bullet h_2^i\right)_{i\in\mathbb{Z}}$ is a chain homotopy for $\alpha_1 \bullet v_1 \bullet v_2 - u_1 \bullet u_2 \bullet \alpha_3$. Hence, the morphism $w_1 \bullet w_2\colon \operatorname{Cone}(\alpha_1) \to \operatorname{Cone}(\alpha_3)$, whose component at $i \in \mathbb{Z}$ is

$$\begin{pmatrix} u_1^{i+1} & h_1^{i+1} \\ 0 & v_1^i \end{pmatrix} \cdot \begin{pmatrix} u_2^{i+1} & h_2^{i+1} \\ 0 & v_2^i \end{pmatrix} = \begin{pmatrix} u_1^{i+1} \bullet u_2^{i+1} & h_1^{i+1} \bullet v_2^i + u_1^{i+1} \bullet h_2^{i+1} \\ 0 & v_1^i \bullet v_2^i \end{pmatrix},$$

belongs to $\operatorname{ConeMors}^{st}_{\alpha_1,\alpha_3}(u_1 \bullet u_2, v_1 \bullet v_2)$. ☐

In a similar way, we can prove the following lemma:

Lemma 5.13. *Let $\mathscr{C}$ be an additive category and $\mathcal{K}^b(\mathscr{C})$ be its homotopy category. Then for any commutative diagram*

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha_1} & B_1 \\ {\scriptstyle u_1}\downarrow\downarrow{\scriptstyle u_2} & \circlearrowleft & {\scriptstyle v_1}\downarrow\downarrow{\scriptstyle v_2} \\ A_2 & \xrightarrow[\alpha_2]{} & B_2 \end{array}$$

if $w_1 \in \mathrm{ConeMors}^{st}_{\alpha_1,\alpha_2}(u_1, v_1)$ and $w_2 \in \mathrm{ConeMors}^{st}_{\alpha_1,\alpha_2}(u_2, v_2)$, then

- $w_1 + w_2 \in \mathrm{ConeMors}^{st}_{\alpha_1,\alpha_2}(u_1 + u_2, v_1 + v_2)$,
- $-w_1 \in \mathrm{ConeMors}^{st}_{\alpha_1,\alpha_2}(-u_1, -v_1)$.

Proof. If $\left(h^i_1 \colon A^i_1 \to B^{i-1}_2\right)_{i\in\mathbb{Z}}$ and $\left(h^i_2 \colon A^i_1 \to B^{i-1}_2\right)_{i\in\mathbb{Z}}$ are chain-homotopies of $\alpha_1 \bullet v_1 - u_1 \bullet \alpha_2$ resp. $\alpha_1 \bullet v_2 - u_2 \bullet \alpha_2$, then $\left(h^i_1 + h^i_2 \colon A^i_1 \to B^{i-1}_2\right)_{i\in\mathbb{Z}}$ is a chain homotopy associated to $\alpha_1 \bullet (v_1 - v_2) - (u_1 - u_2) \bullet \alpha_2$. Similarly, $\left(-h^i_1 \colon A^i_1 \to B^{i-1}_2\right)_{i\in\mathbb{Z}}$ is a chain homotopy associated to w_1. □

Lemma 5.14. *Let $\mathscr{C}$ be an additive category and $\mathcal{K}^b(\mathscr{C})$ be its homotopy category. For any morphism $\alpha \colon A \to B$ and any $\ell \in \mathbb{Z}$, we have*

$$\Sigma^\ell(\mathrm{Cone}(\alpha)) = \mathrm{Cone}\Big(\Sigma^\ell\Big((-1)^\ell \cdot \alpha\Big)\Big) = \mathrm{Cone}\Big((-1)^\ell \cdot \Sigma^\ell(\alpha)\Big).$$

Proof. The differential at $i \in \mathbb{Z}$ of the above complexes is given by

$$A^{i+1+\ell} \oplus B^{i+\ell} \xrightarrow{\begin{pmatrix} (-1)^{\ell+1}\cdot\partial_A^{i+1+\ell} & (-1)^\ell\cdot\alpha^{i+1+\ell} \\ 0 & (-1)^\ell\cdot\partial_B^{i+\ell} \end{pmatrix}} A^{i+2+\ell} \oplus B^{i+1+\ell}.$$

□

Lemma 5.15. *Let $\mathscr{C}$ be an additive category and $\mathcal{K}^b(\mathscr{C})$ be its homotopy category. Then for any commutative diagram*

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha_1} & B_1 \\ {\scriptstyle u}\downarrow & \circlearrowleft & \downarrow{\scriptstyle v} \\ A_2 & \xrightarrow[\alpha_2]{} & B_2 \end{array}$$

if $w \in \mathrm{ConeMors}^{st}_{\alpha_1,\alpha_2}(u, v)$ then $\Sigma^\ell(w) \in \mathrm{ConeMors}^{st}_{(-1)^\ell\cdot\Sigma^\ell(\alpha_1),(-1)^\ell\cdot\Sigma^\ell(\alpha_2)}\Big(\Sigma^\ell(u), \Sigma^\ell(v)\Big)$.

Proof. It follows from Lemma 5.14 that $\Sigma^\ell(\mathrm{Cone}(\alpha_i)) = \mathrm{Cone}\Big((-1)^\ell \cdot \Sigma^\ell(\alpha_i)\Big)$ for $i = 1, 2$. Let $w \colon \mathrm{Cone}(\alpha_1) \to \mathrm{Cone}(\alpha_2)$ be a standard morphism whose component at $i \in \mathbb{Z}$ is given by

$$w^i = A^{i+1}_1 \oplus B^i_1 \xrightarrow{\begin{pmatrix} u^{i+1} & h^{i+1} \\ 0 & v^i \end{pmatrix}} A^{i+1}_2 \oplus B^i_2,$$

where $\left(h^i\colon A_1^i \to B_2^{i-1}\right)_{i\in\mathbb{Z}}$ is a chain homotopy associated to $\alpha_1 \bullet v - u \bullet \alpha_2$. A direct verification shows that $\left(h^{\ell+i}\colon A_1^{\ell+i} \to B_2^{\ell+i-1}\right)_{i\in\mathbb{Z}}$ is a chain homotopy associated to

$$\Big((-1)^\ell \cdot \Sigma^\ell(\alpha_1)\Big) \bullet \Sigma^\ell(v) - \Sigma^\ell(u) \bullet \Big((-1)^\ell \cdot \Sigma^\ell(\alpha_2)\Big),$$

i.e., the associated standard morphism

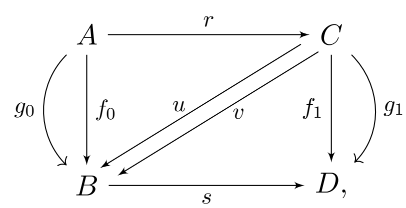
$$t\colon \operatorname{Cone}\Big((-1)^\ell \cdot \Sigma^\ell(\alpha_1)\Big) \to \operatorname{Cone}\Big((-1)^\ell \cdot \Sigma^\ell(\alpha_2)\Big)$$

is given at $i \in \mathbb{Z}$ by

$$t^i = A_1^{\ell+i+1} \oplus B_1^{\ell+i} \xrightarrow{\begin{pmatrix} u^{\ell+i+1} & h^{\ell+i+1} \\ 0 & v^{\ell+i} \end{pmatrix}} A_2^{\ell+i+1} \oplus B_2^{\ell+i},$$

i.e., $t = \Sigma^\ell(w)$ and the assertion follows. □

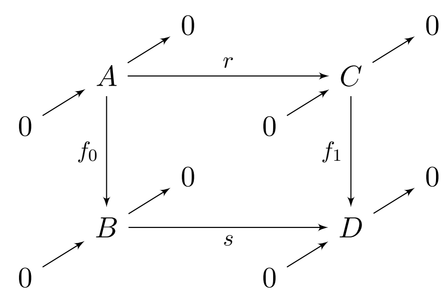
Example 5.16. Let k be a field and $\mathscr{A}$ be the finitely presented category defined by the right quiver



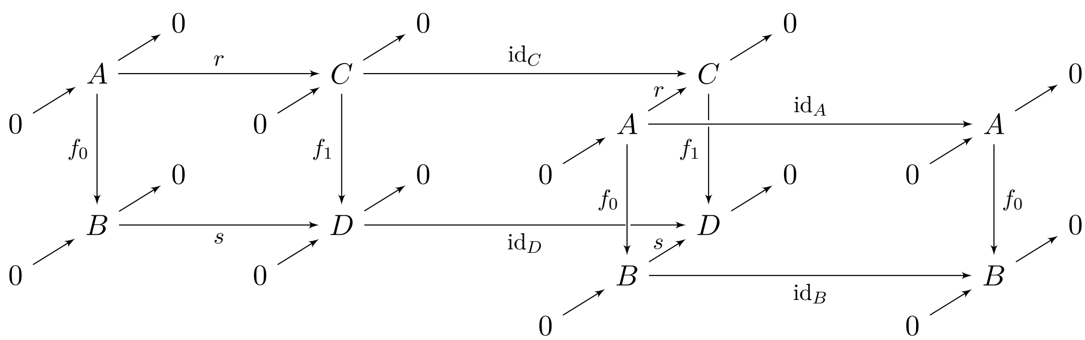
subject to the relations

$$\{rf_1 - f_0 s, rg_1 - g_0 s, f_0 - g_0 - ru, f_1 - g_1 - vs\}.$$

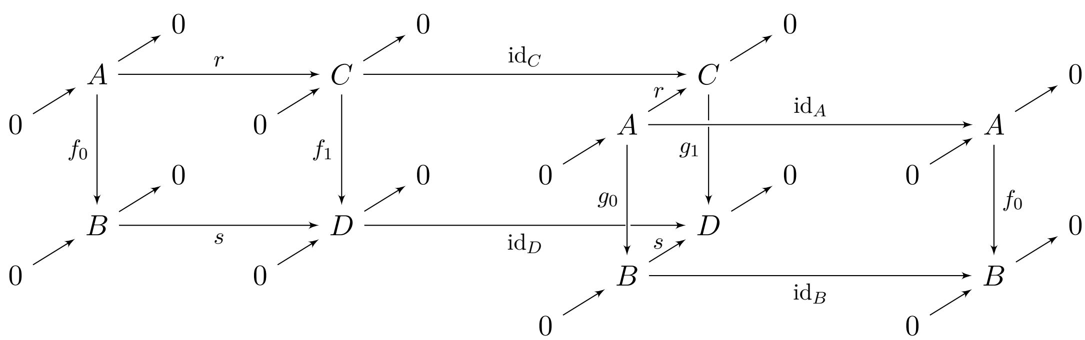
The following commutative square



in $\mathcal{K}^b(\mathscr{A}^\oplus)$ can be completed into a morphism of exact triangles in two different ways:



and



The first morphism is standard, i.e., it belongs to $\mathrm{ConeMors}^{st}_{\lceil r \rfloor_0, \lceil s \rfloor_0}(\lceil f_0 \rfloor_0, \lceil f_1 \rfloor_0)$. However, the second morphism is not standard.

Lemma 5.17. *Let $\mathscr{C}$ be an additive category and $\mathcal{K}^b(\mathscr{C})$ be its homotopy category. Let A_i, B_i for $i = 1, 2$ be objects in $\mathcal{K}^b(\mathscr{C})$ such that $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma(A_1), B_2) = 0$. Then for any commutative diagram*

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha_1} & B_1 \\ {\scriptstyle u_1}\downarrow & \circlearrowleft & \downarrow{\scriptstyle v_1} \\ A_2 & \xrightarrow[\alpha_2]{} & B_2, \end{array}$$

the set $\mathrm{ConeMors}^{st}_{\alpha_1, \alpha_2}(u_1, v_1)$ is a singleton set.

Proof. We will prove that the morphism resulted **TR′ 3** in Theorem 5.9 does not depend on the choice of representatives for u_1 and v_1. Precisely, given u_2 and v_2 with $u_1 = u_2$ and $v_1 = v_2$ we will prove that $\mathrm{ConeMors}^{st}_{\alpha_1, \alpha_2}(u_1, v_1) = \mathrm{ConeMors}^{st}_{\alpha_1, \alpha_2}(u_2, v_2)$.

Since $\alpha_1 \bullet v_1 - u_1 \bullet \alpha_2 = 0$, there exists a family of morphisms $\left(h^i \colon A_1^i \to B_2^{i-1}\right)_{i \in \mathbb{Z}}$ with

$$\alpha_1^i \bullet v_1^i - u_1^i \bullet \alpha_2^i = \partial_{A_1}^i \bullet h^{i+1} + h^i \bullet \partial_{B_2}^{i-1}$$

for all $i \in \mathbb{Z}$. Hence, **TR′ 3** induces the morphism $w_1\colon \operatorname{Cone}(\alpha_1) \to \operatorname{Cone}(\alpha_2)$ defined by

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & A_1^i \oplus B_1^{i-1} & \xrightarrow{\left(\begin{smallmatrix} -\partial_{A_1}^i & \alpha_1^i \\ 0 & \partial_{B_1}^{i-1} \end{smallmatrix}\right)} & A_1^{i+1} \oplus B_1^i & \longrightarrow & \cdots \\
 & & {\scriptstyle w_1^{i-1} := \left(\begin{smallmatrix} u_1^i & h^i \\ 0 & v_1^{i-1} \end{smallmatrix}\right)} \big\downarrow & & \big\downarrow {\scriptstyle \left(\begin{smallmatrix} u_1^{i+1} & h^{i+1} \\ 0 & v_1^i \end{smallmatrix}\right)} & & \\
\cdots & \longrightarrow & A_2^i \oplus B_2^{i-1} & \xrightarrow[\left(\begin{smallmatrix} -\partial_{A_2}^i & \alpha_2^i \\ 0 & \partial_{B_2}^{i-1} \end{smallmatrix}\right)]{} & A_2^{i+1} \oplus B_2^i & \longrightarrow & \cdots
\end{array}$$

Similarly, there exists a family of morphisms $\left(\ell^i\colon A_1^i \to B_2^{i-1}\right)_{i\in\mathbb{Z}}$ with

$$\alpha_1^i \bullet v_2^i - u_2^i \bullet \alpha_2^i = \partial_{A_1}^i \bullet \ell^{i+1} + \ell^i \bullet \partial_{B_2}^{i-1}$$

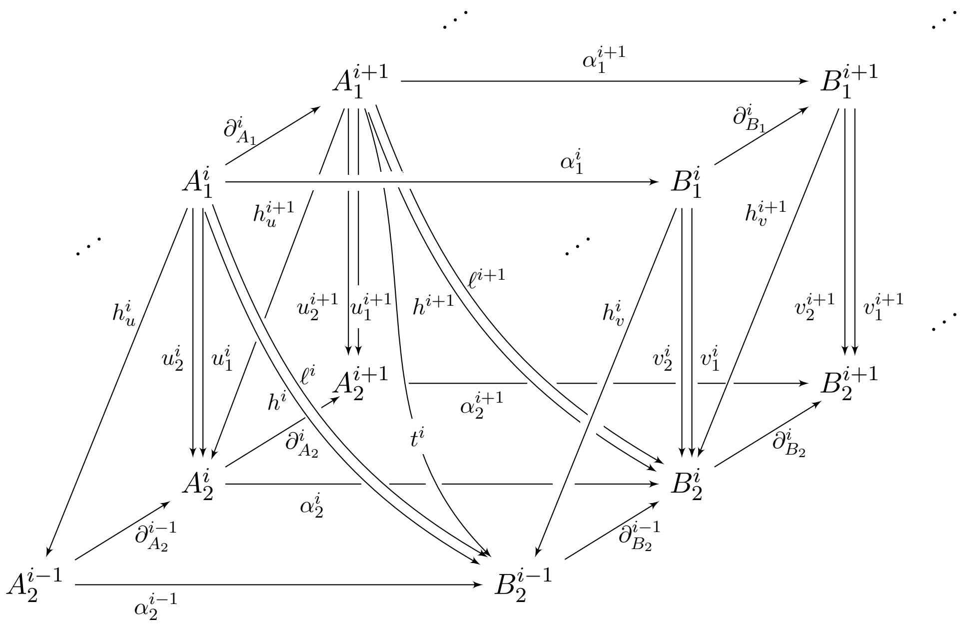
for all $i \in \mathbb{Z}$; and $w_2\colon \operatorname{Cone}(\alpha_1) \to \operatorname{Cone}(\alpha_2)$ is given by

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & A_1^i \oplus B_1^{i-1} & \xrightarrow{\left(\begin{smallmatrix} -\partial_{A_1}^i & \alpha_1^i \\ 0 & \partial_{B_1}^{i-1} \end{smallmatrix}\right)} & A_1^{i+1} \oplus B_1^i & \longrightarrow & \cdots \\
 & & {\scriptstyle w_2^{i-1} := \left(\begin{smallmatrix} u_2^i & \ell^i \\ 0 & v_2^{i-1} \end{smallmatrix}\right)} \big\downarrow & & \big\downarrow {\scriptstyle \left(\begin{smallmatrix} u_2^{i+1} & \ell^{i+1} \\ 0 & v_2^i \end{smallmatrix}\right)} & & \\
\cdots & \longrightarrow & A_2^i \oplus B_2^{i-1} & \xrightarrow[\left(\begin{smallmatrix} -\partial_{A_2}^i & \alpha_2^i \\ 0 & \partial_{B_2}^{i-1} \end{smallmatrix}\right)]{} & A_2^{i+1} \oplus B_2^i & \longrightarrow & \cdots
\end{array}$$

On the other hand, since $u_1 = u_2$ and $v_1 = v_2$, there exists two families of morphisms $\left(h_u^i\colon A_1^i \to A_2^{i-1}\right)_{i\in\mathbb{Z}}$ and $\left(h_v^i\colon B_1^i \to B_2^{i-1}\right)_{i\in\mathbb{Z}}$ with

$$u_1^i - u_2^i = \partial_{A_1}^i \bullet h_u^{i+1} + h_u^i \bullet \partial_{A_2}^{i-1} \text{ and}$$
$$v_1^i - v_2^i = \partial_{B_1}^i \bullet h_v^{i+1} + h_v^i \bullet \partial_{B_2}^{i-1}$$

for all $i \in \mathbb{Z}$.



For each $i \in \mathbb{Z}$, we define $\varphi^i \colon A_1^{i+1} \to B_2^i$ by $h^{i+1} - \ell^{i+1} + h_u^{i+1} \bullet \alpha_2^i - \alpha_1^{i+1} \bullet h_v^{i+1}$. By the following computation

$$\begin{aligned}
-\partial_{A_1}^i \bullet \varphi^i =& -\partial_{A_1}^i \bullet \left(h^{i+1} - \ell^{i+1} + h_u^{i+1} \bullet \alpha_2^i - \alpha_1^{i+1} \bullet h_v^{i+1}\right) \\
=& \left(h^i \bullet \partial_{B_2}^{i-1} + u_1^i \bullet \alpha_2^i - \alpha_1^i \bullet v_1^i\right) + \left(-\ell^i \bullet \partial_{B_2}^{i-1} - u_2^i \bullet \alpha_2^i + \alpha_1^i \bullet v_2^i\right) \\
& + \left(-u_1^i + u_2^i + h_u^i \bullet \partial_{A_2}^{i-1}\right) \bullet \alpha_2^i + \alpha_1^i \bullet \partial_{B_1}^i \bullet h_v^{i+1} \\
=& h^i \bullet \partial_{B_2}^{i-1} + u_1^i \bullet \alpha_2^i - \alpha_1^i \bullet v_1^i - \ell^i \bullet \partial_{B_2}^{i-1} - u_2^i \bullet \alpha_2^i + \alpha_1^i \bullet v_2^i \\
& - u_1^i \bullet \alpha_2^i + u_2^i \bullet \alpha_2^i - h_u^i \bullet \partial_{A_2}^{i-1} \bullet \alpha_2^i + \alpha_1^i \bullet \left(v_1^i - v_2^i - h_v^i \bullet \partial_{B_2}^{i-1}\right) \\
=& h^i \bullet \partial_{B_2}^{i-1} + u_1^i \bullet \alpha_2^i - \alpha_1^i \bullet v_1^i - \ell^i \bullet \partial_{B_2}^{i-1} - u_2^i \bullet \alpha_2^i + \alpha_1^i \bullet v_2^i \\
& - u_1^i \bullet \alpha_2^i + u_2^i \bullet \alpha_2^i - h_u^i \bullet \alpha_2^{i-1} \bullet \partial_{B_2}^{i-1} + \alpha_1^i \bullet v_1^i - \alpha_1^i \bullet v_2^i - \alpha_1^i \bullet h_v^i \bullet \partial_{B_2}^{i-1} \\
=& h^i \bullet \partial_{B_2}^{i-1} - \ell^i \bullet \partial_{B_2}^{i-1} + h_u^i \bullet \alpha_2^{i-1} \bullet \partial_{B_2}^{i-1} - \alpha_1^i \bullet h_v^i \bullet \partial_{B_2}^{i-1} \\
=& \left(h^i - \ell^i + h_u^i \bullet \alpha_2^{i-1} - \alpha_1^i \bullet h_v^i\right) \bullet \partial_{B_2}^{i-1} \\
=& \varphi^{i-1} \bullet \partial_{B_2}^{i-1};
\end{aligned}$$

the family $\left(\varphi^i\right)_{i \in \mathbb{Z}}$ defines a morphism $\varphi \colon \Sigma(A_1) \to B_2$, which should then be zero by the assumption $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma(A_1), B_2) = 0$. Hence, there exists a family of morphisms $\left(t^i \colon A_1^{i+1} \to B_2^{i-1}\right)_{i \in \mathbb{Z}}$ with $\varphi^i = t^i \bullet \partial_{B_2}^{i-1} - \partial_{A_1}^{i+1} \bullet t^{i+1}$ for all $i \in \mathbb{Z}$.

For each $i \in \mathbb{Z}$, we define $r^i \colon A_1^{i+1} \oplus B_1^i \to A_2^i \oplus B_2^{i-1}$ by the matrix $\begin{pmatrix} -h_u^{i+1} & t^i \\ 0 & h_v^i \end{pmatrix}$. The following computation

$$\begin{aligned}
\partial^i_{\mathrm{Cone}(\alpha_1)} \bullet r^{i+1} + r^i \bullet \partial^{i-1}_{\mathrm{Cone}(\alpha_2)} =& \begin{pmatrix} -\partial_{A_1}^{i+1} & \alpha_1^{i+1} \\ 0 & \partial_{B_1}^i \end{pmatrix} \cdot \begin{pmatrix} -h_u^{i+2} & t^{i+1} \\ 0 & h_v^{i+1} \end{pmatrix} + \begin{pmatrix} -h_u^{i+1} & t^i \\ 0 & h_v^i \end{pmatrix} \cdot \begin{pmatrix} -\partial_{A_2}^i & \alpha_2^i \\ 0 & \partial_{B_2}^{i-1} \end{pmatrix} \\
=& \begin{pmatrix} \partial_{A_1}^{i+1} \bullet h_u^{i+2} + h_u^{i+1} \bullet \partial_{A_2}^i & -\partial_{A_1}^{i+1} \bullet t^{i+1} + \alpha_1^{i+1} \bullet h_v^{i+1} - h_u^{i+1} \bullet \alpha_2^i + t^i \bullet \partial_{B_2}^{i-1} \\ 0 & \partial_{B_1}^i \bullet h_v^{i+1} + h_v^i \bullet \partial_{B_2}^{i-1} \end{pmatrix} \\
=& \begin{pmatrix} u_1^{i+1} - u_2^{i+1} & \varphi^i + \alpha_1^{i+1} \bullet h_v^{i+1} - h_u^{i+1} \bullet \alpha_2^i \\ 0 & v_1^i - v_2^i \end{pmatrix} \\
=& \begin{pmatrix} u_1^{i+1} - u_2^{i+1} & h^{i+1} - \ell^{i+1} \\ 0 & v_1^i - v_2^i \end{pmatrix} \\
=& \begin{pmatrix} u_1^{i+1} & h^{i+1} \\ 0 & v_1^i \end{pmatrix} - \begin{pmatrix} u_2^{i+1} & \ell^{i+1} \\ 0 & v_2^i \end{pmatrix} \\
=& w_1^i - w_2^i,
\end{aligned}$$

proves that $w_1 = w_2$, which is the desired conclusion. □

Corollary 5.18. *With the same assumptions as in Lemma 5.17, the set* $\mathrm{CoconeMors}^{st}_{\alpha_1,\alpha_2}(u, v)$ *is a singleton set.*

Proof. By Remark 5.11

$$\mathrm{CoconeMors}^{st}_{\alpha_1,\alpha_2}(u_1, v_1) := \{\Sigma^{-1} w \mid w \in \mathrm{ConeMors}^{st}_{\alpha_1,\alpha_2}(u_1, v_1)\}.$$

□

Corollary 5.19. *Let $\mathscr{C}$ be an additive category and $\mathcal{K}^b(\mathscr{C})$ be its bounded homotopy category. Let S be a class of objects such that* $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma(A), B) = 0$ *for all pairs of objects A, B in S. Then, the standard cone object defines a functor*

$$\mathrm{Cone} \colon \begin{cases} \mathrm{Arr}(S) & \to \mathcal{K}^b(\mathscr{C}), \\ \left(A_1 \xrightarrow{\alpha_1} B_1\right) & \mapsto \mathrm{Cone}(\alpha_1), \\ \left(A_1 \xrightarrow{\alpha_1} B_1\right) \xrightarrow{\{u,v\}} \left(A_2 \xrightarrow{\alpha_2} B_2\right) & \mapsto \textit{the unique element in } \mathrm{ConeMors}^{st}_{\alpha_1,\alpha_2}(u, v); \end{cases}$$

where $\mathrm{Arr}(S)$ *is the category of arrows of the full subcategory generated by S.*

Proof. Follows from Lemma 5.12 and Lemma 5.17. □

5.3. Stable Categories of Frobenius Categories are Triangulated

In this section we give an algorithmic description of the triangulated structure on the stable categories of Frobenius categories (cf. Theorem 5.29). We reproduce the related proofs and constructions in [**Hap88**, Chapter 1] so that they can be directly implemented on the computer. Our primary example of a Frobenius category is the category of finitely presented (graded) left modules over the exterior algebra $E = k[e_0, \dots, e_n]$ for some field k (cf. Example 5.37). If E is equipped with a $\mathbb{Z}$-graded with $\deg e_0 = \deg e_1 = \cdots = \deg e_n = -1$ then the stable category of the Frobenius category E-**fpgrmod** provides a model for the bounded derived category of coherent sheaves over the projective space $\mathbb{P}^n_k$ (cf. [**BGG78**] and [**EFS03**]).

We start by defining exact categories:

Definition 5.20. Let $\mathscr{C}$ be an additive category and let $\mathcal{E}$ be a class of short exact sequences[5] in $\mathscr{C}$. An element (ι, π) in $\mathcal{E}$ will be called **conflation**. The components ι and π of a conflation (ι, π) will be called **inflation** resp. **deflation**[6]. The pair $(\mathscr{C}, \mathcal{E})$ will be called **exact** if the following axioms hold:

EX 1. The class $\mathcal{E}$ is closed under taking isomorphisms.

EX 2. Inflations and deflations are closed under composition.

EX 3. For any pair of objects A, B in $\mathscr{C}$ the canonical sequence

$$A \xhookrightarrow{\mathrm{inf}_A} A \oplus B \xrightarrow{\mathrm{def}_B} B$$

is a conflation.

EX 4. If $\pi\colon A \twoheadrightarrow C$ is a deflation, then for any morphism $\alpha\colon B \to C$ the fiber product

$$\begin{array}{ccc} F_{\pi,\alpha} & \xrightarrow{p_A} & A \\ {\scriptstyle p_B}\downarrow & \ulcorner & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow[\alpha]{} & C \end{array}$$

exists in which p_B is a deflation.

EX 5. If $\iota\colon C \hookrightarrow A$ is an inflation, then for any morphism $\alpha\colon C \to B$ the pushout

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & B \\ {\scriptstyle \iota}\downarrow & \lrcorner & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow[q_A]{} & P_{\iota,\alpha} \end{array}$$

exists in which q_B is an inflation.

EX 6. Let α be a morphism which has a kernel. Then for any morphism β, if $\beta \cdot \alpha$ is a deflation then so is α.

EX 7. Let α be a morphism which has a cokernel. Then for any morphism β, if $\alpha \cdot \beta$ is an inflation then so is α.

Definition 5.21. An additive category $\mathscr{C}$ together with a class $\mathcal{E}$ of short exact sequences in $\mathscr{C}$ is called **computable exact** if the following holds:

(1) The axioms **EX 1, 2, 3, 4, 5, 6, 7** are satisfied and all their existential quantifiers are realized by algorithms.
(2) We have an algorithm which for a given pair of morphisms (ι, π) decides whether the pair is a conflation.
(3) We have an algorithm which for a given inflation ι computes a cokernel projection $\mathrm{def}(\iota)$ of ι, i.e., such that $(\iota, \mathrm{def}(\iota))$ is a conflation.
(4) We have an algorithm which for a given deflation π computes a kernel embedding $\mathrm{inf}(\pi)$ of π, i.e., such that $(\mathrm{inf}(\pi), \pi)$ is a conflation.

[5]A short exact sequence in $\mathscr{C}$ is a bounded cochain complex $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ with a vanishing cohomology in each degree.

[6]Since every conflation froms a short exact sequence, every inflation is a monomorphism and every deflation is an epimorphism.

The concept of $\mathcal{E}$-projective and $\mathcal{E}$-injective objects in an exact category $(\mathscr{C}, \mathcal{E})$ generalizes the concept of projective and injective objects in Abelian categories:

Definition 5.22. Let $(\mathscr{C}, \mathcal{E})$ be an exact category.

(1) An object P in $\mathscr{C}$ is called **$\mathcal{E}$-projective** if for all deflations $\pi\colon B \twoheadrightarrow C$ and all morphisms $\tau\colon P \to C$, there exists a lift morphism of τ along π.
(2) An object I in $\mathscr{C}$ is called **$\mathcal{E}$-injective** if for all inflations $\iota\colon A \hookrightarrow I$ and all morphisms $\tau\colon A \to B$, there exists a colift morphism of ι along τ.
(3) $(\mathscr{C}, \mathcal{E})$ is said to have **enough $\mathcal{E}$-projectives** if for each object A in $\mathscr{C}$ there exists a deflation $\mathrm{def}_A\colon P_A \twoheadrightarrow A$ from some $\mathcal{E}$-projective object P_A.
(4) $(\mathscr{C}, \mathcal{E})$ is said to have **enough $\mathcal{E}$-injectives** if for each object A in $\mathscr{C}$ there exists an inflation $\mathrm{inf}_A\colon A \hookrightarrow I_A$ into some $\mathcal{E}$-injective object I_A.

Example 5.23. If $\mathscr{C}$ is an Abelian category and $\mathcal{E}$ is the class of all short exact sequences, then $(\mathscr{C}, \mathcal{E})$ defines an exact category. Since every epimorphism is a cokernel of its kernel, every epimorphism is a deflation; and since every monomorphism is a kernel of its cokernel, every monomorphism is an inflation. Furthermore, if $\mathscr{C}$ has enough projective or injective objects, then $(\mathscr{C}, \mathcal{E})$ also has enough $\mathcal{E}$-projective resp. $\mathcal{E}$-injective objects. In particular, an object in $\mathscr{C}$ is $\mathcal{E}$-projective resp. $\mathcal{E}$-injective if and only if it is projective resp. injective in the usual sense (cf. Definition 2.71).

A Frobenius category is an exact category with extra structure:

Definition 5.24. An exact category $(\mathscr{C}, \mathcal{E})$ is called a **Frobenius category** if it has enough $\mathcal{E}$-projectives and $\mathcal{E}$-injectives and the classes of $\mathcal{E}$-projective and $\mathcal{E}$-injective objects in $\mathscr{C}$ coincide. Furthermore, if $(\mathscr{C}, \mathcal{E})$ is computable exact and the axioms in Definition 5.22 are realized by algorithms, then $(\mathscr{C}, \mathcal{E})$ is called **computable Frobenius**.

Lemma 5.25. *Let $(\mathscr{C}, \mathcal{E})$ be an exact category.*

(1) If $(\mathscr{C}, \mathcal{E})$ has enough $\mathcal{E}$-projective objects, then the class $\mathcal{L}$ of all $\mathcal{E}$-projective objects is a class of lifting objects in $\mathscr{C}$.
(2) If $(\mathscr{C}, \mathcal{E})$ has enough $\mathcal{E}$-injective objects, then the class $\mathcal{Q}$ of all $\mathcal{E}$-injective objects is a class of colifting objects in $\mathscr{C}$.

Proof. The proof is analogous to Examples 2.60 and 2.62. □

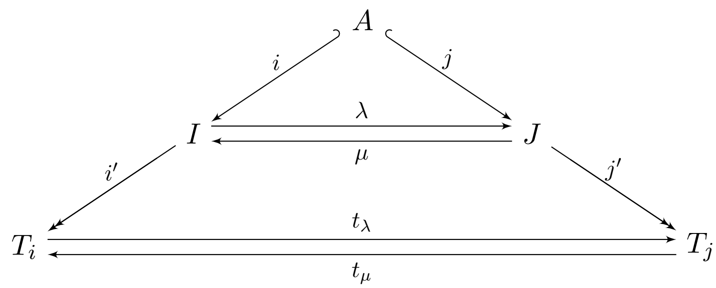
This means if $(\mathscr{C}, \mathcal{E})$ is a Frobenius category, then the stable categories associated to the above classes of lifting and colifting object coincide. In particular, a morphism $[\varphi]\colon [A] \to [B]$ in $\mathscr{C}/\mathcal{L} \cong \mathscr{C}/\mathcal{Q}$ is zero if and only if φ factors through some $\mathcal{E}$-projective object if and only if φ is liftable along the deflation $\mathrm{def}_B\colon P_B \twoheadrightarrow B$ if and only if φ is coliftable along the inflation $\mathrm{inf}_A\colon A \hookrightarrow I_A$.

The Schanuels lemma characterizes isomorphisms in stable categories of exact categories:

Lemma 5.26 (Schanuels Lemma)**.** *Let $(\mathscr{C}, \mathcal{E})$ be an exact category. Given two conflations $A \xhookrightarrow{i} I \xrightarrow{i'} T_i$, $A \xhookrightarrow{j} J \xrightarrow{j'} T_j$ where I and J are $\mathcal{E}$-injective objects, then T_i and T_j are isomorphic in $\mathscr{C}/\mathcal{Q}$. Furthermore, for any morphism $\alpha\colon A \to X$ in $\mathscr{C}$, the pushout objects $P_{i,\alpha}$ and $P_{j,\alpha}$ are isomorphic in $\mathscr{C}/\mathcal{Q}$ as well.*

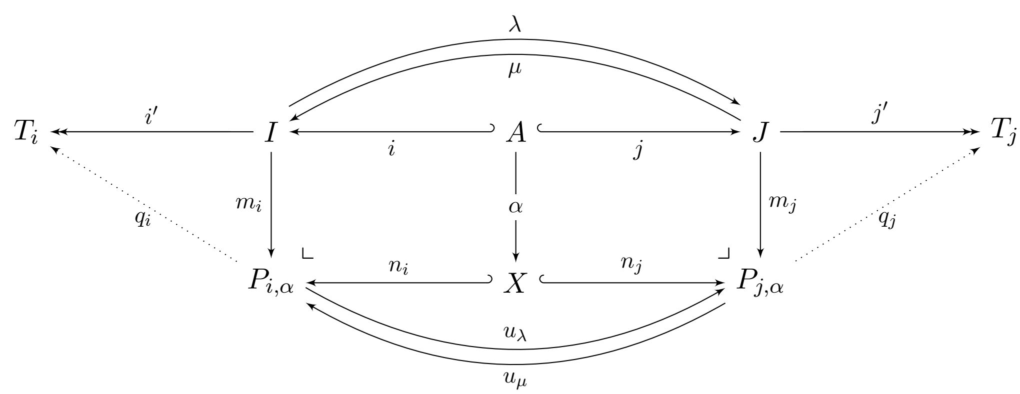
Proof. Let λ, μ be the $\mathcal{E}$-injective colifts of i and j along each other and t_λ, t_μ the induced cokernel colifts along the cokernel projections i' resp. j'. In the following we show that the residue class $[t_\lambda]$ in $\mathscr{C}/\mathcal{Q}$ does not depend on the choice of λ. Let $\lambda'\colon I \to J$ be another $\mathcal{E}$-injective colift of j along i and let $t_{\lambda'}$ be the cokernel colift of $\lambda' \cdot j'$ along i'. Since $i \cdot (\lambda - \lambda') = j - j = 0$, there exists the unique morphism $\zeta\colon T_i \to J$ with $i' \cdot \zeta = \lambda - \lambda'$. It follows that $i' \cdot (t_\lambda - t_{\lambda'}) = (\lambda - \lambda') \cdot j' = i' \cdot \zeta \cdot j'$. Since i' is an epimorphism, $t_\lambda - t_{\lambda'} = \zeta \cdot j'$, hence $[t_\lambda] = [t_{\lambda'}]$ as desired. Similarly, $[t_\mu]$ in $\mathscr{C}/\mathcal{Q}$ does not depend on the choice of μ.

We get the following commutative diagram:



We have $i \cdot (\lambda \cdot \mu - \mathrm{id}_I) = i \cdot \lambda \cdot \mu - i = j \cdot \mu - i = i - i = 0$, hence there exists a cokernel colift $h\colon B \to I$ of $\lambda \cdot \mu - \mathrm{id}_I$ along i', i.e., with $i' \cdot h = \lambda \cdot \mu - \mathrm{id}_I$. Therefore, $i' \cdot h \cdot i' = (\lambda \cdot \mu - \mathrm{id}_I) \cdot i' = \lambda \cdot \mu \cdot i' - i' = \lambda \cdot j' \cdot t_\mu - i' = i' \cdot t_\lambda \cdot t_\mu - i' = i' \cdot (t_\lambda \cdot t_\mu - \mathrm{id}_B)$. Since i' is an epimorphism, we get $h \cdot i' = t_\lambda \cdot t_\mu - \mathrm{id}_B$, hence $[t_\lambda] \cdot [t_\mu] = [\mathrm{id}_B]$. Similarly, $[t_\mu] \cdot [t_\lambda] = [\mathrm{id}_C]$. This proves the first assertion.

By the universal property of pushout objects, λ induces a morphism $u_\lambda\colon P_{i,\alpha} \to P_{j,\alpha}$ with $m_i \cdot u_\lambda = \lambda \cdot m_j$ and $n_i \cdot u_\lambda = n_j$. Similarly, μ induces a morphism $u_\mu\colon P_{j,\alpha} \to P_{i,\alpha}$ with $m_j \cdot u_\mu = \mu \cdot m_i$ and $n_j \cdot u_\mu = n_i$.



Since $i \cdot (\lambda \cdot \mu - \mathrm{id}_I) = 0$, there exists a unique morphism $\ell\colon P_{i,\alpha} \to I$ with $m_i \cdot \ell = \lambda \cdot \mu - \mathrm{id}_I$ and $n_i \cdot \ell = 0$.

It follows from the assumption that $P_{i,\alpha}$ is a pushout object of (i, α) and the following two equalities

(1) $m_i \cdot \left(u_\lambda \cdot u_\mu - \mathrm{id}_{P_{i,\alpha}}\right) = m_i \cdot u_\lambda \cdot u_\mu - m_i = \lambda \cdot m_j \cdot u_\mu - m_i = \lambda \cdot \mu \cdot m_i - m_i = (\lambda \cdot \mu - \mathrm{id}_I) \cdot m_i = m_i \cdot \ell \cdot m_i = m_i \cdot (\ell \cdot m_i)$ and

(2) $n_i \cdot \left(u_\lambda \cdot u_\mu - \mathrm{id}_{P_{i,\alpha}}\right) = n_i \cdot u_\lambda \cdot u_\mu - n_i = n_j \cdot u_\mu - n_i = n_i - n_i = 0 = n_i \cdot (\ell \cdot m_i)$

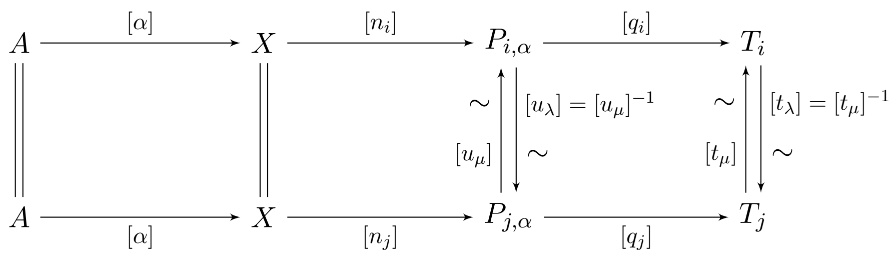
that $u_\lambda \bullet u_\mu - \mathrm{id}_{P_{i,\alpha}} = \ell \bullet m_i$, hence $[u_\lambda] \bullet [u_\mu] = \left[\mathrm{id}_{P_{i,\alpha}}\right]$. Similarly, we can show that $[u_\mu] \bullet [u_\lambda] = \left[\mathrm{id}_{P_{j,\alpha}}\right]$. □

Remark 5.27. By the universal property of pushout objects, there exist two morphisms $q_i \colon P_{i,\alpha} \to B$ with $m_i \bullet q_i = i', n_i \bullet q_i = 0$; and $q_j \colon P_{j,\alpha} \to C$ with $v \bullet q_j = r_j, u \bullet q_j = 0$.

The following computation

(1) $m_i \bullet (q_i \bullet t_\lambda) = i' \bullet t_\lambda = \lambda \bullet j' = \lambda \bullet v \bullet q_j = m_i \bullet u_\lambda \bullet q_j = m_i \bullet (u_\lambda \bullet q_j)$,
(2) $n_i \bullet (q_i \bullet t_\lambda) = 0 \bullet t_\lambda = 0 = n_j \bullet q_j = n_i \bullet u_\lambda \bullet q_j = n_i \bullet (u_\lambda \bullet q_j)$.

shows that $q_i \bullet t_\lambda = u_\lambda \bullet q_j$. Similarly, $q_j \bullet t_\mu = u_\mu \bullet q_i$. In particular, we get the following commutative diagram in $\mathscr{C}/\mathcal{Q}$:



The following is the dual statement:

Lemma 5.28. *Let $(\mathscr{C}, \mathcal{E})$ be an exact category. Given two conflations $S_r \overset{i_r}{\hookrightarrow} P \overset{r}{\twoheadrightarrow} A$, $S_t \overset{i_t}{\hookrightarrow} Q \overset{t}{\twoheadrightarrow} A$ where P and Q are $\mathcal{E}$-projective objects, then S_r and S_t are isomorphic in $\mathscr{C}/\mathcal{L}$. Furthermore, for any morphism $\alpha \colon X \to A$ in $\mathscr{C}$, the pullback objects $F_{r,\alpha}$ and $F_{t,\alpha}$ are isomorphic in $\mathscr{C}/\mathcal{L}$ as well.*

We refer the reader to [**Hap88**, Chapter 1] for the original proof of the following theorem:

Theorem 5.29. *Let $(\mathscr{C}, \mathcal{E})$ be a computable* Frobenius *category, then the stable category $\mathscr{C}/\mathcal{Q}$ is a precomputable triangulated category.*

Proof. We start by constructing the auto-equivalence $\Sigma \colon \mathscr{C}/\mathcal{Q} \to \mathscr{C}/\mathcal{Q}$ and its quasi-inverse. For each object A in $\mathscr{C}$, we fix an inflation $\inf_A \colon A \hookrightarrow I_A$ into some $\mathcal{E}$-injective object I_A. We will refer to the associated deflation of $\inf_A$ by $\operatorname{def}(\inf_A) \colon I_A \twoheadrightarrow T_A$. That is, $\operatorname{def}(\inf_A)$ is a cokernel projection of $\inf_A$. Each morphism $\alpha \colon A \to B$ can be colifted into a morphism $I_\alpha \colon I_A \to I_B$ which in turn can be colifted to a morphism $T_\alpha \colon T_A \to T_B$. In the following we show that the residue class $[T_\alpha]$ depends only on $[\alpha]$. Let $\beta \colon A \to B$ be another representative of $[\alpha]$, i.e., $[\alpha] = [\beta]$. We need to prove that $[T_\alpha] = [T_\beta]$. Since $[\alpha] = [\beta]$, there exists a morphism $\zeta \colon I_A \to B$ such that $\alpha - \beta = \inf_A \bullet \zeta$. We have $\inf_A \bullet (I_\alpha - I_\beta - \zeta \bullet \inf_B) = \alpha \bullet \inf_B - \beta \bullet \inf_B - (\alpha - \beta) \bullet \inf_B = 0$. Hence, there exists a uniquely determined morphism $\tau \colon T_A \to I_B$ such that $\operatorname{def}(\inf_A) \bullet \tau = I_\alpha - I_\beta - \zeta \bullet \inf_B$. We get $\operatorname{def}(\inf_A) \bullet (T_\alpha - T_\beta) = (I_\alpha - I_\beta) \bullet \operatorname{def}(\inf_B) = (I_\alpha - I_\beta - \zeta \bullet \inf_B) \bullet \operatorname{def}(\inf_B) = \operatorname{def}(\inf_A) \bullet \tau \bullet \operatorname{def}(\inf_B)$. Since $\operatorname{def}(\inf_A)$ is an epimorphism, $T_\alpha - T_\beta = \tau \bullet \operatorname{def}(\inf_B)$. Hence, $[T_\alpha] = [T_\beta]$. In particular, the map

$$\Sigma_{A,B} \colon \begin{cases} \operatorname{Hom}(A,B) & \to \operatorname{Hom}(T_A, T_B), \\ \quad [\alpha] & \mapsto [T_\alpha] \end{cases}$$

is well-defined. In fact $\Sigma_{A,B}$ is a bijection. We first prove it is surjective. Let $\mu \colon T_A \to T_B$ be a morphism in $\mathscr{C}$. Since I_A is projective and $\operatorname{def}(\inf_B)$ is an epimorphism, there exists a lift morphism $\delta\colon I_A \to I_B$ of $\operatorname{def}(\operatorname{inf}_A)\cdot\mu$ along $\operatorname{def}(\operatorname{inf}_B)$. Since inf_B is a kernel embedding of $\operatorname{def}(infl_B)$, there exists a uniquely determined lift morphism $\gamma\colon A \to B$ of $\operatorname{inf}_A\cdot\delta$ along inf_B. It follows that $\Sigma_{A,B}([\gamma]) = [\mu]$ hence $\Sigma_{A,B}$ is surjective. Next, we show $\Sigma_{A,B}$ is injective. Let α, β be two morphisms in $\mathscr{C}$ such that $[T_\alpha] = [T_\beta]$. Since $[T_\alpha] = [T_\beta]$ there exists a morphism $\tau\colon T_A \to I_B$ such that $T_\alpha - T_\beta = \tau\cdot\operatorname{def}(\operatorname{inf}_B)$. It follows that $(I_\alpha - I_\beta - \operatorname{def}(\operatorname{inf}_A)\cdot\tau)\cdot\operatorname{def}(\operatorname{inf}_B) = 0$, hence there exists a uniquely determined morphism $\zeta\colon I_A \to B$ such that $I_\alpha - I_\beta - \operatorname{def}(\operatorname{inf}_A)\cdot\tau = \zeta\cdot\operatorname{inf}_B$. It follows that $(\alpha - \beta - \operatorname{inf}_A\cdot\zeta)\cdot\operatorname{inf}_B = \operatorname{inf}_A\cdot I_\alpha - \operatorname{inf}_A\cdot I_\beta - \operatorname{inf}_A\cdot(I_\alpha - I_\beta - \operatorname{def}(\operatorname{inf}_A)\cdot\tau) = 0$. Since inf_B is a monomorphism, $\alpha - \beta - \operatorname{inf}_A\cdot\zeta = 0$, thus, $[\alpha] = [\beta]$ as desired.

Analogously, for each object A in $\mathscr{C}$, we fix a deflation $\operatorname{def}_A\colon P_A \twoheadrightarrow A$ from some $\mathcal{E}$-projective object P_A. The associated inflation of def_A will be denoted by $\operatorname{inf}(\operatorname{def}_A)\colon S_A \hookrightarrow P_A$. A morphism $\alpha\colon A \to B$ can be lifted to morphisms $P_\alpha\colon P_A \to P_B$ and $S_\alpha\colon S_A \to S_B$ where the residue class $[S_\alpha]$ does not depend on the choice of P_α.

$$\begin{array}{ccccccccc}
S_A & \xrightarrow{\operatorname{inf}(\operatorname{def}_A)} & P_A & \xrightarrow{\operatorname{def}_A} & A & \xrightarrow{\operatorname{inf}_A} & I_A & \xrightarrow{\operatorname{def}(\operatorname{inf}_A)} & T_A \\
\downarrow{\scriptstyle S_\alpha} & & \downarrow{\scriptstyle P_\alpha} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle I_\alpha} & & \downarrow{\scriptstyle T_\alpha} \\
S_B & \xrightarrow[\operatorname{inf}(\operatorname{def}_B)]{} & P_B & \xrightarrow[\operatorname{def}_B]{} & B & \xrightarrow[\operatorname{inf}_B]{} & I_B & \xrightarrow[\operatorname{def}(\operatorname{inf}_B)]{} & T_B.
\end{array}$$

This enables us to define two fully faithful functors:

$$\Sigma\colon \begin{cases} \mathscr{C}/\mathcal{Q} & \to \mathscr{C}/\mathcal{Q}, \\ A & \mapsto T_A, \\ [\alpha] & \mapsto [T_\alpha] \end{cases}$$

and

$$\Sigma^{-1}\colon \begin{cases} \mathscr{C}/\mathcal{Q} & \to \mathscr{C}/\mathcal{Q}, \\ A & \mapsto S_A, \\ [\alpha] & \mapsto [S_\alpha]. \end{cases}$$

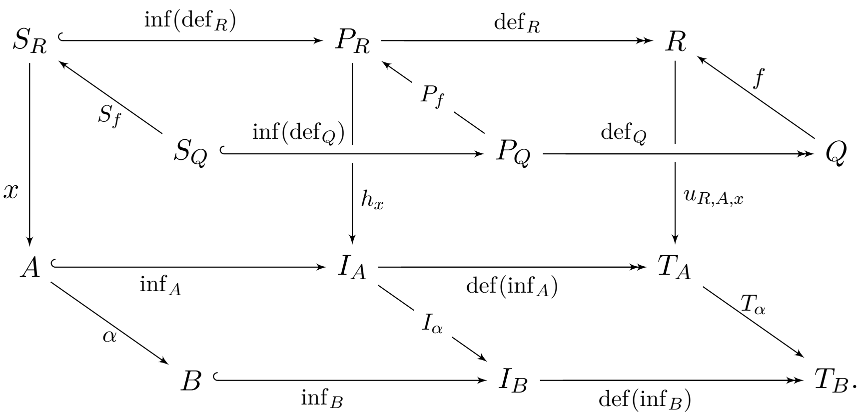
In the following we show that these functors define an adjunction $\Sigma^{-1} \dashv \Sigma$. Let $R,\ A$ be two objects in $\mathscr{C}$. For any morphism $x\colon S_R \to A$ in $\mathscr{C}$ there exists an $\mathcal{E}$-injective colift, h_x, of $x\cdot\operatorname{inf}_A$ along $\operatorname{inf}(\operatorname{def}_R)$ and a cokernel colift $u_{R,A,x}\colon R \to T_A$ of $h_x\cdot\operatorname{def}(\operatorname{inf}_A)$ along def_R as depicted in the following commutative diagram:

$$\begin{array}{ccccc}
S_R & \xrightarrow{\operatorname{inf}(\operatorname{def}_R)} & P_R & \xrightarrow{\operatorname{def}_R} & R \\
\downarrow{\scriptstyle x} & & \downarrow{\scriptstyle h_x} & & \downarrow{\scriptstyle u_{R,A,x}} \\
A & \xrightarrow[\operatorname{inf}_A]{} & I_A & \xrightarrow[\operatorname{def}(\operatorname{inf}_A)]{} & T_A.
\end{array}$$

Similar to the above discussion, $[u_{R,A,x}]$ in $\mathscr{C}/\mathcal{Q}$ depends only on $[x]$. Hence, we can define a map

$$\Phi_{R,A}\colon \begin{cases} \operatorname{Hom}_{\mathscr{C}/\mathcal{Q}}(\Sigma^{-1}(R), A) & \to \operatorname{Hom}_{\mathscr{C}/\mathcal{Q}}(R, \Sigma(A)), \\ [x] & \mapsto [u_{R,A,x}]. \end{cases}$$

For a given triple of morphisms $\alpha\colon A \to B$, $f\colon Q \to R$ and $x\colon S_R \to A$, we can construct the following commutative diagram:



Let $y = S_f \cdot x \cdot \alpha$, then $y \cdot \inf_B$ can be colifted along $\inf(\mathrm{def}_Q)$ via $h_y := P_f \cdot h_x \cdot I_\alpha$ and $h_y \cdot \mathrm{def}(\inf_B)$ can uniquely be colifted along def_Q via $u_{Q,B,y} := f \cdot u_{R,A,x} \cdot T_\alpha$. Hence,

$$\Phi_{Q,B}\Big(\Sigma^{-1}([f]) \cdot [x] \cdot [\alpha]\Big) = [f] \cdot \Phi_{R,A}([x]) \cdot \Sigma([\alpha]).$$

That is, the assignment

$$\Phi\colon \begin{cases} \mathrm{Hom}_{\mathscr{C}/\mathcal{Q}}(\Sigma^{-1}(-), -) & \to \mathrm{Hom}_{\mathscr{C}/\mathcal{Q}}(-, \Sigma(-)), \\ \qquad (R, A) & \mapsto \Phi_{R,A} \end{cases}$$

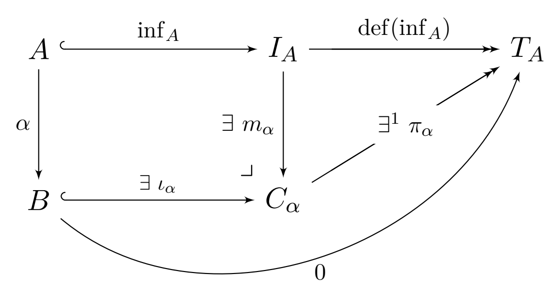
defines a natural transformation. By Lemma A.22, the associated unit η and counit ϵ of the adjunction are natural isomorphisms.

Let $[\alpha]\colon A \to B$ be a morphism in $\mathscr{C}/\mathcal{Q}$. As discussed above, the object A can be used to construct a conflation

$$A \xhookrightarrow{\inf_A} I_A \xrightarrow{\mathrm{def}(\inf_A)} T_A$$

in $\mathscr{C}$ where $\Sigma(A) := T_A$ as object in $\mathscr{C}/\mathcal{Q}$.

Since the axiom **EX 5** is realized by algorithms, we can construct the following commutative diagram:



By setting $\mathrm{Cone}([\alpha]) := C_\alpha$, $\iota([\alpha]) := [\iota_\alpha]$ and $\pi([\alpha]) := [\pi_\alpha]$, we get a triangle

$$A \xrightarrow{[\alpha]} B \xrightarrow{\iota([\alpha])} \mathrm{Cone}([\alpha]) \xrightarrow{\pi([\alpha])} \Sigma(A)$$

in $\mathscr{C}/\mathcal{Q}$. A triangle

$$A \xrightarrow{[\alpha]} B \xrightarrow{[\iota]} C \xrightarrow{[\pi]} \Sigma(A)$$

will be called exact if it is isomorphic to

$$A \xrightarrow{[\alpha]} B \xrightarrow{\iota([\alpha])} \mathrm{Cone}([\alpha]) \xrightarrow{\pi([\alpha])} \Sigma(A).$$

The class of all exact triangles will be denoted by $\triangle$. In the following we prove that $(\mathscr{C}/\mathcal{Q}, \triangle, \Sigma)$ satisfies the axioms of a precomputable triangulated category.

TR′ 0. The computation of the auto-equivalences Σ, Σ^{-1} and the natural isomorphisms η and ϵ can be achieved in any computable Frobenius category.

TR′ 1.
a. A given morphism $[\alpha]\colon A \to B$ can be completed into the exact triangle

$$A \xrightarrow{[\alpha]} B \xrightarrow{\iota([\alpha])} \mathrm{Cone}([\alpha]) \xrightarrow{\pi([\alpha])} \Sigma(A).$$

It is called the standard exact triangle associated to $[\alpha]$.

b. For any object A in $\mathscr{C}$, $\mathrm{Cone}([\mathrm{id}_A]) := C_{\mathrm{id}_A} = I_A$, hence $\mathrm{Cone}([\mathrm{id}_A]) \cong 0$.

c. It is satisfied by Lemma 5.5.

TR′ 2. For a given morphism $[\alpha]\colon A \to B$, we need to construct a morphism $\lambda\colon T_A \to C_{\iota_\alpha}$ which induces an isomorphism of triangles

$$\begin{array}{ccccccc}
B & \xrightarrow{\iota([\alpha])} & \mathrm{Cone}([\alpha]) & \xrightarrow{\pi([\alpha])} & \Sigma(A) & \xrightarrow{-\Sigma([\alpha])} & \Sigma(B) \\
\| & \circlearrowleft & \| & \circlearrowleft & \downarrow{\scriptstyle [\lambda]} & \circlearrowleft & \| \\
B & \xrightarrow[\iota([\alpha])]{} & \mathrm{Cone}([\alpha]) & \xrightarrow[\iota(\iota([\alpha]))]{} & \mathrm{Cone}(\iota([\alpha])) & \xrightarrow[\pi(\iota([\alpha]))]{} & \Sigma(B)
\end{array}$$

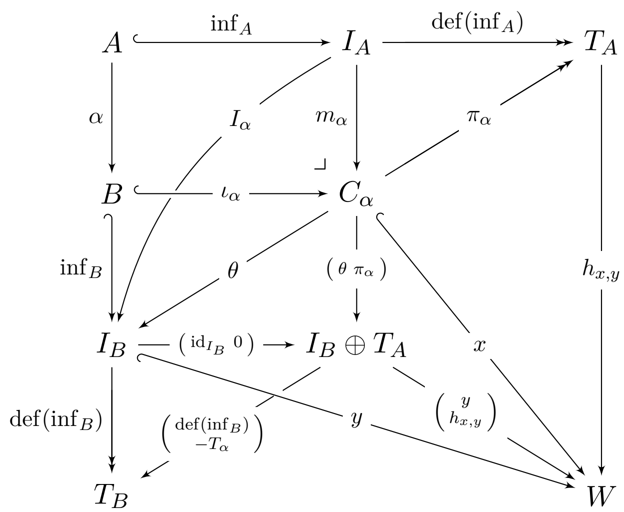
Let $I_\alpha\colon I_A \to I_B$ be a colift morphism of $\alpha \cdot \mathrm{inf}_B$ along inf_A and T_α the cokernel colift of $I_\alpha \cdot \mathrm{def}(\mathrm{inf}_B)$ along $\mathrm{def}(\mathrm{inf}_A)$, i.e., $\Sigma([\alpha]) = [T_\alpha]$.

Since $\mathrm{inf}_A \cdot I_\alpha = \alpha \cdot \mathrm{inf}_B$, there exists a unique morphism $\theta\colon C_\alpha \to I_B$ such that $\iota_\alpha \cdot \theta = \mathrm{inf}_B$ and $m_\alpha \cdot \theta = I_\alpha$.

In the following, we prove that $\begin{pmatrix} \theta & \pi_\alpha \end{pmatrix}\colon C_\alpha \to I_B \oplus T_A$ and $\begin{pmatrix} \mathrm{id}_{I_B} & 0 \end{pmatrix}\colon I_B \to I_B \oplus T_A$ define a pushout diagram of $(\iota_\alpha, \mathrm{inf}_B)$. Suppose $x\colon C_\alpha \to W$ and $y\colon I_B \to W$ are two morphisms with $\iota_\alpha \cdot x = \mathrm{inf}_B \cdot y$. The following equality

$$\begin{aligned}
\mathrm{inf}_A \cdot m_\alpha \cdot (x - \theta \cdot y) &= \mathrm{inf}_A \cdot m_\alpha \cdot x - \mathrm{inf}_A \cdot m_\alpha \cdot \theta \cdot y \\
&= \alpha \cdot \iota_\alpha \cdot x - \mathrm{inf}_A \cdot I_\alpha \cdot y \\
&= \alpha \cdot \iota_\alpha \cdot x - \alpha \cdot \mathrm{inf}_B \cdot y \\
&= \alpha \cdot (\iota_\alpha \cdot x - \mathrm{inf}_B \cdot y) \\
&= 0
\end{aligned}$$

implies the existence a cokernel colift $h_{x,y}\colon T_A \to W$ of $m_\alpha \bullet (x - \theta \bullet y)$ along $\operatorname{def}(\inf_A)$.



Set $u_{x,y} \coloneqq \begin{pmatrix} y \\ h_{x,y} \end{pmatrix}\colon I_B \oplus T_A \to W$. Then, $\begin{pmatrix}\mathrm{id}_{I_B} & 0\end{pmatrix} \bullet u_{x,y} = y$. By the assumption that C_α is a pushout object and the two equalities

1. $\iota_\alpha \bullet \begin{pmatrix}\theta & \pi_\alpha\end{pmatrix} \bullet \begin{pmatrix} y \\ h_{x,y} \end{pmatrix} = \iota_\alpha \bullet \theta \bullet y + \iota_\alpha \bullet \pi_\alpha \bullet h_{x,y} = \inf_B \bullet y + 0 = \iota_\alpha \bullet x$,
2. $m_\alpha \bullet \begin{pmatrix}\theta & \pi_\alpha\end{pmatrix} \bullet \begin{pmatrix} y \\ h_{x,y} \end{pmatrix} = m_\alpha \bullet \theta \bullet y + m_\alpha \bullet \pi_\alpha \bullet h_{x,y} = m_\alpha \bullet \theta \bullet y + \operatorname{def}(\inf_A) \bullet h_{x,y} = m_\alpha \bullet x$;

we get $\begin{pmatrix}\theta & \pi_\alpha\end{pmatrix} \bullet u_{x,y} = x$.

Any other solution to the linear system

$$\begin{pmatrix}\mathrm{id}_{I_B} & 0\end{pmatrix} \bullet \chi = y,\ \begin{pmatrix}\theta & \pi_\alpha\end{pmatrix} \bullet \chi = x$$

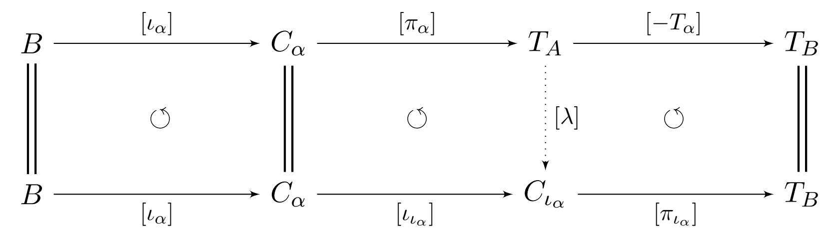
would consist necessarily of y and a cokernel colift of $m_\alpha \bullet (x - \theta \bullet y)$ along $\operatorname{def}(\inf_A)$. By the universal property of cokernel objects we conclude that $u_{x,y}$ is the only solution to the above system. This means, the pair $\begin{pmatrix}\theta & \pi_\alpha\end{pmatrix}\colon C_\alpha \to I_B \oplus T_A$ and $\begin{pmatrix}\mathrm{id}_{I_B} & 0\end{pmatrix}\colon I_B \to I_B \oplus T_A$ is a pushout diagram of $(\iota_\alpha, \inf_B)$.

By the universal property a pushout diagrams there exists a unique solution $U\colon I_B \oplus T_A \to T_B$ to the linear system $\begin{pmatrix}\mathrm{id}_{I_B} & 0\end{pmatrix} \bullet U = \operatorname{def}(\inf_B)$ and $\begin{pmatrix}\theta & \pi_\alpha\end{pmatrix} \bullet U = 0$. We claim that this solution is given by $U \coloneqq \begin{pmatrix} \operatorname{def}(\inf_B) \\ -T_\alpha \end{pmatrix}$. The first equality is evident and the second equality follows by the universal property of the pushout object C_α and the following two equalities:

1. $\iota_\alpha \bullet \begin{pmatrix}\theta & \pi_\alpha\end{pmatrix} \bullet \begin{pmatrix} \operatorname{def}(\inf_B) \\ -T_\alpha \end{pmatrix} = \iota_\alpha \bullet \theta \bullet \operatorname{def}(\inf_B) = \inf_B \bullet \operatorname{def}(\inf_B) = 0$ and
2. $m_\alpha \bullet \begin{pmatrix}\theta & \pi_\alpha\end{pmatrix} \bullet \begin{pmatrix} \operatorname{def}(\inf_B) \\ -T_\alpha \end{pmatrix} = m_\alpha \bullet \theta \bullet \operatorname{def}(\inf_B) - m_\alpha \bullet \pi_\alpha \bullet T_\alpha = I_\alpha \bullet \operatorname{def}(\inf_B) - \operatorname{def}(\inf_A) \bullet T_\alpha = 0$.

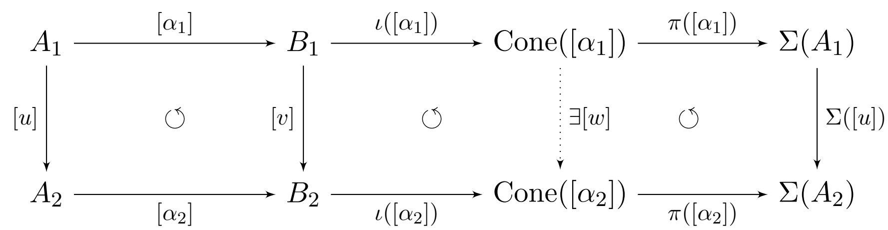
Set $W \coloneqq C_{\iota_\alpha}$, $x \coloneqq \iota_{\iota_\alpha}$ and $y \coloneqq m_{\iota_\alpha}$, then $u_{x,y}\colon I_B \oplus T_A \to C_{\iota_\alpha}$ is an isomorphism. If we denote $h_{x,y}\colon T_A \to C_{\iota_\alpha}$ by λ, then $[\lambda]$ induces the desired isomorphism[7] of triangles

[7]Note that $I_B \oplus T_A \cong T_A$ in $\mathscr{C}/\mathcal{Q}$.



The inverse morphism of $[\lambda]$ can be computed again by the universal property of the pushout object C_{ι_α}. I.e., since $\iota_\alpha \cdot \pi_\alpha = 0 = \inf_B \cdot 0$, there exists a unique morphism $\mu\colon C_\alpha \to T_A$ with $\iota_{\iota_\alpha} \cdot \mu = \pi_\alpha$ and $m_{\iota_\alpha} \cdot \mu = 0$. We have then $[\lambda]^{-1} = [\mu]$.

TR′ 3. For a given quadruple of morphisms $[\alpha_1], [u], [v]$ and $[\alpha_2]$ with $[\alpha_1] \cdot [v] = [u] \cdot [\alpha_2]$, we need to compute a morphism $[w]\colon \mathrm{Cone}([\alpha_1]) \to \mathrm{Cone}([\alpha_2])$ which induces a morphism of exact triangles

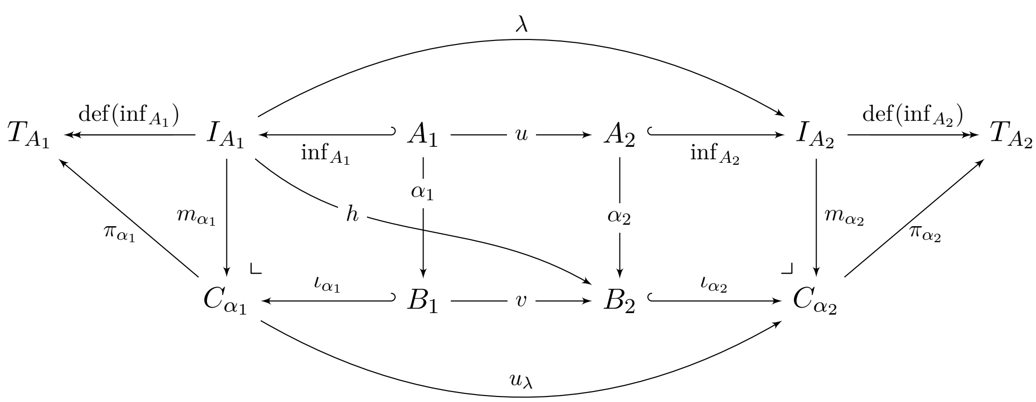


Let $\lambda\colon I_{A_1} \to I_{A_2}$ be an $\mathcal{E}$-injective colift of $u \cdot \inf_{A_2}$ along $\inf_{A_1}$. The equality $[\alpha_1] \cdot [v] - [u] \cdot [\alpha_2] = 0$ implies the existence of a morphism $h\colon I_{A_1} \to B_2$ with $\alpha_1 \cdot v - u \cdot \alpha_2 = \inf_{A_1} \cdot h$. A direct verification shows that

$$\inf_{A_1} \cdot (\lambda \cdot m_{\alpha_2} + h \cdot \iota_{\alpha_2}) = \alpha_1 \cdot v \cdot \iota_{\alpha_2},$$

hence there exists a unique morphism $u_\lambda\colon C_{\alpha_1} \to C_{\alpha_2}$ with

$$m_{\alpha_1} \cdot u_\lambda = \lambda \cdot m_{\alpha_2} + h \cdot \iota_{\alpha_2} \text{ and } \iota_{\alpha_1} \cdot u_\lambda = v \cdot \iota_{\alpha_2}.$$



Furthermore, $\pi_{\alpha_1} \cdot T_u = u_\lambda \cdot \pi_{\alpha_2}$ by the universal property of the pushout object C_{α_1} and the following two equalities:

1. $m_{\alpha_1} \cdot \pi_{\alpha_1} \cdot T_u = \mathrm{def}(\inf_{A_1}) \cdot T_u = \lambda \cdot \mathrm{def}(\inf_{A_2}) = \lambda \cdot m_{\alpha_2} \cdot \pi_{\alpha_2}$
$= (m_{\alpha_1} \cdot u_\lambda - h \cdot \iota_{\alpha_2}) \cdot \pi_{\alpha_2} = m_{\alpha_1} \cdot u_\lambda \cdot \pi_{\alpha_2}$ and

2. $\iota_{\alpha_1} \bullet \pi_{\alpha_1} \bullet T_u = 0 \bullet T_u = 0 = v \bullet 0 = v \bullet \iota_{\alpha_2} \bullet \pi_{\alpha_2} = \iota_{\alpha_1} \bullet u_\lambda \bullet \pi_{\alpha_2}$.

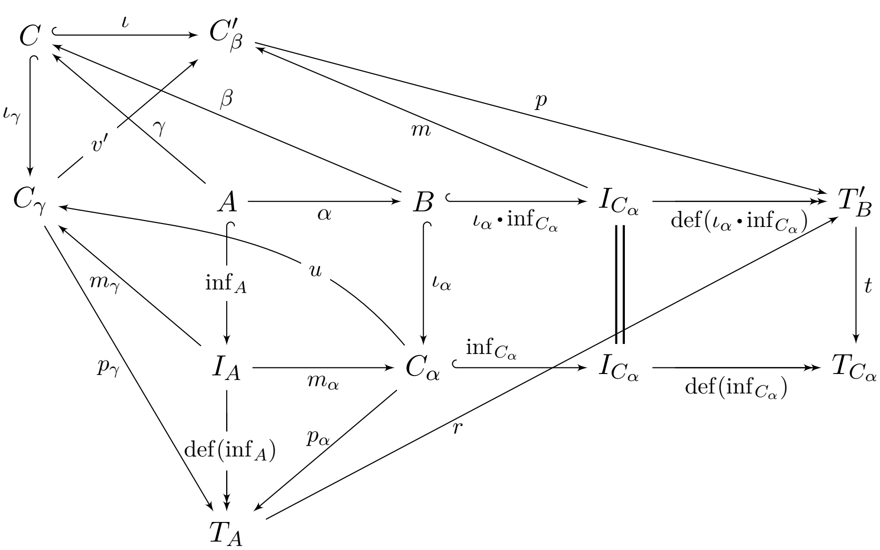
The morphism $[w] \coloneqq [u_\lambda]\colon C_{\alpha_1} \to C_{\alpha_2}$ induces the desired morphism of exact triangles.

TR′ 4. Let $[\alpha]\colon A \to B, [\beta]\colon B \to C$ and $[\gamma]\colon A \to C$ be a triple of morphisms with $[\alpha] \bullet [\beta] = [\gamma]$. Without loss of generality we can assume $\gamma = \alpha \bullet \beta$. Let $\inf_A\colon A \hookrightarrow I_A$ be an inflation into some $\mathcal{E}$-injective object I_A and let $\operatorname{def}(\inf_A)$ be the associated deflation. By the axiom **EX 3**, we can complete the cospan $I_A \xleftarrow{\inf_A} A \xrightarrow{\alpha} B$ via an object C_α and two morphisms $i_\alpha\colon B \to C_\alpha$ and $m_\alpha\colon I_A \to C_\alpha$ into a pushout diagram. Since $\inf_A \bullet \operatorname{def}(\inf_A) = 0$, there exists a unique morphism $p_\alpha\colon C_\alpha \to T_A$ with $m_\alpha \bullet p_\alpha = \operatorname{def}(\inf_A)$ and $\iota_\alpha \bullet p_\alpha = 0$. By a similar discussion for the cospan $I_A \xleftarrow{\inf_A} A \xrightarrow{\gamma} C$ we get a pushout object C_γ and a triple of morphisms i_γ, m_γ and p_γ with $\gamma \bullet \iota_\gamma = \inf_A \bullet m_\gamma$, $m_\gamma \bullet p_\gamma = \operatorname{def}(\inf_A)$ and $\iota_\gamma \bullet p_\gamma = 0$.

Since ι_α and $\inf_{C_\alpha}$ are inflations, their composition $\iota_\alpha \bullet \inf_{C_\alpha}$ is an inflation as well. Its associated deflation will be denoted by $\operatorname{def}(\iota_\alpha \bullet \inf_{C_\alpha})$. Again, the cospan $I_{C_\alpha} \xleftarrow{\iota_\alpha \bullet \inf_{C_\alpha}} B \xrightarrow{\beta} C$ gives rise to a pushout object C'_β and a triple of morphisms ι, m and p where $\beta \bullet \iota = \iota_\alpha \bullet \inf_{C_\alpha} \bullet m$, $m \bullet p = \operatorname{def}(\iota_\alpha \bullet \inf_{C_\alpha})$ and $\iota \bullet p = 0$.

We denote by $t\colon T'_B \to T_{C_\alpha}$ the cokernel colift of $\operatorname{def}(\inf_{C_\alpha})$ along $\operatorname{def}(\iota_\alpha \bullet \inf_{C_\alpha})$ and by $r\colon T_A \to T'_B$ the cokernel colift of $m_\alpha \bullet \inf_{C_\alpha} \bullet \operatorname{def}(\iota_\alpha \bullet \inf_{C_\alpha})$ along $\operatorname{def}(\inf_A)$.

Since $\alpha \bullet (\beta \bullet \iota_\gamma) = \gamma \bullet \iota_\gamma = \inf_A \bullet (m_\gamma)$, there exists a unique morphism $u\colon C_\alpha \to C_\gamma$ with $\iota_\alpha \bullet u = \beta \bullet \iota_\gamma$ and $m_\alpha \bullet u = m_\gamma$. On the other hand, $\gamma \bullet (\iota) = \alpha \bullet \beta \bullet \iota = \alpha \bullet \iota_\alpha \bullet \inf_{C_\alpha} \bullet m = \inf_A \bullet (m_\alpha \bullet \inf_{C_\alpha} \bullet m)$, hence there exists a unique morphism $v'\colon C_\gamma \to C'_\beta$ with $\iota_\gamma \bullet v' = \iota$ and $m_\gamma \bullet v' = m_\alpha \bullet \inf_{C_\alpha} \bullet m$.



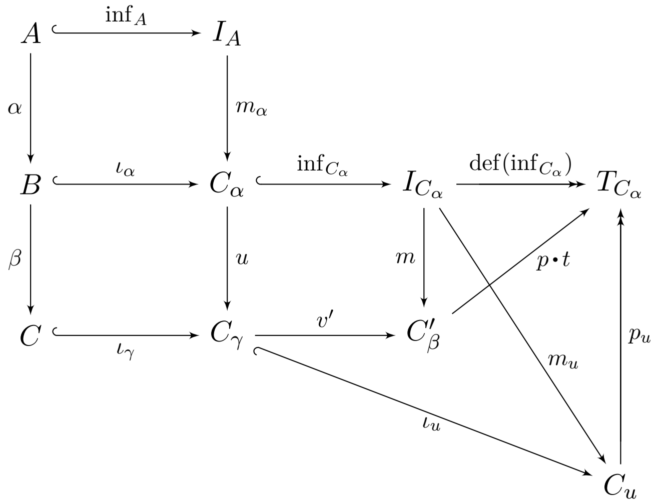
We claim that $u \bullet v' = \inf_{C_\alpha} \bullet m$. The equality follows by the universal property of the pushout object C_α and the following two equalities:

1. $\iota_\alpha \bullet (u \bullet v') = \beta \bullet \iota_\gamma \bullet v' = \beta \bullet \iota = \iota_\alpha \bullet (\inf_{C_\alpha} \bullet m)$ and
2. $m_\alpha \bullet (u \bullet v') = m_\gamma \bullet v' = m_\alpha \bullet (\inf_{C_\alpha} \bullet m)$.

The equality $u \bullet v' = \inf_{C_\alpha} \bullet m$ implies $[u] \bullet [v'] = [u \bullet v'] = 0$.

By a similar argument, we get the following equalities $v' \bullet p = p_\gamma \bullet r$, $u \bullet p_\gamma = p_\alpha$ and $v' \bullet p \bullet t = 0$.

The above morphisms induce the following commutative diagram

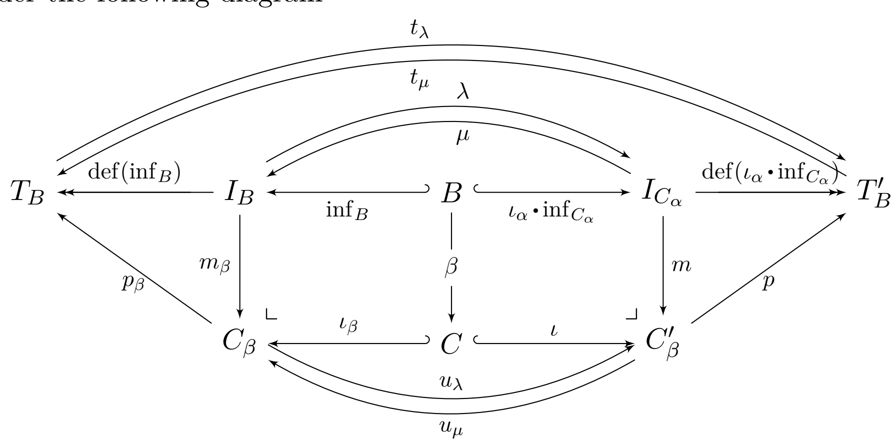


The pair (m_α, ι_α) defines a pushout diagram of the cospan (inf_A, α) and $(m_\gamma = m_\alpha \bullet u, \iota_\gamma)$ defines a pushout diagram of the cospan $(\mathrm{inf}_A, \gamma = \alpha \bullet \beta)$, hence (u, ι_γ) defines a pushout diagram of the cospan (ι_α, β). On the other hand, the pair $(m, \iota = \iota_\gamma \bullet v')$ defines a pushout diagram of the cospan $(\iota_\alpha \bullet \mathrm{inf}_{C_\alpha}, \beta)$, hence the pair (m, v') defines a pushout diagram of the cospan $(\mathrm{inf}_{C_\alpha}, u)$. Furthermore, $v' \bullet p \bullet t = 0$ and $m \bullet p \bullet t = \mathrm{def}(\iota_\alpha \bullet \mathrm{inf}_{C_\alpha}) \bullet t = \mathrm{def}(\mathrm{inf}_{C_\alpha})$, hence the triangle

$$C_\alpha \xrightarrow{[u]} C_\gamma \xrightarrow{[v']} C'_\beta \xrightarrow{[p \bullet t]} T_{C_\alpha}$$

is exact. The isomorphism between C_u and C'_β can be computed by the universal property of the pushout object C_u.

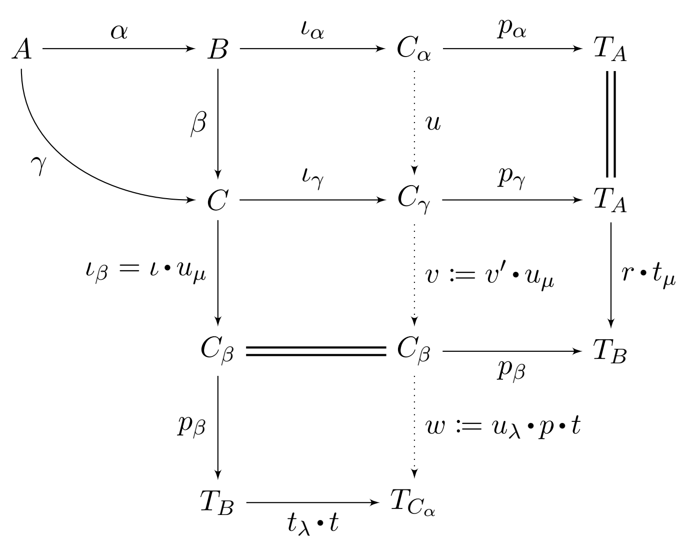
Suppose $\lambda\colon I_B \to I_{C_\alpha}$ and $\mu\colon I_{C_\alpha} \to I_B$ are $\mathcal{E}$-injective colifts of inf_B and $\iota_\alpha \bullet \mathrm{inf}_{C_\alpha}$ along each other. By Remark 5.27, there exist unique morphisms $t_\lambda, u_\lambda, t_\mu$ and t_λ which render the following diagram



commutative and satisfy $[t_\mu] = [t_\lambda]^{-1}$ and $[u_\mu] = [u_\lambda]^{-1}$. In particular, the triangle

$$C_\alpha \xrightarrow{[u]} C_\gamma \xrightarrow{[v' \bullet u_\mu]} C_\beta \xrightarrow{[u_\lambda \bullet p \bullet t]} T_{C_\alpha}$$

is exact. The above data gives rise to the following commutative diagram:



A simple diagram chase shows that $m_\alpha \bullet \inf_{C_\alpha} \bullet \mu \colon I_A \to I_B$ is an $\mathcal{E}$-injective colift of $\alpha \bullet \inf_B \colon A \to I_B$ along $\inf_A \colon A \to I_A$; and $r \bullet t_\mu$ is *the* cokernel colift of $m_\alpha \bullet \inf_{C_\alpha} \bullet \mu \bullet \operatorname{def}(\inf_B) \colon I_A \to T_B$ along $\operatorname{def}(\inf_A) \colon I_A \to T_A$, hence $[r \bullet t_\mu] = \Sigma([\alpha])$. By a similar argument, we can show that $[t_\lambda \bullet t] = \Sigma([\iota_\alpha])$. The octahedral axiom follows by considering the above commutative diagram in the stable category $\mathscr{C}/\mathcal{Q}$.

□

Corollary 5.30. *Let $(\mathscr{C}, \mathcal{E})$ be a computable* FROBENIUS *category equipped with a $\mathscr{D}$-homomorphism structure $(\mathbb{1}, H(-,-), \nu)$ such that*

- *$\mathscr{D}$ is Abelian and has decidable lifts,*
- *$\mathbb{1}$ is a projective object,*
- *$\operatorname{Hom}_{\mathscr{D}}(\mathbb{1}, -)$ is a faithful functor,*

then the stable category $\mathscr{C}/\mathcal{Q}$ is a computable triangulated category.

PROOF. Follows by Lemma 5.4 and Corollary 4.24. □

In the rest of this section we discuss our primary example of a FROBENIUS category: The category of finitely presented (graded) modules over the exterior algebra $E = k[e_0, \dots, e_n]$.

Definition 5.31. An **involution** on a ring R is an anti-isomorphism $\Theta \colon R \to R$ with $\Theta^2 = \mathrm{id}_R$, i.e., Θ is an isomorphism of the underlying Abelian group $(R, +)$ and $\Theta(1) = 1$, $\Theta(\Theta(a)) = a$, and $\Theta(ab) = \Theta(b)\Theta(a)$ for all $a, b \in R$.

Definition 5.32. Let R be a ring with involution $\Theta \colon R \to R$. For a given matrix $\mathtt{M} \in R^{s \times t}$ we denote by $\Theta(\mathtt{M})$ the matrix $(\Theta(a_{ji}))_{ij} \in R^{t \times s}$. For a given compatible pair of matrices[8] $(\mathtt{M}, \mathtt{N})$, we have $\Theta(\mathtt{MN}) = \Theta(\mathtt{N})\Theta(\mathtt{M})$ and $\Theta(\Theta(\mathtt{M})) = \mathtt{M}$.

[8] I.e., the number of columns of $\mathtt{M}$ equals the number of rows of $\mathtt{N}$.

Remark 5.33. Let R be a ring with involution $\Theta\colon R \to R$. Any right R-module M can be turned to a left R-module via $rm := m\Theta(r)$.

Example 5.34. The identity mapping of any commutative ring defines an involution. In this case, the involution of a matrix is simply its transposed matrix.

Remark 5.35. Let R be a G-graded ring and M, N objects in R-**grmod**, then[9]

$$\mathrm{Hom}_{R\text{-}\mathbf{mod}}(M,N) \cong \bigoplus_{d\in G} \mathrm{Hom}_d(M,N).$$

Remark 5.36. Let R, S, Q be three rings. If M is an R-S-bimodule and N is a R-Q-bimodule then $\mathrm{Hom}_R(M,N)$ is an S-Q-bimodule via $(s\varphi q)(m) = \varphi(ms)q$. If we take $N := R$ as an R-R-bimodule, we get, according to Remark 5.35, that $\mathrm{Hom}_{R\text{-}\mathbf{mod}}(M,R)$ is a G-graded right R-module whose d-homogeneous part for $d \in G$ is $\mathrm{Hom}_d(M,R)$.

Example 5.37. Let k be a field. The exterior algebra $E = k[e_0,\dots,e_n]$ can be equipped with the involution

$$\Theta\colon \begin{cases} E & \to E, \\ e_{i_1}e_{i_2}\dots e_{i_m} & \mapsto e_{i_m}\dots e_{i_2}e_{i_1}. \end{cases}$$

For instance, $\Theta(e_0e_2) = e_2e_0 = -e_0e_2$ and $\Theta(e_0e_2e_1) = e_1e_2e_0 = e_0e_1e_2$. For every left E-module M, the Abelian group $\mathrm{Hom}_{E\text{-}\mathbf{mod}}(M,E)$ carries a right E-module structure via

$$\begin{cases} \mathrm{Hom}_{E\text{-}\mathbf{mod}}(M,E)\times E & \to \mathrm{Hom}_{E\text{-}\mathbf{mod}}(M,E), \\ (f,q) & \mapsto fq\colon \begin{cases} M & \to E, \\ m & \mapsto f(m)q. \end{cases} \end{cases}$$

By Remark 5.33, $\mathrm{Hom}_{E\text{-}\mathbf{mod}}(M,E)$ can be turned into a left E-module via

$$\begin{cases} E\times \mathrm{Hom}_{E\text{-}\mathbf{mod}}(M,E) & \to \mathrm{Hom}_{E\text{-}\mathbf{mod}}(M,E), \\ (r,f) & \mapsto rf\colon \begin{cases} M & \to E, \\ m & \mapsto f(m)\Theta(r). \end{cases} \end{cases}$$

The **duality functor** $(-)^*$ is defined by

$$(-)^*\colon \begin{cases} (E\text{-}\mathbf{mod})^{\mathrm{op}} & \to E\text{-}\mathbf{mod}, \\ M & \mapsto M^* := \mathrm{Hom}_{E\text{-}\mathbf{mod}}(M,E), \\ \varphi^{\mathrm{op}}\colon N\to M & \mapsto \varphi^*\colon \begin{cases} N^* & \to M^*, \\ f & \mapsto \varphi\cdot f. \end{cases} \end{cases}$$

For $p \in E$, we define the morphism $\varphi_p\colon E^{1\times 1} \to E^{1\times 1}$, $r \mapsto rp$. Of course, φ_p corresponds in E-**rows** to the morphism $E^{1\times 1} \xrightarrow{(p)} E^{1\times 1}$.

The left E-module $\left(E^{1\times 1}\right)^* = \{\varphi_p \mid p \in E\}$ is generated by φ_1. In particular, $\varphi_p = \Theta(p)\varphi_1$ for all $p \in E$. Furthermore, $\left(E^{1\times 1}\right)^* \cong E^{1\times 1}$ via $\varphi_1 \leftrightarrow 1$.

For any morphism $\varphi_p\colon E^{1\times 1} \to E^{1\times 1}$, we have

$$\Big(\varphi_p^*(\varphi_1)\Big)(r) = (\varphi_p\cdot\varphi_1)(r) = \varphi_1(\varphi_p(r)) = \varphi_1(rp) = (\Theta(p)\varphi_1)(r),$$

[9] For a proof, see [**NVO04**, Section 2.4].

hence φ_p^* corresponds in E-**rows** to the morphism $E^{1\times 1} \xrightarrow{(\Theta(p))} E^{1\times 1}$. Since $(-)^*$ is additive, we can explicitly construct it on E-**rows**.

$$(-)^*\colon \begin{cases} (E\text{-}\mathbf{rows})^{\mathrm{op}} & \to E\text{-}\mathbf{rows}, \\ E^{1\times m} & \mapsto E^{1\times m}, \\ \left(E^{1\times m} \xrightarrow{\mathtt{F}} E^{1\times n}\right)^{\mathrm{op}} & \mapsto E^{1\times n} \xrightarrow{\Theta(\mathtt{F})} E^{1\times m}. \end{cases}$$

Any finitely presented left E-module M fits into an exact sequence

$$E^{1\times m} \xrightarrow{\mathtt{M}} E^{1\times n} \xrightarrow{\pi} M \to 0;$$

and since $(-)^*$ is left exact, we get another exact sequence

$$E^{1\times m} \xleftarrow{\Theta(\mathtt{M})} E^{1\times n} \xleftarrow{\pi^*} M^* \leftarrow 0.$$

Because of the universal property of kernels, $(-)^*$ can be extended to $\mathcal{A}(E\text{-}\mathbf{rows}) \cong E\text{-}\mathbf{fpres} \cong E\text{-}\mathbf{fpmod}$ as follows:

$$(-)^*\colon \begin{cases} (\mathcal{A}(E\text{-}\mathbf{rows}))^{\mathrm{op}} & \to \mathcal{A}(E\text{-}\mathbf{rows}), \\ M := \left(E^{1\times m} \xrightarrow{\mathtt{M}} E^{1\times n}\right)_{\mathcal{A}} & \mapsto M^* := \ker\left(\left(0 \to E^{1\times n}\right)_{\mathcal{A}} \xrightarrow{\Theta(\mathtt{M})} \left(0 \to E^{1\times m}\right)_{\mathcal{A}}\right), \\ \left(M \xrightarrow{\mathtt{F}} N\right)^{\mathrm{op}} & \mapsto \textit{the} \text{ induced kernel lift from } N^* \text{ to } M^*. \end{cases}$$

The exterior algebra is quasi-Frobenius [**Die58**], hence an E-module is projective if and only if it is injective. In particular, E is injective, hence the functor $(-)^* = \mathrm{Hom}_{\mathbf{mod}\text{-}E}(-, E)$ is exact. This means applying it on the above exact sequence yields again another exact sequence

$$E^{1\times m} \xrightarrow{\mathtt{M}} E^{1\times n} \xrightarrow{\pi^{**}} M^{**} \to 0.$$

In particular, we get a natural isomorphism

$$\nu\colon \begin{cases} \mathrm{id}_{E\text{-}\mathbf{mod}} & \to (-)^{**}, \\ M & \mapsto \textit{the} \text{ cokernel colift of } \pi \text{ along } \pi^{**}. \end{cases}$$

This enables us to compute for each M in E-**fpmod** a monomorphism $\inf_M\colon M \hookrightarrow I_M$ where I_M is an injective E-module: We compute an epimorphism $E^{1\times t} \xrightarrow{\tau} M^*$ from some free E-module, then take the composition $M \xrightarrow{\nu(M)} M^{**} \xhookrightarrow{\tau^*} E^{1\times t}$.

To sum up, the category E-**fpmod** $\cong \mathcal{A}(E\text{-}\mathbf{rows}) \cong E$-**fpres** is computable Abelian with enough projectives and injectives. By Example 2.60, the class $\mathcal{L}$ of all projective objects defines a system of lifting objects. Analogously, by Example 2.62, the class $\mathcal{Q}$ of all injective objects defines a system of colifting objects. Since $\mathcal{L} = \mathcal{Q}$, the associated stable categories coincide: $\mathscr{C}/\mathcal{L} \cong \mathscr{C}/\mathcal{Q}$. This means, for a morphism $\varphi\colon M \to N$ in E-**fpmod**, $[\varphi] = 0$ if and only if φ lifts along $\ell_N\colon L_N \twoheadrightarrow N$ if and only if φ colifts along $q_M\colon M \hookrightarrow Q_M$.

This whole discussion can be lifted to the graded case up to minor issues. Let E be a G-graded exterior algebra and M an object in E-**grmod**, then[10]

$$M^* := \mathrm{Hom}_{E\text{-}\mathbf{mod}}(M, E(0)) \cong \bigoplus_{d\in G} \mathrm{Hom}_d(M, E(0)),$$

[10]See Definition 2.7.

hence, M^* still belongs to E-**grmod** and for every $d \in G$ the homogeneous part $(M^*)_d$ consists of the graded morphisms $M \to E(0)$ in E-**mod** of degree d. In particular, $E(d)^*$ is generated as a G-graded left E-module by the map $\varphi_1\colon E(d) \to E(0)$, $r \mapsto r$ whose degree is d and $E(d)^* \cong E(-d)$ via $\varphi_1 \leftrightarrow 1$. Moreover, the dual of a morphism $E(d) \xrightarrow{(p)} E(h)$ in E-**grrows** is the morphism $E(-h) \xrightarrow{(\theta(p))} E(-d)$ in E-**grrows**. Analogously, this can be extended to $\mathcal{A}(E\text{-}\mathbf{grrows}) \cong E\text{-}\mathbf{fpgrmod} \cong E\text{-}\mathbf{grfpres}$ and can be used to compute injective resolutions in these categories.

The category E-**fpmod** is Abelian, hence exact. Since E is a quasi-Frobenius algebra, the classes of projective and injective objects coincide, it is a Frobenius category. Consequently, the associated stable category $E\text{-}\mathbf{fpmod}/\mathcal{Q}$ is triangulated (cf. [**HJR10**]). See Appendix D for a software demonstration of this category.

CHAPTER 6

Tilting Equivalences via Strong Exceptional Sequences

6.1. Overview of Tilting Theory between Algebras

Tilting theory is a mathematical tool introduced in the early seventies to characterize the existence of equivalences between module categories over finite dimensional algebras by means of a class of bimodules and the standard operations of Hom and $\otimes$ functors (see e.g., [**BGfP73**] and [**BB80**]). The derived version of the tilting theory has been initiated in [**Hap88**], [**Bon81**] and [**CPS86**] via the notion of generalized tilting modules which enables the construction of exact equivalences between derived categories of modules in terms of derived functors $-\otimes^{\mathbb{L}} T$ and $\mathbb{R}\operatorname{Hom}(T,-)$. Soon after, Rickard introduced the notion of a tilting complex in his work to characterize the existence of exact equivalences between derived categories of modules (cf. [**Ric89**] and [**Ric91**]).

Remark 6.1. Let S, R, Q be three rings. Then

(1) If M is an S-R-bimodule and N is an S-Q-bimodule then $\operatorname{Hom}_S(M,N)$ is an R-Q-bimodule via $(r\varphi q)(m) = \varphi(mr)q$.
(2) If M is an S-R-bimodule and N is a Q-R-bimodule then $\operatorname{Hom}_R(M,N)$ is an Q-S-bimodule via $(q\varphi s)(m) = q\varphi(sm)$.
(3) If M is an S-R-bimodule and N is an R-Q-bimodule then $M\otimes_R N$ is an S-Q-bimodule via $s(m\otimes n)q = (sm)\otimes(nq)$.

Remark 6.2. Let $\mathscr{C}$ be an additive category and let M be an object in $\mathscr{C}$, then the Abelian group $\operatorname{End}_{\mathscr{C}} M \coloneqq \operatorname{Hom}_{\mathscr{C}}(M,M)$ can be turned into a ring in two different ways:

(1) We define the multiplication of two elements $f, g\colon M\to M$ by their pre-composition, i.e., $f\cdot g \coloneqq f \bullet g$. This will be the default choice for considering $\operatorname{End}_{\mathscr{C}} M$ as a ring.
(2) We define the multiplication of two elements $f, g\colon M\to M$ by their post-composition, i.e., $f\cdot g \coloneqq f\circ g$. The resulted ring is isomorphic to $\operatorname{End}_{\mathscr{C}}^{\mathrm{op}} M$.

Remark 6.3. Any Abelian group M is a right $\operatorname{End}_R M$-module via $m\cdot f \coloneqq f(m)$.

Let R and S be associative unital k-algebras and T an S-R-bimodule[1]. Then we have adjoint functors[2]

$$-\otimes_S T\colon \mathbf{Mod}\text{-}S \rightleftarrows \mathbf{Mod}\text{-}R : \operatorname{Hom}_R(T,-).$$

One variant of Morita's theorems states that these functors are quasi-inverse equivalences if and only if

(1) T is a finitely generated projective right R-module,

[1]An S-R-bimodule is by definition a left S-module and right R-module.
[2]Each pair of adjoint functors between module categories is of this form [**Kel07**].

(2) the canonical map

$$\begin{cases} S & \to \operatorname{End}_R^{\mathrm{op}} T, \\ s & \mapsto \varphi_s \colon \begin{cases} T & \to T, \\ t & \mapsto s \cdot t \end{cases} \end{cases}$$

is an isomorphism, and

(3) the free right R-module of rank one R is a direct summand of a finite direct sum of copies of T.

In this case, we call T a tilting R-module and we say R and S are **Morita equivalent**.

Example 6.4. Let k be a field and A a finite dimensional k-algebra. Suppose $P_1, \dots, P_n$ are the isomorphism classes of indecomposable direct summands of A_A which are necessarily projective right A-modules. The A-module $T = \bigoplus_{i=1}^n P_i$ is an $\operatorname{End}_A^{\mathrm{op}} T$-$A$-bimodule and satisfies the above assumptions, hence it is a tilting A-module. Since P_i's are pairwise non-isomorphic the algebra $\operatorname{End}_A^{\mathrm{op}} T$ is basic. In particular, any finite dimensional k-algebra is Morita equivalent to a basic algebra. If k is algebraically closed, then $\operatorname{End}_A^{\mathrm{op}} T$ is isomorphic to an admissible quiver k-algebra (see, e.g., [**ARS97**], [**DW17**] or [**ASS06**]).

The following is the derived version of Morita's equivalence. For the proof we refer to [**Hap88**, Theorem 2.10], [**Kel07**, Section 4] and [**CPS86**].

Theorem 6.5 (Happel's theorem)**.** *Let R and S be associative unital k-algebras and T an S-R-bimodule. The derived functors*

$$\left(- \otimes_S^{\mathbb{L}} T\right) \colon \mathcal{D}(\mathbf{Mod}\text{-}S) \rightleftarrows \mathcal{D}(\mathbf{Mod}\text{-}R) \colon \mathbb{R}\operatorname{Hom}_R(T, -)$$

are quasi-inverse equivalences if and only if

(1) As a right R-module, T admits a finite resolution

$$0 \to P^{-n} \to \cdots \to P^0 \to T \to 0$$

by finitely generated projective right R-modules P^i,

(2) The canonical map

$$\begin{cases} S & \to \operatorname{End}_R^{\mathrm{op}} T, \\ s & \mapsto \varphi_s \colon \begin{cases} T & \to T, \\ t & \mapsto s \cdot t \end{cases} \end{cases}$$

is an isomorphism and for each $i > 0$, we have $\operatorname{Ext}_R^i(T, T) = 0$, and

(3) There exists an acyclic complex

$$0 \to R \to T^0 \to T^1 \to \cdots \to T^m \to 0$$

where R is considered as a right R-module over itself and the T^i are direct summands of finite direct sums of copies of T.

If these conditions hold and, moreover, S and R are right noetherian, then the derived functors restrict to quasi-inverse equivalences

$$\left(- \otimes_S^{\mathbb{L}} T\right) \colon \mathcal{D}^b(\mathbf{mod}\text{-}S) \rightleftarrows \mathcal{D}^b(\mathbf{mod}\text{-}R) \colon \mathbb{R}\operatorname{Hom}_R(T, -).$$

where **mod**-S *and* **mod**-R *denote the category of finitely generated right S-modules resp. R-modules.*

Definition 6.6. Let R be a ring. A right R-module T will be called a **generalized tilting right R-module** if

(1) T admits a finite resolution

$$0 \to P^{-n} \to \cdots \to P^0 \to T \to 0$$

by finitely generated projective right R-modules P^i,

(2) T has no higher extensions, i.e., $\operatorname{Ext}^i(T,T) = 0$ for all $i > 0$,

(3) There is an acyclic complex

$$0 \to R \to T^0 \to T^1 \to \cdots \to T^m \to 0$$

where R is considered as a right R-module over itself and the T^i are direct summands of finite direct sums of copies of T.

Corollary 6.7. *Let T be a generalized tilting right R-module, then the derived functors*

$$- \otimes^{\mathbb{L}}_{\operatorname{End}^{\mathrm{op}} T} T \colon \mathcal{D}(\mathbf{Mod}\text{-}\operatorname{End}^{\mathrm{op}} T) \rightleftarrows \mathcal{D}(\mathbf{Mod}\text{-}R) : \mathbb{R}\operatorname{Hom}_R(T,-).$$

are quasi-inverse. If in addition, $\operatorname{End}^{\mathrm{op}} T$ *and R are right noetherian, then the derived functors restrict to quasi-inverse equivalences*

$$- \otimes^{\mathbb{L}}_{\operatorname{End}^{\mathrm{op}} T} T \colon \mathcal{D}^b(\mathbf{mod}\text{-}\operatorname{End}^{\mathrm{op}} T) \rightleftarrows \mathcal{D}^b(\mathbf{mod}\text{-}R) : \mathbb{R}\operatorname{Hom}_R(T,-).$$

Remark 6.8. Let k be a field and A be a finite dimensional k-algebra. According to [**Miy86**] and [**Bae88**, Definition 8.1], Axiom 3 in Definition 6.6 can be replaced by the following condition:

($3'$) R belongs to the smallest thick triangulated subcategory of $\mathcal{D}^b(\mathbf{mod}\text{-}R)$ containing T.

This means, instead of verifying 3, we can now verify $3'$ by, e.g., checking whether the counit component

$$\epsilon_R \colon \mathbb{R}\operatorname{Hom}_R(T,R) \otimes^{\mathbb{L}}_{\operatorname{End}^{\mathrm{op}} T} T \to R$$

is an isomorphism (cf. Appendix E).

Remark 6.9. Let k be a field and $\mathbb{A}$ be a finite dimensional k-algebra. Then the category **mod**-$\mathbb{A}$ of finitely generated $\mathbb{A}$-modules coincide with the category **fdmod**-$\mathbb{A}$ of finite dimensional $\mathbb{A}$-modules. Any generalized tilting right $\mathbb{A}$-module T can be resolved by finitely generated projective right $\mathbb{A}$-modules, hence T belongs to **fdmod**-$\mathbb{A}$. In particular, $\operatorname{End} T$ is also a finite dimensional k-algebra. If the indecomposable direct summands of T form a strong exceptional sequence in **fdmod**-$\mathbb{A}$ (cf. Definition 6.19), then $\operatorname{End}^{\mathrm{op}} T$ has finite global dimension (cf. Corollary 6.37).

The adjunction

$$- \otimes_{\operatorname{End}^{\mathrm{op}} T} T \colon \mathbf{mod}\text{-}\operatorname{End}^{\mathrm{op}} T \rightleftarrows \mathbf{mod}\text{-}\mathbb{A} : \operatorname{Hom}_{\mathbb{A}}(T,-)$$

can naturally be extended to the bounded homotopy categories

$$- \otimes_{\operatorname{End}^{\mathrm{op}} T} T \colon \mathcal{K}^b(\mathbf{mod}\text{-}\operatorname{End}^{\mathrm{op}} T) \rightleftarrows \mathcal{K}^b(\mathbf{mod}\text{-}\mathbb{A}) : \operatorname{Hom}_{\mathbb{A}}(T,-).$$

If $\mathbb{A}$ has finite global dimension, then by Corollary 3.38 and Remark 3.40, the localization functors are adjoint to the natural embedding functors:

$$\iota \colon \mathcal{K}^b(\mathbf{proj}\text{-}\operatorname{End}^{\mathrm{op}} T) \rightleftarrows \mathcal{K}^b(\mathbf{mod}\text{-}\operatorname{End}^{\mathrm{op}} T) : \mathcal{P}$$

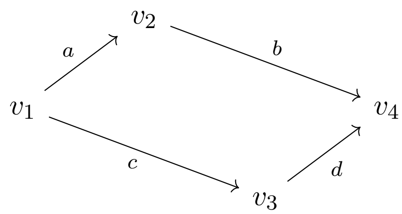
and

$$\mathcal{I} \colon \mathcal{K}^b(\mathbf{mod}\text{-}\mathbb{A}) \rightleftarrows \mathcal{K}^b(\mathbf{inj}\text{-}\mathbb{A}) : \iota.$$

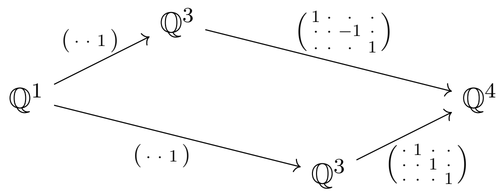
The composition of the above three adjunctions defines a pair of adjoint exact equivalences:

$$\iota \bullet (- \otimes_{\operatorname{End}^{\mathrm{op}} T}) \bullet \mathcal{I} \colon \mathcal{K}^b(\mathbf{proj}\text{-}\operatorname{End}^{\mathrm{op}} T) \rightleftarrows \mathcal{K}^b(\mathbf{inj}\text{-}\mathbb{A}) : \iota \bullet \operatorname{Hom}_{\mathbb{A}}(T,-) \bullet \mathcal{P}\,.$$

Example 6.10. Let $\mathfrak{q}$ be the right quiver:



and let $\mathbb{A}$ be the k-algebra $\mathbb{Q}\mathfrak{q}/\langle\rho\rangle$ where $\mathbb{Q}\mathfrak{q}$ is the path $\mathbb{Q}$-algebra of q and $\langle\rho\rangle \lhd \mathbb{Q}\mathfrak{q}$ is the two-sided admissible ideal generated by the relation $\rho = \{ab - cd\}$. According to Theorem 2.70, $\mathbf{mod}\text{-}\mathbb{Q}\mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle \simeq \mathbf{mod}\text{-}\mathbb{A}$. The object T



in $\mathbf{mod}\text{-}\mathbb{Q}\mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle$ is a generalized tilting object, hence induces a derived equivalences

$$\mathcal{D}^b(\mathbf{mod}\text{-}\operatorname{End}^{\mathrm{op}} T) \simeq \mathcal{D}^b(\mathbf{mod}\text{-}\mathbb{Q}\mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle).$$

The indecomposable direct summands of T form a strong exceptional sequence. For details we refer to Appendix E.

6.2. The Abstraction Algebroid of a Strong Exceptional Sequence

This section is devoted to review the definition of strong exceptional sequences in k-linear triangulated categories. We develop algorithms to compute some of their invariants. For example, an algorithm to compute an isomorphism between a strong exceptional sequence $\mathscr{E}$ and a k-linear finitely presented category $\mathbf{A}_{\mathscr{E}}$ defined by an acyclic quiver $\mathfrak{q}_{\mathscr{E}}$ subject to an admissible set of relations $\rho \subset k\mathcal{F}_{\mathfrak{q}}$. For detailed background we refer to [**BvdB03**], [**Bon89**] and [**Huy06**].

Definition 6.11. Let $\mathfrak{T}$ be a triangulated category and let $\{T_i\}_{i\in I}$ be a family of objects in $\mathfrak{T}$. The **triangulated hull** of the family $\{T_i\}_{i\in I}$, denoted by $\langle T_i\rangle_{i\in I}$, is the smallest triangulated subcategory of $\mathfrak{T}$ containing all objects of the family.

Remark 6.12. The triangulated hull of the family $\{T_i\}_{i\in I}$ can be obtained as the full additive subcategory whose objects belong to the smallest collection with the following properties:

(1) It contains the family $\{T_i\}_{i\in I}$.
(2) For any object T in the collection, $\Sigma^i(T)$ belongs to the collection for all $i \in \mathbb{Z}$.
(3) If $A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$ is an exact triangle in $\mathfrak{T}$ and A, B are in the collection, then C is also in the collection.

Definition 6.13. Let $\mathfrak{T}$ be a triangulated category and let $\{T_i\}_{i\in I}$ be a family of objects in $\mathfrak{T}$. We say that the family $\{T_i\}_{i\in I}$ generates $\mathfrak{T}$ if its triangulated hull is $\mathfrak{T}$.

Definition 6.14. Let k be a field and $\mathfrak{T}$ a k-linear Hom-finite triangulated category.

- A full subcategory $\mathscr{E} \subset \mathfrak{T}$ is called **strong exceptional** if the following hold:
  (1) It is skeletal and has finitely many objects.

(2) $\operatorname{Hom}_{\mathfrak{T}}(E, \Sigma^{\ell}(E')) = 0$ for all $E, E' \in \mathscr{E}$ and $0 \neq \ell \in \mathbb{Z}$.
(3) $\operatorname{End}_{\mathfrak{T}} E \cong k$ for all $E \in \mathscr{E}$.
(4) There exists a total ordering $\preccurlyeq$ on the objects of $\mathscr{E}$ such that $E \neq E'$ and $E \preccurlyeq E'$ implies $\operatorname{Hom}_{\mathfrak{T}}(E', E) = 0$.

- A strong exceptional subcategory $\mathscr{E} \subset \mathfrak{T}$ is called **complete** (or **full**) if its objects generate $\mathfrak{T}$.
- A sequence of objects $(E_1, \ldots, E_n)$ is called **strong exceptional sequence** in $\mathfrak{T}$ if the full subcategory generated by these objects is strong exceptional in $\mathfrak{T}$ and $E_1 \preccurlyeq \cdots \preccurlyeq E_n$.

Definition 6.15. Let $\mathfrak{T}$ a triangulated category and T in $\mathfrak{T}$. For $n \geq 1$, we define $\langle T \rangle_n$ by the full subcategory of objects in $\mathfrak{T}$ which, up to isomorphism, can be obtained from T by taking finite direct sums, direct summands, shifts and at most $n-1$ cones. It can be shown that $\bigcup_{n \geq 1} \langle T \rangle_n$ is the smallest thick triangulated subcategory of $\mathfrak{T}$ containing T. We call T a

(1) **classical generator** for $\mathfrak{T}$ if $\bigcup_{n \geq 1} \langle T \rangle_n = \mathfrak{T}$,
(2) **strong generator** for $\mathfrak{T}$ if there exists an integer $n \geq 1$ such that $\langle T \rangle_n = \mathfrak{T}$,
(3) **weak generator** for $\mathfrak{T}$ if $\operatorname{Hom}_{\mathfrak{T}}(T, \Sigma^i(U)) = 0$ for all $i \in \mathbb{Z}$ implies $U \cong 0$.

Lemma 6.16. *Let $\mathfrak{T}$ be a triangulated category. Let T, U be objects in $\mathfrak{T}$. The following statements are equivalent:*

(1) $\operatorname{Hom}_{\mathfrak{T}}(T, \Sigma^i(U)) = 0$ *for all $i \in \mathbb{Z}$,*
(2) $\operatorname{Hom}_{\mathfrak{T}}(E, \Sigma^i(U)) = 0$ *for all $i \in \mathbb{Z}$ and $E \in \bigcup_{n \geq 1} \langle T \rangle_n$.*

PROOF. We will prove the assertion by induction on n. Let $E \in \langle T \rangle_n$ for some $n \in \mathbb{Z}$. If $E \in \langle T \rangle_1$, then the assertion is obvious. If $E \in \langle T \rangle_n$ for some $n > 1$, then by the definition of $\langle T \rangle_n$, there exist two objects $E_1 \in \langle T \rangle_{n_1}, E_2 \in \langle T \rangle_{n_2}$ with $n_1, n_2 < n$ and a morphism $\alpha \colon E_1 \to E_2$ that can be completed to an exact triangle

$$E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\iota} E \xrightarrow{\pi} \Sigma(E_1).$$

Suppose there exists $i \in \mathbb{Z}$ such that $\operatorname{Hom}_{\mathfrak{T}}(E, \Sigma^i(U)) \neq 0$ and let $\varphi \colon E \to \Sigma^i(U)$ be a nonzero morphism. It follows by the induction hypothesis that $\iota \bullet \varphi = 0$. By **TR 3**, the pair $E_2 \to 0$ and $E \xrightarrow{\varphi} \Sigma^i(U)$ can be extended via a morphism $\psi \colon E_1 \to \Sigma^{i-1}(U)$ to a morphism of exact triangles from

$$\begin{array}{ccccccc}
E_1 & \xrightarrow{\alpha} & E_2 & \xrightarrow{\iota} & E & \xrightarrow{\pi} & \Sigma(E_1) \\
{\scriptstyle\psi}\downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow{\scriptstyle\varphi} & \circlearrowleft & \downarrow{\scriptstyle\Sigma(\psi)} \\
\Sigma^{i-1}(U) & \longrightarrow & 0 & \longrightarrow & \Sigma^i(U) & \xrightarrow[\mathrm{id}_{\Sigma^i(U)}]{} & \Sigma^i(U)
\end{array}$$

By the induction hypothesis $\psi = 0$, hence $\varphi = \varphi \bullet \mathrm{id}_{\Sigma^i(U)} = \pi \bullet \Sigma(\psi) = \pi \bullet 0 = 0$, which is the desired conclusion. The converse follows since $T \in \langle T \rangle_1$. □

The following is an immediate consequence of the above lemma.

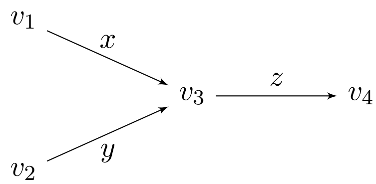
Corollary 6.17. *Every classical generator is weak.*

Example 6.18. Let $\mathscr{E} = (E_i \mid i = 1, \ldots, n)$ be a strong exceptional sequence in $\mathfrak{T}$. We refer to the object $T_{\mathscr{E}} := \bigoplus_{i=1}^n E_i$ as the tilting object associated to $\mathscr{E}$. If $\mathscr{E}$ is complete, then $T_{\mathscr{E}}$ is a classical generator to $\mathfrak{T}$.

Definition 6.19. Let $\mathscr{C}$ be an Abelian category. A full subcategory $\mathscr{E}$ in $\mathscr{C}$ is called (complete) strong exceptional in $\mathscr{C}$ if its embedding in $\mathcal{D}^b(\mathscr{C})$ is (complete) strong exceptional.

Example 6.20. Let k be a field and $\mathfrak{q}$ an acyclic quiver. Let $A := k\mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle$ be a k-linear finitely presented category defined by $\mathfrak{q}$ subject to an admissible set of relations ρ. The image of the Yoneda embedding $A \hookrightarrow A$-**mod** is complete strong exceptional. Details can be found in [**Bon89**, Lemma 5.5].

Example 6.21. Let $k = \mathbb{Q}$ and $\mathfrak{q}$ be the quiver



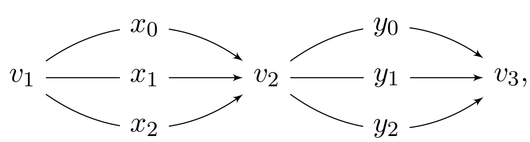
and let $\mathbf{A}_{\mathfrak{q}}$ be the $\mathbb{Q}$-linear finitely presented category defined by $\mathfrak{q}$. Consider in $\mathcal{K}^b\left(\mathbf{A}_{\mathfrak{q}}^{\oplus}\right) \cong \mathcal{D}^b(\mathbf{A}_{\mathfrak{q}}\text{-}\mathbf{mod})$ the following objects

$$\mathcal{V}_0 := \qquad 0 \longrightarrow v_1 \oplus v_2 \xrightarrow[0]{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)} v_3 \longrightarrow 0,$$

$$\mathcal{V}_1 := \lceil v_1 \rfloor_0, \qquad \mathcal{V}_2 := \lceil v_2 \rfloor_0, \qquad \mathcal{V}_3 := \lceil v_3 \rfloor_0, \qquad \mathcal{V}_4 := \lceil v_4 \rfloor_0,$$

then $\mathcal{E}_1 = (\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4)$ and $\mathcal{E}_2 = (\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_4)$ are both complete strong exceptional sequences (cf. Appendix E).

Example 6.22. Let $\mathcal{O}$ be the quiver



and let $\mathbf{A}_{\mathcal{O}}$ be the $\mathbb{Q}$-linear finitely presented category defined by $\mathcal{O}$ subject to the admissible relations $\rho = \{x_i y_j - x_j y_i \mid 0 \leq i, j \leq 2\}$. Let $\mathbf{A}_{\mathcal{O}}^{\oplus}$ be the additive closure of $\mathbf{A}_{\mathcal{O}}$ and $\mathcal{K}^b\left(\mathbf{A}_{\mathcal{O}}^{\oplus}\right)$ its bounded homotopy category. The Yoneda embedding

$$\mathbf{A}_{\mathcal{O}} \hookrightarrow \mathbf{A}_{\mathcal{O}}\text{-}\mathbf{proj} \subset \mathbf{A}_{\mathcal{O}}\text{-}\mathbf{mod}$$

can be extended to an exact equivalence

$$\mathcal{K}^b\left(\mathbf{A}_{\mathcal{O}}^{\oplus}\right) \xrightarrow{\sim} \mathcal{K}^b(\mathbf{A}_{\mathcal{O}}\text{-}\mathbf{proj}) \xrightarrow{\sim} \mathcal{D}^b(\mathbf{A}_{\mathcal{O}}\text{-}\mathbf{mod}).$$

Consider in $\mathcal{K}^b\left(\mathbf{A}_{\mathcal{O}}^{\oplus}\right)$ the following six objects:

$$\mathcal{O}_{-1} := \qquad 0 \longrightarrow v_1^3 \xrightarrow[0]{\begin{pmatrix} x_1 & -x_0 & 0 \\ x_2 & 0 & -x_0 \\ 0 & x_2 & -x_1 \end{pmatrix}} v_2^3 \xrightarrow{\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}} v_3 \longrightarrow 0,$$

$$\Omega_1 := \qquad 0 \longrightarrow v_1^3 \xrightarrow[0]{\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}} v_2 \longrightarrow 0,$$

$$\mathcal{O}_0 := \lceil v_1 \rfloor_0, \qquad \mathcal{O}_1 := \lceil v_2 \rfloor_0, \qquad \mathcal{O}_2 := \lceil v_3 \rfloor_0,$$

$$\mathcal{O}_3 := \qquad 0 \longrightarrow v_1 \xrightarrow[-2]{\begin{pmatrix} -x_0 & x_1 & -x_2 \end{pmatrix}} v_2^{\oplus 3} \xrightarrow{\begin{pmatrix} 0 & y_2 & -y_1 \\ y_2 & 0 & -y_0 \\ y_1 & -y_0 & 0 \end{pmatrix}} v_3^{\oplus 3} \longrightarrow 0.$$

Then $\mathcal{E}_1 := (\mathcal{O}_{-1}, \mathcal{O}_0, \mathcal{O}_1)$, $\mathcal{E}_2 := (\mathcal{O}_{-1}, \Omega_1, \mathcal{O}_0)$, $\mathcal{E}_3 := (\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2)$ and $\mathcal{E}_4 := (\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ are complete strong exceptional sequences. The objects $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ and $\mathcal{O}_4$ as objects in $\mathcal{D}^b(\mathbf{A}_{\mathcal{O}}\textbf{-mod})$ are isomorphic to their cohomologies at index 0, hence $\mathcal{E}_3$ and $\mathcal{E}_4$ live in the Abelian heart $\mathbf{A}_{\mathcal{O}}\textbf{-mod} \subset \mathcal{D}^b(\mathbf{A}_{\mathcal{O}}\textbf{-mod})$.

Definition 6.23. Let $\mathfrak{T}$ be a Hom-finite k-linear triangulated category and $\mathscr{E} = (E_i \mid i = 1, \dots, n)$ be a strong exceptional sequence in $\mathfrak{T}$. For indices $1 \leq i \leq \ell \leq j \leq n$, we denote by $\mathscr{E}_{i\ell j}$ the k-vector subspace of $\mathrm{Hom}_{\mathfrak{T}}(E_i, E_j)$ generated by all morphisms that factor through E_ℓ.

Example 6.24. Let $\mathscr{E} = (E_i \mid i = 1, \dots, n)$ be a strong exceptional sequence. Then

(1) $\mathscr{E}_{iij} = \mathscr{E}_{ijj} = \mathrm{Hom}_{\mathfrak{T}}(E_i, E_j)$.
(2) If $j < i$, $\ell < i$ or $j < \ell$ then $\mathscr{E}_{i\ell j} = 0$ (cf. Definition 6.14).

Remark 6.25. Any strong exceptional sequence $\mathscr{E} = (E_i \mid i = 1, \dots, n)$ is locular (cf. Definition A.34). Furthermore, for all $1 \leq i, j \leq n$ we have

$$\mathrm{rad}_{\mathscr{E}}(E_i, E_j) = \begin{cases} \mathrm{Hom}_{\mathscr{E}}(E_i, E_j) & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

and

$$\mathrm{rad}^2_{\mathscr{E}}(E_i, E_j) = \begin{cases} \Sigma_{\ell=i+1}^{j-1} \mathscr{E}_{i\ell j} & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

Notation 6.26. For a pair of indices $1 \leq i \neq j \leq n$ we denote by $\mathcal{B}^2_{ij}$ a basis of $\mathrm{rad}^2_{\mathscr{E}}(E_i, E_j)$ and by $\mathcal{B}_{ij}$ a basis of a complementary k-vector space of $\mathrm{rad}^2_{\mathscr{E}}(E_i, E_j)$ in $\mathrm{rad}_{\mathscr{E}}(E_i, E_j)$. In particular, the set $\{b + \mathrm{rad}^2_{\mathscr{E}}(E_i, E_j) \mid b \in \mathcal{B}_{ij}\}$ forms a basis for the space of irreducible morphisms $\mathrm{irr}_{\mathscr{E}}(E_i, E_j) := \mathrm{rad}_{\mathscr{E}}(E_i, E_j) / \mathrm{rad}^2_{\mathscr{E}}(E_i, E_j)$ (cf. Definition A.34).

The identity morphisms in $\mathscr{E}$ will be called the **paths of length** 0 in $\mathscr{E}$. For $i \neq j$, we call the elements of $\mathcal{B}_{ij}$ the **paths of length** 1 or **arrows** in $\mathscr{E}$. Compositions of arrows are called **paths of length greater than one** in $\mathscr{E}$. Since the quiver of ϵ is acyclic, there is a finite number of paths in $\mathscr{E}$. It is obvious that $\mathcal{B}_{ij} \cup \mathcal{B}^2_{ij}$ forms a basis for $\mathrm{Hom}_{\mathscr{E}}(E_i, E_j)$ for all $1 \leq i \neq j \leq n$.

The following lemma implies that the compositions of elements of $\mathcal{B}_{ij}$, $i \leq j$ completely determine the morphism spaces of $\mathscr{E}$:

Lemma 6.27. *For all $1 \leq i \neq j \leq n$, the k-vector space $\mathrm{rad}^2_{\mathscr{E}}(E_i, E_j)$ is generated by the set of all paths of length greater than one from E_i to E_j.*

Proof. A path of length greater than one from E_i to E_j factors through some object E_ℓ with $i < \ell < j$, hence lies in $\mathscr{E}_{i\ell j} \subseteq \operatorname{rad}^2_{\mathscr{E}}(E_i, E_j)$. That is, all paths of length greater than one already belong to $\operatorname{rad}^2_{\mathscr{E}}(E_i, E_j)$.

We prove the assertion by induction on $j - i$. In the case $j - i = 1$ there is no paths of length greater than one from E_i to E_j, hence the assertion holds since $\operatorname{rad}^2_{\mathscr{E}}(E_i, E_j) = 0$. For $j - i > 1$, we have $\operatorname{rad}^2_{\mathscr{E}}(E_i, E_j) = \Sigma_{\ell=i+1}^{j-1} \mathscr{E}_{i\ell j}$, i.e., it is sufficient to prove that each of $\mathscr{E}_{i\ell j}, i < \ell < j$ is generated by a set of paths of length greater than one from E_i to E_j. Let $\varphi\colon E_i \to E_j \in \mathscr{E}_{i\ell j}$ and $\varphi = \varphi_{i\ell} \cdot \varphi_{\ell j}$ for $\varphi_{i\ell} \in \operatorname{Hom}_k(E_i, E_\ell)$ and $\varphi_{\ell j} \in \operatorname{Hom}_{\mathscr{E}}(E_\ell, E_j)$. Since $\ell - i < j - i$ and $j - \ell < j - i$, it follows by the induction hypothesis that $\mathcal{B}^2_{i\ell}$ and $\mathcal{B}^2_{\ell j}$ can be chosen to be sets of paths of length greater than one. This means $\operatorname{Hom}_{\mathscr{E}}(E_i, E_\ell)$ and $\operatorname{Hom}_{\mathscr{E}}(E_\ell, E_j)$ are generated by paths of length one or greater. Consequently, φ can be written as a linear combination of paths of length greater than one. □

The proof of the above lemma translates to the following algorithms for computing $\mathcal{B}_{ij}$ and $\mathcal{B}^2_{ij}$:

Algorithm 1: Computing the set of arrows $\mathcal{B}_{ij}$

Input: A strong exceptional sequence $E = (E_i \mid i = 1, \ldots, n)$ and two indices $1 \leq i < j \leq n$
Output: $\mathcal{B}_{ij}$
if $j - i = 1$ **then**
  return *a basis for* $\operatorname{Hom}_{\mathscr{E}}(E_i, E_j)$ `// e.g., via BasisOfExternalHom(`E_i, E_j`)`
else
  Perform the next algorithm to compute $\mathcal{B}^2_{ij}$
  Compute a set $\mathcal{B}$ such that $\mathcal{B} \cup \mathcal{B}^2_{ij}$ is a basis of $\operatorname{Hom}_{\mathfrak{T}}(E_i, E_j)$
  return $\mathcal{B}$

Algorithm 2: Computing the set of paths of length greater than one $\mathcal{B}^2_{ij}$

Input: A strong exceptional sequence $E = (E_i \mid i = 1, \ldots, n)$ and two indices $1 \leq i < j \leq n$
Output: $\mathcal{B}^2_{ij}$
if $j - i = 1$ **then**
  return $\emptyset$
else
  Compute $\mathcal{B}_{i\ell}$, $\mathcal{B}_{\ell j}$ and $\mathcal{B}^2_{\ell j}$ for all $i < \ell < j$;   `// inductively`
  Compute a generating set for $\operatorname{rad}^2_{\mathscr{E}}(E_i, E_j)$ by $B \coloneqq \bigcup_{\ell=i+1}^{j-1} \mathcal{B}_{i\ell} \cdot \Big(\mathcal{B}_{\ell j} \cup \mathcal{B}^2_{\ell j}\Big)$;
  `//` $U \cdot V = \{u \cdot v \mid u \in U, v \in V\}$
  Compute a maximal set of independent elements $\mathcal{B}$ in B;   `// e.g., via RelationsBetweenMorphisms(-)`
  return $\mathcal{B}$

Remark 6.28. If $\mathfrak{T}$ (or $\mathscr{E}$) is equipped with a k-**mat**-homomorphism structure via $(1, H(-,-), \nu)$, then we can deduce the k-linear relations between morphisms $f_1, f_2, \dots, f_m : E_i \to E_j$ in $\mathscr{E}$ from the kernel of the map

$$1^{\oplus m} \xrightarrow{\begin{pmatrix} \nu(f_1) \\ \vdots \\ \nu(f_m) \end{pmatrix}} H(E_i, E_j).$$

Notation 6.29. In the following $T_{\mathscr{E}}$ denotes the object $\bigoplus_{i=1}^n E_i$, ϵ_{E_i} denotes the natural injection $E_i \hookrightarrow T_{\mathscr{E}}$, and π_{E_i} denotes the natural projection $T_{\mathscr{E}} \twoheadrightarrow E_i$.

Remark 6.30. Any endomorphism $\varphi\colon T_{\mathscr{E}} \to T_{\mathscr{E}}$ is given by a matrix

$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} & \cdots & \varphi_{1n} \\ 0 & \varphi_{22} & \varphi_{23} & \cdots & \varphi_{2n} \\ 0 & 0 & \varphi_{33} & \cdots & \varphi_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \varphi_{nn} \end{pmatrix},$$

where $\varphi_{ij} \in \mathrm{Hom}_{\mathfrak{T}}(E_i, E_j)$ for $i, j = 1, \dots, n$. Therefore, any morphism $\varphi_{ij}\colon E_i \to E_j$ can be identified with the element $\pi_{E_i} \cdot \varphi_{ij} \cdot \epsilon_{E_j}$ in $\mathrm{End}\, T_{\mathscr{E}}$;

From now on, we will consider the bases $\mathcal{B}_{ij}$ and $\mathcal{B}^2_{ij}$ for $1 \le i \ne j \le n$ as subsets of $\mathrm{End}\, T_{\mathscr{E}}$.

The previous identification justifies the following lemma:

Lemma 6.31. *The set*

$$\{\mathrm{id}_{E_1}, \dots, \mathrm{id}_{E_n}\} \cup \bigcup_{i \le j} \left(\mathcal{B}_{ij} \cup \mathcal{B}^2_{ij}\right)$$

is a basis for the endomorphism algebra $\mathrm{End}\, T_{\mathscr{E}}$. *Consequently,* $\mathrm{End}\, T_{\mathscr{E}}$ *is generated as k-algebra by the set of all paths of length zero or one.*

Proof. Since $\mathscr{E}$ is strong exceptional

$$\mathrm{End}\, T_{\mathscr{E}} = \bigoplus_{1 \le i,j \le n} \mathrm{Hom}_{\mathfrak{T}}(E_i, E_j) = \bigoplus_{1 \le i \le j \le n} \mathrm{Hom}_{\mathfrak{T}}(E_i, E_j).$$

The assertion follows from the fact that the basis of $\mathrm{Hom}_{\mathfrak{T}}(E_i, E_j)$ is $\{\mathrm{id}_{E_i}\}$ if $i = j$ and $\mathcal{B}_{ij} \cup \mathcal{B}^2_{ij}$ if $i < j$. The second assertion follows from Lemma 6.27. ☐

Notation 6.32. Let $\mathfrak{q}_{\mathscr{E}}$ be the quiver associated to some strong exceptional sequence $\mathscr{E}$. The vertex which corresponds to E_i will be labeled by v_i and the arrow from v_i to v_j which corresponds to the ℓ'th element of $\mathcal{B}_{ij}$ will be labeled by $\alpha_{ij\ell}$.

Lemma 6.33. *Every strong exceptional sequence $\mathscr{E} = (E_i | i = 1, \dots, n)$ is isomorphic to a k-linear finitely presented category $k\mathcal{F}_{\mathfrak{q}_{\mathscr{E}}}/\langle\rho\rangle$ for an admissible set of relations ρ in $k\mathcal{F}_{\mathfrak{q}_{\mathscr{E}}}$.*

Proof. By the universal property of k-linear closure categories, there exists a k-linear functor $F\colon k\mathcal{F}_{\mathfrak{q}_{\mathscr{E}}} \to \mathscr{E}$ which maps the i'th arrow from v_i to v_j to the i'th arrow from E_i to E_j, i.e., the i'th element of $\mathcal{B}_{ij}$.

It follows from Lemma 6.31 that the k-linear maps

$$F_{i,j}\colon \mathrm{Hom}_{k\mathcal{F}_{\mathfrak{q}_{\mathscr{E}}}}(v_i, v_j) \twoheadrightarrow \mathrm{Hom}_{\mathscr{E}}(E_i, E_j)$$

are surjective for all $1 \le i, j \le n$.

We define $\rho := \bigcup_{i \leq j} \mathcal{B}(\ker(F_{i,j}))$ where $\mathcal{B}(\ker(F_{i,j}))$ is a basis for the k-vector space $\ker(F_{i,j})$. Since the quiver is acyclic, $k\mathcal{F}_{\mathfrak{q}_{\mathscr{E}}}$ is Hom-finite, hence ρ is a finite set. To show that ρ is admissible, it is sufficient to prove that any element in ρ is a k-linear combination of paths of length greater than one.

We prove the claim by induction on $j - i$. If $j - i \in \{0, 1\}$, then $F_{i,j}$ is an isomorphism, consequently $\ker(F_{i,j}) = 0$ and the claim holds. Suppose now that $j-i > 1$ and let p be an element in $\ker(F_{i,j})$. Since $p \in \mathrm{Hom}_{k\mathcal{F}_{\mathfrak{q}_{\mathscr{E}}}}(v_i, v_j)$, it can be written as a sum $p_1 + p_2$ where $p_1 = \Sigma_{\ell=1}^{n_{ij}} c_\ell \alpha_{ij\ell}$, $n_{i,j} = |\mathcal{B}_{ij}|$ and p_2 is a k-linear combination of paths of length greater than one. By the definition of $F_{i,j}$, we have $F_{i,j}(p_1) = \langle \mathcal{B}_{ij} \rangle$ and $F_{i,j}(p_2) \in \langle \mathcal{B}^2_{ij} \rangle$. Since $\mathrm{Hom}_{\mathfrak{T}}(E_i, E_j) = \langle \mathcal{B}_{ij} \rangle \oplus \langle \mathcal{B}^2_{ij} \rangle$ and $F_{i,j}(p) = 0$, we have $F_{i,j}(p_1) = 0$ and $F_{i,j}(p_2) = 0$; which implies that $c_\ell = 0$ for all $1 \leq \ell \leq n_{i,j}$. Therefore, p is a k-linear combination of paths of length greater than one. That is, every element of ρ is a linear combination of paths of length greater than one and the lemma follows. □

Notation 6.34. The k-algebroid $k\mathcal{F}_{\mathfrak{q}_{\mathscr{E}}}/\langle \rho \rangle$ will be called the **abstraction k-algebroid** of $\mathscr{E}$ and will be denoted by $\mathbf{A}_{\mathscr{E}}$.

The proof of the above lemma translates to the following algorithm to compute ρ:

Algorithm 3: The abstraction k-algebroid of a strong exceptional sequence

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Input: A strong exceptional sequence $\mathscr{E} = (E_i \mid i = 1, \dots, n)$
Output: An admissible set of relations $\rho \subset k\mathcal{F}_{\mathfrak{q}_{\mathscr{E}}}$ with $\mathscr{E} \cong \mathbf{A}_{\mathscr{E}} := k\mathcal{F}_{\mathfrak{q}_{\mathscr{E}}}/\langle \rho \rangle$
Compute $\mathfrak{q}_{\mathscr{E}}$
$\rho \leftarrow \emptyset$
for $i = 1, \dots, n-2$ **do**
  for $j = i+2, \dots, n$ **do**
    Compute a basis S_{ij} for $\mathrm{Hom}_{k\mathcal{F}_{\mathfrak{q}_{\mathscr{E}}}}(v_i, v_j)$ `// i.e., all paths from` v_i `to` v_j
    Apply F_{ij} on S_{ij} to compute a generating set G_{ij} for $\mathrm{im}(F_{ij})$
    Compute the set of k-linear relations ρ_{ij} between the elements of G_{ij}
      `// e.g., via RelationsBetweenMorphisms(-)`
    $\rho \leftarrow \rho \cup \rho_{ij}$
return ρ

Lemma 6.35. *Let $\mathscr{E} = (E_i | i = 1, \dots, n)$ be a strong exceptional sequence and let $T_{\mathscr{E}} = \bigoplus_{i=1}^n E_i$. Then $\mathrm{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, -)\colon \mathfrak{T} \to \mathbf{Ab}$ factors along the embedding*

$$\mathbf{A}_{\mathscr{E}}\text{-}\mathbf{mod} \simeq \mathrm{End}\, T_{\mathscr{E}}\text{-}\mathbf{mod} \hookrightarrow \mathbf{Ab}.$$

Furthermore, the restriction of the functor on $\mathscr{E}$ defines an isomorphism

$$\mathrm{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, -)\colon \mathscr{E} \xrightarrow{\sim} \mathbf{ind}_0(\mathbf{A}_{\mathscr{E}}\text{-}\mathbf{proj})$$

where $\mathbf{ind}_0(\mathbf{A}_{\mathscr{E}}\text{-}\mathbf{proj})$ is the skeletal full subcategory generated by the indecomposable projective objects in $\mathbf{A}_{\mathscr{E}}$-$\mathbf{mod}$.

Proof. The equivalence $\mathbf{A}_{\mathscr{E}}$-$\mathbf{mod} \simeq \mathrm{End}\, T_{\mathscr{E}}$-$\mathbf{mod}$ follows from Lemma 6.33 and Theorem 2.70. For any object U in $\mathfrak{T}$, $\mathrm{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, U)$ is a left $\mathrm{End}\, T_{\mathscr{E}}$-module via $f \cdot m := f \bullet m$. An easy verification shows that for any morphism $\varphi\colon U \to V$ the map

$$\mathrm{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, \varphi)\colon \begin{cases} \mathrm{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, U) & \to \mathrm{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, V), \\ \qquad \psi & \mapsto \psi \bullet \varphi \end{cases}$$

is a left $\operatorname{End} T_{\mathscr{E}}$-module homomorphism. Hence,

$$\operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, -)\colon \mathfrak{T} \to \operatorname{End} T_{\mathscr{E}}\textbf{-mod}$$

is indeed well-defined.

Since $\operatorname{End} T_{\mathscr{E}} \cong k\mathfrak{q}_{\mathscr{E}}/\langle \rho \rangle$, it follows from Theorem 2.79 that the indecomposable projective left $\operatorname{End} T_{\mathscr{E}}$-modules are, up to isomorphism, the cyclic modules $\operatorname{End} T_{\mathscr{E}} \cdot (\pi_{E_i} \bullet \epsilon_{E_i}) \cong \operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, E_i)$ for $i = 1, \ldots, n$ where $\epsilon_{E_i}\colon E_i \hookrightarrow T_{\mathscr{E}}$ is the natural injection of E_i in the direct sum $T_{\mathscr{E}}$ and $\pi_{E_i}\colon T_{\mathscr{E}} \twoheadrightarrow E_i$ is the natural projection from the direct sum $T_{\mathscr{E}}$ on E_i.

It remains to show that the restriction of $\operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, -)$ to $\mathscr{E}$ induces a fully faithful functor

$$\operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, -)\colon \mathscr{E} \to \operatorname{End} T_{\mathscr{E}}\textbf{-mod}.$$

Let $\varphi_{ij}\colon E_i \to E_j$ be a morphism with $\operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, \varphi_{ij}) = 0$, then

$$0 = \operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, \varphi_{ij})(\pi_{E_i}) = \pi_{E_i} \bullet \varphi_{ij} = \begin{pmatrix} \vdots \\ 0 \\ \varphi_{ij} \\ 0 \\ \vdots \end{pmatrix},$$

i.e., $\varphi_{ij} = 0$. This shows that the restriction of $\operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, -)$ to $\mathscr{E}$ is faithful.

Let $\lambda\colon \operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, E_i) \to \operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, E_j)$ be a homomorphism of $\operatorname{End} T_{\mathscr{E}}$-modules. We claim that $\operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, \epsilon_{E_i} \bullet \lambda(\pi_{E_i})) = \lambda$ where ϵ_{E_i} be the natural injection $E_i \hookrightarrow T_{\mathscr{E}}$. For any $\psi \in \operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, E_i)$, we have

$$\begin{aligned}
\lambda(\psi) &= \lambda(\psi \bullet \epsilon_{E_i} \bullet \pi_{E_i}) \\
&= \lambda((\psi \bullet \epsilon_{E_i}) \cdot \pi_{E_i}) \\
&= (\psi \bullet \epsilon_{E_i}) \cdot \lambda(\pi_{E_i}) \\
&= \psi \bullet \epsilon_{E_i} \bullet \lambda(\pi_{E_i}) \\
&= \operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, \epsilon_{E_i} \bullet \lambda(\pi_{E_i}))(\psi);
\end{aligned}$$

i.e., $\lambda = \operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, \epsilon_{E_i} \bullet \lambda(\pi_{E_i}))$. This shows that the restriction of $\operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, -)$ on $\mathscr{E}$ is full. ☐

Remark 6.36. Analogously to Theorem 2.70 the explicit construction of $\operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, -)$ is given by

$$\operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, -)\colon \begin{cases}
\mathfrak{T} & \to \mathbf{A}_{\mathscr{E}}\textbf{-mod}, \\
U & \mapsto \operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, U)\colon \begin{cases}
\mathbf{A}_{\mathscr{E}}^{\mathrm{op}} & \to k\textbf{-mat}, \\
v_i^{\mathrm{op}} & \mapsto \dim_k \operatorname{Hom}_{\mathfrak{T}}(E_i, U), \\
\alpha_{ij\ell}^{\mathrm{op}} & \mapsto \begin{array}{l} \text{the matrix of the map} \\ \begin{cases} \operatorname{Hom}_{\mathfrak{T}}(E_j, U) & \to \operatorname{Hom}_{\mathfrak{T}}(E_i, U), \\ g & \mapsto \alpha_{ij\ell}^{\mathscr{E}} \bullet g \end{cases} \\ \text{w.r.t. } \mathcal{B}(\operatorname{Hom}_{\mathfrak{T}}(E_j, U)),\ \mathcal{B}(\operatorname{Hom}_{\mathfrak{T}}(E_i, U)). \end{array}
\end{cases} \\
\varphi\colon U \to V & \mapsto \operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, \varphi)\colon \begin{cases}
\operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, U) & \to \operatorname{Hom}_{\mathfrak{T}}(T_{\mathscr{E}}, V), \\
v_i^{\mathrm{op}} & \mapsto \begin{array}{l} \text{the matrix of the map} \\ \begin{cases} \operatorname{Hom}_{\mathfrak{T}}(E_i, U) & \to \operatorname{Hom}_{\mathfrak{T}}(E_i, V), \\ g & \mapsto g \bullet \varphi \end{cases} \\ \text{w.r.t. } \mathcal{B}(\operatorname{Hom}_{\mathfrak{T}}(E_i, U)),\ \mathcal{B}(\operatorname{Hom}_{\mathfrak{T}}(E_i, V)) \end{array}
\end{cases}
\end{cases}$$

where $\alpha_{ij\ell}^{\mathscr{E}}$ denotes the ℓ'th element of $\mathcal{B}_{ij}$ and $\mathcal{B}(\operatorname{Hom}_{\mathfrak{T}}(E_i, U))$ denotes a particular choice of a basis for $\operatorname{Hom}_{\mathfrak{T}}(E_i, U)$.

Corollary 6.37. *Let $\mathscr{E} = (E_i \mid i = 1, \dots, n)$ be a strong exceptional sequence in $\mathfrak{T}$. We have the following equivalences*

$$\mathcal{K}^b(\mathscr{E}^{\oplus}) \cong \mathcal{K}^b\Big(\mathbf{A}_{\mathscr{E}}^{\oplus}\Big) \simeq \mathcal{K}^b(\mathbf{A}_{\mathscr{E}}\text{-}\mathbf{proj}) \simeq \mathcal{D}^b(\mathbf{A}_{\mathscr{E}}\text{-}\mathbf{mod}) \simeq \mathcal{D}^b(\mathrm{End}\ T_{\mathscr{E}}\text{-}\mathbf{mod}).$$

Proof. The first isomorphism follows by extending the isomorphism in Lemma 6.33 to additive closures then to bounded homotopy categories. The second equivalence follows by extending the Yoneda isomorphism in Corollary 2.90 to additive closures then to bounded homotopy categories. The third equivalence follows by Theorem 3.61 and the fact that $\mathbf{A}_{\mathscr{E}}$-**mod** has a finite global dimension (cf. Corollary 2.96). The last equivalence follows by Theorem 2.70. □

6.3. The Convolution Functor F

Let $\mathfrak{T}$ be a triangulated category and $\mathscr{E}$ a strong exceptional sequence in $\mathfrak{T}$. The induced exact equivalences in Corollary 6.37 can be depicted in the right side of following diagram:

$$\begin{array}{ccccc}
\mathcal{K}^b(\mathfrak{T}) & \hookleftarrow & \mathcal{K}^b(\mathscr{E}^{\oplus}) & \xrightarrow{\sim} & \mathcal{K}^b\Big(\mathbf{A}_{\mathscr{E}}^{\oplus}\Big) \\
 & & & & \downarrow \wr \\
\big\uparrow & & & & \mathcal{K}^b(\mathbf{A}_{\mathscr{E}}\text{-}\mathbf{proj}) \\
 & & & & \downarrow \wr \\
\mathfrak{T} & & & & \mathcal{D}^b(\mathbf{A}_{\mathscr{E}}\text{-}\mathbf{mod})
\end{array}$$

where the left side is merely the natural embedding functor. In fact, $\mathscr{E}$ is a complete strong exceptional sequence in $\mathcal{K}^b(\mathscr{E}^{\oplus})$. This follows from the fact that any object in $\mathcal{K}^b(\mathscr{E}^{\oplus})$ with lower bound ℓ is a standard cone object of a morphism between objects with common lower bound $\ell + 1$ (cf. Construction 6.62).

Suppose $\mathfrak{T} \coloneqq \mathcal{D}^b(\mathbf{mod}\text{-}\mathbb{A})$ for some finite dimensional k-algebra $\mathbb{A}$ and T is a generalized tilting object in **mod**-$\mathbb{A}$ whose direct summands $E_1, \dots, E_n$ form a complete strong exceptional sequence $\mathscr{E}$. Let $T_{\mathscr{E}} \coloneqq \bigoplus_{i=1}^n E_i \cong T$. Then Happel's theorem states that the derived functors

$$- \otimes^{\mathbb{L}} T_{\mathscr{E}} : \mathcal{D}^b(\mathbf{mod}\text{-}\,\mathrm{End}^{\mathrm{op}}\, T_{\mathscr{E}}) \leftrightarrows \mathcal{D}^b(\mathbf{mod}\text{-}\mathbb{A}) : \mathbb{R}\mathrm{Hom}_{\mathbf{mod}\text{-}\mathbb{A}}(T_{\mathscr{E}}, -).$$

are exact quasi-inverses. In particular, the composition

$$\mathcal{K}^b(\mathscr{E}^{\oplus}) \hookrightarrow \mathcal{K}^b(\mathbf{mod}\text{-}\mathbb{A}) \xrightarrow{Q} \mathcal{D}^b(\mathbf{mod}\text{-}\mathbb{A})$$

defines an exact equivalence of categories where Q is the natural localization functor (cf. Remark 3.42).

For arbitrary triangulated category $\mathfrak{T}$ finding a functor from $\mathcal{K}^b(\mathscr{E}^{\oplus})$ to $\mathfrak{T}$ is a tricky task. The convolution construction (cf. [**GM03**], [**Orl97**], [**BBHR09**]) associates to an object

$$T \coloneqq \quad \cdots \to T^{-1} \to T^0 \to T^1 \to \cdots$$

in $\mathcal{C}^b(\mathfrak{T})$ a set $\mathrm{tot}(T)$ of "totalizations" for T all of which belong to the triangulated hull $\{T^i | i \in \mathbb{Z}\}^{\triangle} \subseteq \mathfrak{T}$. In general, the set $\mathrm{tot}(T)$ might be empty or might contain one or several non-isomorphic objects.

However, if all $T^i, i \in \mathbb{Z}$ live in the image of the natural embedding $\mathscr{E}^{\oplus} \subset \mathfrak{T}$, then T has, up to a non-canonical isomorphism, only one convolution in $\mathfrak{T}$ (cf. [**GM03**] and [**BBHR09**]). If S is another complex whose components also belong to the image of the embedding $\mathscr{E}^{\oplus} \subset \mathfrak{T}$, then every morphism from T to S lifts to a morphism between the corresponding convolutions.

As we will see later, computing the convolution is based on an iterative computation of (co)cone objects. The non-functoriality of the (co)cone construction in triangulated categories prevents the convolution construction from being functorial.

In this section, we introduce an approach to rectify this limitation in the case where $\mathfrak{T}$ is the bounded homotopy category $\mathcal{K}^b(\mathscr{C})$ of a k-linear additive category $\mathscr{C}$. Achieving the functoriality is based on extending complexes over $\mathcal{K}^b(\mathscr{C})$ and their morphisms to what we call *standard* objects and morphisms in the category of Postnikov systems over $\mathfrak{T}$. We call them standard because their construction depends on an iterative computation of chain homotopies. Let us first state the theorem

Theorem 6.38. *Let k be a field and $\mathscr{C}$ a* Hom*-finite k-linear additive category. Then any strong exceptional sequence $\mathscr{E} \subset \mathcal{K}^b(\mathscr{C})$ induces a fully faithful exact functor*

$$\mathbf{F}\colon \mathcal{K}^b(\mathscr{E}^{\oplus}) \to \mathcal{K}^b(\mathscr{C})$$

whose essential image is the triangulated hull $\mathscr{E}^{\triangle}$ of $\mathscr{E}$ in $\mathcal{K}^b(\mathscr{C})$.

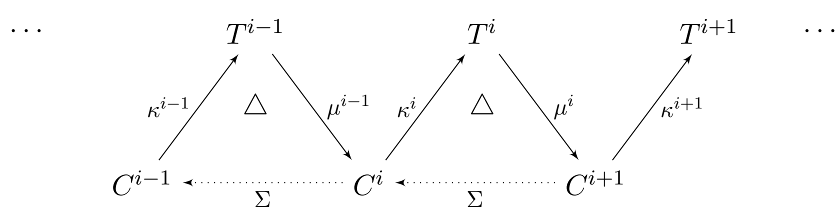
The asserted functor will be called the *standard convolution functor*. This means, if $\mathscr{E}$ is complete, then $\mathbf{F}$ becomes an exact equivalence of triangulated categories. The next two sections are devoted to the proof of the above theorem. For an implementation of the presented concepts we refer to the `GAP` packages `TriangulatedCategories` [**Sal21f**] and `HomotopyCategories` [**Sal21d**].

Throughout this section $\mathfrak{T}$ will be a triangulated category equipped with a shift automorphism (cf. Remark B.4). We start by defining the category of Postnikov systems over $\mathfrak{T}$:

Definition 6.39. Let $\mathfrak{T}$ be a triangulated category. A **Postnikov system** P over $\mathfrak{T}$ is a family of exact triangles

$$P = \left(C^i \xrightarrow{\kappa^i} T^i \xrightarrow{\mu^i} C^{i+1} \xrightarrow{\rho^i} \Sigma\left(C^i\right)\right)_{i\in\mathbb{Z}}$$

depicted in the following diagram:



We say P is **bounded below** if there exists $\ell \in \mathbb{Z}$ with $T^i = 0$ for all $i < \ell$. Similarly, P is **bounded above** if there exists $u \in \mathbb{Z}$ with $T^i = 0$ for all $i > u$.

Remark 6.40. The cone and cocone[3] objects of any morphism are unique up to (non-canonical) isomorphism, hence

$$\mathrm{Cone}\left(\kappa^{i-1}\right) \cong C^i \cong \mathrm{Cocone}\left(\mu^i\right).$$

[3]For a morphism φ, we have $\mathrm{Cocone}(\varphi) := \Sigma^{-1}(\mathrm{Cone}(\varphi))$ (cf. Remark 5.10).

Remark 6.41. Suppose ℓ is a lower bound of P. We can extend the morphism $0 \to C^\ell$ to the exact triangle

$$\Sigma^{-1}\Big(C^\ell\Big) \to 0 \to C^\ell \to C^\ell.$$

Hence $C^{\ell-i} \cong \Sigma^{-i}\Big(C^\ell\Big)$ for all $i \geq 0$. Similarly, if u is an upper bound for P, then we can extend $C^{u+1} \to 0$ to the exact triangle

$$C^{u+1} \to 0 \to \Sigma\Big(C^{u+1}\Big) \to \Sigma\Big(C^{u+1}\Big).$$

Hence $C^{u+1+i} \cong \Sigma^i\big(C^{u+1}\big)$ for all $i \geq 0$.

Remark 6.42. The composition of any two consecutive morphisms of an exact triangle is zero, thus the family $\big(\partial_T^i := \mu^i \bullet \kappa^{i+1}\big)_{i\in\mathbb{Z}}$ defines a complex over $\mathfrak{T}$:

$$T := \qquad \cdots \longrightarrow T^{i-1} \xrightarrow{\ \partial_T^\ell\ } T^i \xrightarrow{\ \partial_T^{\ell+1}\ } T^{i+1} \longrightarrow \cdots$$

We call T **the underlying complex** of P.

In the following we define the category of Postnikov systems:

Definition 6.43. Let $\mathfrak{T}$ be a triangulated category. The category of Postnikov systems over $\mathfrak{T}$ consists of the following data:

(1) The objects are the Postnikov systems over $\mathfrak{T}$.
(2) A morphism from P_1 to P_2 is defined by a pair of families:

$$\varphi = \Big(\varphi^i\colon T_1^i \to T_2^i\Big)_{i\in\mathbb{Z}} \text{ and } \varphi_C = \Big(\varphi_C^i\colon C_1^i \to C_2^i\Big)_{i\in\mathbb{Z}}$$

which induce morphisms of exact triangles:

$$\begin{array}{ccccccc}
C_1^i & \xrightarrow{\ \kappa_1^i\ } & T_1^i & \xrightarrow{\ \mu_1^i\ } & C_1^{i+1} & \longrightarrow & \Sigma(C_1^i) \\
\downarrow{\scriptstyle \varphi_C^i} & \circlearrowleft & \downarrow{\scriptstyle \varphi^i} & \circlearrowleft & \downarrow{\scriptstyle \varphi_C^{i+1}} & \circlearrowleft & \downarrow{\scriptstyle \Sigma(\varphi_C^i)} \\
C_2^i & \xrightarrow[\ \kappa_2^i\ ]{} & T_2^i & \xrightarrow[\ \mu_2^i\ ]{} & C_2^{i+1} & \longrightarrow & \Sigma(C_2^i)
\end{array}$$

for all $i \in \mathbb{Z}$.
(3) The identity morphisms and the composition are inherited from $\mathfrak{T}$.

Remark 6.44. Let $[\varphi, \varphi_C]\colon P_1 \to P_2$ be a morphism of Postnikov systems and T_1 and T_2 the underlying complexes of P_1 resp. P_2. Since $\partial_{T_1}^i = \mu_1^i \bullet \kappa_1^{i+1}$ and $\partial_{T_2}^i = \mu_2^i \bullet \kappa_2^{i+1}$ for all $i \in \mathbb{Z}$, the family $\varphi = (\varphi^i\colon T_1^i \to T_2^i)_{i\in\mathbb{Z}}$ defines a morphism from T_1 to T_2 in $\mathcal{C}^b(\mathfrak{T})$. We call $\varphi\colon T_1 \to T_2$ the **underlying complex morphism** of $[\varphi, \varphi_C] : P_1 \to P_2$.

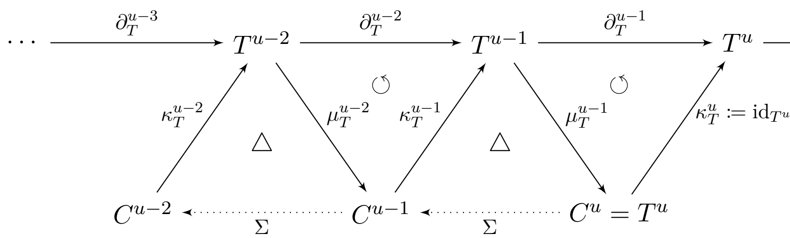
Definition 6.45. Let P be a Postnikov system bounded below by ℓ. We define the **convolution object** $\mathbf{F}(P)$ of P by $\Sigma^{-\ell}\Big(C^\ell\Big)$. Similarly, let $[\varphi, \varphi_C]\colon P_1 \to P_2$ be a morphism of bounded below Postnikov systems. We define the **convolution morphism** $\mathbf{F}([\varphi, \varphi_C])$ by $\Sigma^{-\ell}\Big(\varphi^\ell\Big)$ where ℓ is a common lower bound of P_1 and P_2.

Definition 6.46. Let T be an object in $\mathcal{C}^b(\mathfrak{T})$ with an upper bound u. A Postnikov system P_T over $\mathfrak{T}$:

$$\left(C_T^i \xrightarrow{\kappa_T^i} T^i \xrightarrow{\mu_T^i} C_T^i \xrightarrow{\rho_T^i} \Sigma\left(C_T^i\right)\right)_{i\in\mathbb{Z}}$$

is called **extension** of T to a Postnikov system if

(1) $\mu_T^i \bullet \kappa_T^{i+1} = \partial_T^i$ for $i \in \mathbb{Z}$ and
(2) $\kappa_T^i = \mathrm{id}_{T^i}$ for $i \geq u$.



We define the set of convolutions associated to T by the set of convolutions of all extensions of T to Postnikov systems.

By axioms **TR 1** and **TR 2** in Definition B.1, Lemma B.5, and Corollary B.15, any morphism $\alpha\colon T \to B$ in $\mathfrak{T}$ can be extended to the exact triangle

$$\mathrm{Cocone}(\alpha) \xrightarrow{\Sigma^{-1}(\pi(\alpha))} T \xrightarrow{\alpha} B \xrightarrow{-\iota(\alpha)} \mathrm{Cone}(\alpha).$$

The equality $\kappa_T^u = \mathrm{id}_{T^u}$ in the above definition implies $\mu_T^{u-1} = \partial_T^{u-1}$. We can take κ_T^{u-1} to be $\Sigma^{-1}\left(\pi\left(\mu_T^{u-1}\right)\right)$. If we do so, the next step toward computing a Postnikov system is the computation of μ_T^{u-2}, which is equivalent to solving the two-sided linear system

$$\chi \bullet \kappa_T^{u-1} = \partial_T^{u-2}, \qquad \partial_T^{u-3} \bullet \chi = 0$$

where the first equation is justified by Definition 6.46 and the second equation is justified by the fact that the composition of any two consecutive morphisms in an exact triangle is trivial.

If such solution exists, we might continue with same procedure. The number of convolutions associated to T depends on how many solutions we get in each iteration.

Definition 6.47. Let T and S be objects in $\mathcal{C}^b(\mathfrak{T})$. Let P_T and P_S extensions of T resp. S to Postnikov systems. An **extension** of a morphism $\varphi\colon T \to S$ to a morphism from P_T to P_S is a morphism $[\varphi, \varphi_C]\colon P_T \to P_S$ whose underlying morphism of complexes is φ.

We define the set of convolutions associated to φ by the set of convolutions of all such extensions of φ.

Remark 6.48. Let $[\varphi, \varphi_C]\colon P_T \to P_S$ be an extension of morphism $\varphi\colon T \to S$ in $\mathcal{C}^b(\mathfrak{T})$. Due to Lemma B.11, we can use induction to prove that φ is an isomorphism if and only $[\varphi, \varphi_C]$ is.

Definition 6.49. Let $\mathscr{C}$ be an additive category and $\mathfrak{T} = \mathcal{K}^b(\mathscr{C})$ its bounded homotopy category.

(1) An extension P_T of an object T in $\mathcal{C}^b(\mathfrak{T})$ to a Postnikov system is called **standard** if $C_T^i := \mathrm{Cocone}^{st}(\mu_T^i)$ and $\kappa_T^i := \Sigma^{-1}(\pi(\mu_T^i))$ for all $i \in \mathbb{Z}$ (cf. Definition 5.7 and

Remark 5.10). The set of **standard convolutions** associated to T is defined by the set of all non-isomorphic convolutions of all standard extensions of T to Postnikov systems[4].

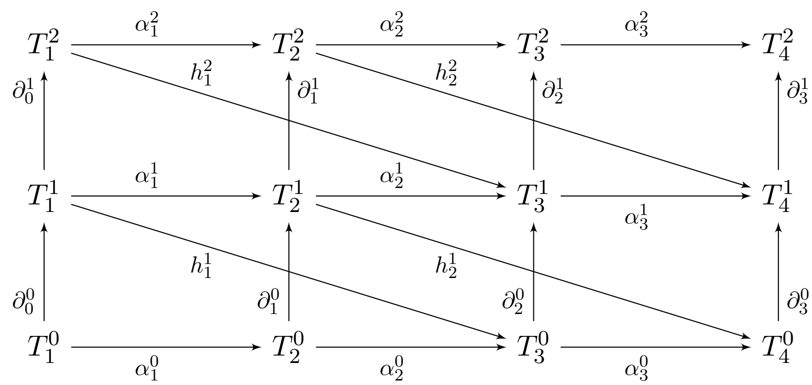
(2) An extension $[\varphi, \varphi_C]\colon P_T \to P_S$ of a morphism $\varphi\colon T \to S$ in $\mathcal{C}^b(\mathfrak{T})$ to a morphism from P_T to P_S is called **standard** if P_T and P_S are standard and

$$\varphi^i_C \in \mathrm{CoconeMors}^{st}_{\mu^i_T, \mu^i_S}\left(\varphi^i, \varphi^{i+1}_C\right)$$

for all $n \in \mathbb{Z}$. The set of **standard convolutions** associated to φ, P_T and P_S is defined by the set of convolutions of all standard extensions of φ to morphisms from P_T to P_S.

An object or morphism in $\mathcal{C}^b(\mathfrak{T})$ may have no associated standard convolutions or may have one or several non-isomorphic associated standard convolutions:

Example 6.50. Let k be a field and $\mathfrak{q}$ the following right quiver



Let $\mathscr{A}$ the k-linear finitely presented category defined by $\mathfrak{q}$ subject to the relations:

$$\{\partial^0_j \cdot \partial^1_j \mid 1 \le j \le 4\} \cup \{\partial^i_j \cdot \alpha^{i+1}_j - \alpha^i_j \cdot \partial^i_{j+1} \mid 0 \le i \le 1, 1 \le j \le 3\}$$

$$\cup\{\partial^0_j \cdot h^1_j - \alpha^0_j \cdot \alpha^0_{j+1} \mid 1 \le j \le 2\} \cup \{h^2_j \cdot \partial^1_{j+2} - \alpha^2_j \cdot \alpha^2_{j+1} \mid 1 \le j \le 2\}$$

$$\cup\{\partial^1_j \cdot h^2_j + h^1_j \cdot \partial^0_{j+2} - \alpha^1_j \cdot \alpha^1_{j+1} \mid 1 \le j \le 2\}$$

i.e., the sets defined by taking the sum of all paths between vertices T^i_j and $T^{i+j-l+2}_l$ after replacing each α^i_j in each path by $(-1)^{i+j-1}\alpha^i_j$. Let $\mathscr{A}^\oplus$ be the additive closure of $\mathscr{A}$ and $\mathcal{K}^b(\mathscr{A}^\oplus)$ be the bounded homotopy category of $\mathscr{A}^\oplus$. For every j with $0 \le j \le 3$, define T_j by the object of $\mathcal{K}^b(\mathscr{A}^\oplus)$ whose differential at index $0 \le i \le 1$ is ∂^i_j. For every j with $0 \le j \le 2$, define $\alpha_j\colon T_j \to T_{j+1}$ by the morphism whose component at index i is α^i_j.

Let T be the object in $\mathcal{C}^b\Big(\mathcal{K}^b(\mathscr{A}^\oplus)\Big)$ defined by the sequence

$$0 \to T_1 \xrightarrow{\alpha_1} T_2 \xrightarrow{\alpha_2} T_3 \xrightarrow{\alpha_3} T_4 \to 0$$

where T_1 is concentrated at index 1.

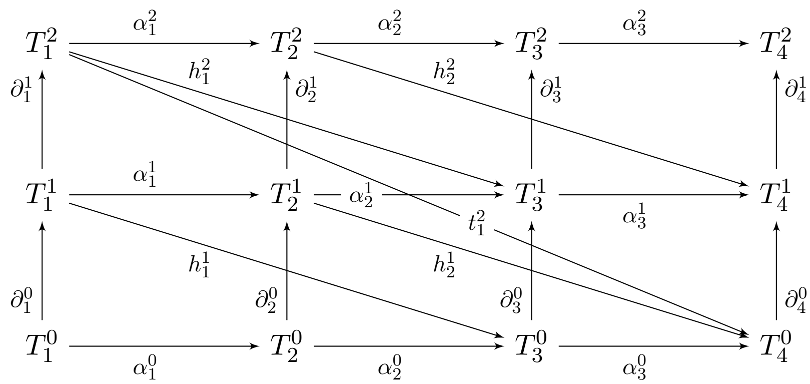
There exists a unique lift morphism of α_2 along $\kappa^3_T = \Sigma^{-1}(\pi(\alpha_3))\colon \mathrm{Cocone}^{st}(\alpha_3) \to T_3$, say $\mu^2_T\colon T_2 \to \mathrm{Cocone}^{st}(\alpha_3)$. However, $\alpha_1 \bullet \mu^2_T \neq 0$; hence T can not be extended to a standard

[4] All convolutions of a standard extension P_T of T are equal, regardless of which lower bound we use.

Postnikov system. In particular, the set of associated standard convolutions of T is empty[5]. The computations in this example can be performed analog to the example in Appendix F.

To rectify the situation in Example 6.50 we must add an extra arrow and impose more relations:

Example 6.51. Let k be a field and $\mathfrak{q}$ be the following quiver



Let $\mathscr{A}$ the k-linear finitely presented category defined by $\mathfrak{q}$ subject to the relations

$$\{\partial_j^0 \cdot \partial_j^1 \mid 1 \le j \le 4\} \cup \{\partial_j^i \cdot \alpha_j^{i+1} - \alpha_j^i \cdot \partial_{j+1}^i \mid 0 \le i \le 1, 1 \le j \le 3\}$$

$$\cup\{\partial_j^0 \cdot h_j^1 - \alpha_j^0 \cdot \alpha_{j+1}^0 \mid 1 \le j \le 2\} \cup \{h_j^2 \cdot \partial_{j+2}^1 - \alpha_j^2 \cdot \alpha_{j+1}^2 \mid 1 \le j \le 2\}$$

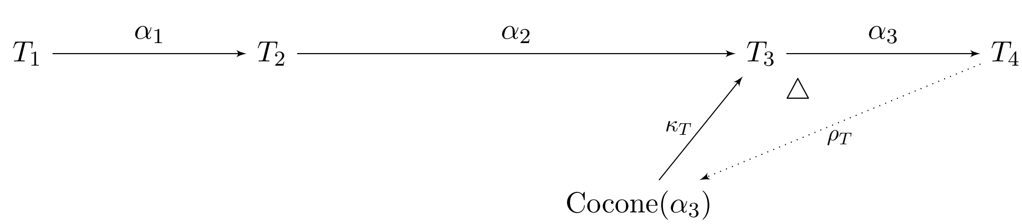
$$\cup\{\partial_j^1 \cdot h_j^2 + h_j^1 \cdot \partial_{j+2}^0 - \alpha_j^1 \cdot \alpha_{j+1}^1 \mid 1 \le j \le 2\}$$

$$\cup\{\partial_1^1 \cdot t_1^2 + h_1^1 \cdot \alpha_3^0 - \alpha_1^1 \cdot h_2^1\} \cup \{\alpha_1^2 \cdot h_2^2 + t_1^2 \cdot \partial_4^0 - h_1^2 \cdot \alpha_3^1\};$$

and let T be defined as in Example 6.50. Then T can be extended to a standard Postnikov system. In fact, this trick enables us to construct as many non-isomorphic standard convolutions associated to T as we want: we simply add similar arrows and relations. For a software demonstration of this example we refer to Appendix F.

Example 6.52. Example 5.16 can be used to construct a morphism with two associated convolutions only one of which is standard.

Lemma 6.53. *Let $\mathfrak{T}$ be a triangulated category. Let $T_i, i = 1, 2, 3, 4$ be objects in $\mathfrak{T}$ and $\alpha_i \colon T_i \to T_{i+1}, i = 1, 2, 3$ morphisms with $\alpha_i \bullet \alpha_{i+1} = 0$. If $\mathrm{Hom}_{\mathfrak{T}}(\Sigma(T_2), T_4) = 0$, then for every diagram*



[5]By the end of this section, we will see that this occurs because $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{A}^\oplus)}(\Sigma(T_1), T_4) \neq 0$.

there exists a unique morphism $\mu_T : T_2 \to \mathrm{Cocone}(\alpha_3)$ *such that* $\mu_T \bullet \kappa_T = \alpha_2$*. Furthermore, if* $\mathrm{Hom}_{\mathfrak{T}}(\Sigma(T_1), T_4) = 0$*, then* $\alpha_1 \bullet \mu_T = 0$*.*

Proof. Since the triangle

$$\mathrm{Cocone}(\alpha_3) \xrightarrow{\kappa_T} T_3 \xrightarrow{\alpha_3} T_4 \xrightarrow{\rho_T} \mathrm{Cone}(\alpha_3)$$

is exact and the functor $\mathrm{Hom}_{\mathfrak{T}}(-, T_3)$ is a cohomological, we get the long exact sequence

$$\cdots \xrightarrow{-\bullet(-\Sigma^{-1}(\kappa_T))} \mathrm{Hom}_{\mathfrak{T}}(T_2, \Sigma^{-1}(T_3)) \xrightarrow{-\bullet(-\Sigma^{-1}(\alpha_3))} \overbrace{\mathrm{Hom}_{\mathfrak{T}}\Big(T_2, \Sigma^{-1}(T_4)\Big)}^{=0} \xrightarrow{-\bullet(-\Sigma^{-1}(\rho_T))}$$

$$\mathrm{Hom}_{\mathfrak{T}}(T_2, \mathrm{Cocone}(\alpha_3)) \xhookrightarrow{-\bullet\kappa_T} \mathrm{Hom}_{\mathfrak{T}}(T_2, T_3) \xrightarrow{-\bullet\alpha_3} \mathrm{Hom}_{\mathfrak{T}}(T_2, T_4) \longrightarrow \cdots$$

from which we can easily deduce that $-\bullet\kappa_T$ is a monomorphism. Since $\alpha_2 \in \ker(-\bullet\alpha_3) = \mathrm{im}(-\bullet\kappa_T)$, there exists a unique morphism $\mu_T : T_2 \to \mathrm{Cocone}(\alpha_3)$ such that $\mu_T \bullet \kappa_T = \alpha_2$.

The morphism μ_T can be constructed by applying axiom **TR 4** on the following diagram:

$$\begin{array}{ccccccc}
T_2 & \xrightarrow{\mathrm{id}_{T_2}} & T_2 & \longrightarrow & 0 & \longrightarrow & \Sigma(T_2) \\
 & & \downarrow{\scriptstyle \alpha_2} & \circlearrowleft & \downarrow & & \\
\mathrm{Cocone}(\alpha_3) & \xrightarrow[\kappa_T]{} & T_3 & \xrightarrow[\alpha_3]{} & T_4 & \xrightarrow[\rho_T]{} & \mathrm{Cone}(\alpha_3)
\end{array}$$

It remains to verify that $\alpha_1 \bullet \mu_T = 0$. Again by **TR 4**, there exists a morphism $\lambda : \Sigma(T_1) \to T_4$ that renders the following diagram

$$\begin{array}{ccccccc}
T_1 & \longrightarrow & 0 & \longrightarrow & \Sigma(T_1) & \xrightarrow{\mathrm{id}_{\Sigma(T_1)}} & \Sigma(T_1) \\
{\scriptstyle \alpha_1\bullet\mu_T}\downarrow & \circlearrowleft & \downarrow & \circlearrowleft & {\scriptstyle \lambda}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \Sigma(\alpha_1\bullet\mu_T)} \\
\mathrm{Cocone}(\alpha_3) & \xrightarrow[\kappa_T]{} & T_3 & \xrightarrow[\alpha_3]{} & T_4 & \xrightarrow[\rho_T]{} & \mathrm{Cone}(\alpha_3)
\end{array}$$

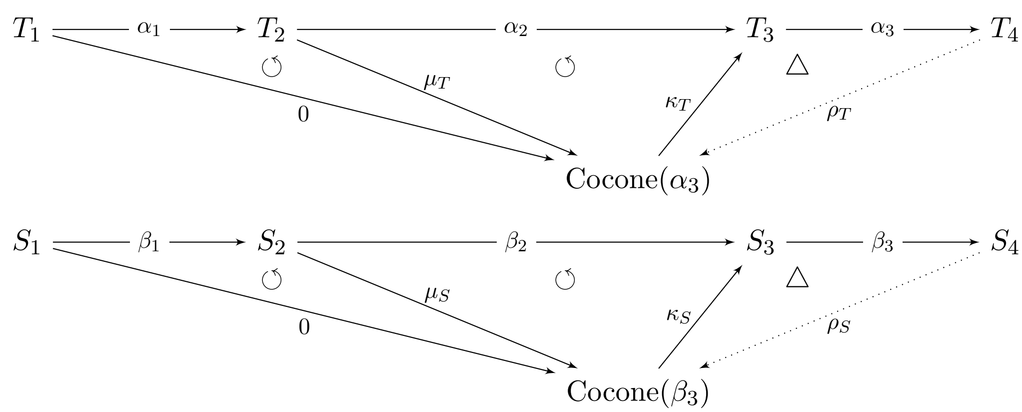
commutative, i.e., $\lambda \bullet \rho_T = \Sigma(\alpha_1 \bullet \mu_T)$. It follows from the assumption $\mathrm{Hom}_{\mathfrak{T}}(\Sigma(T_1), T_4) = 0$ that $\lambda = 0$, hence $\Sigma(\alpha_1 \bullet \mu_T) = 0$; and consequently $\alpha_1 \bullet \mu_T = 0$ as desired. $\square$

Lemma 6.54. *Let* $\mathfrak{T}$ *be a triangulated category. Let* $T_i, S_i, i = 1, 2, 3, 4$ *be objects in* $\mathfrak{T}$ *and* $\alpha_i : T_i \to T_{i+1}$*,* $\beta_i : S_i \to S_{i+1}$ *be morphisms with* $\alpha_i \bullet \alpha_{i+1} = 0$ *and* $\beta_i \bullet \beta_{i+1} = 0$ *for* $i = 1, 2, 3$*.*

Furthermore, let $\varphi\colon T_i \to S_i, i = 1,2,3,4$ *be four morphisms that render the following diagram*

$$\begin{array}{ccccccc} T_1 & \xrightarrow{\alpha_1} & T_2 & \xrightarrow{\alpha_2} & T_3 & \xrightarrow{\alpha_3} & T_4 \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \downarrow{\scriptstyle \varphi_4} \\ S_1 & \xrightarrow[\beta_1]{} & S_2 & \xrightarrow[\beta_2]{} & S_3 & \xrightarrow[\beta_3]{} & S_4 \end{array}$$

commutative. For a given pair of commutative diagrams



there exists a morphism $\tau\colon \mathrm{Cocone}(\alpha_3) \to \mathrm{Cocone}(\beta_3)$ *giving rise to a morphism of exact triangles:*

$$\begin{array}{ccccccc} \mathrm{Cocone}(\alpha_3) & \xrightarrow{\kappa_T} & T_3 & \xrightarrow{\alpha_3} & T_4 & \xrightarrow{\rho_T} & \Sigma(\mathrm{Cocone}(\alpha_3)) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \varphi_3} & & \downarrow{\scriptstyle \varphi_4} & & \downarrow{\scriptstyle \Sigma(\tau)} \\ \mathrm{Cocone}(\beta_3) & \xrightarrow[\kappa_S]{} & S_3 & \xrightarrow[\beta_3]{} & S_4 & \xrightarrow[\rho_S]{} & \Sigma(\mathrm{Cocone}(\beta_3)) \end{array}$$

Furthermore, if $\mathrm{Hom}_{\mathfrak{T}}(\Sigma(T_2), S_4) = 0$*, then any such* τ *renders the following square*

$$\begin{array}{ccc} T_2 & \xrightarrow{\mu_T} & \mathrm{Cocone}(\alpha_3) \\ \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \tau} \\ S_2 & \xrightarrow[\mu_S]{} & \mathrm{Cocone}(\beta_3) \end{array}$$

commutative.

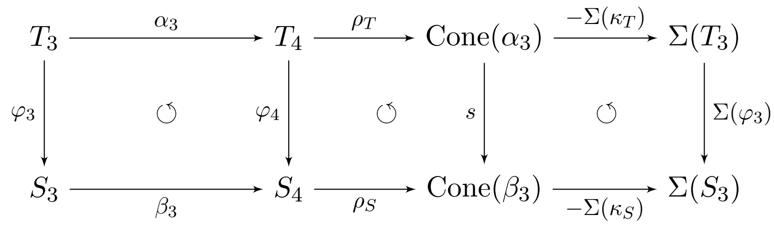
PROOF. By axiom **TR 2** we get two exact triangles:

$$T_3 \xrightarrow{\alpha_3} T_4 \xrightarrow{\rho_T} \mathrm{Cone}(\alpha_3) \xrightarrow{-\Sigma(\kappa_T)} \Sigma(T_3)$$

and

$$S_3 \xrightarrow{\beta_3} S_4 \xrightarrow{\rho_S} \mathrm{Cone}(\beta_3) \xrightarrow{-\Sigma(\kappa_S)} \Sigma(S_3).$$

By **TR 3**, there exists a morphism $s\colon \mathrm{Cone}(\alpha_3) \to \mathrm{Cone}(\beta_3)$ inducing a morphism a exact triangles:

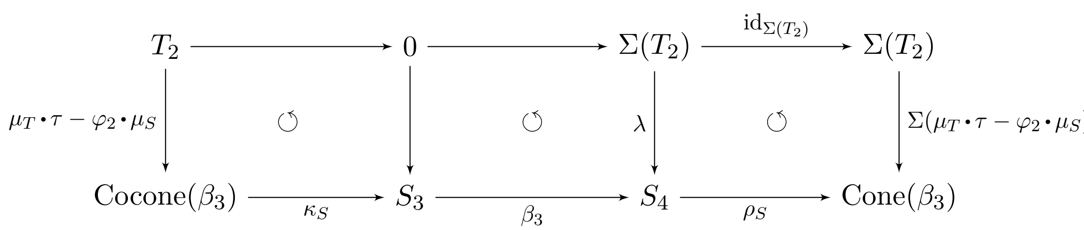


The morphism $\tau \coloneqq \Sigma^{-1}(s)$ satisfies the required assertions.

Now suppose $\mathrm{Hom}_{\mathfrak{T}}(\Sigma(T_2), S_4) = 0$. The computation

$$\begin{aligned}(\mu_T \bullet \tau - \varphi_2 \bullet \mu_S) \bullet \kappa_S &= \mu_T \bullet \tau \bullet \kappa_S - \varphi_2 \bullet \mu_S \bullet \kappa_S \\ &= \mu_T \bullet \kappa_T \bullet \varphi_3 - \varphi_2 \bullet \mu_S \bullet \kappa_S \\ &= \alpha_2 \bullet \varphi_3 - \varphi_2 \bullet \beta_2 \\ &= 0\end{aligned}$$

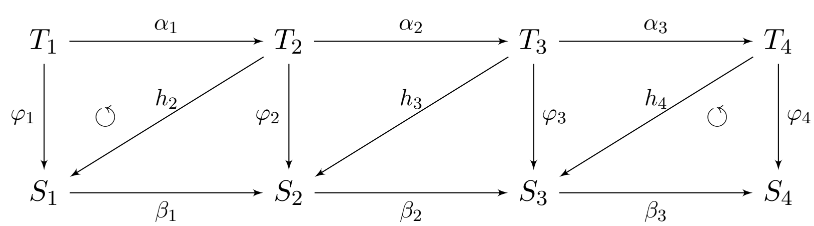
implies the existence of a morphism $\lambda\colon \Sigma(T_2) \to S_4$ that renders the following diagram



commutative, i.e., $\Sigma(\mu_T \bullet \tau - \varphi_2 \bullet \mu_S) = \lambda \bullet \rho_S$. Since $\mathrm{Hom}_{\mathfrak{T}}(\Sigma(T_2), S_4) = 0$, it follows that $\lambda = 0$, i.e., $\Sigma(\mu_T \bullet \tau - \varphi_2 \bullet \mu_S) = 0$; which holds if and only if $\mu_T \bullet \tau - \varphi_2 \bullet \mu_S = 0$. ☐

The following lemma implies that our iterative construction of the convolution respects null-homotopic morphisms.

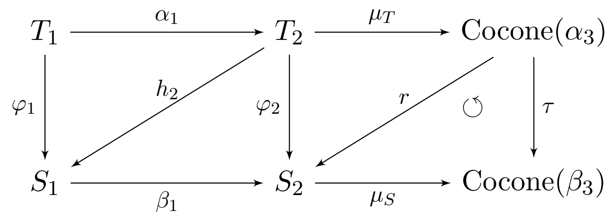
Lemma 6.55. *Let $\mathfrak{T}$ be a triangulated category. Let T_i, S_i, $i = 1, 2, 3, 4$, $\alpha_i\colon T_i \to T_{i+1}, \beta_i\colon S_i \to S_{i+1}, i = 1, 2, 3$, $\varphi_i\colon T_i \to S_i$, $i = 1, 2, 3, 4$ and $h_i\colon T_i \to S_{i-1}, i = 2, 3, 4$ be cells in $\mathfrak{T}$. Suppose that $\alpha_i \bullet \alpha_{i+1} = 0, \beta_i \bullet \beta_{i+1} = 0$ for $i = 1, 2, 3$ and $\alpha_i \bullet \varphi_{i+1} = \varphi_i \bullet \beta_i$ for $i = 1, 2, 3$.*



If $\varphi_2 = \alpha_2 \bullet h_3 + h_2 \bullet \beta_1$, $\varphi_3 = \alpha_3 \bullet h_4 + h_3 \bullet \beta_2$ and $\varphi_4 = h_4 \bullet \beta_3$, then the asserted morphism

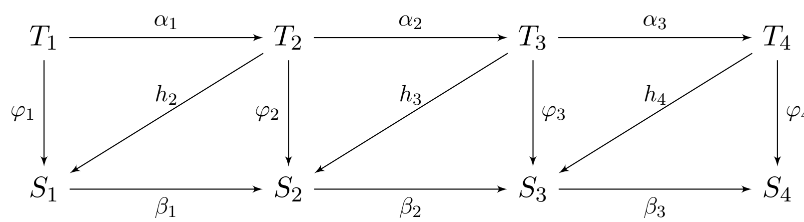
$$\tau\colon \mathrm{Cocone}(\alpha_3) \to \mathrm{Cocone}(\beta_3)$$

in Lemma 6.54 can be chosen together with a morphism $r\colon \mathrm{Cocone}(\alpha_3) \to S_2$ *such that* $\varphi_2 = \mu_T \bullet r + h_2 \bullet \beta_1$ *and* $\tau = r \bullet \mu_S$*.*



Proof. An easy diagram chase shows that $r \coloneqq \kappa_T \bullet h_3$ and $\tau \coloneqq r \bullet \mu_S$ satisfy all the above assertions. ☐

Lemma 6.56. *Let* $\mathscr{C}$ *be an additive category and* $\mathcal{K}^b(\mathscr{C})$ *its bounded homotopy category. Let* $T_i, S_i,\ i = 1,2,3,4,\ \alpha_i\colon T_i \to T_{i+1}, \beta_i\colon S_i \to S_{i+1}, i = 1,2,3,\ \varphi_i\colon T_i \to S_i,\ i = 1,2,3,4$ *and* $h_i\colon T_i \to S_{i-1}, i = 2,3,4$ *be cells in* $\mathcal{K}^b(\mathscr{C})$*. Suppose that* $\alpha_i \bullet \alpha_{i+1} = 0, \beta_i \bullet \beta_{i+1} = 0$ *for* $i = 1,2,3$ *and* $\alpha_i \bullet \varphi_{i+1} = \varphi_i \bullet \beta_i$ *for* $i = 1,2,3$*.*



If we have

(1) $\mathrm{Hom}_{\mathfrak{T}}(\Sigma(T_1), T_4) \cong \mathrm{Hom}_{\mathfrak{T}}(\Sigma(T_2), T_4) = 0$,
(2) $\mathrm{Hom}_{\mathfrak{T}}(\Sigma(S_1), S_4) \cong \mathrm{Hom}_{\mathfrak{T}}(\Sigma(S_2), S_4) = 0$,
(3) $\mathrm{Hom}_{\mathfrak{T}}(\Sigma(T_2), S_4) \cong \mathrm{Hom}_{\mathfrak{T}}(\Sigma(T_3), S_4) = 0$,
(4) $\kappa_T = \Sigma^{-1}(\pi(\alpha_3))$, $\rho_T = -\iota(\alpha_3)$, $\kappa_S = \Sigma^{-1}(\pi(\beta_3))$ *and* $\rho_S = -\iota(\beta_3)$*;*

then the morphism $\tau\colon \mathrm{Cocone}(\alpha_3) \to \mathrm{Cocone}(\beta_3)$ *asserted by Lemma 6.54 can always be chosen to be* ***the*** *element of* $\mathrm{CoconeMors}^{st}_{\alpha_3,\beta_3}(\varphi_3, \varphi_4)$*. Furthermore, if we have* $\varphi_2 = \alpha_2 \bullet h_3 + h_2 \bullet \beta_1$*,* $\varphi_3 = \alpha_3 \bullet h_4 + h_3 \bullet \beta_2$ *and* $\varphi_4 = h_4 \bullet \beta_3$*, then*[6] $\kappa_T \bullet h_3 \bullet \mu_S = \tau$*.*

Proof. By Corollary 5.18, $\mathrm{CoconeMors}^{st}_{\alpha_3,\beta_3}(\varphi_3, \varphi_4)$ is a singleton set. By Remark 5.11, the morphism $\tau\colon \mathrm{Cocone}(\alpha_3) \to \mathrm{Cocone}(\beta_3)$ is given at index $i \in \mathbb{Z}$ by

$$\tau^i \coloneqq \qquad T_3^i \oplus T_4^{i-1} \xrightarrow{\begin{pmatrix} \varphi_3^i & s_3^i \\ 0 & \varphi_4^{i-1} \end{pmatrix}} S_3^i \oplus S_4^{i-1}$$

where $\left(s_3^i\colon T_3^i \to S_4^{i-1}\right)_{i\in\mathbb{Z}}$ is a chain homotopy associated to $\alpha_3 \bullet \varphi_4 - \varphi_3 \bullet \beta_3$.

According to the proof of Lemma 6.53, $\mu_S\colon S_2 \to \mathrm{Cocone}(\beta_3)$ is given at index $i \in \mathbb{Z}$ by

$$\mu_S^i \coloneqq \qquad S_2^i \xrightarrow{\left(\begin{smallmatrix} \beta_2^i & -h_{S_2}^i \end{smallmatrix}\right)} S_3^i \oplus S_4^{i-1}$$

[6] I.e., this choice satisfies the assertion of Lemma 6.55.

where $\left(h^i_{S_2}\colon S^i_2 \to S^{i-1}_4\right)_{i\in\mathbb{Z}}$ is a chain homotopy associated to $\beta_2 \bullet \beta_3$.

Hence the morphism $\sigma := \kappa_T \bullet h_3 \bullet \mu_S$ is given at index $i \in \mathbb{Z}$ by

$$\sigma^i := \qquad T^i_3 \oplus T^{i-1}_4 \xrightarrow{\begin{pmatrix} h^i_3 \bullet \beta^i_2 & -h^i_3 \bullet h^i_{S_2} \\ 0 & 0 \end{pmatrix}} S^i_3 \oplus S^{i-1}_4$$

Let $\left(t^i_3\colon T^i_3 \to S^{i-1}_3\right)_{i\in\mathbb{Z}}$ and $\left(t^i_4\colon T^i_4 \to S^{i-1}_4\right)_{i\in\mathbb{Z}}$ be chain homotopies of $\alpha_3 \bullet h_4 + h_3 \bullet \beta_2 - \varphi_3$ resp. $h_4 \bullet \beta_3 - \varphi_4$.

It can be shown that the family

$$\left(\lambda^i := t^{i+1}_3 \bullet \beta^i_3 - h^{i+1}_3 \bullet h^{i+1}_{S_2} - \alpha^{i+1}_3 \bullet t^{i+1}_4 - s^{i+1}_3 \colon T^{i+1}_3 \to S^i_4\right)_{i\in\mathbb{Z}}$$

defines a morphism $\lambda\colon \Sigma(T_3) \to S_4$. It follows from the assumption $\mathrm{Hom}_{\mathfrak{T}}(\Sigma(T_3), S_4) = 0$ that $\lambda = 0$. Let $\left(y^i\colon T^{i+1}_3 \to S^{i-1}_4\right)_{i\in\mathbb{Z}}$ be a chain homotopy associated to λ.

For $i \in \mathbb{Z}$, we define

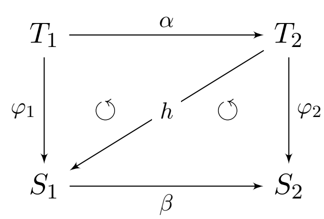
$$h^i := \qquad T^i_3 \oplus T^{i-1}_4 \xrightarrow{\begin{pmatrix} t^i_3 & -y^{i-1} \\ -h^{i-1}_4 & t^{i-1}_4 \end{pmatrix}} S^{i-1}_3 \oplus S^{i-2}_4$$

A straightforward verification shows that

$$\partial^i_{\mathrm{Cocone}(\alpha_3)} \bullet h^{i+1} + h^i \bullet \partial^{i-1}_{\mathrm{Cocone}(\beta_3)} = \sigma^i - \tau^i$$

for all $i \in \mathbb{Z}$, hence $\sigma = \tau$. □

Example 6.57. Let $\mathscr{C}$ be an additive category and let $\mathcal{K}^b(\mathscr{C})$ be its bounded homotopy category. Let



be commutative square such that $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma(T_1), S_2) = 0$. A direct consequence of Lemmas 6.55 and 6.56 is

$$\mathrm{CoconeMors}^{st}_{\alpha,\beta}(\varphi_1, \varphi_2) = \{\mathrm{Cocone}(\alpha) \xrightarrow{0} \mathrm{Cocone}(\beta)\}.$$

Lemma 6.58. *Let $\mathscr{C}$ be an additive category and let $\mathcal{K}^b(\mathscr{C})$ be its bounded homotopy category. Let T and S be objects in $\mathcal{C}^b\left(\mathcal{K}^b(\mathscr{C})\right)$ with common lower and upper bounds ℓ resp. u such that*

(1) $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma^r(T^i), T^j) \cong \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma^r(S^i), S^j) = 0$ for all $r > 0$ and $i < j$ with $(i,j) \neq (u-1, u)$,

(2) $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma^r(T^i), S^j) = 0$ for all $r > 0$ and $i < j$.

Then

*(1) For any standard Postnikov systems P_T and P_S of T resp. S, any morphism $\varphi\colon T \to B$ can be extended to exactly one standard[7] morphism of Postnikov systems from P_T to P_S. Furthermore, if φ is null-homotopic, then **the** associated standard convolution of φ is zero.*

(2) The object T can be extended to a standard Postnikov system. Furthermore, any two such extensions are isomorphic.

PROOF. We use induction on $u - \ell \geq 0$.

(1) In the case $u-\ell = 0$, the morphism φ is an ℓ-stalk morphism and its standard convolution is $\Sigma^{-\ell}\left(\varphi^{\ell}\right)$. An ℓ-stalk morphism is null-homotopic if and only if $\varphi^{\ell} = 0$, which holds if and only if $\Sigma^{-\ell}\left(\varphi^{\ell}\right)$ is zero. Suppose now that $u - \ell > 0$. By Lemma 6.56, there exists a unique morphism $\tau\colon C_T^{u-1} \to C_S^{u-1}$ in $\mathrm{CoconeMors}^{st}_{\partial_T^{u-1},\partial_S^{u-1}}(\varphi^{u-1},\varphi^{u})$ with $\mu_T^{u-2} \bullet \varphi^{\ell+2} = \tau \bullet \mu_S^{u-2}$ inducing a morphism $\varphi'\colon A' \to Y$:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & T^{u-3} & \xrightarrow[\ell+1]{\partial_T^{u-3}} & T^{u-2} & \xrightarrow{\mu_T^{u-2}} & \mathrm{Cocone}\left(\partial_T^{u-1}\right) & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \varphi^{u-3}} & \circlearrowleft & \Big\downarrow {\scriptstyle \varphi^{u-2}} & \circlearrowleft & \Big\downarrow {\scriptstyle \tau} & & \\
\cdots & \longrightarrow & S^{u-3} & \xrightarrow[\partial_S^{u-3}]{} & S^{u-2} & \xrightarrow[\mu_S^{u-2}]{} & \mathrm{Cocone}\left(\partial_S^{u-1}\right) & \longrightarrow & 0
\end{array}$$

If, in addition, φ is null-homotopic, then so is φ' by Lemmas 6.55 and 6.56. By Lemma B.10

- $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}\left(\Sigma^r(T^i), \mathrm{Cocone}\left(\partial_T^{u-1}\right)\right) \cong \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}\left(\Sigma^r(S^i), \mathrm{Cocone}\left(\partial_S^{u-1}\right)\right) = 0$,
- $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}\left(\Sigma^r(T^i), \mathrm{Cocone}\left(\partial_S^{u-1}\right)\right) = 0$

for all $r > 0$ and $i \leq u-2$, i.e., φ' satisfies the assumptions of the lemma. Since τ is uniquely determined, there is one-to-one correspondence between extensions of φ and φ' to morphisms of Postnikov systems. Hence, the assertion follows by the induction hypothesis.

(2) Let ℓ_T and u_T be lower resp. upper bounds of T. We will prove the existence by induction on $u_T - \ell_T$. By the definition of a standard Postnikov system, $\mu_T^u \coloneqq T^u \to 0$, hence $\kappa_T^u \coloneqq \Sigma^{-1}(\pi(\mu_T^u)) = \mathrm{id}_{T^u}$ and $\mu_T^{u-1} = \partial_T^{u-1}\colon T^{u-1} \to T^u$. If $u_T - \ell_T = 0$, then we are done. Suppose now that $u_T - \ell_T > 0$. By Lemma 6.53, there exists a unique morphism $\mu_T^{u-2}\colon T^{u-2} \to \mathrm{Cocone}\left(\partial_T^{u-1}\right)$ with $\mu_T^{u-2} \bullet \Sigma^{-1}\left(\pi\left(\mu_T^{u-1}\right)\right) = \partial_T^{u-2}$ and $\partial_T^{u-3} \bullet \mu_T^{u-2} = 0$. By Lemma B.10, we have $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}\left(\Sigma^r(T^i), \mathrm{Cocone}\left(\partial_T^{u-1}\right)\right) = 0$ for all $r > 0$ and $i \leq u-2$, i.e., the complex A' defined by

$$\cdots \longrightarrow T^{u-3} \xrightarrow{\partial_T^{u-3}} T^{u-2} \xrightarrow{\mu_T^{u-2}} \mathrm{Cocone}\left(\partial_T^{u-1}\right) \xrightarrow[u-1]{} 0$$

fulfils the assumption of the lemma. Hence, the existence follows by the induction hypothesis. The morphism μ_T^{u-2} is uniquely determined, however it might be represented

[7] φ might be extended in different ways to morphisms from P_T to P_S, however only one of them is standard.

by different morphisms in $\mathcal{C}^b(\mathscr{C})$, which leads to different (but isomorphic) cocone objects in the next iteration, i.e., to different standard Postnikov systems. Suppose ζ_T^{u-2} is another representative of μ_T^{u-2}. Then we still have the following morphism:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & T^{u-3} & \xrightarrow[\ell+1]{\partial_T^{u-3}} & T^{u-2} & \xrightarrow{\mu_T^{u-2}} & \operatorname{Cocone}\left(\partial_T^{u-1}\right) & \longrightarrow & 0 \\
 & & \Vert & \circlearrowleft & \Vert & \circlearrowleft & \Vert & & \\
\cdots & \longrightarrow & T^{u-3} & \xrightarrow[\partial_T^{u-3}]{} & T^{u-2} & \xrightarrow[\zeta_T^{u-2}]{} & \operatorname{Cocone}\left(\partial_T^{u-1}\right) & \longrightarrow & 0
\end{array}$$

which, by (1), can be extended to a morphism between the corresponding different Postnikov systems. The assertion follows now by Remarks 6.48 and B.12.

□

Let T be an object as in Lemma 6.58 and P_T a standard Postnikov system of T. Since no assumption has been made on $\operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma(T^{u-1}), T^u)$, the first assertion of the lemma does not apply to the identity morphism of T. In other words, the set

$$\operatorname{CoconeMors}^{st}_{\partial_T^{u-1}, \partial_T^{u-1}}(\mathrm{id}_{T^{u-1}}, \mathrm{id}_{T^u})$$

might contain more than one element. That is, id_T might be extended to different standard isomorphisms on P_T which restricts the convolution construction from being functorial. We can rectify this situation by imposing more assumptions:

Construction 6.59. Let $\mathscr{C}$ be an additive category and let $\mathscr{D} \subseteq \mathcal{K}^b(\mathscr{C})$ be an additive full subcategory such that $\operatorname{Hom}_{\mathscr{D}}(\Sigma^r(X), Y) = 0$ for all $r > 0$ and X, Y in $\mathscr{D}$. Since the assumptions of Lemma 6.58 hold on all objects and morphisms of $\mathcal{K}^b(\mathscr{D}) \subset \mathcal{K}^b\left(\mathcal{K}^b(\mathscr{C})\right)$, every morphism can be extended to exactly one standard morphism between standard Postnikov systems. It follows from Lemmas 5.12 and 5.13 that the composition, addition and additive inverses of standard morphisms are also standard. Hence, the standard convolution construction becomes functorial. We call the induced functor

$$\mathbf{F}\colon \mathcal{K}^b(\mathscr{D}) \to \mathcal{K}^b(\mathscr{C})$$

the **convolution functor**.

The consideration of Lemma 6.58 gives rise to the following two algorithms for computing standard Postnikov systems and their convolutions. Both algorithms are implemented in the `GAP` package `HomotopyCategories` [**Sal21d**].

Algorithm 4: Standard convolution of an object

Input: An additive category $\mathscr{C}$ and an object $T \in \mathcal{K}^b\Big(\mathcal{K}^b(\mathscr{C})\Big)$ with the assumptions as in Lemma 6.58.

Output: The standard convolution $\mathbf{F}(T)$

$\ell :=$ some lower bound of T

$u :=$ some upper bound of T

if $u = \ell$ **then**

  return $\Sigma^{-\ell}\Big(T^{\ell}\Big)$

else

- Compute a chain homotopy $\Big(h_T^{u-2,i} \colon T^{u-2,i} \to T^{u,i-1}\Big)_{i\in\mathbb{Z}}$ of $\partial_T^{u-2} \bullet \partial_T^{u-1}$

  `/* See Remark 3.19 */`

- Define $\mu_T^{u-2} \colon T^{u-2} \to \operatorname{Cocone}\Big(\partial_T^{u-1}\Big)$ by the morphism whose component at $i \in \mathbb{Z}$ is

$$\mu_T^{u-2,i} := T^{u-2,i} \xrightarrow{\left(\,\partial_T^{u-2,i} \;\; -h_T^{u-2,i}\,\right)} T^{u-1,i} \oplus T^{u,i-1}$$

  `/* According to Lemma 6.53,` μ_T^{u-2} `is the only morphism in` $\mathcal{K}^b(\mathscr{C})$ `which satisfies */`

  `/*` $\partial_T^{u-3} \bullet \mu_T^{u-2} = 0$ `and` $\mu_T^{u-2} \bullet \Sigma^{-1}\big(\pi\big(\partial_T^{u-1}\big)\big) = \partial_T^{u-2}$ `*/`

- Redefine T by

$$P_T^{\leq u-1} := \quad \cdots \longrightarrow T^{u-3} \xrightarrow{\partial_T^{u-3}} T^{u-2} \xrightarrow{\mu_T^{u-2}} \operatorname{Cocone}\Big(\partial_T^{u-1}\Big) \xrightarrow[u-1]{} 0$$

- **return** $\mathbf{F}(T)$

Algorithm 5: Standard convolution of a morphism

Input: An additive category $\mathscr{C}$ and a morphism $\varphi\colon T \to S \in \mathcal{K}^b\Big(\mathcal{K}^b(\mathscr{C})\Big)$ with the same assumptions in Lemma 6.58.

Output: the standard convolution of φ

- $\ell :=$ Some common lower bound of T and S
- $u :=$ Some common upper bound of T and S

if $u = \ell$ **then**

  return $\Sigma^{-\ell}\Big(\varphi^{\ell}\Big)$

else

- Compute a chain homotopy $\Big(h_\varphi^{u-1,i}\colon T^{u-1,i} \to S^{u,i-1}\Big)_{i\in\mathbb{Z}}$ of $\partial_T^{u-1}\cdot\varphi^{u} - \varphi^{u-1}\cdot\partial_S^{u-1}$

  `/* See Remark 3.19 */`

- Define $\varphi_C^{u-1}\colon \mathrm{Cocone}\Big(\partial_T^{u-1}\Big) \to \mathrm{Cocone}\Big(\partial_S^{u-1}\Big)$ by the morphism whose component at $i \in \mathbb{Z}$ is

$$\varphi_C^{u-1,i} := T^{u-1,i} \oplus T^{u,i-1} \xrightarrow{\begin{pmatrix} \varphi^{u-1,i} & h_\varphi^{u-1,i} \\ 0 & \varphi^{u,i-1} \end{pmatrix}} S^{u-1,i} \oplus S^{u,i-1}$$

  `/* By Lemma 6.56,` φ_C^{u-1} `is the unique element in` $\mathrm{CoconeMors}^{st}_{\partial_T^{u-1},\partial_S^{u-1}}\left(\varphi^{u-1},\varphi^{u}\right)$ `*/`

  `/* and` $\mu_T^{u-2}\cdot\varphi_C^{u-1} = \varphi^{u-2}\cdot\mu_S^{u-2}$. `*/`

- Redefine φ by

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & T^{u-3} & \xrightarrow{\partial_T^{u-3}} & T^{u-2} & \xrightarrow{\mu_T^{u-2}} & \mathrm{Cocone}\Big(\partial_T^{u-1}\Big) & \xrightarrow[u-1]{} & 0 \\
 & & \downarrow{\scriptstyle \varphi^{u-3}} & & \downarrow{\scriptstyle \varphi^{u-2}} & & \downarrow{\scriptstyle \varphi_C^{u-1}} & & \\
\cdots & \longrightarrow & S^{u-3} & \xrightarrow[\partial_S^{u-3}]{} & S^{u-2} & \xrightarrow[\mu_S^{u-2}]{} & \mathrm{Cocone}\Big(\partial_S^{u-1}\Big) & \longrightarrow & 0
\end{array}$$

- **return** $\mathbf{F}(\varphi)$

Let $\mathscr{C}$ and $\mathscr{D}$ be defined as in Construction 6.59.

Lemma 6.60. *The convolution functor* $\mathbf{F}\colon \mathcal{K}^b(\mathscr{D}) \to \mathcal{K}^b(\mathscr{C})$ *commutes up to a natural isomorphism with the shift functors.*

Proof. For an object T in $\mathcal{K}^b(\mathscr{D})$, we define $T^{\ominus}$ by the object in $\mathcal{K}^b(\mathscr{D})$ whose differential at $i \in \mathbb{Z}$ is $\partial^i_{T^{\ominus}} := -\partial^i_T$. In fact, this construction is functorial: for a morphism $\varphi\colon T \to S$, $\varphi^{\ominus}\colon T^{\ominus} \to S^{\ominus}$ is defined at $i \in \mathbb{Z}$ by φ^i. Of course, $T \cong T^{\ominus}$ via the natural isomorphism $\epsilon_T\colon T \to T^{\ominus}$ defined at $i \in \mathbb{Z}$ by $\epsilon^i_T := (-1)^{i+1} \cdot \mathrm{id}_{T^i}$. It follows that $\epsilon_T \bullet \varphi^{\ominus} = \varphi \bullet \epsilon_S$ and $\epsilon_T^{-1} = \epsilon_{T^{\ominus}}$.

The morphisms φ and $\Sigma(\varphi)^{\ominus}$ consist of the same differentials and morphisms and differ only in the lower and upper bounds, hence, $\mathbf{F}\Big((\Sigma(\varphi))^{\ominus}\Big) = \Sigma(\mathbf{F}(\varphi))$.

Functors sends isomorphisms to isomorphisms, hence

$$\eta_T := \mathbf{F}\Big(\epsilon_{\Sigma(T)}\Big)\colon \mathbf{F}(\Sigma(T)) \to \mathbf{F}\Big(\Sigma(T)^{\ominus}\Big) = \Sigma(\mathbf{F}(T))$$

is an isomorphism. By applying $\mathbf{F}$ to the equation

$$\epsilon_{\Sigma(T)} \bullet (\Sigma(\varphi))^{\ominus} = \Sigma(\varphi) \bullet \epsilon_{\Sigma(S)}$$

we get a commutative diagram

$$\begin{array}{ccccc}
\mathbf{F}(\Sigma(T)) & \xrightarrow[\sim]{\eta_T := \mathbf{F}\left(\epsilon_{\Sigma(T)}\right)} & \mathbf{F}\Big(\Sigma(T)^{\ominus}\Big) & = & \Sigma(\mathbf{F}(T)) \\
\Big\downarrow{\scriptstyle \mathbf{F}(\Sigma(\varphi))} & \circlearrowleft & \Big\downarrow{\scriptstyle \mathbf{F}\left(\Sigma(\varphi)^{\ominus}\right)} & \circlearrowleft & \Big\downarrow{\scriptstyle \Sigma(\mathbf{F}(\varphi))} \\
\mathbf{F}(\Sigma(S)) & \xrightarrow[\eta_S := \mathbf{F}\left(\epsilon_{\Sigma(S)}\right)]{\sim} & \mathbf{F}\Big(\Sigma(S)^{\ominus}\Big) & = & \Sigma(\mathbf{F}(S))
\end{array}$$

which translates to the naturality of the following assignment:

$$\eta\colon \begin{cases} \Sigma \bullet \mathbf{F} & \to \mathbf{F} \bullet \Sigma, \\ \quad T & \mapsto \eta_T := \mathbf{F}\Big(\epsilon_{\Sigma(T)}\Big). \end{cases}$$

$\square$

Lemma 6.61. *Let $\mathscr{C}$ be an additive category and $\mathcal{K}^b(\mathscr{C})$ its bounded homotopy category. Let T and S be objects in $\mathcal{K}^b\Big(\mathcal{K}^b(\mathscr{C})\Big)$ with a common upper bound u and the following properties:*

(1) $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma^r(T^i), T^j) \cong \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma^r(S^i), S^j) = 0$ *for all $r > 0$ and $i < j$,*

(2) $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma^r(T^i), S^j) \cong \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma^r(S^i), T^j) = 0$ *for all $r > 0$ and $i \le j$ with $(i, j) \neq (u, u)$.*

Then for any morphism $\varphi\colon T \to S$, running Algorithm 5 on the sequence

$$T \xrightarrow{\varphi} S \xrightarrow{\iota(\varphi)} \mathrm{Cone}(\varphi) \xrightarrow{\pi(\varphi)} \Sigma(T) \xrightarrow[\sim]{\epsilon_{\Sigma(T)}} \Sigma(T)^{\ominus}$$

yields a sequence of standard morphisms of Postnikov systems

$$P_T \xrightarrow{P_\varphi} P_S \xrightarrow{P_{\iota(\varphi)}} P_{\mathrm{Cone}(\varphi)} \xrightarrow{P_{\pi(\varphi)}} P_{\Sigma(T)} \xrightarrow[\sim]{P_{\epsilon_{\Sigma(T)}}} P_{\Sigma(T)^{\ominus}}$$

whose convolutions

$$\mathbf{F}(T) \xrightarrow{\mathbf{F}(\varphi)} \mathbf{F}(S) \xrightarrow{\mathbf{F}(\iota(\varphi))} \mathbf{F}(\mathrm{Cone}(\varphi)) \xrightarrow{\mathbf{F}(\pi(\varphi))\bullet\eta_T} \mathbf{F}\Big(\Sigma(T)^{\ominus}\Big) = \Sigma(\mathbf{F}(T))$$

form an exact triangle in $\mathcal{K}^b(\mathscr{C})$*.*

Proof. The sequence

$$T \xrightarrow{\varphi} S \xrightarrow{\iota(\varphi)} \mathrm{Cone}(\varphi) \xrightarrow{\pi(\varphi)} \Sigma(T) \xrightarrow{\epsilon_{\Sigma(T)}} \Sigma(T)^{\ominus}$$

is depicted in the following diagram:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & T^{u-3} & \xrightarrow{\partial_T^{u-3}} & T^{u-2} & \xrightarrow{\partial_T^{u-2}} & T^{u-1} & \xrightarrow{\partial_T^{u-1}} & T^u \xrightarrow{\;{\scriptscriptstyle u}\;} 0 \\
 & & \downarrow{\scriptstyle \varphi^{u-3}} & & \downarrow{\scriptstyle \varphi^{u-2}} & & \downarrow{\scriptstyle \varphi^{u-1}} & & \downarrow{\scriptstyle \varphi^{u}} \\
\cdots & \longrightarrow & S^{u-3} & \xrightarrow{\partial_S^{u-3}} & S^{u-2} & \xrightarrow{\partial_S^{u-2}} & S^{u-1} & \xrightarrow{\partial_S^{u-1}} & S^u \longrightarrow 0 \\
 & & \downarrow{\scriptstyle \begin{pmatrix} 0 & \mathrm{id}_{S^{u-3}} \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} 0 & \mathrm{id}_{S^{u-2}} \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} 0 & \mathrm{id}_{S^{u-1}} \end{pmatrix}} & & \downarrow{\scriptstyle \mathrm{id}_{S^u}} \\
 & & T^{u-2}\oplus B^{u-3} & \xrightarrow{\begin{pmatrix} -\partial_T^{u-2} & \varphi^{u-2} \\ 0 & \partial_S^{u-3} \end{pmatrix}} & T^{u-1}\oplus B^{u-2} & \xrightarrow{\begin{pmatrix} -\partial_T^{u-1} & \varphi^{u-1} \\ 0 & \partial_S^{u-2} \end{pmatrix}} & T^{u}\oplus B^{u-1} & \xrightarrow{\begin{pmatrix} \varphi^{u} \\ \partial_S^{u-1} \end{pmatrix}} & S^u \longrightarrow 0 \\
 & & \downarrow{\scriptstyle \begin{pmatrix} \mathrm{id}_{T^{u-2}} \\ 0 \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} \mathrm{id}_{T^{u-1}} \\ 0 \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} \mathrm{id}_{T^{u}} \\ 0 \end{pmatrix}} & & \\
\cdots & \longrightarrow & T^{u-2} & \xrightarrow{-\partial_T^{u-2}} & T^{u-1} & \xrightarrow{-\partial_T^{u-1}} & T^u & \longrightarrow & 0 \longrightarrow 0 \\
 & & \downarrow{\scriptstyle (-1)^{u-2}\cdot \mathrm{id}_{T^{u-2}}} & & \downarrow{\scriptstyle (-1)^{u-1}\cdot \mathrm{id}_{T^{u-1}}} & & \downarrow{\scriptstyle (-1)^{u}\cdot \mathrm{id}_{T^{u}}} & & \\
\cdots & \longrightarrow & T^{u-2} & \xrightarrow{\partial_T^{u-2}} & T^{u-1} & \xrightarrow{\partial_T^{u-1}} & T^u & \longrightarrow & 0 \longrightarrow 0
\end{array}$$

from which we observe that φ, $\iota(\varphi)$, $\pi(\varphi)$ and $\epsilon_{\Sigma(T)}$ satisfy the assumptions of Lemma 6.58, hence they can be extended to a sequence of standard morphisms

$$P_T \xrightarrow{P_\varphi} P_S \xrightarrow{P_{\iota(\varphi)}} P_{\mathrm{Cone}(\varphi)} \xrightarrow{P_{\pi(\varphi)}} P_{\Sigma(T)} \xrightarrow[\sim]{P_{\epsilon_{\Sigma(T)}}} P_{\Sigma(T)^{\ominus}}.$$

Computing the preceding sequence relies on computing chain homotopies of zero morphisms in $\mathcal{K}^b(\mathscr{C})$ which are usually not uniquely determined. Let

$$Q_T \xrightarrow{Q_\varphi} Q_S \xrightarrow{Q_{\iota(\varphi)}} Q_{\mathrm{Cone}(\varphi)} \xrightarrow{Q_{\pi(\varphi)}} Q_{\Sigma(T)} \xrightarrow[\sim]{Q_{\epsilon_{\Sigma(T)}}} Q_{\Sigma(T)^{\ominus}}.$$

be another extension to standard morphisms of Postnikov systems where Q_T and $Q_{\Sigma(T)^{\ominus}}$ consist of the same exact triangles[8]. By Lemma 6.58, the morphisms id_T, id_S, $\mathrm{id}_{\Sigma(T)}$ and $\mathrm{id}_{\Sigma(T)^{\ominus}}$ can be extended uniquely to standard isomorphisms $I_T\colon P_T \xrightarrow{\sim} Q_T$, $I_S\colon P_S \xrightarrow{\sim} Q_S$, $I_{\Sigma(T)}\colon P_{\Sigma(T)} \xrightarrow{\sim} Q_{\Sigma(T)}$ and $I_{\Sigma(T)^{\ominus}}\colon P_{\Sigma(T)^{\ominus}} \xrightarrow{\sim} Q_{\Sigma(T)^{\ominus}}$. However, since no assumptions have been made about $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T^u, S^u)$, the morphism $\mathrm{id}_{\mathrm{Cone}(\varphi)}$ may be extended in different ways to standard isomorphisms between $P_{\mathrm{Cone}(\varphi)}$ and $Q_{\mathrm{Cone}(\varphi)}$. Let $\zeta\colon P_{\mathrm{Cone}(\varphi)} \xrightarrow{\sim} Q_{\mathrm{Cone}(\varphi)}$ be one them. Due to

[8]The exact triangle in $Q_{\Sigma(T)^{\ominus}}$ at index $i \in \mathbb{Z}$ is identical to the exact triangle of Q_T at index $i+1$.

the fact that standard morphisms are closed under composition, any such ζ renders the following diagram

$$\begin{array}{ccccccccc}
P_T & \xrightarrow{P_\varphi} & P_S & \xrightarrow{P_{\iota(\varphi)}} & P_{\mathrm{Cone}(\varphi)} & \xrightarrow{P_{\pi(\varphi)}} & P_{\Sigma(T)} & \xrightarrow[\sim]{P_{\epsilon_{\Sigma(T)}}} & P_{\Sigma(T)^\ominus} \\
\sim\downarrow I_T & \circlearrowleft & \sim\downarrow I_S & \circlearrowleft & \sim\downarrow \zeta & \circlearrowleft & \sim\downarrow I_{\Sigma(T)} & \circlearrowleft & \sim\downarrow I_{\Sigma(T)^\ominus} \\
Q_T & \xrightarrow[Q_\varphi]{} & Q_S & \xrightarrow[Q_{\iota(\varphi)}]{} & Q_{\mathrm{Cone}(\varphi)} & \xrightarrow[Q_{\pi(\varphi)}]{} & Q_{\Sigma(T)} & \xrightarrow[Q_{\epsilon_{\Sigma(T)}}]{\sim} & Q_{\Sigma(T)^\ominus}
\end{array}$$

commutative. Therefore, the associated convolutions form isomorphic triangles (after composing the last two morphisms in each sequence). In other words, while proving the assertion, we can use arbitrary chain homotopies as long as the triangle formed by the convolutions is well-defined.

We will prove the lemma using induction on $u - \ell \geq 0$ where ℓ is a common lower bound for both T and S. If $u - \ell = 0$, then $T = \lceil T^u \rfloor_u$ and $S = \lceil S^u \rfloor_u$. The sequence

$$T \xrightarrow{\varphi} S \xrightarrow{\iota(\varphi)} \mathrm{Cone}(\varphi) \xrightarrow{\pi(\varphi)} \Sigma(T) \xrightarrow[\sim]{\epsilon_{\Sigma(T)}} \Sigma(T)^\ominus$$

is illustrated in the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & T^u & \xrightarrow[u]{} & 0 \\
 & & & & \downarrow \varphi^u & & \\
0 & \longrightarrow & 0 & \longrightarrow & S^u & \longrightarrow & 0 \\
 & & & & \downarrow \mathrm{id}_{S^u} & & \\
0 & \longrightarrow & T^u & \xrightarrow{\varphi^u} & S^u & \longrightarrow & 0 \\
 & & \downarrow \mathrm{id}_{T^u} & & & & \\
0 & \longrightarrow & T^u & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow (-1)^u \cdot \mathrm{id}_{T^u} & & & & \\
0 & \longrightarrow & T^u & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

The first iteration of Algorithm 5 yields the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Sigma^{-1}(T^u) & \xrightarrow[u-1]{} & 0 & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \Sigma^{-1}(\varphi^u)} & & & & \\
0 & \longrightarrow & \Sigma^{-1}(S^u) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \left(0 \ \mathrm{id}_{S^{u,i-1}} \right)_{i\in\mathbb{Z}}} & & & & \\
0 & \longrightarrow & \mathrm{Cocone}(\varphi^u) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \left(\begin{smallmatrix} \mathrm{id}_{T^{u,i}} \\ 0 \end{smallmatrix} \right)_{i\in\mathbb{Z}}} & & & & \\
0 & \longrightarrow & T^u & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle (-1)^u \cdot \mathrm{id}_{T^u}} & & & & \\
0 & \longrightarrow & T^u & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

The associated convolutions form the following sequence:

$$\Sigma^{-(u-1)}(\Sigma^{-1}(T^u)) \xrightarrow{\Sigma^{-(u-1)}\left(\Sigma^{-1}(\varphi^u)\right)} \Sigma^{-(u-1)}(\Sigma^{-1}(S^u)) \xrightarrow{\left(0 \ \mathrm{id}_{S^{u,i-1-(u-1)}} \right)_{i\in\mathbb{Z}}}$$
$$\Sigma^{-(u-1)}(\mathrm{Cocone}(\varphi^u)) \xrightarrow{\left(\begin{smallmatrix} \mathrm{id}_{T^{u,i-(u-1)}} \\ 0 \end{smallmatrix} \right)_{i\in\mathbb{Z}}} \Sigma^{-(u-1)}(T^u) \xrightarrow[\sim]{(-1)^u \mathrm{id}_{\Sigma^{-(u-1)}(T^u)}} \Sigma^{-(u-1)}(T^u)$$

which can be simplified to

$$\Sigma^{-u}(T^u) \xrightarrow{\Sigma^{-u}(\varphi^u)} \Sigma^{-u}(S^u) \xrightarrow{\Sigma^{-u}(\iota(\varphi^u))}$$
$$\Sigma^{-u}(\mathrm{Cone}(\varphi^u)) \xrightarrow{\Sigma^{-u}(\pi(\varphi^u))} \Sigma^{-u}(\Sigma(T^u)) \xrightarrow[\sim]{(-1)^u \Sigma^{-u}\left(\mathrm{id}_{\Sigma(T^u)}\right)} \Sigma^{-u}(\Sigma(T^u)) \ .$$

By composing the last two morphisms, we get an exact triangle according to Lemma B.18, since

$$T^u \xrightarrow{\varphi^u} S^u \xrightarrow{\iota(\varphi)} \mathrm{Cone}(\varphi) \xrightarrow{\pi(\varphi)} \Sigma(T)$$

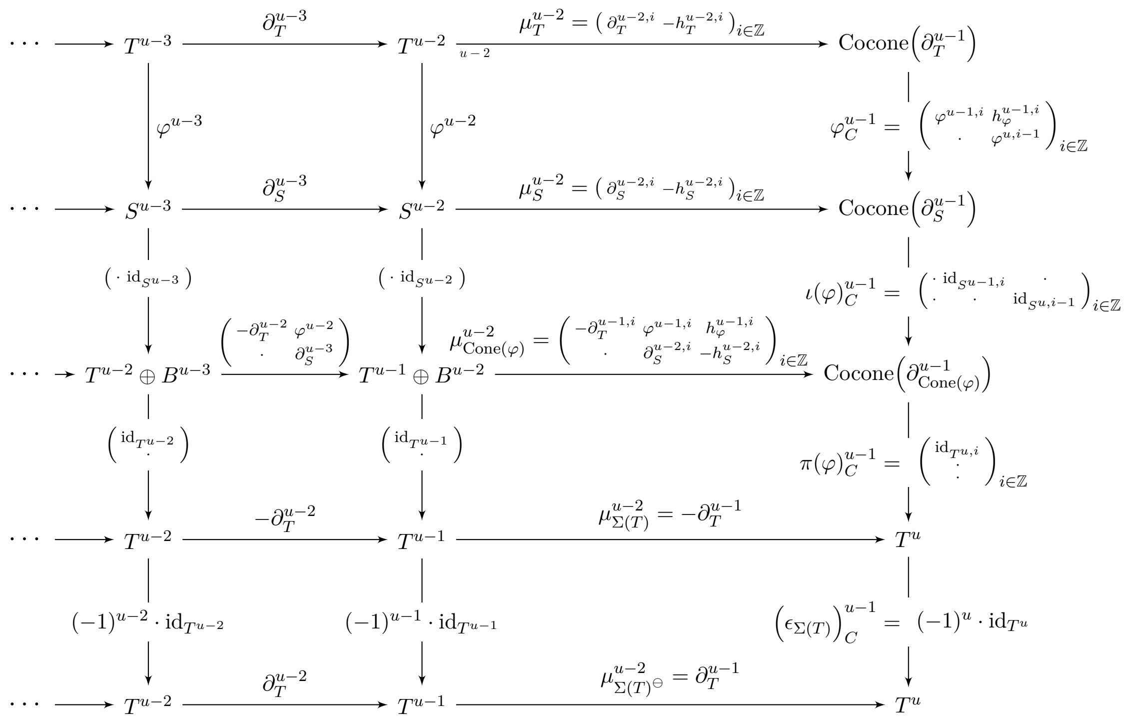
is exact.

Suppose that $u - \ell > 0$. Let $\left(h_T^{m,i} \colon T^{m,i} \to T^{m+2,i-1}\right)_{i\in\mathbb{Z}}$, $\left(h_S^{m,i} \colon S^{m,i} \to S^{m+2,i-1}\right)_{i\in\mathbb{Z}}$ and $\left(h_\varphi^{m,i} \colon T^{m,i} \to S^{m+1,i-1}\right)_{i\in\mathbb{Z}}$ be chain homotopies of $\partial_T^m \bullet \partial_T^{m+1}$, $\partial_S^m \bullet \partial_S^{m+1}$ resp. $\partial_T^m \bullet \varphi^{m+1} - \varphi^m \bullet \partial_S^m$ for every $m \in \mathbb{Z}$. We can use these chain homotopies to compute chain homotopies of $\partial^m_{\mathrm{Cone}(\varphi)} \bullet \partial^{m+1}_{\mathrm{Cone}(\varphi)}$ and $\partial^m_{\Sigma(T)} \bullet \partial^{m+1}_{\Sigma(T)}$ for every $m \in \mathbb{Z}$. That is, they can be used to compute standard Postnikov systems $Q_{\mathrm{Cone}(\varphi)}$, $Q_{\Sigma(T)}$ and $Q_{\Sigma(T)^\ominus}$ of $\mathrm{Cone}(\varphi)$, $\Sigma(T)$ resp. $\Sigma(T)^\ominus$.

The first iteration of the Algorithm 5 on φ, $\iota(\varphi)$, $\pi(\varphi)$ and $\epsilon_{\Sigma(T)}$ (whose common upper bound is u) yields four[9] morphisms $Q_\varphi^{\leq u-1} \colon Q_T^{\leq u-1} \to Q_S^{\leq u-1}$, $Q_{\iota(\varphi)}^{\leq u-1} \colon P_S^{\leq u-1} \to Q_{\mathrm{Cone}(\varphi)}^{\leq u-1}$,

[9]The morphism $\epsilon_{\Sigma(T)}$ remains unaffected because its upper bound is $u - 1$.

$Q^{\leq u-1}_{\pi(\varphi)} \colon Q^{\leq u-1}_{\mathrm{Cone}(\varphi)} \to Q^{\leq u-1}_{\Sigma(T)}$ with $Q^{\leq u-1}_{\Sigma(T)} = \Sigma(T)$, and $Q^{\leq u-1}_{\epsilon_{\Sigma(T)}} = \epsilon_{\Sigma(T)}$; depicted in the following commutative diagram:



A straightforward verification shows that the families

$$\Big(h_T^{u-3,i+1} \bullet \partial_T^{u-1,i} - \partial_T^{u-3,i+1} \bullet h_T^{u-2,i} \colon T^{u-3,i+1} \to T^{u,i}\Big)_{i\in\mathbb{Z}}$$

and

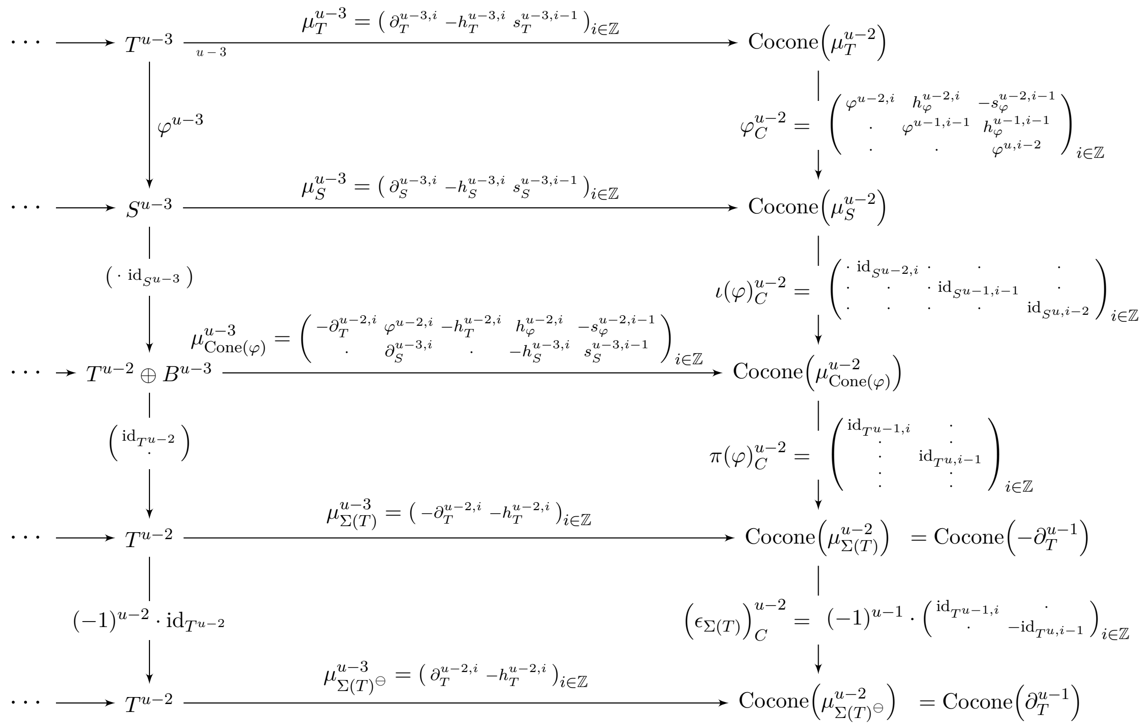
$$\Big(h_S^{u-3,i+1} \bullet \partial_S^{u-1,i} - \partial_S^{u-3,i+1} \bullet h_S^{u-2,i} \colon S^{u-3,i+1} \to S^{u,i}\Big)_{i\in\mathbb{Z}}$$

define morphisms $t_T \colon \Sigma\big(T^{u-3}\big) \to T^u$ resp. $t_S \colon \Sigma\big(S^{u-3}\big) \to S^u$. It follows from the assumption $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}\big(\Sigma\big(T^{u-3}\big), T^u\big) \cong \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}\big(\Sigma\big(S^{u-3}\big), S^u\big) = 0$ that $t_T = 0$ and $t_S = 0$. Let $(s_T^{u-3,i} \colon T^{u-3,i+1} \to T^{u,i-1})_{i\in\mathbb{Z}}$ and $(s_S^{u-3,i} \colon S^{u-3,i+1} \to S^{u,i-1})_{i\in\mathbb{Z}}$ be chain homotopies of t_T resp. t_S. Similarly, the family

$$\Big(h_\varphi^{u-2,i+1} \bullet \partial_S^{u-1,i} - h_T^{u-2,i+1} \bullet \varphi^{u,i} + \varphi^{u-2,i+1} \bullet h_S^{u-2,i+1} + \partial_T^{u-2,i+1} \bullet h_\varphi^{u-1,i+1} \colon T^{u-2,i+1} \to S^{u,i}\Big)_{i\in\mathbb{Z}}$$

defines a morphism $t_\varphi \colon \Sigma\big(T^{u-2}\big) \to S^u$ which, due to $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}\big(T^{u-2}, S^u\big) = 0$, should also be zero. Let $(s_\varphi^{u-2,i} \colon T^{u-2,i+1} \to S^{u,i-1})_{i\in\mathbb{Z}}$ be a chain homotopy associated to t_φ.

The second iteration yields the morphisms $Q^{\leq u-2}_{\varphi} \colon Q^{\leq u-2}_{T} \to Q^{\leq u-2}_{S}$, $Q^{\leq u-2}_{\iota(\varphi)} \colon Q^{\leq u-2}_{S} \to Q^{\leq u-2}_{\mathrm{Cone}(\varphi)}$, $Q^{\leq u-2}_{\pi(\varphi)} \colon Q^{\leq u-2}_{\mathrm{Cone}(\varphi)} \to Q^{\leq u-2}_{\Sigma(T)}$ and $Q^{\leq u-2}_{\epsilon_{\Sigma(T)}} \colon Q^{\leq u-2}_{\Sigma(T)} \xrightarrow{\sim} Q^{\leq u-2}_{\Sigma(T)^{\ominus}}$ depicted in the following commutative diagram:

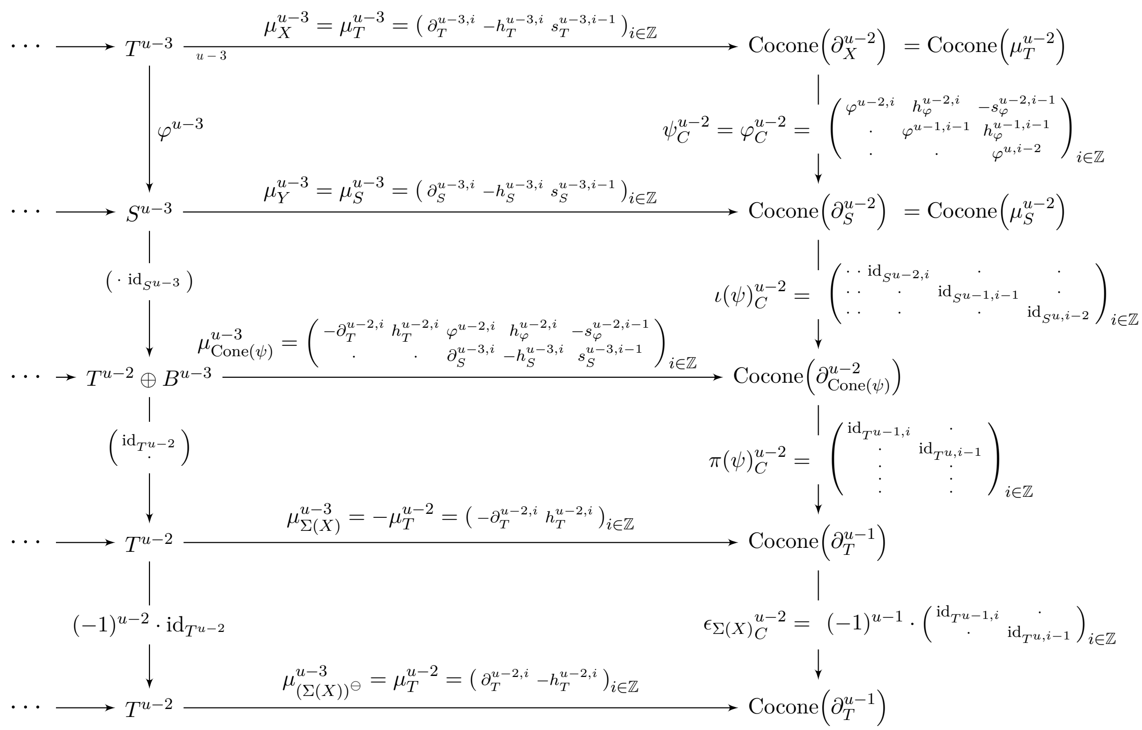


Set $X := P_T^{\leq u-1}$, $Y := P_S^{\leq u-1}$ and $\psi := P_\varphi^{\leq u-1} \colon X \to Y$. By Lemma B.10, X and Y satisfy the assumptions:

(1) $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma^r(X^i), X^j) \cong \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma^r(Y^i), Y^j) = 0$ for all $r > 0$ and $i < j$,
(2) $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma^r(X^i), Y^j) \cong \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma^r(Y^i), X^j) = 0$ for all $r > 0$ and $i \leq j$ with $(i,j) \neq (u-1, u-1)$,

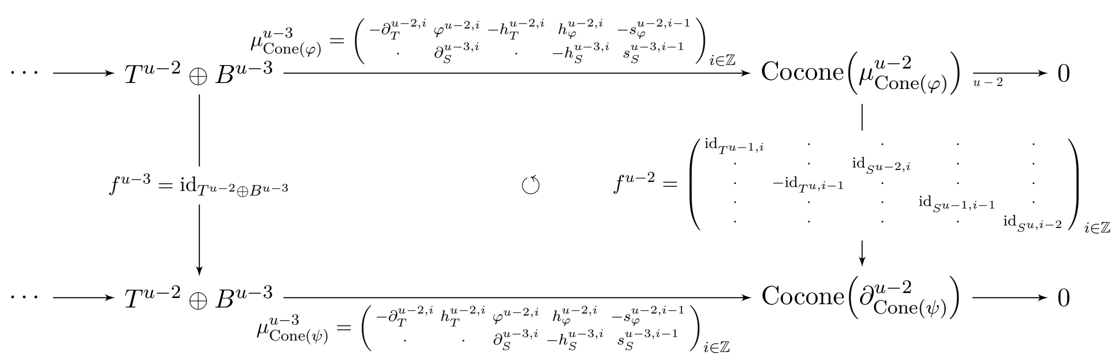
and since their common upper bound is $u-1$, the induction hypothesis applies for ψ.

Applying Algorithm 5 on $\psi, \iota(\psi), \pi(\psi)$ and $\epsilon_{\Sigma(X)}$ yields the following four morphisms: $Q_\psi^{\leq u-2} = Q_\varphi^{\leq u-2}$, $Q_{\iota(\psi)}^{\leq u-2} \colon Q_Y^{\leq u-2} \to Q_{\mathrm{Cone}(\psi)}^{\leq u-2}$, $Q_{\pi(\psi)}^{\leq u-2} \colon Q_{\mathrm{Cone}(\psi)}^{\leq u-2} \to Q_{\Sigma(X)}^{\leq u-2}$, $Q_{\epsilon_{\Sigma(X)}}^{\leq u-2} \colon Q_{\Sigma(X)}^{\leq u-2} \xrightarrow{\sim} Q_{\Sigma(X)^\ominus}^{\leq u-2}$; depicted in the following commutative diagram:



When we compare the morphisms in the above two diagrams, we see that they are identical up to the order of some objects, e.g., the difference between $\mathrm{Cocone}\Big(\mu^{u-2}_{\mathrm{Cone}(\varphi)}\Big)$ and $\mathrm{Cocone}\Big(\partial^{u-2}_{\mathrm{Cone}(\psi)}\Big)$; or up to the multiplication of certain morphisms by -1 , e.g., the right-lower morphisms.

The following two isomorphisms: $f\colon Q^{\leq u-2}_{\mathrm{Cone}(\varphi)} \to Q^{\leq u-2}_{\mathrm{Cone}(\psi)}$



and $g\colon Q^{\leq u-2}_{\Sigma(T)} \to Q^{\leq u-2}_{\Sigma(X)}$

$$
\begin{array}{ccccccc}
\cdots & \xrightarrow{-\partial_T^{u-3}} & T^{u-2} & \xrightarrow{\mu^{u-3}_{\Sigma(T)} = \left(-\partial_T^{u-2,i} \; -h_T^{u-2,i} \right)_{i\in\mathbb{Z}}} & \mathrm{Cocone}\Big(\mu^{u-2}_{\Sigma(T)}\Big) & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle g^{u-2} = \mathrm{id}_{T^{u-2}}} & \circlearrowleft & \Big\downarrow {\scriptstyle g^{u-2} = \begin{pmatrix} \mathrm{id}_{T^{u-1,i}} & 0 \\ 0 & -\mathrm{id}_{T^{u,i-1}} \end{pmatrix}_{i\in\mathbb{Z}}} & & \\
\cdots & \xrightarrow[-\partial_T^{u-3}]{} & T^{u-2} & \xrightarrow[\mu^{u-3}_{\Sigma(X)} = -\mu^{u-2}_T = \left(-\partial_T^{u-2,i} \; h_T^{u-2,i} \right)_{i\in\mathbb{Z}}]{} & \mathrm{Cocone}\Big(\partial_T^{u-1}\Big) & \longrightarrow & 0
\end{array}
$$

induce a commutative diagram:

$$
\begin{array}{ccccccccccc}
Q^{\leq u-2}_T & \xrightarrow{Q^{\leq u-2}_{\varphi}} & Q^{\leq u-2}_S & \xrightarrow{Q^{\leq u-2}_{\iota(\varphi)}} & Q^{\leq u-2}_{\mathrm{Cone}(\varphi)} & \xrightarrow{Q^{\leq u-2}_{\pi(\varphi)}} & Q^{\leq u-2}_{\Sigma(T)} & \xrightarrow{Q^{\leq u-2}_{\epsilon_{\Sigma(T)}}} & Q^{\leq u-2}_{\Sigma(T)^{\ominus}} \\
\Vert & \circlearrowleft & \Vert & \circlearrowleft & \sim \Big\downarrow f & \circlearrowleft & \sim \Big\downarrow g & \circlearrowleft & \Vert \\
Q^{\leq u-2}_X & \xrightarrow[Q^{\leq u-2}_{\psi}]{} & Q^{\leq u-2}_Y & \xrightarrow[Q^{\leq u-2}_{\iota(\psi)}]{} & Q^{\leq u-2}_{\mathrm{Cone}(\psi)} & \xrightarrow[Q^{\leq u-2}_{\pi(\psi)}]{} & Q^{\leq u-2}_{\Sigma(X)} & \xrightarrow[Q^{\leq u-2}_{\epsilon_{\Sigma(X)}}]{} & Q^{\leq u-2}_{\Sigma(X)^{\ominus}}
\end{array}
$$

Hence, by repeatedly applying Algorithms 4 and 5, we get a commutative diagram of standard Postnikov systems:

$$
\begin{array}{ccccccccc}
Q_T & \xrightarrow{Q_{\varphi}} & Q_S & \xrightarrow{Q_{\iota(\varphi)}} & Q_{\mathrm{Cone}(\varphi)} & \xrightarrow{Q_{\pi(\varphi)}} & Q_{\Sigma(T)} & \xrightarrow{Q_{\epsilon_{\Sigma(T)}}} & Q_{\Sigma(T)^{\ominus}} \\
\Vert & \circlearrowleft & \Vert & \circlearrowleft & \sim \Big\downarrow Q_f & \circlearrowleft & \sim \Big\downarrow Q_g & \circlearrowleft & \Vert \\
Q_X & \xrightarrow[Q_{\psi}]{} & Q_Y & \xrightarrow[Q_{\iota(\psi)}]{} & Q_{\mathrm{Cone}(\psi)} & \xrightarrow[Q_{\pi(\psi)}]{} & Q_{\Sigma(X)} & \xrightarrow[Q_{\epsilon_{\Sigma(X)}}]{} & Q_{\Sigma(X)^{\ominus}}
\end{array}
$$

By taking convolutions, we get isomorphic triangles one of which is exact by the induction hypothesis, hence so is the other as desired. □

Construction 6.62. Let T be an object in $\mathcal{K}^b\Big(\mathcal{K}^b(\mathscr{C})\Big)$ and let ℓ be a lower bound for T. Let $\lceil T^\ell \rfloor_{\ell+1}$ the $(\ell+1)$-stalk complex defined by T^ℓ and let $T^{\geq \ell+1}$ be the brutal truncation of T below $\ell+1$. We denote by $\tau_{T,\ell}$ the morphism

$$
\begin{array}{ccccccccccc}
\lceil T^\ell \rfloor_{\ell+1} \coloneqq & 0 & \longrightarrow & T^\ell & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\Big\downarrow {\scriptstyle \tau_{T,\ell}} & & & \Big\downarrow {\scriptstyle \partial_T^\ell} & & & & & & \\
T^{\geq \ell+1} \coloneqq & 0 & \longrightarrow & T^{\ell+1} & \xrightarrow{\partial_T^{\ell+1}} & T^{\ell+2} & \xrightarrow{\partial_T^{\ell+2}} & T^{\ell+3} & \longrightarrow & \cdots
\end{array}
$$

It follows immediately that $\mathrm{Cone}(\tau_{T,\ell}) = T$.

Theorem 6.63. *Let $\mathscr{C}$ be an additive category and $\mathcal{K}^b(\mathscr{C})$ its bounded homotopy category. Let $\mathscr{D}$ be an additive full subcategory with $\mathrm{Hom}_{\mathscr{D}}(\Sigma^r(X), Y) = 0$ for all $r > 0$ and X, Y in $\mathscr{D}$. Then*

$$\mathbf{F}\colon \mathcal{K}^b(\mathscr{D}) \to \mathcal{K}^b(\mathscr{C})$$

is exact and fully faithful.

PROOF. The exactness follows by Lemma 6.61. The functor is fully faithful if and only if

$$\mathbf{F}\colon \operatorname{Hom}_{\mathcal{K}^b(\mathscr{D})}(T,S) \to \operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(\mathbf{F}(T),\mathbf{F}(S))$$

is an isomorphism for all objects T and S in $\mathcal{K}^b(\mathscr{D})$. We proceed by induction on $N = u_T - \ell_T + u_S - \ell_S$ where ℓ_T, ℓ_S, u_T and u_S are lower resp. upper bounds for T and S. If $N = 0$, then $\ell_T = u_T$ and $\ell_S = u_S$. This means $\mathbf{F}(T) = \Sigma^{-\ell_T}\left(T^{\ell_T}\right)$ and $\mathbf{F}(S) = \Sigma^{-\ell_S}\left(S^{\ell_S}\right)$. In the case $\ell_T = \ell_S$, the assertion follows by the fact that the shift functor Σ is an autoequivalence. Assume $\ell_T \neq \ell_S$. Then $\operatorname{Hom}_{\mathcal{K}^b(\mathscr{D})}(T,S) = 0$ because their common object-support is empty. On the other hand, $\operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(\mathbf{F}(T),\mathbf{F}(S)) \cong \operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}\left(\Sigma^{\ell_S-\ell_T}\left(T^{\ell_T}\right), S^{\ell_S}\right) = 0$, since T^ℓ and B^ℓ belong to $\mathscr{D}$.

Suppose that $N > 0$, then either $u_T - \ell_T > 0$ or $u_S - \ell_S > 0$. If $u_T - \ell_T > 0$, then by Construction 6.62, we can create a standard exact triangle

$$T_1 \xrightarrow{\tau} T_2 \xrightarrow{\iota(\tau)} T \xrightarrow{\pi(\tau)} \Sigma(T_1)$$

such that $u_{T_1} - \ell_{T_1} = 0$ and $u_{T_2} - \ell_{T_2} = u_T - \ell_T - 1$. The rotation of the above exact triangle is

$$T_2 \xrightarrow{\iota(\tau)} T \xrightarrow{\pi(\tau)} \Sigma(T_1) \xrightarrow{-\Sigma(\tau)} \Sigma(T_2),$$

hence the triangle

$$T_2 \xrightarrow{\iota(\tau)} T \xrightarrow{\pi(\tau)\bullet\epsilon_{\Sigma(T_1)}} \Sigma(T_1)^{\ominus} \xrightarrow{\epsilon^{-1}_{\Sigma(T_1)}\bullet(-\Sigma(\tau))} \Sigma(T_2)$$

is exact as well. These data incorporates into the following commutative diagram:

$$
\begin{array}{ccccc}
\operatorname{Hom}_{\mathcal{K}^b(\mathscr{D})}(T_1,S) & = & \operatorname{Hom}_{\mathcal{K}^b(\mathscr{D})}(T_1,S) & \xrightarrow[\sim]{\mathbf{F}} & \operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(\mathbf{F}(T_1),\mathbf{F}(S)) \\
\uparrow {\scriptstyle \varphi\bullet -} & & \uparrow {\scriptstyle \varphi\bullet -} & & \uparrow {\scriptstyle \mathbf{F}(\varphi)\bullet -} \\
\operatorname{Hom}_{\mathcal{K}^b(\mathscr{D})}(T_2,S) & = & \operatorname{Hom}_{\mathcal{K}^b(\mathscr{D})}(T_2,S) & \xrightarrow[\sim]{\mathbf{F}} & \operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(\mathbf{F}(T_2),\mathbf{F}(S)) \\
\uparrow {\scriptstyle \iota(\varphi)\bullet -} & & \uparrow {\scriptstyle \iota(\varphi)\bullet -} & & \uparrow {\scriptstyle \mathbf{F}(\iota(\varphi))\bullet -} \\
\operatorname{Hom}_{\mathcal{K}^b(\mathscr{D})}(T,S) & = & \operatorname{Hom}_{\mathcal{K}^b(\mathscr{D})}(T,S) & \xrightarrow{\mathbf{F}} & \operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(\mathbf{F}(T),\mathbf{F}(S)) \\
\uparrow {\scriptstyle \pi(\varphi)\bullet -} & & \uparrow {\scriptstyle \pi(\varphi)\bullet\epsilon_{\Sigma(T_1)}\bullet -} & & \uparrow {\scriptstyle \mathbf{F}\left(\pi(\varphi)\bullet\epsilon_{\Sigma(T_1)}\right)\bullet -} \\
\operatorname{Hom}_{\mathcal{K}^b(\mathscr{D})}(\Sigma(T_1),S) & \xleftarrow[\sim]{\epsilon_{\Sigma(T_1)}\bullet -} & \operatorname{Hom}_{\mathcal{K}^b(\mathscr{D})}\left((\Sigma(T_1))^{\ominus},S\right) & \xrightarrow[\sim]{\mathbf{F}} & \operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}\left(\mathbf{F}\left((\Sigma(T_1))^{\ominus}\right),\mathbf{F}(S)\right) \\
\uparrow {\scriptstyle -\Sigma(\varphi)\bullet -} & & \uparrow {\scriptstyle \epsilon^{-1}_{\Sigma(T_1)}\bullet(-\Sigma(\varphi))\bullet\epsilon_{\Sigma(T_2)}\bullet -} & & \uparrow {\scriptstyle \mathbf{F}\left(\epsilon^{-1}_{\Sigma(T_1)}\bullet(-\Sigma(\varphi))\bullet\epsilon_{\Sigma(T_2)}\right)\bullet -} \\
\operatorname{Hom}_{\mathcal{K}^b(\mathscr{D})}(\Sigma(T_2),S) & \xleftarrow[\sim]{\epsilon_{\Sigma(T_2)}\bullet -} & \operatorname{Hom}_{\mathcal{K}^b(\mathscr{D})}\left((\Sigma(T_2))^{\ominus},S\right) & \xrightarrow[\sim]{\mathbf{F}} & \operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}\left(\mathbf{F}\left((\Sigma(T_2))^{\ominus}\right),\mathbf{F}(S)\right)
\end{array}
$$

Since $\operatorname{Hom}_{\mathcal{K}^b(\mathscr{D})}(-,S)$ is a cohomological functor, the left hand side column is exact; hence, so is the isomorphic middle column. The right hand side column is exact as well because $\mathbf{F}$ is exact by the first assertion. By the induction hypothesis, the right upper and lower two morphisms are isomorphisms. Hence, by the 5-Lemma, the right middle morphism is also an isomorphism. The same trick can be used for the remaining case, i.e., when $u_T - \ell_T = 0$ and $u_S - \ell_S > 0$. □

Corollary 6.64. *Let k be a field, $\mathscr{C}$ a k-linear* Hom*-finite additive category. Let $\mathscr{E}$ a strong exceptional sequence in $\mathcal{K}^b(\mathscr{C})$ and $\mathscr{E}^\oplus$ the additive closure of $\mathscr{E}$. Then the convolution functor*

$$\mathbf{F}\colon \mathcal{K}^b(\mathscr{E}^\oplus) \to \mathcal{K}^b(\mathscr{C}).$$

is fully faithful and exact.

6.4. The Replacement Functor G

Let k be a field, $\mathscr{C}$ a k-linear additive Hom-finite category and $\mathscr{E} = (E_i \mid i = 1, \dots, n)$ a (complete) strong exceptional sequence in $\mathcal{K}^b(\mathscr{C})$. This section is devoted to constructing a right adjoint $\mathbf{G}$ to the convolution functor $\mathbf{F}$ introduced in Theorem 6.63. The constructions and proofs in this section are inspired by the theory of derived tilting equivalences [**Ric89**], [**KZ98**].

In order to construct this functor we need the concept of $\mathscr{E}$-approximations of objects in $\mathcal{K}^b(\mathscr{C})$.

Definition 6.65. An $\mathscr{E}$-**approximation** of an object A in $\mathcal{K}^b(\mathscr{C})$ consists of an object $P_{A,\mathscr{E}}$ in the the image of the embedding $\mathscr{E}^\oplus \hookrightarrow \mathcal{K}^b(\mathscr{C})$ and a morphism $\pi_{A,\mathscr{E}}\colon P_{A,\mathscr{E}} \to A$ such that the map

$$\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_\mathscr{E}, \pi_{A,\mathscr{E}})\colon \begin{cases} \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_\mathscr{E}, P_{A,\mathscr{E}}) & \to \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_\mathscr{E}, A), \\ \qquad f & \mapsto f \cdot \pi_{A,\mathscr{E}} \end{cases}$$

is surjective in the Abelian category $\mathbf{A}_\mathscr{E}$**-mod** $\simeq \mathrm{End}\, T_\mathscr{E}$**-mod**. An $\mathscr{E}$-approximation is called $\mathscr{E}$-**cover** if $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_\mathscr{E}, \pi_{A,\mathscr{E}})$ is a projective cover[10] for the left $\mathrm{End}\, T_\mathscr{E}$-module $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_\mathscr{E}, A)$ (cf. Lemma 6.35).

A detailed construction of the functor

$$\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_\mathscr{E}, -)\colon \mathcal{K}^b(\mathscr{C}) \to \mathbf{A}_\mathscr{E}\textbf{-mod}$$

is given in Remark 6.36. We prove in Remark 6.71 that computing $\mathscr{E}$-covers in $\mathcal{K}^b(\mathscr{C})$ amounts to computing projective covers in $\mathbf{A}_\mathscr{E}$**-mod** which is easy due to Theorem 2.95.

Example 6.66. Let A be an object in $\mathcal{K}^b(\mathscr{C})$ and $\{f_1, \dots, f_m\}$ a basis of of the k-vector space $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_\mathscr{E}, A)$. Then the morphism

$$T_\mathscr{E}^m \xrightarrow{\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}} A$$

is an $\mathscr{E}$-approximation of A.

We will see that computing $\mathbf{G}(A)$ for an object A in $\mathcal{K}^b(\mathscr{C})$ is based on an iterative construction which starts with the input

$$A' \coloneqq \Sigma^{u_{A,\mathscr{E}}}(A),$$

and each iteration returns an intermediate value

$$A' \coloneqq \Sigma^{-1}(\mathrm{Cone}(\pi_{A',\mathscr{E}}))$$

[10]By Lemma 6.35, $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_\mathscr{E}, P_{A,\mathscr{E}})$ is a projective object in $\mathbf{A}_\mathscr{E}$**-mod** $\simeq \mathrm{End}\, T_\mathscr{E}$**-mod**.

for some $\mathscr{E}$-cover $\pi_{A',\mathscr{E}}$ of A'. The iteration terminates as we eventually get an object A' whose set of $\mathscr{E}$-exceptional shifts is empty (cf. Definition 6.67 and Construction 6.72).

It turns out that A belongs to $\mathscr{E}^{\triangle} \subseteq \mathcal{K}^b(\mathscr{C})$ if and only if the last obtained A' (while computing $\mathbf{G}(A)$) is zero, i.e., if and only if A' is a contractible complex over $\mathscr{C}$. This observation is algorithmically very important, because if $\mathcal{K}^b(\mathscr{C})$ is generated as a triangulated category by a finite set of objects $\mathcal{B} = \{B_1, \ldots, B_m\}$, then checking $\mathcal{B} \subset \mathscr{E}^{\triangle}$ enables us to decide whether $\mathscr{E}$ is complete or not. Checking whether $\mathscr{E}$ is strong exceptional is also straightforward if $\mathscr{C}$, and consequently $\mathcal{K}^b(\mathscr{C})$, is equipped with a ($k$-**mat**)-homomorphism structure (cf. Chapter 4).

Definition 6.67. The set of **$\mathscr{E}$-exceptional shifts** of an object A in $\mathcal{K}^b(\mathscr{C})$ is defined by

$$\Omega_{A,\mathscr{E}} \coloneqq \{i \in \mathbb{Z} \mid \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}\Big(T_{\mathscr{E}}, \Sigma^i(A)\Big) \neq 0\}.$$

The **maximal $\mathscr{E}$-exceptional shift** of A, denoted by $u_{A,\mathscr{E}}$, is defined by $\max \Omega_{A,\mathscr{E}}$ if $\Omega_{A,\mathscr{E}} \neq \emptyset$, and by $-\infty$ otherwise. Analogously, the **minimal $\mathscr{E}$-exceptional shift** of A, denoted by $\ell_{A,\mathscr{E}}$, is defined by $\min \Omega_{A,\mathscr{E}}$ if $\Omega_{A,\mathscr{E}} \neq \emptyset$, and by $+\infty$ otherwise.

Example 6.68.
(1) Since $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^r(T_{\mathscr{E}})) = 0$ for all $r \neq 0$, we have $u_{\Sigma^r(T_{\mathscr{E}}),\mathscr{E}} = \ell_{\Sigma^r(T_{\mathscr{E}}),\mathscr{E}} = -r$.
(2) Since $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^r(0)) = 0$ for all $r \in \mathbb{Z}$, we have $u_{0,\mathscr{E}} = -\infty$ and $\ell_{0,\mathscr{E}} = +\infty$.
(3) For any object A, we have $\ell_{\Sigma^r(A),\mathscr{E}} = \ell_{A,\mathscr{E}} - r$ and $u_{\Sigma^r(A),\mathscr{E}} = u_{A,\mathscr{E}} - r$ for all $r \in \mathbb{Z}$.

Lemma 6.69. *Let A be an object in $\mathcal{K}^b(\mathscr{C})$, ℓ_A a lower bound of A and u_A an upper bound of A. Then*

$$\ell_A - u_{T_{\mathscr{E}}} \leq i \leq u_A - \ell_{T_{\mathscr{E}}}$$

for all $i \in \Omega_{A,\mathscr{E}}$.

PROOF. There exists a nonzero morphism between two objects in $\mathcal{K}^b(\mathscr{C})$ only if their supports overlap. Hence, for $i \in \mathbb{Z}$, $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^i(A)) \neq 0$ only if $u_{T_{\mathscr{E}}} \geq \ell_{\Sigma^i(A)}$ and $\ell_{T_{\mathscr{E}}} \leq u_{\Sigma^i(A)}$, i.e., $u_{T_{\mathscr{E}}} \geq \ell_A - i$ and $\ell_{T_{\mathscr{E}}} \leq u_A - i$, hence $\ell_A - u_{T_{\mathscr{E}}} \leq i \leq u_A - \ell_{T_{\mathscr{E}}}$. □

The following corollary highlights the relation between the exceptional shifts of an object in $\mathscr{E}^{\triangle}$ and the property of being isomorphic to the zero object.

Corollary 6.70. *Let A be an object in the triangulated hull $\mathscr{E}^{\triangle}$. Then the following statements are equivalent:*

(1) $A \ncong 0$,
(2) $u_{A,\mathscr{E}} \neq -\infty$,
(3) $\ell_{A,\mathscr{E}} \neq +\infty$,
(4) $\ell_A - u_{T_{\mathscr{E}}} \leq \ell_{A,\mathscr{E}} \leq u_{A,\mathscr{E}} \leq u_A - \ell_{T_{\mathscr{E}}}$.

PROOF. By Example 6.18, $T_{\mathscr{E}}$ is a classical generator for $\mathscr{E}^{\triangle}$, thus a weak generator by Corollary 6.17. This implies that $A \ncong 0$ if and only if there exists an integer $i \in \mathbb{Z}$ with $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^i(A)) \neq 0$. The assertions follow by Lemma 6.69. □

For each object E_i in $\mathscr{E}$, we denote by π_i the natural projection $T_{\mathscr{E}} \twoheadrightarrow E_i$. The following remark enables us to compute better $\mathscr{E}$-approximations:

Remark 6.71. Let A be an object in $\mathcal{K}^b(\mathscr{C})$ and

$$\{f_j\colon E_{i_j} \to A, j = 1, \ldots, m\} \subset \bigcup_{i=1}^{n} \operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(E_i, A)$$

a generating set of $\operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, A) \cong \bigoplus_{i=1}^{n} \operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(E_i, A)$ as a left $\operatorname{End} T_{\mathscr{E}}$-module, i.e., for every morphism $g\colon T_{\mathscr{E}} \to A$, there exist endomorphisms $\lambda_j\colon T_{\mathscr{E}} \to T_{\mathscr{E}}, j = 1, \ldots, m$ such that $g = \sum_{j=1}^{m} \lambda_j \cdot \left(\pi_{i_j} \cdot f_j\right) = \sum_{j=1}^{m} \lambda_j \cdot \pi_{i_j} \cdot f_j$. Then applying the functor $\operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, -)$ on the composition

$$T_{\mathscr{E}}^m \xrightarrow{\pi := \begin{pmatrix} \pi_{i_1} & & \\ & \ddots & \\ & & \pi_{i_m} \end{pmatrix}} \bigoplus_{j=1}^{m} E_{i_j} \xrightarrow{f := \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}} A,$$

yields a surjection $\operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \pi \cdot f)$, i.e., $\pi \cdot f\colon T_{\mathscr{E}}^m \to A$ is an $\mathscr{E}$-approximation of U. Since π is a split-epimorphism, $\operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \pi)$ is also a split-epimorphism, thus surjective. The surjectivity of $\operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \pi \cdot f)$ and $\operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \pi)$ implies the surjectivity of $\operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, f)$, i.e., f is also an $\mathscr{E}$-approximation.

The above discussion gives rise to an algorithm for computing an $\mathscr{E}$-cover of the object A. According to Lemma 6.35, $F := \operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, A)$ can be considered as an object in

$$\operatorname{End} T_{\mathscr{E}}\text{-}\mathbf{mod} \simeq \mathbf{A}_{\mathscr{E}}\text{-}\mathbf{mod} := [\mathbf{A}_{\mathscr{E}}^{\mathrm{op}}, k\text{-}\mathbf{mat}]$$

which is Abelian and has computable projective covers (cf. Theorem 2.95). Let

$$\lambda_F\colon \bigoplus_{i=1}^{n} P_{v_i}^{m_i} \twoheadrightarrow F$$

be a projective cover in $\mathbf{A}_{\mathscr{E}}$-**mod** of F where v_i's are the objects of $\mathbf{A}_{\mathscr{E}}^{\mathrm{op}}$. The morphism λ_F can be used to obtain a minimal generating set of F by simply applying λ_F to *the* generator of each P_{v_i} that appears in $\bigoplus_{i=1}^{n} P_{v_i}^{m_i}$. In particular, for each object $v_i \in \mathbf{A}_{\mathscr{E}}^{\mathrm{op}}$, we get m_i elements in $F(v_i) \cong \operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(E_i, A)$. This yields an $\mathscr{E}$-approximation

$$f\colon \bigoplus_{i=1}^{n} E_i^{m_i} \to A$$

of A. Applying the functor $\operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, -)$ on f yields the epimorphism

$$\operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, f)\colon \bigoplus_{i=1}^{n} P_{v_i}^{m_i} \twoheadrightarrow F.$$

Since λ_F is a projective cover, the lift morphism, say $\tau \in \operatorname{End} \bigoplus_{i=1}^{n} P_{v_i}^{m_i}$, of $\operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, f)$ along λ_F is an epimorphism. On the other hand, τ is an endomorphism of a finite dimensional k-vector space, hence τ is an isomorphism. This means $\operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, f)$ is a projective cover, hence f defines an $\mathscr{E}$-cover of U.

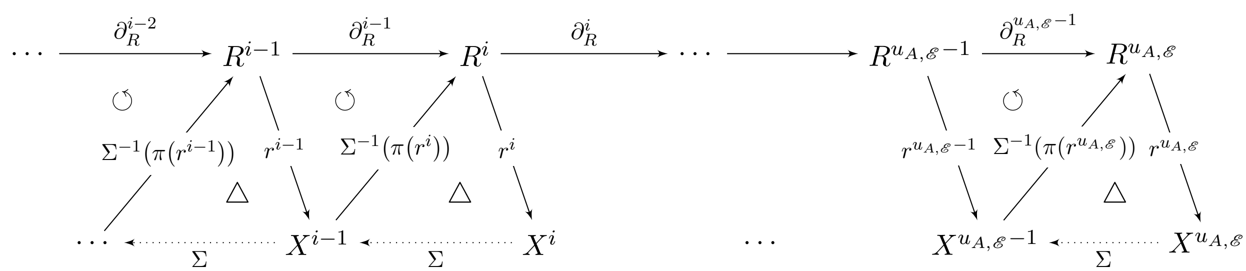
Construction 6.72. For an object A in $\mathcal{K}^b(\mathscr{C})$, we recursively define the sequences $\left(X^i\right)_{i \le u_{A,\mathscr{E}}}$ and $\left(R^i\right)_{i \le u_{A,\mathscr{E}}}$ of objects in $\mathcal{K}^b(\mathscr{C})$, together with a sequence of exact triangles

$$\left(X^{i-1} \xrightarrow{\Sigma^{-1}\left(\pi\left(r^i\right)\right)} R^i \xrightarrow{r^i} X^i \xrightarrow{-\iota\left(r^i\right)} \Sigma\left(X^{i-1}\right)\right)_{i \in \mathbb{Z}}$$

according to the following steps:

(1) Compute the maximal $\mathscr{E}$-exceptional shift $u_{A,\mathscr{E}} \in \mathbb{Z} \cup \{-\infty\}$.
(2) For each $i \in \mathbb{Z}$ define $r^i \colon R^i \to X^i$ as follows:
- if $i > u_{A,\mathscr{E}}$, then
  - define $X^i \coloneqq \Sigma^i(A)$, $R^i \coloneqq 0$ and $r^i \coloneqq R^i \xrightarrow{0} X^i$
- if $i \leq u_{A,\mathscr{E}}$, then
  - define $X^i \coloneqq \mathrm{Cocone}(r^{i+1})$,
  - if $\Omega_{X^i,\mathscr{E}} = \emptyset$, then terminate the computation,
  - otherwise, compute an $\mathscr{E}$-approximation $\pi_{X^i,\mathscr{E}} \colon P_{X^i,\mathscr{E}} \to X^i$ for X^i,
  - define $R^i \coloneqq P_{X^i,\mathscr{E}}$ and $r^i \coloneqq \pi_{X^i,\mathscr{E}}$.

We use these sequences to construct a complex R whose upper bound is $u_{A,\mathscr{E}}$ and whose differential at index $i \in \mathbb{Z}$ is $\partial_R^i \coloneqq r^i \bullet \Sigma^{-1}\big(\pi(r^{i+1})\big)$. The complex R is called an $\mathscr{E}$**-replacement** of A (or $\mathscr{E}$**-resolution**). It is called **minimal** $\mathscr{E}$**-replacement** if r^i is an $\mathscr{E}$-cover of X^i for all $i \in \mathbb{Z}$. The computation steps can be depicted in the following diagram:



The following lemma states that the maximal $\mathscr{E}$-shifts of all X^i's in an $\mathscr{E}$-replacement of an nonzero object A are non-positive.

Lemma 6.73. *Let $A \ncong 0$ be an object $\mathcal{K}^b(\mathscr{C})$ and R an $\mathscr{E}$-replacement of A. We have $u_{X^i,\mathscr{E}} \leq 0$ for all $i \leq u_{A,\mathscr{E}}$.*

Proof. We will use backward induction on $i \leq u_{A,\mathscr{E}}$. For $i = u_{A,\mathscr{E}}$, we have $X^{u_{A,\mathscr{E}}} \coloneqq \Sigma^{u_{A,\mathscr{E}}}(A)$, i.e., $u_{X^i,\mathscr{E}} = 0$. Now, suppose the assertion holds for some $i \leq u_{A,\mathscr{E}}$, and let us show it holds for $i-1$. By the construction of an $\mathscr{E}$-replacement, we have an exact triangle

$$X^{i-1} \to R^i \xrightarrow{r^i} X^i \to \Sigma\Big(X^{i-1}\Big)$$

where r^i is an $\mathscr{E}$-approximation of X^i, i.e., $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, r^i) = - \bullet r^i$ is surjective.

Since $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, -)$ is a cohomological functor, we get a long exact sequence:

$$\cdots \longrightarrow \overbrace{\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^{-1}(R^i))}^{=0} \xrightarrow{-\bullet(-\Sigma^{-1}(r^i))} \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^{-1}(X^i))$$

$$\longrightarrow \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, X^{i-1}) \longrightarrow \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, R^i) \xrightarrow{-\bullet r^i} \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, X^i)$$

$$\longrightarrow \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^1(X^{i-1})) \longrightarrow \underbrace{\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^1(R^i))}_{=0} \xrightarrow{-\bullet(-\Sigma^1(r^i))} \underbrace{\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^1(X^i))}_{=0}$$

$$\longrightarrow \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^2(X^{i-1})) \longrightarrow \underbrace{\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^2(R^i))}_{=0} \xrightarrow[-\bullet\Sigma^2(r^i)]{} \underbrace{\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^2(X^i))}_{=0},$$

in which the zeros in the right column are due to the induction hypothesis; and the zeros in middle column are due to the fact that $T_{\mathscr{E}}$ is the tilting object associated to a strong exceptional sequence. Since the sequence is exact and $-\bullet r^i$ is surjective, we get $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^n(X^{i-1})) = 0$ for all $n \geq 1$. Consequently $u_{X^{i-1},\mathscr{E}} \leq 0$ as desired. □

While the previous lemma investigates the behavior of the maximal $\mathscr{E}$-exceptional shifts of X^i's in an $\mathscr{E}$-replacement of A, the following lemma investigates the behavior of their minimal $\mathscr{E}$-exceptional shifts. The lemma asserts that these shifts increase with each iteration until they become 0 after a finite number of iterations. Of course, these shifts can not exceed 0 as long as $X^i \ncong 0$ (cf. Lemma 6.73).

Lemma 6.74. *Let $A \ncong 0$ be an object $\mathcal{K}^b(\mathscr{C})$ and R an $\mathscr{E}$-replacement of A. For all $i \leq u_{A,\mathscr{E}}$, if $\ell_{X^i,\mathscr{E}} < 0$, then $\ell_{X^i,\mathscr{E}} < \ell_{X^{i-1},\mathscr{E}}$.*

Proof. Analogously to Lemma 6.73, we can create a long exact sequence:

$$\cdots \longrightarrow \overbrace{\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^{\ell_{X^i,\mathscr{E}}-2}(R^i))}^{=0} \longrightarrow \overbrace{\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^{\ell_{X^i,\mathscr{E}}-2}(X^i))}^{=0}$$

$$\longrightarrow \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^{\ell_{X^i,\mathscr{E}}-1}(X^{i-1})) \longrightarrow \underbrace{\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^{\ell_{X^i,\mathscr{E}}-1}(R^i))}_{=0} \longrightarrow \underbrace{\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^{\ell_{X^i,\mathscr{E}}-1}(X^i))}_{=0}$$

$$\longrightarrow \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^{\ell_{X^i,\mathscr{E}}}(X^{i-1})) \longrightarrow \underbrace{\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^{\ell_{X^i,\mathscr{E}}}(R^i))}_{=0} \longrightarrow \underbrace{\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^{\ell_{X^i,\mathscr{E}}}(X^i))}_{\neq 0};$$

in which the zeros in the right column are due to the assumption of the lemma; and the zeros in the middle column are due to the fact that $T_{\mathscr{E}}$ is a tilting object associated to a strong exceptional sequence. Since the sequence is exact, it follows easily that $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^n(X^{i-1})) = 0$ for all $n \leq \ell_{X^i,\mathscr{E}}$. Consequently, $\ell_{X^i,\mathscr{E}} < \ell_{X^{i-1},\mathscr{E}}$ as desired. □

Lemma 6.75. *Let $A \ncong 0$ be an object $\mathcal{K}^b(\mathscr{C})$ and R an $\mathscr{E}$-replacement of A. For all $i \leq u_{A,\mathscr{E}}$, if $\ell_{X^i,\mathscr{E}} = 0$, then*

(1) $\ell_{X^{i-1},\mathscr{E}} = 0$ or $\ell_{X^{i-1},\mathscr{E}} = +\infty$.

(2) Applying the functor $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, -)$ on $X^{i-1} \xrightarrow{\Sigma^{-1}(\pi(r^i))} R^i \xrightarrow{r^i} X^i$ yields a short exact sequence of left $\mathrm{End}\, T_{\mathscr{E}}$-modules

$$0 \to \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, X^{i-1}) \xhookrightarrow{-\bullet\Sigma^{-1}(\pi(r^i))} \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, R^i) \xrightarrow{-\bullet r^i} \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, X^i) \to 0.$$

(3) The morphism

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{-\bullet\partial_R^{i-3}} & \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, R^{i-2}) & \xrightarrow{-\bullet\partial_R^{i-2}} & \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, R^{i-1}) & \xrightarrow{-\bullet\partial_R^{i-1}} & \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, R^{i}) & \longrightarrow & 0 \\
& & & & & \downarrow{\scriptstyle -\bullet r^i} & & \\
& \cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, X^{i}) & \longrightarrow & 0
\end{array}$$

is a quasi-isomorphism, i.e., a projective resolution of $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, X^i)$. Furthermore, if R is a minimal $\mathscr{E}$-resolution for A, then the above induced projective resolution is also minimal.

Proof. Analogously to Lemma 6.73, we can create a long exact sequence:

$$\begin{array}{ccccc}
\cdots & \longrightarrow & \overbrace{\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^{-1}(R^i))}^{=0} & \longrightarrow & \overbrace{\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^{-1}(X^i))}^{=0} \\
\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, X^{i-1}) & \xhookrightarrow{-\bullet\Sigma^{-1}(\pi(r^i))} & \underbrace{\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, R^i)}_{\neq 0} & \xrightarrow{-\bullet r^i} & \underbrace{\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, X^i)}_{\neq 0} \\
\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma(X^{i-1})) & \longrightarrow & \underbrace{\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma(R^i))}_{=0} & \longrightarrow & \underbrace{\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma(X^i))}_{=0}
\end{array}$$

in which the zeros in the right column are due to the assumption $\ell_{X^i,\mathscr{E}} = 0$ and Lemma 6.73; and the zeros in the middle column are due to the fact that $T_{\mathscr{E}}$ is a tilting object associated to a strong exceptional sequence. Hence, $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_{\mathscr{E}}, \Sigma^n(X^{i-1})) = 0$ for all $n \neq 0$, which implies the assertions (1) and (2).

In order to prove the third assertion, we need to show that $-\bullet r^i$ is a cokernel projection for $-\bullet\partial_R^{i-1}$ and for each $j \leq i-2$, the unique lift morphism of $-\bullet\partial_R^{j}$ along the kernel embedding of $-\bullet\partial_R^{j+1}$ is an epimorphism (See Remark 3.7). By definition $\partial_R^{i-1} = r^{i-1}\bullet\Sigma^{-1}(\pi(r^i))$, hence $-\bullet\partial_R^{i-1} = (-\bullet r^{i-1})\bullet(-\bullet\Sigma^{-1}(\pi(r^i)))$. It follows from the second assertion that $-\bullet r^{i-1}$ is an epimorphism and $-\bullet r^i$ is a cokernel projection for $-\bullet\Sigma^{-1}(\pi(r^i))$, hence $-\bullet r^i$ is a cokernel projection of $-\bullet\partial_R^{i-1}$ as well. Similarly, for each $j \leq i-2$, the morphism $-\bullet\Sigma^{-1}(\pi(r^{j+2}))$ is a monomorphism and $-\bullet\Sigma^{-1}(\pi(r^{j+1}))$ is a kernel embedding of $-\bullet r^{j+1}$, hence $-\bullet\Sigma^{-1}(\pi(r^{j+1}))$ is a kernel embedding for $-\bullet\partial_R^{j+1}$ as well. This means $-\bullet\partial_R^j$ lifts along the kernel embedding of $-\bullet\partial_R^j$ via the epimorphism $-\bullet r^j$, i.e., the morphism is indeed a quasi-isomorphism resp. a projective resolution. If R is a minimal $\mathscr{E}$-resolution of A, then each $-\bullet r^j$ is a projective cover for X^j, hence the induced projective resolution is minimal. □

The algorithmic content of the following lemma allows us to detect whether a given object A lives in the triangulated hull $\mathscr{E}^\triangle$ of $\mathscr{E}$. In other words, computing an $\mathscr{E}$-replacement of A can be thought of as a kind of iterative reduction of A modulo $\mathscr{E}^\triangle$, such that A belongs to $\mathscr{E}^\triangle$ if and only if the remainder is zero.

Lemma 6.76. *Let A be an object in $\mathcal{K}^b(\mathscr{C})$ and let R be a minimal $\mathscr{E}$-resolution of A as introduced in Construction 6.72. Then R is bounded, i.e., R belongs to $\mathcal{K}^b(\mathscr{E}^\oplus)$. Furthermore, if ℓ is a lower bound of R, then A belongs to $\mathscr{E}^\triangle$ if and only if $X^{\ell-1} \cong 0$.*

Proof. Lemma 6.73 states that $u_{X^i,\mathscr{E}} \leq 0$ for all $i \leq u_{A,\mathscr{E}}$, i.e., for all $i \leq u_{A,\mathscr{E}}$ either $\ell_{X^i,\mathscr{E}} = +\infty$ or $\ell_{X^i,\mathscr{E}} \leq u_{X^i,\mathscr{E}} \leq 0$. On the other hand, by Lemma 6.74, we see that after at most $u_{A,\mathscr{E}} - \ell_{A,\mathscr{E}}$ iterations, we reach an integer $i \in \mathbb{Z}$ for which either $\ell_{X^i,\mathscr{E}} = +\infty$ or $\ell_{X^i,\mathscr{E}} = 0$.

Now if $\ell_{X^i,\mathscr{E}} = +\infty$, then the $\mathscr{E}$-cover r^i is given by $0 \to X^i$ and $X^{i-1} = \operatorname{Cocone}(r^i) = \Sigma^{-1}(X^i)$ whose lower $\mathscr{E}$-exceptional shift is again $+\infty$, hence by induction we find that $R^j = 0$ and $X^{j-1} = \Sigma^{-1}(X^j)$ for all $j \leq i$. Hence, R is bounded below by $i+1$.

If $\ell_{X^i,\mathscr{E}} = 0$ then by Lemma 6.75, $\ell_{X^j,\mathscr{E}} \in \{0, +\infty\}$ for all $j \leq i-1$ and we get a minimal projective resolution

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{-\bullet\partial_R^{i-3}} & \operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_\mathscr{E}, R^{i-2}) & \xrightarrow{-\bullet\partial_R^{i-2}} & \operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_\mathscr{E}, R^{i-1}) & \xrightarrow{-\bullet\partial_R^{i-1}} & \operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_\mathscr{E}, R^{i}) & \longrightarrow & 0 \\
 & & & & & & \Big\downarrow{\scriptstyle -\bullet r^i} & & \\
 & & \cdots & \longrightarrow & 0 & \longrightarrow & \operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_\mathscr{E}, X^{i}) & \longrightarrow & 0
\end{array}$$

of $\operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_\mathscr{E}, X^i)$. By Lemma 6.33 and Corollary 2.96, we see that the global dimension of $\operatorname{End} T_\mathscr{E}$-**fdmod** is finite and is bounded by the number of vertices of the quiver $\mathfrak{q}_\mathscr{E}$ of $\mathscr{E}$, which is exactly the number of objects of $\mathscr{E}$, i.e., n. By Corollary 3.58, $\operatorname{Hom}_{\mathcal{K}^b(\mathscr{C})}(T_\mathscr{E}, X^{i-n})$ is a projective object, hence isomorphic to the zero object because otherwise the above projective resolution would not be minimal. It follows that $\ell_{X^{i-n},\mathscr{E}} = +\infty$, hence similar to the above discussion we conclude that R is bounded below by $i-n+1$.

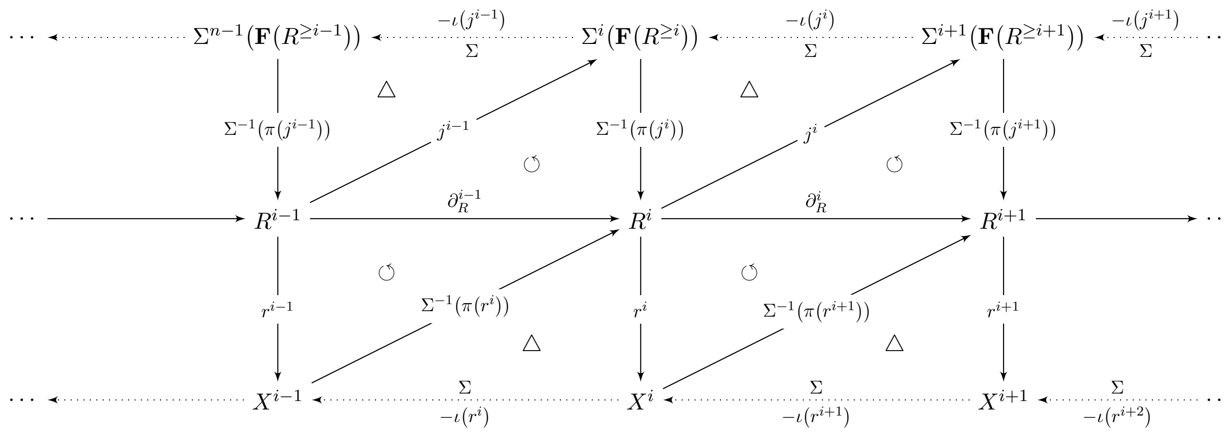
Suppose that A belongs to $\mathscr{E}^\triangle$, then $X^{u_{A,\mathscr{E}}} = \Sigma^{u_{A,\mathscr{E}}}(A)$ also belongs to $\mathscr{E}^\triangle$. Since each X^j is the cocone object of a morphism from R^{j+1} to X^{j+1}, we can inductively prove that X^j belongs to $\mathscr{E}^\triangle$ for all $j \leq u_{A,\mathscr{E}}$. Since ℓ is a lower bound of R, the $\mathscr{E}$-cover $r^{\ell-1}$ is given by $0 \to X^{\ell-1}$ and $\ell_{X^{\ell-1},\mathscr{E}} = +\infty$, hence $X^{\ell-1} \cong 0$ by Corollary 6.70. The converse statment follows by induction since $X^{\ell-1} \cong 0$ and each X^j is the cone object of a morphism from X^{j-1} to R^j. □

Lemma 6.77. *Let A be an object $\mathcal{K}^b(\mathscr{C})$ and R in $\mathcal{K}^b(\mathscr{E}^\oplus)$ be an $\mathscr{E}$-resolution of A. Then A belongs to $\mathscr{E}^\triangle$ if and only if $\mathbf{F}(R) \cong A$.*

PROOF. Suppose that A belongs to $\mathscr{E}^{\triangle}$. By Lemma 6.60, we can assume without loss of generality that $u_{A,\mathscr{E}} = 0$. We denote the brutal truncation of R below some integer $i \in \mathbb{Z}$

$$\cdots \longrightarrow 0 \longrightarrow R^i \xrightarrow[i]{\partial_R^i} R^{i+1} \xrightarrow{\partial_R^{i+1}} \cdots$$

by $R^{\geq i}$. We can depict the computation of R and $\mathbf{F}(R)$ by the following diagram:

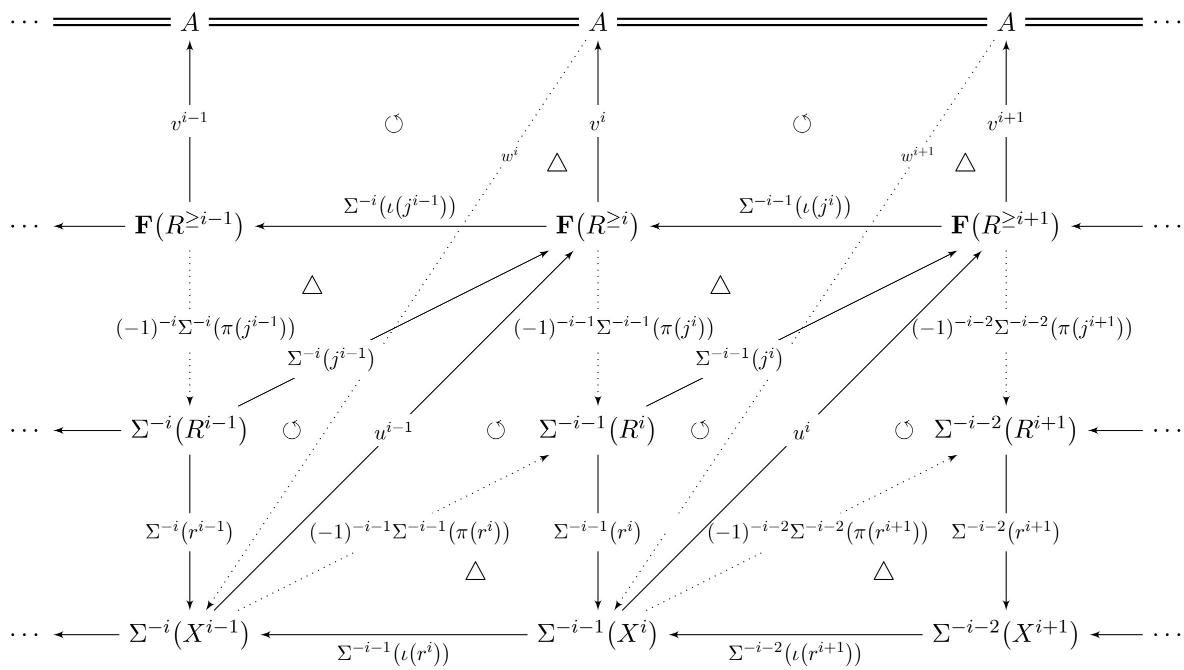


It follows from Construction 6.72 that $X^0 = A$, and if ℓ is a lower bound of R, then $X^{\ell-1} \cong 0$ by Lemma 6.76. By definition j^0 is given by the zero morphism $R^0 \to 0$, hence $\pi(j^0) = \mathrm{id}_{\Sigma(R^0)}$. We will prove the lemma by constructing a morphism $\mathbf{F}(R^{\geq i}) \xrightarrow{v^i} A$ for each $i \leq -1$, then proving that these morphisms will eventually become isomorphisms.

We claim the existence of a family of exact triangles

$$\left(\Sigma^{-i-1}\left(X^i\right) \xrightarrow{u^i} \mathbf{F}\left(R^{\geq i+1}\right) \xrightarrow{v^{i+1}} A \xrightarrow{w^{i+1}} \Sigma^{-i}\left(X^i\right)\right)_{i \leq -1}$$

that fit together into a commutative diagram:



We will construct the family by a backward induction on $i \leq -1$. For $i = -1$, we define u^{-1} by $X^{-1} \xrightarrow{\Sigma^{-1}(\pi(r^0))} R^0$, v^0 by $R^0 \xrightarrow{r^0} A$ and w^0 by $A \xrightarrow{-\iota(r^0)} \Sigma(X^{-1})$, hence

$$X^{-1} \xrightarrow{u^{-1}} R^0 \xrightarrow{v^0} A \xrightarrow{w^0} \Sigma\Big(X^{-1}\Big)$$

is exact triangle by the rotation axiom. Moreover, the asserted equalities

1. $\Sigma^{-i-1}(r^i) \bullet u^i = \Sigma^{-i-1}(j^i)$ and
2. $u^i \bullet ((-1)^{-i-2}\Sigma^{-i-2}(\pi(j^{i+1}))) = (-1)^{-i-2}\Sigma^{-i-2}(\pi(r^{i+1}))$,

for $i = -1$ hold because $\Sigma^{-1}(\pi(j^0)) = \mathrm{id}_{R^0}$ and $r^{-1} \bullet \Sigma^{-1}(\pi(r^0)) = \partial_R^{-1} = j^{-1}$.

Now, suppose we have computed the asserted exact triangle

$$\Sigma^{-i-1}\Big(X^i\Big) \xrightarrow{u^i} \mathbf{F}\Big(R^{\geq i+1}\Big) \xrightarrow{v^{i+1}} A \xrightarrow{w^{i+1}} \Sigma^{-i}\Big(X^i\Big),$$

for some $i < -1$ and let us compute the asserted exact triangle for $i - 1$.

Applying Lemma B.18 on the standard exact triangles

$$R^i \xrightarrow{r^i} X^i \xrightarrow{\iota(r^i)} \Sigma\Big(X^{i-1}\Big) \xrightarrow{\pi(r^i)} \Sigma\Big(R^i\Big)$$

and

$$R^i \xrightarrow{j^i} \Sigma^{i+1}\Big(\mathbf{F}\Big(R^{\geq i+1}\Big)\Big) \xrightarrow{\iota(j^i)} \Sigma^{i+1}\Big(\mathbf{F}\Big(R^{\geq i}\Big)\Big) \xrightarrow{\pi(j^i)} \Sigma\Big(R^i\Big)$$

yields two exact triangles

$$\Sigma^{-i-1}\Big(R^i\Big) \xrightarrow{\Sigma^{-i-1}(r^i)} \Sigma^{-i-1}\Big(X^i\Big) \xrightarrow{\Sigma^{-i-1}(\iota(r^i))} \Sigma^{-i}\Big(X^{i-1}\Big) \xrightarrow{(-1)^{-i-1}\Sigma^{-i-1}(\pi(r^i))} \Sigma^{-i-1}\Big(R^i\Big)$$

and

$$\Sigma^{-i-1}\Big(R^i\Big) \xrightarrow{\Sigma^{-i-1}(j^i)} \mathbf{F}\Big(R^{\geq i+1}\Big) \xrightarrow{\Sigma^{-i-1}(\iota(j^i))} \mathbf{F}\Big(R^{\geq i}\Big) \xrightarrow{(-1)^{-i-1}\Sigma^{-i-1}(\pi(j^i))} \Sigma^{-i}\Big(R^i\Big).$$

By the Octahedral Axiom **TR 4**, there exists an exact triangle

$$\Sigma^{-i}\Big(X^{i-1}\Big) \xrightarrow{u^{i-1}} \mathbf{F}\Big(R^{\geq i}\Big) \xrightarrow{v^i} A \xrightarrow{w^i} \Sigma^{-i+1}\Big(X^{i-1}\Big)$$

rendering the diagram

$$\begin{array}{ccccccc}
\Sigma^{-i-1}(R^i) & \xrightarrow{\Sigma^{-i-1}(r^i)} & \Sigma^{-i-1}(X^i) & \xrightarrow{\Sigma^{-i-1}(\iota(r^i))} & \Sigma^{-i}(X^{i-1}) & \xrightarrow{(-1)^{-i-1}\Sigma^{-i-1}(\pi(r^i))} & \Sigma^{-i}(R^i) \\
 & \circlearrowright & \downarrow{\scriptstyle u^i} & \circlearrowright & \downarrow{\scriptstyle \exists u^{i-1}} & \circlearrowright & \Vert \\
 & \searrow{\scriptstyle \Sigma^{-i-1}(j^i)} & \mathbf{F}(R^{\geq i+1}) & \xrightarrow{\Sigma^{-i-1}(\iota(j^i))} & \mathbf{F}(R^{\geq i}) & \xrightarrow{(-1)^{-i-1}\Sigma^{-i-1}(\pi(j^i))} & \Sigma^{-i}(R^i) \\
 & & \downarrow{\scriptstyle v^{i+1}} & \circlearrowright & \downarrow{\scriptstyle \exists v^i} & \circlearrowright & \downarrow{\scriptstyle \Sigma^{-i}(r^i)} \\
 & & A & = & A & \xrightarrow{w^{i+1}} & \Sigma^{-i}(X^i) \\
 & & \downarrow{\scriptstyle w^{i+1}} & \circlearrowright & \downarrow{\scriptstyle \exists w^i} & & \\
 & & \Sigma^{-i}(X^i) & \xrightarrow{\Sigma^{-i}(\iota(r^i))} & \Sigma^{-i+1}(X^{i-1}) & &
\end{array}$$

commutative. All claimed equalities can easily be read from the diagram except for

$$\Sigma^{-i}\Big(r^{i-1}\Big) \bullet u^{i-1} = \Sigma^{-i}\Big(j^{i-1}\Big).$$

The following computation

$$\begin{aligned}
\Sigma^{-i}\Big(r^{i-1}\Big) \bullet u^{i-1} \bullet (-1)^{-i-1}\Sigma^{-i-1}\Big(\pi\Big(j^i\Big)\Big) &= \Big(\Sigma^{-i}\Big(r^{i-1}\Big)\Big) \bullet (-1)^{-i-1}\Sigma^{-i-1}\Big(\pi\Big(r^i\Big)\Big) \\
&= (-1)^{-i-1}\Sigma^{-i}\Big(\Big(r^{i-1} \bullet \Sigma^{-1}\Big(\pi\Big(r^i\Big)\Big)\Big)\Big) \\
&= (-1)^{-i-1}\Sigma^{-i}\Big(\partial_R^{i-1}\Big) \\
&= (-1)^{-i-1}\Sigma^{-i}\Big(\Big(j^{i-1} \bullet \Sigma^{-1}\Big(\pi\Big(j^i\Big)\Big)\Big)\Big) \\
&= \Big(\Sigma^{-i}\Big(j^{i-1}\Big)\Big) \bullet (-1)^{-i-1}\Sigma^{-i-1}\Big(\pi\Big(j^i\Big)\Big)
\end{aligned}$$

shows that $\Sigma^{-i}(r^{i-1}) \bullet u^{i-1} - \Sigma^{-i}(j^{i-1})$ lies in the kernel of the morphism

$$\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma^{-i}(R^{i-1}), \mathbf{F}(R^{\geq i})) \xrightarrow{-\bullet(-1)^{-i-1}\Sigma^{-i-1}(\pi(j^i))} \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma^{-i}(R^{i-1}), \Sigma^{-i}(R^i));$$

i.e., to prove the equality, it is sufficient to show $-\bullet(-1)^{-i-1}\Sigma^{-i-1}(\pi(j^i))$ is a monomorphism. Let $\lceil R^{i-1} \rfloor_i$ be the i-stalk object in $\mathcal{K}^b(\mathscr{E}^{\oplus})$ defined by R^{i-1}, then $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{E}^{\oplus})}(\lceil R^{i-1} \rfloor_i, R^{\geq i+1}) = 0$

because the object-supports of $\lceil R^{i-1} \rfloor_i$ and $R^{\geq n+1}$ do not overlap, hence by Theorem 6.63,

$$\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma^{-i}(R^{i-1}), \mathbf{F}(R^{\geq i+1})) = \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\mathbf{F}(\lceil R^{i-1} \rfloor_i), \mathbf{F}(R^{\geq i+1})) = 0.$$

Therefore, the claim that $-\bullet(-1)^{-i-1}\Sigma^{-i-1}(\pi(j^i))$ is a monomorphism follows easily from the long exact sequence resulted by applying $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\Sigma^{-i}(R^{i-1}), -)$ on the exact triangle

$$\Sigma^{-i-1}\left(R^i\right) \xrightarrow{\Sigma^{-i-1}(j^i)} \mathbf{F}\left(R^{\geq i+1}\right) \xrightarrow{\Sigma^{-i-1}(\iota(j^i))} \mathbf{F}\left(R^{\geq i}\right) \xrightarrow{(-1)^{-i-1}\Sigma^{-i-1}(\pi(j^i))} \Sigma^{-i}\left(R^i\right).$$

Let ℓ be a lower bound of R. By Remark 6.41, $\mathbf{F}(R) = \mathbf{F}\left(R^{\geq \ell}\right)$. It follows from Lemma 6.76 that $X^{\ell-1} \cong 0$, consequently $\mathbf{F}\left(R^{\geq \ell}\right) \xrightarrow{v^\ell} A$ is an isomorphism by Lemma B.22.

Conversely, if $A \cong \mathbf{F}(R)$, then A belongs to $\mathscr{E}^\triangle$ because $\mathbf{F}(R)$ is constructed by iterated computation of cocone objects of morphisms that already belong to $\mathscr{E}^\triangle$ and by shifting the last cocone according to its cohomological index. □

Corollary 6.78. *Let A be an object in $\mathscr{E}^\triangle$, then all $\mathscr{E}$-resolutions in $\mathcal{K}^b(\mathscr{E}^\oplus)$ of A are isomorphic.*

Proof. Let R_A and R'_A be two bounded $\mathscr{E}$-resolutions for A. By the previous lemma, there we have isomorphisms $\lambda_A \colon \mathbf{F}(R_A) \to A$ and $\lambda'_A \colon \mathbf{F}(R'_A) \to A$, hence $\mathbf{F}(R_A)$ and $\mathbf{F}(R'_A)$ are isomorphic via $\varphi := \lambda_A \bullet (\lambda'_A)^{-1}$. By Theorem 6.63, $\mathbf{F}$ is fully faithful, therefore, there are two morphisms $R_A \xrightarrow{i} R'_A$ and $R'_A \xrightarrow{j} R_A$ such that $\varphi = \mathbf{F}(i)$ and $\varphi^{-1} = \mathbf{F}(j)$. Consequently, $\mathbf{F}(i \bullet j - \mathrm{id}_{R_A}) = 0$ and $\mathbf{F}\left(j \bullet i - \mathrm{id}_{R'_A}\right) = 0$, i.e., $i \bullet j = \mathrm{id}_{R_A}$ and $j \bullet i = \mathrm{id}_{R'_A}$. □

Lemma 6.79. *Let $\mathscr{E}$ be a complete strong exceptional sequence in $\mathcal{K}^b(\mathscr{C})$. Then the convolution functor $\mathbf{F} \colon \mathcal{K}^b(\mathscr{E}^\oplus) \to \mathcal{K}^b(\mathscr{C})$ has a right adjoint functor.*

Proof. For each object B in $\mathcal{K}^b(\mathscr{C})$, we fix a bounded $\mathscr{E}$-resolution R_B for B and an isomorphism $\lambda_B \colon \mathbf{F}(R_B) \to B$; and for each object Q in $\mathcal{K}^b(\mathscr{E}^\oplus)$, we denote by $\Phi_{Q,B}$ the composition of the two isomorphisms

$$\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\mathbf{F}(Q), B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\mathbf{F}(Q), \mathbf{F}(R_B)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{K}^b(\mathscr{E}^\oplus)}(Q, R_B),$$

where the first isomorphism is given by $- \bullet \lambda_B^{-1}$ and the second follows from the fact that $\mathbf{F}$ is fully faithful; hence, $\Phi_{Q,B}(\varphi) = \psi$ if and only if $\mathbf{F}(\psi) = \varphi \bullet \lambda_B^{-1}$.

We define the **Replacement functor**

$$\mathbf{G} \colon \mathcal{K}^b(\mathscr{C}) \to \mathcal{K}^b(\mathscr{E}^\oplus)$$

as follows

- an object A is mapped to some bounded $\mathscr{E}$-resolution R_A for A.
- a morphism $\alpha \colon A \to B$ is mapped to $\Phi_{R_A,B}(\lambda_A \bullet \alpha)$, i.e., to the unique morphism $\mathbf{G}(\alpha) \colon \mathbf{G}(A) \to \mathbf{G}(B)$ whose convolution renders the following diagram

$$\begin{array}{ccc} \mathbf{F}(\mathbf{G}(A)) & \xrightarrow{\quad\lambda_A\quad} & A \\ {\scriptstyle \mathbf{F}(\mathbf{G}(\alpha))}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \alpha} \\ \mathbf{F}(\mathbf{G}(B)) & \xrightarrow[\quad\lambda_B\quad]{} & B \end{array}$$

commutative.

We still need to show that for any morphism $\alpha\colon A \to B$ in $\mathcal{K}^b(\mathscr{C})$ and any morphism $f\colon Q \to P$ in $\mathcal{K}^b(\mathscr{E}^{\oplus})$, the following diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\mathbf{F}(P), A) & \xrightarrow{\quad\Phi_{P,A}\quad} & \mathrm{Hom}_{\mathcal{K}^b(\mathscr{E}^{\oplus})}(P, \mathbf{G}(A)) \\ {\scriptstyle \mathbf{F}(f)\bullet - \bullet\alpha}\downarrow & & \downarrow{\scriptstyle f\bullet - \bullet\mathbf{G}(\alpha)} \\ \mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\mathbf{F}(Q), B) & \xrightarrow[\quad\Phi_{Q,B}\quad]{} & \mathrm{Hom}_{\mathcal{K}^b(\mathscr{E}^{\oplus})}(Q, \mathbf{G}(B)) \end{array}$$

is commutative. Let φ be any morphism in $\mathrm{Hom}_{\mathcal{K}^b(\mathscr{C})}(\mathbf{F}(P), A)$, then

$$\begin{aligned} \mathbf{F}(f\bullet\Phi_{P,A}(\varphi)\bullet\mathbf{G}(\alpha)) &= \mathbf{F}(f)\bullet\mathbf{F}(\Phi_{P,A}(\varphi))\bullet\mathbf{F}(\mathbf{G}(\alpha)) \\ &= \mathbf{F}(f)\bullet\varphi\bullet\lambda_A^{-1}\bullet\mathbf{F}(\mathbf{G}(\alpha)) \\ &= \mathbf{F}(f)\bullet\varphi\bullet\lambda_A^{-1}\bullet\lambda_A\bullet\alpha\bullet\lambda_B^{-1} \\ &= \mathbf{F}(f)\bullet\varphi\bullet\alpha\bullet\lambda_B^{-1}; \end{aligned}$$

i.e., $\Phi_{Q,B}(\mathbf{F}(f)\bullet\varphi\bullet\alpha) = f\bullet\Phi_{P,A}(\varphi)\bullet\mathbf{G}(\alpha)$, consequently, the above diagram is commutative as desired. □

Remark 6.80. By Definition A.20, the unit of the adjunction $\mathbf{F} \dashv \mathbf{G}$ is the natural transformation

$$\eta\colon \begin{cases} \mathrm{id}_{\mathcal{K}^b(\mathscr{E}^{\oplus})} & \to \mathbf{F}\bullet\mathbf{G}, \\ \quad Q & \mapsto \eta_Q := \Phi_{Q,\mathbf{F}(Q)}\Big(\mathrm{id}_{\mathbf{F}(Q)}\Big)\colon Q \to \mathbf{G}(\mathbf{F}(Q)), \end{cases}$$

i.e., Q is mapped to the unique morphism η_Q for which $\mathbf{F}(\eta_Q) = \lambda_{\mathbf{F}(Q)}^{-1}$. By Lemma A.15, $\mathbf{F}$ is conservative, hence η_Q is an isomorphism.

The counit is the natural transformation

$$\epsilon\colon \begin{cases} \mathbf{G}\bullet\mathbf{F} & \to \mathrm{id}_{\mathcal{K}^b(\mathscr{C})}, \\ \quad A & \mapsto \epsilon_A := \Phi_{\mathbf{G}(A),A}^{-1}\Big(\mathrm{id}_{\mathbf{G}(A)}\Big)\colon \mathbf{F}(\mathbf{G}(A)) \to A \end{cases}$$

i.e., $\Phi_{\mathbf{G}(A),A}(\epsilon_A) = \mathrm{id}_{\mathbf{G}(A)}$, hence $\mathbf{F}\Big(\mathrm{id}_{\mathbf{G}(A)}\Big) = \epsilon_A\bullet\lambda_A^{-1}$ and $\epsilon_A = \lambda_A$.

This implies the following:

Corollary 6.81. *The functors* $\mathbf{F}$ *and* $\mathbf{G}$ *are quasi-inverse.*

Corollary 6.82. *The replacement functor*

$$\mathbf{G}\colon \mathcal{K}^b(\mathscr{C}) \to \mathcal{K}^b(\mathscr{E}^{\oplus})$$

defined in Lemma 6.79 is fully faithful and exact.

PROOF. Since the counit ϵ of the adjunction $\mathbf{F} \dashv \mathbf{G}$ is a natural isomorphism, it follows by Lemma A.22 that $\mathbf{G}$ is fully faithful. According to Lemma 6.61, $\mathbf{F}$ is an exact functor; and to Lemma B.30, any right adjoint of an exact functor is also exact. Hence $\mathbf{G}$ is exact. □

APPENDIX A

First Steps Toward Constructive Category Theory in Cap

The stringent interpretation of the phrase "there exists" as "we can construct" distinguishes *constructive mathematics* from the *classical mathematics*. In classical mathematics, one can demonstrate the existence of a mathematical object without explicitly "constructing" it by assuming its non-existence and then deriving a contradiction from that assumption. Following a constructive approach to verify a mathematical statement means we must reinterpret not just the existential quantifiers, but also all the logical disjunctions utilized in proving the statement [**BP18**]. For example, to prove the statement "$\exists x P(x)$" we must construct an object x and prove that $P(x)$ holds, and to prove that $P \vee Q$ we must either have a proof of P or a proof of Q. In particular, the law of excluded middle: "For every statement P, either P or $\neg P$ holds" is not an axiom from the viewpoint of constructive mathematics. The constructiveness concept is usually exemplified by the following proposition:

"*There exists a pair of irrational numbers a, b such that a^b is rational*".

Consider the following argument: Either $\sqrt{2}^{\sqrt{2}}$ is rational, in which case $a = \sqrt{2}$ and $b = \sqrt{2}$ satisfy the desired property; or $\sqrt{2}^{\sqrt{2}}$ is irrational, in which case $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$ satisfy the property. However, as written, this argument does not enable us to determine which of the pairs satisfies the property, hence the argument is not correct from the point view of constructive mathematics. A constructive proof can be established by providing an instance of such pair, e.g., $a = \sqrt{2}$ and $b = 2\log_2(3)$. Further details about constructive mathematics can be found in [**BP18**], [**MRR88**] and [**nLa20**].

Following a constructive approach to category theory was the primary motivation behind Cap [**GSP22**]. Cap stands for *Categories, Algorithms, Programming*, is an open source software project for constructive category theory written in the computer algebra system `GAP` [**GAP21**]. The development of Cap started in December 2013 by Sebastian Gutsche and Sebastian Posur followed by major contributions of Øystein Skartsæterhagen in 2015 and Fabian Zickgraf since 2018. Cap was developed to facilitate the implementation of categories and categorical algorithms on the computer.

From the constructive viewpoint of Cap, a category $\mathscr{C}$ which belongs to a doctrine[1] $\mathcal{D}$ is determined

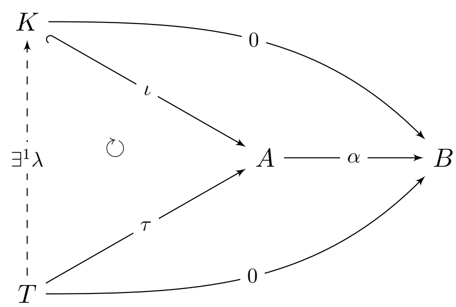
(1) by data structures for the objects $\mathrm{Obj}_{\mathscr{C}}$ and the morphisms in $\mathrm{Hom}_{\mathscr{C}}(A, B)$, along with operations for associatively composing morphisms, deciding their mathematical equality and constructing the identity morphisms $\mathrm{id}_A \in \mathrm{Hom}_{\mathscr{C}}(A, A)$, where $A, B \in \mathrm{Obj}_{\mathscr{C}}$;

[1]We use the term "doctrine" to describe a class of categories with specified additional properties or structures, e.g., additive, Abelian, monoidal, etc.

(2) a collection of *categorical algorithms* realizing the defining axioms of the doctrine $\mathcal{D}$. This is accomplished by formulating all the existential quantifiers and disjunctions of the doctrine's axioms in terms of explicit algorithms[2].

Let us illustrate the concept of formulating quantifiers and disjunctions in terms of operations with a concrete example:

The standard category theory textbooks define kernels of morphisms[3] in preadditive categories as follows: A *kernel*[4] of a morphism $\alpha\colon A \to B$ in a category $\mathscr{C}$ is an object K in $\mathscr{C}$ and a morphism $\iota\colon K \to A$ such that $\iota \cdot \alpha = 0$ with the following universal property: Given any $\tau\colon T \to A$ such that $\tau \cdot \alpha = 0$, there exists a unique[5] morphism $\lambda\colon T \to K$ such that $\lambda \cdot \iota = \tau$. The definition can be depicted in the diagram:



The constructive interpretation of the preceding definition demands algorithms to perform the following categorical operations:

(1) Given $\alpha\colon A \to B$, compute an object $\texttt{KernelObject}(\alpha)$ in $\mathrm{Obj}_{\mathscr{C}}$.
(2) Given $\alpha\colon A \to B$, compute a morphism

$$\texttt{KernelEmbedding}(\alpha)\colon \texttt{KernelObject}(\alpha) \to A$$

such that $\texttt{KernelEmbedding}(\alpha) \cdot \alpha = 0$.
(3) Given $\alpha\colon A \to B$ and $\tau\colon T \to A$ such that $\tau \cdot \alpha = 0$, compute a uniquely determined morphism

$$\texttt{KernelLift}(\alpha, \tau)\colon T \to \texttt{KernelObject}(\alpha)$$

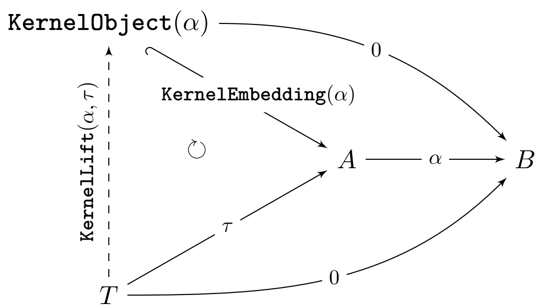
such that $\texttt{KernelLift}(\alpha, \tau) \cdot \texttt{KernelEmbedding}(\alpha) = \tau$.

[2] This procedure is usually called a skolemization of the axioms [**RV01**].

[3] The existence of (co)kernels of morphisms is required in the doctrine of Abelian categories (cf. Definition A.44).

[4] The dual concept of the kernel is the *cokernel*.

[5] This assumption implies that ι is a monomorphism.



Let us perform the above operations in a concrete category. The category $\mathbb{Q}$-**mat** of matrices[6] over (the field of rational numbers) $\mathbb{Q}$ consists of the following data:

(1) $\mathrm{Obj}_{\mathbb{Q}\text{-}\mathbf{mat}} := \mathbb{N}_0$.
(2) For two objects m, n in $\mathbb{Q}$-**mat**, we define $\mathrm{Hom}_{\mathbb{Q}\text{-}\mathbf{mat}}(m, n)$ by $\mathbb{Q}^{m \times n}$.
(3) The composition of morphisms is just matrix multiplication and the identity morphism of an object m is the $m \times m$ identity matrix over $\mathbb{Q}$.

In the following we use the JULIA package `CapAndHomalg` [**CAP21a**] to compute the kernel data of the morphism:

$$\alpha := 3 \xrightarrow{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} 1$$

```
julia> using CapAndHomalg
CapAndHomalg v1.1.8
Imported OSCAR's components GAP and Singular_jll
Type: ?CapAndHomalg for more information

julia> LoadPackage( "LinearAlgebraForCAP" )

julia> ℚ = HomalgFieldOfRationals( )
Q

julia> ℚmat = MatrixCategory( ℚ )
Category of matrices over Q

julia> α = HomalgMatrix( "[ [ 1 ], [ 2 ], [ 3 ] ]", 3, 1, ℚ ) / ℚmat
<A morphism in category of matrices over Q>

julia> K = KernelObject( α )
<A vector space object over Q of dimension 2>

julia> ι = KernelEmbedding( α )
<A split monomorphism in category of matrices over Q>
```

[6]This category is equivalent to the category $\mathrm{vec}_{\mathbb{Q}}$ of finite dimensional $\mathbb{Q}$-vector spaces (cf. Example 2.16). More precisely, an object $m \in \mathbb{N}_0$ in $\mathbb{Q}$-**mat** corresponds to the $\mathbb{Q}$-vector space $\mathbb{Q}^m$ in $\mathrm{vec}_{\mathbb{Q}}$.

```
julia> Display( ι )
[ [  -2,   1,   0 ],
  [  -3,   0,   1 ] ]

A split monomorphism in category of matrices over Q
```

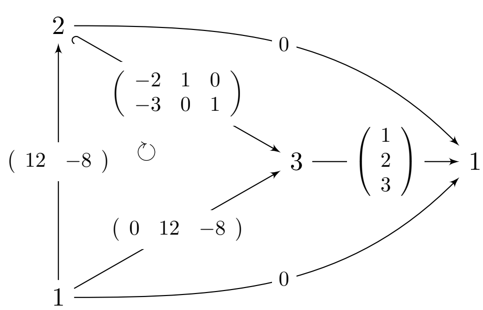
The morphism

$$\tau := 1 \xrightarrow{\left(\begin{array}{ccc} \cdot & 12 & -8 \end{array}\right)} 3$$

satisfies $\tau \cdot \alpha = 0$, hence τ is uniquely liftable along ι. Let us compute the induced kernel lift:

```
julia> τ = HomalgMatrix( "[ [ 0, 12, -8 ] ]", 1, 3, ℚ ) / ℚmat
<A morphism in category of matrices over Q>

julia> IsZeroForMorphisms( PreCompose( τ, α ) )
true

julia> λ = KernelLift( α, τ )
<A morphism in category of matrices over Q>

julia> Display( λ )
[ [  12,  -8 ] ]

A morphism in category of matrices over Q
```

We depict the outputs by the following diagram:



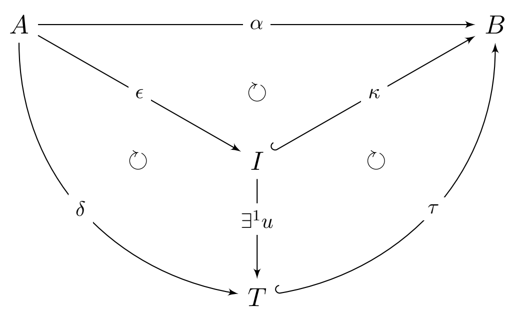
The kernel object $K = 2$ encodes the dimension of the space of row-syzygies[7] of α and the rows of the kernel embedding ι encodes an actual basis of this space. The kernel lift expresses every matrix (in this case τ) containing row-syzygies of α as $\mathbb{Q}$-linear combinations of the basis given by the rows of ι.

One of CAP's most distinguishing features is its *derivation mechanism*, which facilitates deriving categorical algorithms from other existing algorithms by utilizing the constructive proofs in the standard text books. We illustrate this concept by deriving an algorithm to compute the *images* of morphisms in Abelian categories, then using this "derived algorithm" to derive an algorithm to compute the *homology objects* of differential pairs.

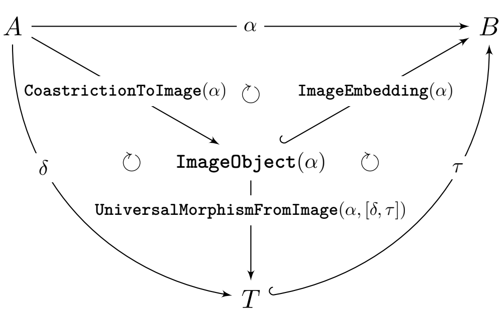
Let us first state the categorical definition of the image objects:

[7] I.e., the $\mathbb{Q}$-relations between the rows of α.

Example A.1 (The computation of images in Abelian categories)**.** The **image**[8] of a morphism $\alpha\colon A \to B$ consists of an object I and two morphisms $\epsilon\colon A \to I$ and $\iota\colon I \to B$ such that κ is a monomorphism and $\epsilon \cdot \kappa = \alpha$ with the following universal property: Given any other triple $(T, \delta\colon A \to T, \tau\colon T \to B)$ with τ a monomorphism and $\delta \cdot \tau = \alpha$, there exists a unique morphism $u\colon I \to T$ such that $\epsilon \cdot u = \delta$ and $u \cdot \tau = \kappa$.



The constructive interpretation of the definition of the image is depicted in the diagram:



The constructive proof of Lemma A.2 enables us to compute the image data as follows:

(1) The image object can be computed by the CAP formula:

$$I := \texttt{KernelObject}(\texttt{CokernelProjection}(\alpha)),$$

(2) The image embedding can be computed by the CAP formula:

$$\kappa := \texttt{KernelEmbedding}(\texttt{CokernelProjection}(\alpha)).$$

(3) The coastriction morphism $\epsilon\colon A \to I$ can be computed by the CAP formula:

$$\epsilon := \texttt{KernelLift}(\texttt{CokernelProjection}(\alpha), \alpha)$$

(4) Consider a triple $(T, \delta\colon A \to T, \tau\colon T \to B)$ with τ a monomorphism and $\delta \cdot \tau = \alpha$. The universal morphism $u\colon I \to T$ with $\epsilon \cdot u = \delta$ and $u \cdot \tau = \kappa$ can be computed by the CAP formula:

$$u := \texttt{ColiftAlongEpimorphism}(\epsilon, \delta).$$

where **`ColiftAlongEpimorphism`** is the operation which corresponds to the second axiom in the definition of Abelian categories (cf. Definition A.44).

[8]The dual concept of the image is called the *coimage*.

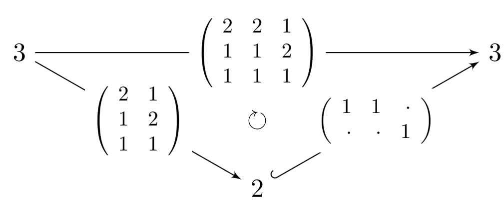
Let us illustrate this by computing the image data of the morphism

$$\alpha := 3 \xrightarrow{\begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}} 3$$

in $\mathbb{Q}$-**mat**:

```
julia> α = HomalgMatrix( "[[ 2, 2, 1 ], [ 1, 1, 2 ], [ 1, 1, 1 ]]", 3, 3, ℚ ) / matℚ
<A morphism in category of matrices over Q>

julia> ι = ImageEmbedding( α )
<A split monomorphism in category of matrices over Q>

julia> Display( ι )
[ [  1,  1,  0 ],
  [  0,  0,  1 ] ]

A split monomorphism in Category of matrices over Q

julia> ϵ = CoastrictionToImage( α )
<A morphism in Category of matrices over Q>

julia> Display( ϵ )
[ [  2,  1 ],
  [  1,  2 ],
  [  1,  1 ] ]

A morphism in Category of matrices over Q
```

We depict the outputs in the following diagram:



The image object $I = 2$ encodes the dimension of the row-space of α and the image embedding ι outputs an actual basis for this space while the coastriction morphism expresses the rows of α as $\mathbb{Q}$-linear combinations of the basis given by the rows of ι.

Example A.2 (The computation of homology in Abelian categories)**.** Let $\mathscr{C}$ be an Abelian category and $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ a differential pair of morphisms, i.e., with $\alpha \cdot \beta = 0$. The homology object of this pair is defined by the cokernel object of the canonical embedding of the image object of α in the kernel object of β (cf. Definition A.51). Hence, it can be computed by the CAP formula

$$\texttt{CokernelObject}(\texttt{KernelLift}(\beta, \texttt{ImageEmbedding}(\alpha))).$$

Let us illustrate this by computing the homology object of the differential pair

$$2 \xrightarrow{\begin{pmatrix} \frac{7}{4} & -\frac{1}{2} & -1 & 1 & 1 \\ -5 & 2 & -1 & \cdot & -1 \end{pmatrix}} 5 \xrightarrow{\begin{pmatrix} -2 & 2 \\ -5 & 3 \\ 1 & -2 \\ 3 & -2 \\ -1 & -2 \end{pmatrix}} 2$$

in $\mathbb{Q}$-**mat**:

```
julia> α = HomalgMatrix( "[ [7/4, -1/2, -1, 1,  1 ],
                           [ -5,    2, -1, 0, -1 ] ]", 2, 5, Q ) / Qmat
<A morphism in category of matrices over Q>

julia> β = HomalgMatrix( "[ [ -2,  2 ],
                           [ -5,  3 ],
                           [  1, -2 ],
                           [  3, -2 ],
                           [ -1, -2 ] ]" , 5, 2, Q ) / Qmat
<A morphism in category of matrices over Q>

julia> HomologyObject( α, β )
<A vector space object over Q of dimension 1>

julia> CokernelObject( KernelLift( β, ImageEmbedding( α ) ) )
<A vector space object over Q of dimension 1>
```

The current implementation of the category $\mathbb{Q}$-**mat** is accomplished by directly implementing methods for 66 categorical operations[9], and a total of 329 operations become available thanks to the derivation mechanism (cf. [**BP19a**] or [**GP21a**]).

```
julia> InfoOfInstalledOperationsOfCategory( ℚmat )
66 primitive operations were used to derive 329 operations for this category which
* IsEquippedWithHomomorphismStructure
* IsLinearCategoryOverCommutativeRing
* IsAbelianCategoryWithEnoughInjectives
* IsRigidSymmetricClosedMonoidalCategory
* IsClosedMonoidalCategory
* IsAbelianCategoryWithEnoughProjectives
```

A.1. Categories, Functors and Natural Transformations

In our constructive setting, we are going to work with *categories with* Hom-*setoids*. Before defining this type of categories, let us first review the classical definition of a category.

Definition A.3 (Category, classical definition)**.** A **(locally small) category** $\mathscr{C}$ consists of the following data:

[9] The majority of them are expressed in terms of algorithms afforded by computable rings (cf. Definition 2.32), which in the case of fields are ultimately based on the GAUSSIAN algorithm.

(1) A class $\mathrm{Obj}_{\mathscr{C}}$ (**objects**).
(2) Depending on $A, B \in \mathrm{Obj}_{\mathscr{C}}$ a set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ (**morphisms**).
(3) Each object $A \in \mathrm{Obj}_{\mathscr{C}}$ has a specified morphism id_A (**identity morphisms**).
(4) For any pair of morphisms $\alpha \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, and $\beta \in \mathrm{Hom}_{\mathscr{C}}(B, C)$, there exists a specified morphism $\alpha \cdot \beta \in \mathrm{Hom}_{\mathscr{C}}(A, C)$ (**composition**).

These data are subject to the following two axioms:

(1) For any morphism $\alpha \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, the compositions $\mathrm{id}_A \cdot \alpha$ and $\alpha \cdot \mathrm{id}_B$ are both equal to α.
(2) For any triple of morphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D$ the composites $\alpha \cdot (\beta \cdot \gamma)$ and $(\alpha \cdot \beta) \cdot \gamma$ are equal[10].

The mathematical equality of a pair $\alpha, \beta \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ of morphisms is implicitly inherited in the preceding definition by the assumption that $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is a set. However, in the constructive setting of CAP the mathematical equality of morphisms is provided as an additional algorithm which acts on pairs of morphisms. In other words, the classical tautology "For any two morphisms $\alpha, \beta \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, either $(\alpha = \beta)$ or $(\alpha \neq \beta)$ holds" should be interpreted constructively. To this end, CAP adopts a slightly more general notion of a category: The homomorphism sets $\mathrm{Hom}_{\mathscr{C}}(A, B)$ are not just sets; they are **setoids**, i.e., a set with an equivalence relation as an additional datum. The following is the formal definition of this type of categories (cf. [**Gut17**] and [**Pos17**]):

Definition A.4 (Category with Hom-setoids)**.** A **(locally small) category (with** Hom**-setoids)** $\mathscr{C}$ consists of the following data:

(1) A class $\mathrm{Obj}_{\mathscr{C}}$ (**objects**).
(2) Depending on $A, B \in \mathrm{Obj}_{\mathscr{C}}$ a set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ (**morphisms**), equipped with an equivalence relation "$=_{A,B}$" (**congruence of morphisms**). If $\alpha =_{A,B} \beta$ for two morphisms α, β in $\mathrm{Hom}_{\mathscr{C}}(A, B)$, we say they are **congruent**.
(3) An algorithm that computes for given A, B and C in $\mathrm{Obj}_{\mathscr{C}}$, $\alpha \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, and $\beta \in \mathrm{Hom}_{\mathscr{C}}(B, C)$ a morphism $\alpha \cdot \beta \in \mathrm{Hom}_{\mathscr{C}}(A, C)$ (**composition**) such that
   (a) The composition is compatible with the congruence relation, i.e., if $\alpha, \alpha' \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, $\beta, \beta' \in \mathrm{Hom}_{\mathscr{C}}(B, C)$ with $\alpha =_{A,B} \alpha'$ and $\beta =_{B,C} \beta'$, then $\alpha \cdot \beta =_{A,C} \alpha' \cdot \beta'$.
   (b) For any triple of morphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D$, we require
   $$\alpha \cdot (\beta \cdot \gamma) =_{A,D} (\alpha \cdot \beta) \cdot \gamma \quad (\textbf{associativity}).$$
(4) An algorithm that constructs for given $B \in \mathrm{Obj}_{\mathscr{C}}$ a morphism $\mathrm{id}_B \in \mathrm{Hom}_{\mathscr{C}}(B, B)$ (**identities**). Furthermore, for any pair of morphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$, we require
   $$\alpha \cdot \mathrm{id}_B =_{A,B} \alpha \text{ and } \mathrm{id}_B \cdot \beta =_{B,C} \beta.$$

Remark A.5. The above definition encompasses two notions of morphism equality:

- The "syntactic equality" or "naive equality" which signifies that the two morphisms are treated identically as elements in the morphism sets. This is often accomplished by checking that the two morphisms are defined by the same data, i.e., that they are represented similarly on the computer. Due to its simplicity, this syntactic equality is usually easily verified and is mostly utilized for compatibility purposes. Because any

[10] Thus denoted by $\alpha \cdot \beta \cdot \gamma$.

form of comparison between morphisms needs comparing their sources and ranges, this naive equality necessitates establishing another naive equality on the objects, which is also often accomplished by checking that the two objects are defined by the same data.

- The "semantic equality" or "mathematical equality" signifies that the two morphisms are congruent in the sense of the above definition. That is, in CAP, a morphism might have (syntactically) different representations on the computer. That is, the classical mathematical interpretation of the set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ can be recovered as the factor set $\mathrm{Hom}_{\mathscr{C}}(A, B)/{=_{A,B}}$ (see [**Pos17**, The CAP Project]). Implementing the mathematical equality (i.e., verifying the congruence of morphisms) is typically the first obstacle we encounter when implementing a new category on the computer, as it typically requires a non-trivial computation that produces an additional datum[11], commonly referred to as a *witness for morphism equality*. When determining morphism congruence, we utilize naive equality of objects to determine if they have the same source and range. In other words, we make no effort to introduce mathematical equality on objects. All categorical invariants and properties can be transmitted from one object to another once an isomorphism exists between them. As a result, there is no "categorical need" to verify any kind of mathematical equality on objects.

Convention. Unless otherwise specified, whenever we use the term "equality of morphisms" or the notation "$\alpha = \beta$" (for two morphisms α and β), we mean the mathematical equality.

Flipping all the morphisms in a category $\mathscr{C}$ defines another category:

Definition A.6. The **opposite category** $\mathscr{C}^{\mathrm{op}}$ of a category $\mathscr{C}$ consists of the same objects and morphisms as $\mathscr{C}$ after interchanging the source and range of every morphism.

Typically, categories may be enriched with additional structure, transforming them into instances of various doctrines, for example, additive, Abelian, triangulated, and so on. (see Definitions A.24, A.38 and A.44).

Definition A.7. A category $\mathscr{C}$ is said to have **decidable equality of morphisms** (alternatively, $\mathscr{C}$ **is computable**) if we can algorithmically decide the congruence between morphisms with the same source and range. A category $\mathscr{C}$ is called **computable as instance of a doctrine** $\mathcal{D}$ if all the existential quantifiers and disjunctions in the defining axioms of $\mathcal{D}$ are realized by algorithms.

Lift and colift morphisms are ubiquitous in category theory. They are essential ingredients for defining many categorical concepts such as kernels, cokernels[12], projective objects, injective objects, etc.

Definition A.8. Let $\mathscr{C}$ be a category.

(1) $\mathscr{C}$ is said to have **decidable lifts** if we have an algorithm which for a cospan

$$A \xrightarrow{\alpha} B \xleftarrow{\gamma} C$$

[11] See e.g., the equality of morphisms in the FREYD categories and bounded homotopy categories (cf. Section 2.1.1 resp. Section 3.2).

[12] And more generally, limits and colimits.

decides the solvability of the equation $\chi \cdot \gamma = \alpha$, and in affirmative case computes a particular solution $\chi\colon A \to C$. If such χ exists, we say that α is **liftable along**[13] γ and we call χ a **lift morphism**[14]. of α along γ.

(2) $\mathscr{C}$ is said to have **decidable colifts** if we have an algorithm which for a span

$$A \xleftarrow{\alpha} B \xrightarrow{\gamma} C$$

decides the solvability of the equation $\gamma \cdot \chi = \alpha$, and in affirmative case computes a particular solution $\chi\colon C \to A$. If such χ exists, we say that α is **coliftable along**[15] γ; and we call χ a **colift morphism**[16] of α along γ.

A functor between two categories is a mapping on objects and morphisms which respects composition and identity morphisms.

Definition A.9. A (covariant) **functor** F from a category $\mathscr{C}$ to a category $\mathscr{D}$ consists of the following data:

(1) An algorithm that computes for a given A in $\mathscr{C}$ an object $F(A)$ in $\mathscr{D}$.
(2) An algorithm that computes for a given morphism $\alpha\colon A \to B$ in $\mathscr{C}$ a morphism $F(\alpha)\colon F(A) \to F(B)$.
(3) For a given object A in $\mathscr{C}$, we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.
(4) For given objects A, B, C and a pair of morphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$, we have

$$F(\alpha \cdot \beta) = F(\alpha) \cdot F(\beta).$$

Remark A.10. A **contravariant functor** F from $\mathscr{C}$ to $\mathscr{D}$ is a functor $F\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$.

Similar to the notions of injections, surjections, and bijections between sets, functors between categories carry analogous notions.

Definition A.11. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor.

(1) F is called **faithful** (resp. **full**, **fully faithful**) if

$$F_{A,B}\colon \operatorname{Hom}_{\mathscr{C}}(A,B) \to \operatorname{Hom}_{\mathscr{D}}(F(A),F(B))$$

is injective (resp. surjective, bijective) for all objects A, B in $\mathscr{C}$.
(2) F is called **essentially surjective on objects** if for each B in $\mathscr{D}$, there exist an object A in $\mathscr{C}$ and an isomorphism $F(A) \xrightarrow{\sim} B$.
(3) F is called **conservative** if for any morphism f in $\mathscr{C}$, $F(f)$ being an isomorphism implies that f is an isomorphism.
(4) F is called **embedding** if it is faithful and injective on objects, i.e., F is injective on morphisms. In this case F identifies $\mathscr{C}$ with a subcategory of $\mathscr{D}$.
(5) F is called **isomorphism** if it is fully faithful and bijective on objects.

Remark A.12. The properties in the previous definition are closed under composition of functors. In other words, if two composable functors satisfy one of the previous properties, then so does their composition.

[13] See the CAP operation `IsLiftable`.
[14] See the CAP operation `Lift`
[15] See the CAP operation `IsColiftable`.
[16] See the CAP operation `Colift`.

Example A.13. (1) The embedding functor **Ab** $\hookrightarrow$ **Grp** from the category of Abelian groups in the category of groups is fully faithful, but it is not essentially surjective on objects.
(2) The forgetful functor **Ring** $\to$ **Ab** from the category of rings to the category of Abelian groups is faithful, but neither full nor essentially surjective on objects.

Lemma A.14. *Let $F\colon \mathscr{C} \to \mathscr{D}$ be a faithful functor and $f\colon A \to B$ a morphism in $\mathscr{C}$. If $F(f)$ is an epimorphism (resp. monomorphism), then f is also an epimorphism (resp. monomorphism).*

Proof. Suppose that $F(f)$ is an epimorphism and let $g, h\colon B \to C$ be a pair of morphisms such that $f \bullet g = f \bullet h$. Then $F(f) \bullet F(g) = F(f) \bullet F(h)$. Since $F(f)$ is epimorphism, it follows that $F(g) = F(h)$. Since F is faithful, it follows that $g = h$. The case of monomorphism is treated similarly. $\square$

Lemma A.15. *Let $F\colon \mathscr{C} \to \mathscr{D}$ be a fully faithful functor. Then F is conservative.*

Proof. Let $f\colon A \to B$ be a morphism such that $F(f)$ is an isomorphism. Since F is full, there exists a morphism $g\colon B \to A$ with $F(g) = F(f)^{-1}$. It follows that $F(f \bullet g) = F(\mathrm{id}_A)$ and $F(g \bullet f) = F(\mathrm{id}_B)$. Since F is faithful, we have $f \bullet g = \mathrm{id}_A$ and $g \bullet f = \mathrm{id}_B$. $\square$

There are also "maps" between functors:

Definition A.16. Given categories $\mathscr{C}$ and $\mathscr{D}$ and functors $F, G\colon \mathscr{C} \to \mathscr{D}$, a **natural transformation** $\eta\colon F \to G$ consists of an algorithm that computes for a given object A in $\mathscr{C}$ a morphism $\eta_A\colon F(A) \to G(A)$ so that for any morphism $\varphi\colon A \to B$ in $\mathscr{C}$ induces the following commutative diagram:

$$\begin{array}{ccc} F(A) & \xrightarrow{\quad \eta_A \quad} & G(A) \\ {\scriptstyle F(\varphi)}\big\downarrow & \circlearrowleft & \big\downarrow{\scriptstyle G(\varphi)} \\ F(B) & \xrightarrow[\quad \eta_B \quad]{} & G(B) \end{array}$$

The morphisms η_A are called the **components** of η. A natural transformation η is called **natural isomorphism** if all its components are isomorphisms. In this case, we depict η as $\eta : F \cong G$.

In the following we define the equivalence of categories:

Definition A.17. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories.

- An **equivalence between** $\mathscr{C}$ and $\mathscr{D}$ consists of the following data:
  (1) A functor $F : \mathscr{C} \to \mathscr{D}$,
  (2) A functor $G : \mathscr{D} \to \mathscr{C}$,
  (3) A natural isomorphism $\eta : \mathrm{id}_{\mathscr{C}} \xrightarrow{\sim} F \bullet G$,
  (4) A natural isomorphism $\epsilon : G \bullet F \xrightarrow{\sim} \mathrm{id}_{\mathscr{D}}$.

  In this case, we write $F : \mathscr{C} \xrightarrow{\sim} \mathscr{D}$, and we say that F and G are quasi-inverse.
- An **isomorphism between** $\mathscr{C}$ and $\mathscr{D}$ is an equivalence for which all components of η and ϵ are identity morphisms.

When two categories are equivalent, then all categorical information available in one of them can be realized unchanged in the other category. In other words, they are categorically identical. The only difference that could happen, is that they could contain different numbers of isomorphic "copies" of the same objects.

Equivalences of categories can be characterized as follows:

Proposition A.18. *Any equivalence of categories is fully faithful and essentially surjective on objects. Assuming the axiom of choice, any functor with these properties defines an equivalence of categories.*

Proof. See [**Rie16**, Theorem 1.5.9]. □

Definition A.19. A category $\mathscr{C}$ is called **skeletal** if isomorphic objects in $\mathscr{C}$ are equal. The **skeleton** category $\mathrm{sk}(\mathscr{C})$ of a category $\mathscr{C}$ is the unique (up to a natural isomorphism) skeletal category that is equivalent to $\mathscr{C}$.

An adjunction consists of a pair of functors that are related to each other in a particular way. They are ubiquitous in mathematics and often arise from constructions which enjoy universal properties. For instance taking the free groups over sets or the free categories over quivers are adjoint to the corresponding forgetful functors.

Definition A.20. An **adjunction** from a category $\mathscr{C}$ to a category $\mathscr{D}$ is a pair of functors

$$F\colon \mathscr{C} \rightleftarrows \mathscr{D} \colon G$$

and, for all P in $\mathscr{C}$ and A in $\mathscr{D}$, a bijection

$$\Phi_{P,A}\colon\ \mathrm{Hom}_{\mathscr{D}}(F(P),A) \to \mathrm{Hom}_{\mathscr{C}}(P,G(A)),$$

which is natural in the sense that for every $\alpha\colon A \to B$ and $f\colon Q \to P$, the following diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathscr{D}}(F(P),A) & \xrightarrow{\Phi_{P,A}} & \mathrm{Hom}_{\mathscr{C}}(P,G(A)) \\
{\scriptstyle F(f)\cdot - \cdot \alpha}\downarrow & \circlearrowleft & \downarrow{\scriptstyle f\cdot - \cdot G(\alpha)} \\
\mathrm{Hom}_{\mathscr{D}}(F(Q),B) & \xrightarrow[\Phi_{Q,B}]{} & \mathrm{Hom}_{\mathscr{C}}(Q,G(B))
\end{array}$$

is commutative. The above adjunction is denoted by $F \dashv G$; and F is called the **left adjoint** of G, while G is called the **right adjoint** of F. The **unit** of the adjunction is the natural transformation $\eta\colon \mathrm{id}_{\mathscr{C}} \to F \cdot G$ whose component at an object P in $\mathscr{C}$ is

$$\eta_P := \Phi_{P,F(P)}\Big(\mathrm{id}_{F(P)}\Big)\colon P \to G(F(P));$$

and the **counit** is the natural transformation $\epsilon\colon G \cdot F \to \mathrm{id}_{\mathscr{D}}$ whose component at object A in $\mathscr{D}$ is

$$\epsilon_A := \Phi^{-1}_{G(A),A}\Big(\mathrm{id}_{G(A)}\Big)\colon F(G(A)) \to A.$$

Even though the unit and counit of an adjunction are images of identity morphisms under the adjunction bijection and its inverse, they completely determine the adjunction:

Lemma A.21. *Let $F \dashv G$ be an adjunction, with unit η and counit ϵ. Then*

$$\Phi_{P,A}(\alpha) = \eta_P \cdot G(\alpha)$$

for any morphism $\alpha\colon F(P) \to A$, *and*

$$\Phi^{-1}_{P,A}(f) = F(f)\cdot\epsilon_A$$

for any morphism $f\colon P \to G(A)$.

The following Lemma highlights a very useful relation between adjoint pairs and the associated unit and counit (see e.g., [**Rie16**, Lemma 4.5.13]).

Lemma A.22. *Let* $F \dashv G$ *be an adjunction. Then*

(1) F *is fully faithful if and only if the unit* η *is a natural isomorphism.*
(2) G *is fully faithful if and only if the counit* ϵ *is a natural isomorphism.*

Example A.23. Let $F\colon \mathbf{Set} \to \mathbf{Grp}$ be the functor assigning to each set Y the free group generated by the elements of Y, and let $G\colon \mathbf{Grp} \to \mathbf{Set}$ be the forgetful functor, which assigns to each group X its underlying set. Then F is left adjoint to G.

A.2. From (pre)Additive Categories to (pre)Abelian Categories

This section provides an constructive approach to the preadditive, linear, additive, pre-Abelian and Abelian categories. In this section we formulate the existential quantifiers and disjunctions in the classical definitions of these concepts in terms of explicit algorithms. We refer to [**Gut17**] and [**Pos17**] for a more in-depth constructive treatment of these concepts.

A category is preadditive if it is enriched over the category **Ab** of Abelian groups.

Definition A.24. A category $\mathscr{C}$ is called **preadditive** or **Ab-category** if we have

(1) An algorithm that computes for a given pair of morphisms $\alpha, \beta\colon A \to B$ in $\mathscr{C}$ a morphism $\alpha + \beta\colon A \to B$ (**addition**).
(2) An algorithm that constructs for a given pair of objects A, B in $\mathscr{C}$ a morphism $0\colon A \to B$ (**zero morphism**).
(3) An algorithm that constructs for a given morphism $\alpha\colon A \to B$ a morphism $-\alpha\colon A \to B$ (**additive inverse**).
(4) For all objects A, B in $\mathscr{C}$, the given algorithms turn $\mathrm{Hom}_{\mathscr{C}}(A, B)$ into an Abelian group.
(5) The composition of morphisms is bilinear, i.e., we have
    a. $(\alpha + \alpha')\cdot\beta = \alpha\cdot\beta + \alpha'\cdot\beta$,
    b. $\alpha\cdot(\beta + \beta') = \alpha\cdot\beta + \alpha\cdot\beta'$ and
    for all $\alpha, \alpha'\colon A \to B$, $\beta, \beta'\colon B \to C$.

A functor $F\colon \mathscr{C} \to \mathscr{D}$ between two preadditive categories is called **additive** if for any two objects A, B in $\mathscr{C}$ the induced map

$$F_{A,B}\colon \begin{cases} \mathrm{Hom}_{\mathscr{C}}(A,B) & \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B)), \\ \quad\alpha & \mapsto F(\alpha) \end{cases}$$

defines a group homomorphism.

Example A.25. Every ring R can be interpreted as a preadditive category $\mathscr{C}(R)$ consisting of only one object, say $*$, whose endomorphisms are the elements of R (cf. Section 2.1.2).

Linear categories are preadditive categories which are enriched over a category of modules.

Definition A.26. Let k be a commutative ring. A preadditive category $\mathscr{C}$ will be called k-**linear category** if we have

(1) An algorithm that constructs for a given element $r \in k$ and morphism $\alpha\colon A \to B$ a morphism $r \cdot \alpha\colon A \to B$ (**ring action on morphisms**). Furthermore, the ring action turns $\mathrm{Hom}_{\mathscr{C}}(A, B)$ into a k-module.
(2) For all $r \in k, \alpha\colon A \to B$, $\beta\colon B \to C$ we have $r \cdot (\alpha \bullet \beta) = (r \cdot \alpha) \bullet \beta = \alpha \bullet (r \cdot \beta)$ (**k-bilinearity of the composition**).

A functor $F\colon \mathscr{C} \to \mathscr{D}$ between two k-linear categories is called **k-linear** if for all pairs A, B in $\mathscr{C}$ the induced map

$$F_{A,B}\colon \begin{cases} \mathrm{Hom}_{\mathscr{C}}(A, B) & \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B)), \\ \qquad \alpha & \mapsto F(\alpha) \end{cases}$$

is a k-module homomorphism.

Example A.27. If the ring R is a k-algebra for some commutative ring k, then $\mathscr{C}(R)$ in Example A.25 is k-linear.

Example A.28. For any commutative ring k and any preadditive category $\mathscr{C}$, there exists a k-linear category $k\mathscr{C}$ and an embedding $\mathscr{C} \xrightarrow{\iota} k\mathscr{C}$ such that any additive functor from $\mathscr{C}$ to a k-linear category factors uniquely along ι (cf. Section 2.2.1).

Definition A.29. The **endomorphism k-algebra**[17] of a k-linear category $\mathscr{C}$ is the (possibly nonunital) associative k-algebra

$$\mathrm{End}\,\mathscr{C} \coloneqq \bigoplus_{A,B \in \mathscr{C}} \mathrm{Hom}_{\mathscr{C}}(A, B)$$

whose multiplication is defined by the bilinear extension of the following product

$$\alpha \cdot \beta \coloneqq \begin{cases} \alpha \bullet \beta & \text{if } \mathrm{Range}(\alpha) = \mathrm{Source}(\beta) \\ 0 & \text{otherwise.} \end{cases}$$

Remark A.30. Any morphism $\alpha\colon A \to B$ in $\mathscr{C}$ can be interpreted as an element in $\mathrm{End}\,\mathscr{C}$. Precisely, we identify α with $i_{A,B}(\alpha)$ where $i_{A,B}$ is the natural injection of $\mathrm{Hom}_{\mathscr{C}}(A, B)$ in $\mathrm{End}\,\mathscr{C}$.

Remark A.31. If $\mathscr{C}$ has finitely many objects then $\mathrm{End}\,\mathscr{C}$ is a unital algebra whose unit is given by $1 \coloneqq \oplus_{A \in \mathscr{C}} \mathrm{id}_A$.

Example A.32. The path k-algebra of a quiver $\mathfrak{q}$ is the endomorphism k-algebra of the k-linear closure category $k\mathcal{F}_{\mathfrak{q}}$ of the free category $\mathcal{F}_{\mathfrak{q}}$ defined by $\mathfrak{q}$ (cf. Example 2.50).

Definition A.33. A k-linear category $\mathscr{C}$ is called Hom-**finite** if $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is finitely generated as a k-module for all objects A, B in $\mathscr{C}$.

The notion of a locular k-linear category $\mathscr{C}$ allows us to visualize $\mathscr{C}$ in terms of quiver $\mathfrak{q}_{\mathscr{C}}$. For the original treatment we refer to [**GR92**, §3], [**ARS97**, Ch. 9] or [**Kel07**].

Definition A.34. A k-linear category $\mathscr{C}$ is called **locular** if it is small, skeletal and the endomorphism algebra of every object is local[18]. It can be shown that in a locular category the set of non-invertible morphisms forms a two-sided ideal of morphisms in $\mathscr{C}$, which we call the **radical ideal** of $\mathscr{C}$ and denote by $\mathrm{rad}_{\mathscr{C}}$. More precisely, $\mathrm{rad}_{\mathscr{C}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(A, B)$ if $A \neq B$,

[17] The name is justified by the fact that if $\mathscr{C}$ has finitely many objects then $\mathrm{End}\,\mathscr{C} \cong \mathrm{End}_{\mathscr{C}^{\oplus}} \bigoplus_{A \in \mathscr{C}} A$ where $\mathscr{C}^{\oplus}$ is the additive closure of $\mathscr{C}$ (cf. Definition 2.24).

[18] A ring is called local if the non-invertible elements form an ideal.

whereas $\mathrm{rad}_{\mathscr{C}}(A,A)$ is the maximal ideal[19] of the local algebra $\mathrm{End}_{\mathscr{C}} A$. For $n > 1$, we denote by $\mathrm{rad}^n_{\mathscr{C}}$ the two-sided ideal of morphisms generated the compositions of n morphisms in $\mathrm{rad}_{\mathscr{C}}$. For two objects A, B in $\mathscr{C}$, the space of **irreducible morphisms** is defined by

$$\mathrm{irr}_{\mathscr{C}}(A,B) := \mathrm{rad}_{\mathscr{C}}(A,B)/\,\mathrm{rad}^2_{\mathscr{C}}(A,B).$$

Definition A.35. Let k be a field and $\mathscr{C}$ a locular k-linear category. The **generating quiver** $\mathfrak{q}_{\mathscr{C}}$ of $\mathscr{C}$ consists of the following data:

- The vertices of $\mathfrak{q}_{\mathscr{C}}$ are labeled by the objects of $\mathscr{C}$.
- The number of arrows from a vertex v_A to v_B is given by $\dim \mathrm{irr}_{\mathscr{C}}(A,B)$ where $\mathrm{irr}_{\mathscr{C}}(A,B)$ is the k-vector space of irreducible morphisms from A to B.

Example A.36. Any strong exceptional sequence $\mathscr{E}$ in a k-linear triangulated category $\mathfrak{T}$ is locular (cf. Definition 6.14).

Example A.37. Let k be a field and $\mathscr{C}$ a k-linear category. Suppose $\mathscr{C}$ is **multilocular**, i.e., each object is a finite direct sum of indecomposables[20] with local endomorphism algebras. Then the skeleton of the full subcategory of $\mathscr{C}$ generated by indecomposable objects is locular (cf. [**GR92**, §3]). Examples include the category of finite-dimensional modules over finite-dimensional algebras (cf. [**GR92**]) and the category of coherent sheaves on a projective variety X (cf. [**Ser55**]). Furthermore, their bounded derived categories are also multilocular.

An additive category is a preadditive category which admits all finitary biproducts.

Definition A.38. A preadditive category $\mathscr{C}$ is **additive** if it is equipped with an algorithm which for a given finite (possibly empty) list of objects $A_1, \ldots, A_n$ in $\mathscr{C}$ computes their **direct sum**, i.e., an object $\bigoplus_{i=1}^n A_i$ in $\mathscr{C}$ together with pairs of morphisms

$$A_j \xrightarrow{\iota_j} \bigoplus_{i=1}^{n} A_i \xrightarrow{\pi_j} A_j$$

for each $j \in \{1, \ldots, n\}$, such that the identities

a. $\sum_{i=1}^n \pi_i \cdot \iota_i = \mathrm{id}_{\bigoplus_{i=1}^n A_i}$,
b. $\iota_i \cdot \pi_i = \mathrm{id}_{A_i}$ and
c. $\iota_i \cdot \pi_j = 0$

hold for all $i, j = 1, \ldots, n$ and $i \neq j$.

Remark A.39. By [**Bor94a**, Proposition 1.3.4], a functor $F\colon \mathscr{C} \to \mathscr{D}$ between additive categories is additive if and only if it preserves finite direct sums.

Example A.40. Let R be a ring. The full subcategory of R-**mod** generated by the free R-modules of finite rank is additive (cf. Section 2.1.3).

Example A.41. For any preadditive category $\mathscr{C}$, there exists an additive category $\mathscr{C}^{\oplus}$ and an embedding functor $\mathscr{C} \xhookrightarrow{\iota} \mathscr{C}^{\oplus}$ such that any additive functor from $\mathscr{C}$ to some additive category factors uniquely along ι (cf. Section 2.2.2).

[19] *The* maximal ideal of a local ring is formed by the non-invertible elements of the ring (see, e.g., [**AM69**]).

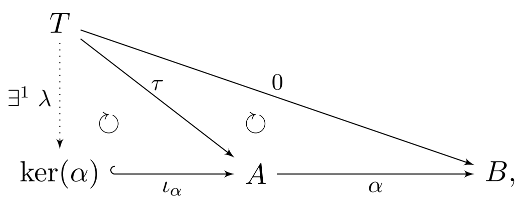
[20] An object in a linear category is called indecomposable if it is not a direct sum of two non-trivial objects.

The primary doctrine over which homological algebra can be developed is the doctrine of Abelian categories. Mac Lane proposed the concept and the term in [**Mac50**], while Grothendieck is credited with the modern axiomatization in [**Gro57**].

In our constructive setting we adopt the following definitions of pre-Abelian resp. Abelian categories.

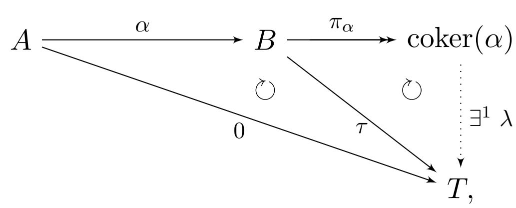
Definition A.42. An additive category $\mathscr{C}$ is called **pre-Abelian** if

(1) We have an algorithm that computes for a given morphism $\alpha\colon A \to B$
   (a) an object $\ker(\alpha)$ in $\mathscr{C}$ (**kernel object**),
   (b) a monomorphism $\iota_\alpha\colon \ker(\alpha) \to A$ such that $\iota_\alpha \cdot \alpha = 0$ (**kernel embedding**) and
   (c) for any morphism $\tau\colon T \to A$ with $\tau \cdot \alpha = 0$, the algorithm computes a lift for τ along ι_α, i.e., a morphism $\lambda\colon T \to \ker(\alpha)$ such that $\lambda \cdot \iota_\alpha = \tau$ (**kernel lift**)



   and furthermore, the morphism λ is uniquely determined (up to the equality $=$) by this property.

(2) We have an algorithm that computes for a given morphism $\alpha\colon A \to B$
   (a) an object $\operatorname{coker}(\alpha)$ in $\mathscr{C}$ (**cokernel object**),
   (b) an epimorphism $\pi_\alpha\colon B \to \operatorname{coker}(\alpha)$ such that $\alpha \cdot \pi_\alpha = 0$ (**cokernel projection**) and
   (c) for any morphism $\tau\colon B \to T$ with $\alpha \cdot \tau = 0$, the algorithm computes a colift for τ along π_α, i.e., a morphism $\lambda\colon \operatorname{coker}(\alpha) \to T$ such that $\pi_\alpha \cdot \lambda = \tau$ (**cokernel colift**)



   and furthermore, the morphism λ is uniquely determined (up to the equality $=$) by this property.

The following proposition is an immediate consequence of the preceding definition.

Proposition A.43. *Every morphism $\alpha\colon A \to B$ in a pre-Abelian category has a canonical decomposition*

$$A \xrightarrow{\pi_{\iota_\alpha}} \operatorname{coker}(\iota_\alpha) \xrightarrow{\overline{\alpha}} \ker(\pi_\alpha) \xrightarrow{\iota_{\pi_\alpha}} B$$

where π_{ι_α} is the cokernel projection of the kernel embedding ι_α of α and ι_{π_α} is the kernel embedding of the cokernel projection π_α of α.

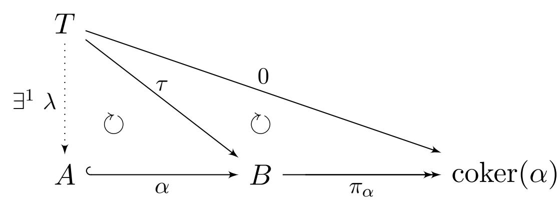
Usually, an Abelian category is defined as a pre-Abelian category satisfying the following equivalent conditions:

- For every morphism $\varphi\colon A \to B$, the canonical morphism $\overline{\alpha}\colon \operatorname{coker}(\iota_\alpha) \to \ker(\pi_\alpha)$ in Proposition A.43 is an isomorphism.
- Every monomorphism is a kernel embedding of its cokernel projection and every epimorphism is a cokernel projection of its kernel embedding.

By unwrapping the second condition and using the same notations as in Definition A.42, we get the following definition of Abelian categories.

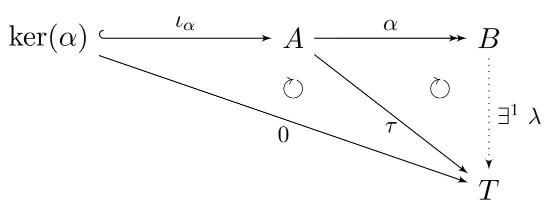
Definition A.44. A pre-Abelian category $\mathscr{C}$ is called **Abelian** if the following holds:

(1) We have an algorithm[21] which computes for a given monomorphism $\alpha\colon A \hookrightarrow B$ and given morphism $\tau\colon T \to B$ with $\tau \cdot \pi_\alpha = 0$ a lift of τ along α, i.e., a morphism $\lambda\colon T \to A$ with $\lambda \cdot \alpha = \tau$:



The lift morphism λ is then uniquely determined because α is a monomorphism.

(2) We have an algorithm[22] which computes for a given epimorphism $\alpha\colon A \twoheadrightarrow B$ and given morphism $\tau\colon A \to T$ with $\iota_\alpha \cdot \tau = 0$ a colift of τ along α, i.e., a morphism $\lambda\colon B \to T$ with $\alpha \cdot \lambda = \tau$:



The colift morphism λ is then uniquely determined because α is an epimorphism.

The following is an immediate consequence of the definition.

Corollary A.45. *Let $\mathscr{C}$ be an Abelian category.*

(1) A morphism $\tau\colon T \to B$ is liftable along a monomorphism $\alpha\colon A \hookrightarrow B$ if and only if $\tau \cdot \pi_\alpha = 0$ where $\pi_\alpha\colon B \twoheadrightarrow \operatorname{coker}(\alpha)$ is the cokernel projection of α.

(2) A morphism $\tau\colon A \to T$ is coliftable along an epimorphism $\alpha\colon A \twoheadrightarrow B$ if and only if $\iota_\alpha \cdot \tau = 0$ where $\iota_\alpha\colon \ker(\alpha) \hookrightarrow A$ is the kernel embedding of α.

Remark A.46. The preceding corollary enables us to enhance the *derivation mechanism* in CAP with a *derivation rule in Abelian categories* for the operation `IsLiftableAlongMonomorphism` from the two operations `CokernelProjection` and `IsZeroForMorphisms`. This kind of derivations is called **doctrine-based derivation**.

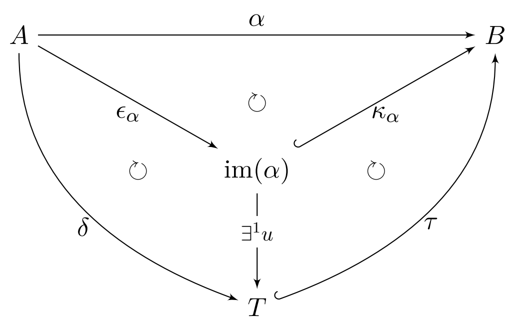
In the following we briefly sketch the construction of *images* in Abelian categories. Images and their dual notion *coimages* are essential for many homological computations, e.g., the homology objects of differential pairs and left and right derived functors.

[21]See the CAP operation `LiftAlongMonomorphism`.

[22]See the CAP operation `ColiftAlongEpimorphism`.

Definition A.47. We say a category $\mathscr{C}$ has **images** if we have an algorithm that computes for a given morphism $\alpha\colon A \to B$ in $\mathscr{C}$

(1) an object $\mathrm{im}(\alpha)$ in $\mathscr{C}$ (**image object**),
(2) a monomorphism $\kappa_\alpha\colon \mathrm{im}(\alpha) \hookrightarrow B$ (**image embedding**) and a morphism $\epsilon_\alpha\colon A \to \mathrm{im}(\alpha)$ (**coastriction morphism**) such that $\epsilon_\alpha \cdot \kappa_\alpha = \alpha$.
(3) Given any triple $(T, \delta\colon A \to T, \tau\colon T \to B)$ with τ a monomorphism and $\delta \cdot \tau = \varphi$, the algorithm computes a morphism $u\colon \mathrm{im}(\alpha) \to T$ such that $\epsilon_\alpha \cdot u = \delta$ and $u \cdot \tau = \kappa_\alpha$ (**universal morphism from image object**).



The dual concept of images is called **coimages**.

The constructive proof of the following lemma can easily be turned to algorithms (cf. Example A.1).

Lemma A.48. *Let $\mathscr{C}$ be an abelian category. Then $\mathscr{C}$ has images and coimages.*

PROOF. Define $\mathrm{im}(\alpha) \coloneqq \ker(\pi_\alpha)$ where $\pi_\alpha\colon B \twoheadrightarrow \mathrm{coker}(\alpha)$ is the cokernel projection of α, and set $\kappa_\alpha \coloneqq \iota_{\pi_\alpha}\colon \mathrm{im}(\alpha) \hookrightarrow B$ where ι_{π_α} is the kernel embedding of π_α.

We set the coastriction morphism $\epsilon_\alpha\colon A \to \mathrm{im}(\alpha)$ to the composition of the first two components of the canonical decomposition of α (cf. Proposition A.43), hence, ϵ is an epimorphism (since $\mathscr{C}$ is Abelian). In fact, since $\alpha \cdot \pi_\alpha = 0$, ϵ_α is *the* kernel lift of α along $\iota_{\pi_\alpha}\colon \mathrm{im}(\alpha) \hookrightarrow B$.

Consider a triple $(T, \delta\colon A \to T, \tau\colon T \to B)$ with τ a monomorphism and $\delta \cdot \tau = \alpha$. We need to compute a morphism $u\colon \mathrm{im}(\alpha) \to T$ with $\epsilon_\alpha \cdot u = \delta$ and $u \cdot \tau = \kappa_\alpha$. The coastriction morphism ϵ_α is an epimorphism, thus, a cokernel projection of its kernel embedding. Let ι_{ϵ_α} be the kernel embedding of ϵ_α. Then, $\iota_{\epsilon_\alpha} \cdot \delta \cdot \tau = \iota_{\epsilon_\alpha} \cdot \alpha = \iota_{\epsilon_\alpha} \cdot \epsilon_\alpha \cdot \kappa_\alpha = 0 \cdot \kappa_\alpha = 0$, and since τ is a monomorphism, $\iota_{\epsilon_\alpha} \cdot \delta = 0$ as well, i.e., there exists a unique colift morphism $u\colon \mathrm{im}(\alpha) \to T$ of δ along ϵ_α, i.e., with $\epsilon_\alpha \cdot u = \delta$ (cf. Definition A.44). That is, $\epsilon_\alpha \cdot (\kappa_\alpha - u \cdot \tau) = \epsilon_\alpha \cdot \kappa_\alpha - \epsilon_\alpha \cdot u \cdot \tau = \alpha - \delta \cdot \tau = \alpha - \alpha = 0$, i.e., $\kappa_\alpha = u \cdot \tau$ because ϵ_α is an epimorphism. The existence of coimages follows by a similar argument. □

Remark A.49. In fact, any category with *universal epi-mono factorization* has images and coimages (cf. [**Pos17**, Lemma 1.36]).

Remark A.50. For any morphism α in an Abelian category the objects $\mathrm{im}(\alpha)$ and $\mathrm{coim}(\alpha)$ are isomorphic.

The following definition can also be turned into an algorithm (cf. Example A.2).

Definition A.51. Let $\mathscr{C}$ be an Abelian category and let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be a *differential pair* of morphisms, i.e., they satisfy $\alpha \cdot \beta = 0$. Let $A \overset{\epsilon_\alpha}{\twoheadrightarrow} \mathrm{im}(\alpha) \overset{\kappa_\alpha}{\hookrightarrow} B$ be an image factorization of α

and let $\iota_\beta\colon \ker(\beta) \hookrightarrow B$ be a kernel embedding of β. Since $\alpha \cdot \beta = 0$ and ϵ_α is an epimorphism, we get $\kappa_\alpha \cdot \beta = 0$, i.e., there exists a unique kernel lift, say $\xi_{\alpha,\beta}\colon \mathrm{im}(\alpha) \to \ker(\beta)$, of κ_α along ι_β. Since $\xi_{\alpha,\beta} \cdot \iota_\beta = \kappa_\alpha$ and both ι_β and κ_α are monomorphisms, $\xi_{\alpha,\beta}$ is a monomorphism as well. We call $\xi_{\alpha,\beta}$ the canonical embedding of $\mathrm{im}(\alpha)$ in $\ker(\beta)$. The **homology object** or **defect of exactness** of the differential pair (α, β) is defined by $\mathrm{coker}(\xi_{\alpha,\beta})$.

The following two examples are the primary examples of Abelian categories in this thesis.

Example A.52. Let R be a left coherent ring (cf. Definition 2.32). Then, the category R-**fpmod** of finitely presented left R-modules is Abelian. We model this category on the computer using Freyd categories (cf. Section 2.2.3).

Example A.53. Let k be a commutative ring and $\mathscr{A}$ a k-linear category. The category $[\mathscr{A}, \mathscr{C}]$ of k-linear functors from $\mathscr{A}$ into an Abelian category $\mathscr{C}$ is also Abelian (cf. Section 2.2.7).

APPENDIX B

Background from Triangulated Categories

In this chapter we give a compressed account of the triangulated categories and their properties. For a more extensive treatment we refer to the standard sources, for example [**HJR10**], [**Hap88**], [**May01**], [**GM03**] and [**Nee01**].

Definition B.1. Let $\mathfrak{T}$ be an additive category. A structure of a **triangulated category** $(\mathfrak{T}, \triangle, \Sigma)$ on $\mathfrak{T}$ is given by the following data:

(1) An additive autoequivalence $\Sigma\colon \mathfrak{T} \to \mathfrak{T}$, called the **shift autoequivalence** of $\mathfrak{T}$,
(2) A class $\triangle$ of **exact** triangles

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A),$$

subject to the following axioms:

TR 1. The following holds:

(a) Any morphism $\alpha\colon A \to B$ can be completed by an object $\mathrm{Cone}(\alpha)$ in $\mathfrak{T}$ and two morphisms $\iota(\alpha)$ and $\pi(\alpha)$ to an exact triangle

$$A \xrightarrow{\alpha} B \xrightarrow{\iota(\alpha)} \mathrm{Cone}(\alpha) \xrightarrow{\pi(\alpha)} \Sigma(A),$$

called the **standard exact triangle** associated to α. The object $\mathrm{Cone}(\alpha)$ is called the **standard cone object** associated to α.

(b) Any triangle of the form $A \xrightarrow{\mathrm{id}_A} A \to 0 \to \Sigma(A)$ is exact.
(c) Any triangle isomorphic to an exact triangle is itself exact.

TR 2. For any exact triangle

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A),$$

the triple

$$B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A) \xrightarrow{-\Sigma(\alpha)} \Sigma(B).$$

defines an exact triangle. We refer to this axiom as the **rotation axiom**.

TR 3. If the rows of the following diagram are exact triangles and u, v are morphisms with $\alpha_1 \bullet v = u \bullet \alpha_2$, then there exists a morphism w (not necessarily unique) that renders the following diagram

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\iota_1} & C_1 & \xrightarrow{\pi_1} & \Sigma(A_1) \\
\downarrow{\scriptstyle u} & \circlearrowleft & \downarrow{\scriptstyle v} & \circlearrowleft & \vdots{\scriptstyle \exists w} & \circlearrowleft & \downarrow{\scriptstyle \Sigma(u)} \\
A_2 & \xrightarrow[\alpha_2]{} & B_2 & \xrightarrow[\iota_2]{} & C_2 & \xrightarrow[\pi_2]{} & \Sigma(A_2)
\end{array}$$

commutative.

TR 4. Any triple of exact triangles

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} U \xrightarrow{\pi} \Sigma(A),$$

$$B \xrightarrow{\beta} C \xrightarrow{\xi} V \xrightarrow{\tau} \Sigma(B)$$

and

$$A \xrightarrow{\gamma} C \xrightarrow{\epsilon} V \xrightarrow{\lambda} \Sigma(A)$$

with $\gamma = \alpha \bullet \beta$ can be completed via an exact triangle

$$U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} \Sigma(U)$$

into a commutative diagram:

$$\begin{array}{ccccccc}
A & \xrightarrow{\alpha} & B & \xrightarrow{\iota} & U & \xrightarrow{\pi} & \Sigma(A) \\
 & \circlearrowleft \; \gamma \searrow & \downarrow{\scriptstyle \beta} & \circlearrowleft & \vdots{\scriptstyle u} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} \\
 & & C & \xrightarrow{\epsilon} & V & \xrightarrow{\lambda} & \Sigma(A) \\
 & & \downarrow{\scriptstyle \xi} & \circlearrowleft & \vdots{\scriptstyle v} & \circlearrowleft & \downarrow{\scriptstyle \Sigma(\alpha)} \\
 & & W & \xrightarrow[\mathrm{id}]{} & W & \xrightarrow[\tau]{} & \Sigma(B) \\
 & & \downarrow{\scriptstyle \tau} & \circlearrowleft & \vdots{\scriptstyle w} & & \\
 & & \Sigma(B) & \xrightarrow[\Sigma(\iota)]{} & \Sigma(U) & &
\end{array}$$

Remark B.2. Since $\Sigma\colon \mathfrak{T} \to \mathfrak{T}$ is an autoequivalence, there is an endofunctor $\Sigma^{-1}\colon \mathfrak{T} \to \mathfrak{T}$ and an adjunction $\Sigma^{-1} \dashv \Sigma$ whose unit $\eta\colon \mathrm{id}_{\mathfrak{T}} \Rightarrow \Sigma^{-1} \bullet \Sigma$ and counit $\epsilon\colon \Sigma \bullet \Sigma^{-1} \Rightarrow \mathrm{id}_{\mathfrak{T}}$ are natural isomorphisms.

Remark B.3. By the unit-counit triangle identities, we have for each object A in $\mathfrak{T}$ the equalities $\mathrm{id}_{\Sigma^{-1}(A)} = \big(\Sigma^{-1}(\eta_A)\big) \bullet \epsilon_{\Sigma^{-1}(A)}$ and $\mathrm{id}_{\Sigma(A)} = \eta_{\Sigma(A)} \bullet \Sigma(\epsilon_A)$. In other words $\Sigma^{-1}(\eta_A) = \epsilon^{-1}_{\Sigma^{-1}(A)}$ and $\Sigma(\epsilon_A) = \eta^{-1}_{\Sigma(A)}$.

Remark B.4. Every triangulated category is equivalent to a triangulated category whose shift functor is an automorphism, i.e., Σ and Σ^{-1} are inverses to each other "on the nose". For details we refer the reader to [**Ver96**].

Lemma B.5. *Let $\mathfrak{T}$ be a triangulated category and let*

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$$

be an exact triangle in $\mathfrak{T}$. Then the following triangles

$$A \xrightarrow{-\alpha} B \xrightarrow{-\iota} C \xrightarrow{\pi} \Sigma(A),$$

$$A \xrightarrow{\alpha} B \xrightarrow{-\iota} C \xrightarrow{-\pi} \Sigma(A)$$

and

$$A \xrightarrow{-\alpha} B \xrightarrow{\iota} C \xrightarrow{-\pi} \Sigma(A)$$

are also exact.

Proof. It is easy to prove that each of the given three triangles is isomorphic to

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A).$$

For instance, the isomorphism from

$$A \xrightarrow{-\alpha} B \xrightarrow{\iota} C \xrightarrow{-\pi} \Sigma(A),$$

is given by the triple $(\mathrm{id}_A, -\mathrm{id}_B, -\mathrm{id}_C)$. □

Lemma B.6. *Let $\mathfrak{T}$ be a triangulated category and let*

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$$

be an exact triangle in $\mathfrak{T}$. Then, for any object U in $\mathfrak{T}$, the two sequences

$$\mathrm{Hom}_{\mathfrak{T}}(U, A) \xrightarrow{-\bullet\alpha} \mathrm{Hom}_{\mathfrak{T}}(U, B) \xrightarrow{-\bullet\iota} \mathrm{Hom}_{\mathfrak{T}}(U, C)$$

and

$$\mathrm{Hom}_{\mathfrak{T}}(A, U) \xleftarrow{\alpha\bullet -} \mathrm{Hom}_{\mathfrak{T}}(B, U) \xleftarrow{\iota\bullet -} \mathrm{Hom}_{\mathfrak{T}}(C, U)$$

are exact.

Proof. Since $\mathrm{Hom}_{\mathfrak{T}}(U, -)$ is an additive functor and $\alpha \bullet \iota = 0$, it follows that $\mathrm{im}(-\bullet\alpha) \subseteq \ker(-\bullet\iota)$. Let u be some morphism in $\ker(-\bullet\iota)$, i.e., $u \bullet \iota = 0$, then by **TR 2** and **TR 3**, there exists a morphism $w\colon \Sigma(U) \to \Sigma(A)$ inducing a morphism of exact triangles:

$$\begin{array}{ccccccc}
U & \longrightarrow & 0 & \longrightarrow & \Sigma(U) & \xrightarrow{-\mathrm{id}_{\Sigma(U)}} & \Sigma(U) \\
{\scriptstyle u}\downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \vdots\,{\scriptstyle \exists w} & \circlearrowleft & \downarrow{\scriptstyle \Sigma(u)} \\
B & \xrightarrow[\iota]{} & C & \xrightarrow[\pi]{} & \Sigma(A) & \xrightarrow[-\Sigma(\alpha)]{} & \Sigma(B)
\end{array}$$

If we take $\chi\colon U \to A$ to be *the*[1] morphism which satisfies $\Sigma(\chi) = w$, then $u = \chi \bullet \alpha$ and $u \in \mathrm{im}(-\bullet\alpha)$. Hence, $\ker(-\bullet\iota) \subseteq \mathrm{im}(-\bullet\alpha)$ and the first sequence is exact. The exactness of the second sequence follows by a similar argument. □

Definition B.7. Let $\mathfrak{T}$ be a triangulated category and $\mathscr{A}$ an Abelian category. An additive functor $H\colon \mathfrak{T} \to \mathscr{A}$ is called **homological functor** if for every exact triangle

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$$

the sequence

$$H(A) \xrightarrow{H(\alpha)} H(B) \xrightarrow{H(\iota)} H(C)$$

is exact. Similarly, a contravariant functor $L\colon \mathfrak{T} \to \mathscr{A}$ is called **cohomological functor** if for every exact triangle

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$$

[1]It can be computed using the unit of the adjunction $\Sigma^{-1} \dashv \Sigma$.

the sequence

$$L(A) \xleftarrow{L(\alpha)} L(B) \xleftarrow{L(\iota)} L(C)$$

is exact. In other words, a cohomological functor from $\mathfrak{T}$ to $\mathscr{A}$ is a homological functor from $\mathfrak{T}$ to $\mathscr{A}^{\mathrm{op}}$.

Example B.8. Let U be an object in a triangulated category $\mathfrak{T}$, then the functor $\mathrm{Hom}_{\mathfrak{T}}(U, -)\colon \mathfrak{T} \to$ **Ab** is a homological functor, while $\mathrm{Hom}_{\mathfrak{T}}(-, U)\colon \mathfrak{T} \to$ **Ab** is a cohomological functor. This is an immediate consequence of Lemma B.6.

Example B.9. Let $\mathscr{C}$ be an Abelian category. Then its homotopy category $\mathcal{K}(\mathscr{C})$ is triangulated and the 0-cohomology functor $H^0\colon \mathcal{K}(\mathscr{C}) \to \mathscr{C}$ is cohomological.

Lemma B.10. *Let $\mathfrak{T}$ be a triangulated category and let A, B, C be objects in $\mathfrak{T}$. If*

$$\mathrm{Hom}_{\mathfrak{T}}(\Sigma^r(C), A) \cong \mathrm{Hom}_{\mathfrak{T}}(\Sigma^r(C), B) = 0$$

for all $r > 0$, then

$$\mathrm{Hom}_{\mathfrak{T}}(\Sigma^r(C), \mathrm{Cocone}(\alpha)) = 0$$

for all $r > 0$ and morphisms $\alpha\colon A \to B$.

Proof. Since Σ is an autoequivalence, the assumption $\mathrm{Hom}_{\mathfrak{T}}(\Sigma^r(C), A) \cong \mathrm{Hom}_{\mathfrak{T}}(\Sigma^r(C), B) = 0$ for all $r > 0$ is equivalent to $\mathrm{Hom}_{\mathfrak{T}}(C, \Sigma^r(A)) \cong \mathrm{Hom}_{\mathfrak{T}}(C, \Sigma^r(B)) = 0$ for all $r < 0$. Since $\mathrm{Hom}_{\mathfrak{T}}(C, -)$ is a homological functor, we get the following long exact sequence

$$\cdots \xrightarrow{-\bullet(-\Sigma^{-3}(\alpha))} \mathrm{Hom}_{\mathfrak{T}}(C, \Sigma^{-3}(B)) \xrightarrow{-\bullet(-\Sigma^{-3}(\iota))} \mathrm{Hom}_{\mathfrak{T}}(C, \Sigma^{-3}(\mathrm{Cone}(\alpha))) \xrightarrow{-\bullet(-\Sigma^{-1}(\pi))}$$

$$\mathrm{Hom}_{\mathfrak{T}}(C, \Sigma^{-2}(A)) \xrightarrow{-\bullet\Sigma^{-2}(\alpha)} \mathrm{Hom}_{\mathfrak{T}}(C, \Sigma^{-2}(B)) \xrightarrow{-\bullet\Sigma^{-2}(\iota)} \mathrm{Hom}_{\mathfrak{T}}(C, \Sigma^{-2}(\mathrm{Cone}(\alpha))) \xrightarrow{-\bullet\pi}$$

$$\mathrm{Hom}_{\mathfrak{T}}(C, \Sigma^{-1}(A)) \xrightarrow{-\bullet(-\Sigma^{-1}(\alpha))} \mathrm{Hom}_{\mathfrak{T}}(C, \Sigma^{-1}(B)) \xrightarrow{-\bullet(-\Sigma^{-1}(\iota))} \mathrm{Hom}_{\mathfrak{T}}(C, \Sigma^{-1}(\mathrm{Cone}(\alpha))) \xrightarrow{-\bullet(-\Sigma^{1}(\pi))}$$

$$\mathrm{Hom}_{\mathfrak{T}}(C, A) \xrightarrow[-\bullet\alpha]{} \mathrm{Hom}_{\mathfrak{T}}(C, B) \xrightarrow[-\bullet\iota]{} \cdots$$

from which we find out that $\mathrm{Hom}_{\mathfrak{T}}(C, \Sigma^r(\mathrm{Cone}(\alpha))) = 0$ for all $r < -1$, which holds if and only if $\mathrm{Hom}_{\mathfrak{T}}(C, \Sigma^r(\mathrm{Cocone}(\alpha))) = 0$ for all $r < 0$, which is equivalent to the desired assertion: $\mathrm{Hom}_{\mathfrak{T}}(\Sigma^r(C), \mathrm{Cocone}(\alpha)) = 0$ for all $r > 0$. $\square$

Lemma B.11. *If the morphisms u, v in* **TR 3** *are isomorphisms, then so is w.*

Proof. By the **TR 2**, we can extend the morphism of exact triangles in **TR 3** to the following commutative diagram:

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\iota_1} & C_1 & \xrightarrow{\pi_1} & \Sigma(A_1) & \xrightarrow{-\Sigma(\alpha_1)} & \Sigma(B_1) \\
\downarrow u & & \downarrow v & & \downarrow \exists w & & \downarrow \Sigma(u) & & \downarrow \Sigma(v) \\
A_2 & \xrightarrow[\alpha_2]{} & B_2 & \xrightarrow[\iota_2]{} & C_2 & \xrightarrow[\pi_2]{} & \Sigma(A_2) & \xrightarrow[-\Sigma(\alpha_2)]{} & \Sigma(B_2)
\end{array}$$

For an arbitrary object U in $\mathfrak{T}$, applying $\mathrm{Hom}_{\mathfrak{T}}(U, -)$ gives rise to a commutative diagram whose rows are exact sequences and whose first and last two columns are isomorphisms. Hence, by the 5-lemma, the morphism $- \bullet w\colon \mathrm{Hom}_{\mathfrak{T}}(U, C_1) \to \mathrm{Hom}_{\mathfrak{T}}(U, C_2)$ is also an isomorphism. Hence, by Corollary 2.89, w is also an isomorphism. □

Remark B.12. By repeatedly applying **TR 2** and the fact that Σ is conservative (see Lemma A.15), we can prove that if two out of u, v and w are isomorphisms, then so is the third.

Lemma B.13. *Let $\mathfrak{T}$ be a triangulated category. A triangle*

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$$

is exact if and only if

$$B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A) \xrightarrow{-\Sigma(\alpha)} \Sigma(B)$$

is.

Proof. The direct implication follows from **TR 2**. To prove the converse, we need to construct an exactness witness, i.e., an isomorphism $\psi\colon C \xrightarrow{\sim} \mathrm{Cone}(\alpha)$ with $\iota \bullet \psi = \iota(\alpha)$ and $\psi \bullet \pi(\alpha) = \pi$.

Applying **TR 2** multiple times to

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$$

yields the following two exact triangles:

$$\Sigma(A) \xrightarrow{-\Sigma(\alpha)} \Sigma(B) \xrightarrow{-\Sigma(\iota)} \Sigma(C) \xrightarrow{-\Sigma(\pi)} \Sigma^2(A)$$

and

$$\Sigma(A) \xrightarrow{-\Sigma(\alpha)} \Sigma(B) \xrightarrow{-\Sigma(\iota(\alpha))} \Sigma(\mathrm{Cone}(\alpha)) \xrightarrow{-\Sigma(\pi(\alpha))} \Sigma^2(A).$$

By **TR 3** and Lemma B.11, there exists an isomorphism $w\colon \Sigma(C) \to \Sigma(\mathrm{Cone}(\alpha))$ with $\Sigma(\iota) \bullet w = \Sigma(\iota(\alpha))$ and $w \bullet \Sigma(\pi(\alpha)) = \Sigma(\pi)$. The functor Σ is fully faithful, hence by Lemma A.15, it is also conservative, i.e., there exists a (unique) isomorphism $\psi\colon C \to \mathrm{Cone}(\alpha)$ with $\Sigma(\psi) = w$. Since Σ is faithful, we have $\iota \bullet \psi = \iota(\alpha)$ and $\psi \bullet \pi(\alpha) = \pi$ as desired. □

Lemma B.14. *Let $\mathfrak{T}$ be a triangulated category. A triangle*

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$$

is exact if and only if its ***inverse rotation*** *triangle*

$$\Sigma^{-1}(C) \xrightarrow{\left(-\Sigma^{-1}(\pi)\right) \bullet \epsilon_A} A \xrightarrow{\alpha} B \xrightarrow{\iota \bullet \eta_C} \Sigma\Big(\Sigma^{-1}(C)\Big)$$

is.

Proof. The assertion follows by Lemma B.13 and by the fact that the rotation of the triangle

$$\Sigma^{-1}(C) \xrightarrow{\left(-\Sigma^{-1}(\pi)\right)\bullet\epsilon_A} A \xrightarrow{\alpha} B \xrightarrow{\iota\bullet\eta_C} \Sigma\Big(\Sigma^{-1}(C)\Big)$$

is isomorphic to the triangle

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$$

via the following isomorphism of triangles[2]:

$$\begin{array}{ccccccc}
A & \xrightarrow{\quad\alpha\quad} & B & \xrightarrow{\quad\iota\quad} & C & \xrightarrow{\qquad\pi\qquad} & \Sigma(A) \\
\Big\downarrow{\scriptstyle \mathrm{id}_A} & \circlearrowleft & \Big\downarrow{\scriptstyle \mathrm{id}_B} & \circlearrowleft & \Big\downarrow{\scriptstyle \eta_C} & \circlearrowleft & \Big\downarrow{\scriptstyle \mathrm{id}_{\Sigma(A)}} \\
A & \xrightarrow[\alpha]{\qquad} & B & \xrightarrow[\iota\bullet\eta_C]{} & \Sigma(\Sigma^{-1}(C)) & \xrightarrow[-\Sigma(-\Sigma^{-1}(\pi))\bullet\Sigma(\epsilon_A)]{} & \Sigma(A)
\end{array}$$

□

Corollary B.15. *Let $\mathfrak{T}$ be a triangulated category with shift automorphism Σ. Then a triangle*

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$$

is exact if and only if its inverse rotation

$$\Sigma^{-1}(C) \xrightarrow{-\Sigma^{-1}(\pi)} A \xrightarrow{\alpha} B \xrightarrow{\iota} C$$

is also exact.

Corollary B.16. *Let $\mathfrak{T}$ be a triangulated category with shift automorphism Σ and let*

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$$

be an exact triangle in $\mathfrak{T}$. Then, for any object U in $\mathfrak{T}$, the following long sequences

$$\cdots \xrightarrow{-\bullet(-\Sigma^{-1}(\alpha))} \mathrm{Hom}_{\mathfrak{T}}(U,\Sigma^{-1}(B)) \xrightarrow{-\bullet(-\Sigma^{-1}(\iota))} \mathrm{Hom}_{\mathfrak{T}}(U,\Sigma^{-1}(C))$$

$$\xrightarrow{-\bullet(-\Sigma^{-1}(\pi))} \mathrm{Hom}_{\mathfrak{T}}(U,A) \xrightarrow{-\bullet\alpha} \mathrm{Hom}_{\mathfrak{T}}(U,B) \xrightarrow{-\bullet\iota} \mathrm{Hom}_{\mathfrak{T}}(U,C)$$

$$\xrightarrow{-\bullet\pi} \mathrm{Hom}_{\mathfrak{T}}(U,\Sigma(A)) \xrightarrow{-\bullet(-\Sigma(\alpha))} \mathrm{Hom}_{\mathfrak{T}}(U,\Sigma(B)) \xrightarrow{-\bullet(-\Sigma(\iota))} \cdots$$

[2]See Remark B.3

and

$$\cdots \xleftarrow{(-\Sigma^{-1}(\alpha))\bullet -} \mathrm{Hom}_{\mathfrak{T}}(\Sigma^{-1}(B),U) \xleftarrow{(-\Sigma^{-1}(\iota))\bullet -} \mathrm{Hom}_{\mathfrak{T}}(\Sigma^{-1}(C),U)$$

$$\mathrm{Hom}_{\mathfrak{T}}(\Sigma^{-1}(C),U) \xleftarrow{(-\Sigma^{-1}(\pi))\bullet -} \mathrm{Hom}_{\mathfrak{T}}(A,U) \xleftarrow{\alpha\bullet -} \mathrm{Hom}_{\mathfrak{T}}(B,U) \xleftarrow{\iota\bullet -} \mathrm{Hom}_{\mathfrak{T}}(C,U)$$

$$\mathrm{Hom}_{\mathfrak{T}}(C,U) \xleftarrow{\pi\bullet -} \mathrm{Hom}_{\mathfrak{T}}(\Sigma(A),U) \xleftarrow{(-\Sigma(\alpha))\bullet -} \mathrm{Hom}_{\mathfrak{T}}(\Sigma(B),U) \xleftarrow{(-\Sigma(\iota))\bullet -} \cdots$$

are exact.

PROOF. By repeatedly applying the axiom **TR 2** and Corollary B.15, we can extend the exact triangle

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$$

into a so called **helix**, i.e., an infinite sequence of morphisms in $\mathfrak{T}$ where each three consecutive morphisms form an exact triangle. Henceforth, the assertions follow by applying the functors $\mathrm{Hom}_{\mathfrak{T}}(U,-)$ resp. $\mathrm{Hom}_{\mathfrak{T}}(-,U)$ on the morphisms of the helix, and then by Lemma B.6. □

Notation B.17. For any integer $i > 0$, we denote by

(1) Σ^i the i-fold composition of the autoequivalence Σ,
(2) Σ^{-i} the i-fold composition of the autoequivalence Σ^{-1} and
(3) Σ^0 the identity functor $\mathrm{id}_{\mathfrak{T}}$.

Lemma B.18. *Let $\mathfrak{T}$ be a triangulated category with shift automorphism Σ and let*

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$$

be an exact triangle in $\mathfrak{T}$. Then for all $i \in \mathbb{Z}$, the triangle

$$\Sigma^i(A) \xrightarrow{\Sigma^i(\alpha)} \Sigma^i(B) \xrightarrow{\Sigma^i(\iota)} \Sigma^i(C) \xrightarrow{(-1)^i\Sigma^i(\pi)} \Sigma^{i+1}(A)$$

is also exact.

PROOF. By applying the the rotation Axiom **TR 2** three times on the given exact triangle, we get the exact triangle:

$$\Sigma(A) \xrightarrow{-\Sigma(\alpha)} \Sigma(B) \xrightarrow{-\Sigma(\iota)} \Sigma(C) \xrightarrow{-\Sigma(\pi)} \Sigma^2(A).$$

Hence, by Lemma B.5 the triangle

$$\Sigma(A) \xrightarrow{\Sigma(\alpha)} \Sigma(B) \xrightarrow{\Sigma(\iota)} \Sigma(C) \xrightarrow{-\Sigma(\pi)} \Sigma^2(A)$$

is also exact. Similarly, by applying Corollary B.15 three times, we get the exact triangle:

$$\Sigma^{-1}(A) \xrightarrow{-\Sigma^{-1}(\alpha)} \Sigma^{-1}(B) \xrightarrow{-\Sigma^{-1}(\iota)} \Sigma^{-1}(C) \xrightarrow{-\Sigma^{-1}(\pi)} A,$$

i.e., the triangle

$$\Sigma^{-1}(A) \xrightarrow{\Sigma^{-1}(\alpha)} \Sigma^{-1}(B) \xrightarrow{\Sigma^{-1}(\iota)} \Sigma^{-1}(C) \xrightarrow{-\Sigma^{-1}(\pi)} A$$

is also exact. Therefore, the assertion follows by a forward resp. backward induction on the values of $i \geq 0$ resp. $i \leq 0$. $\square$

Corollary B.19. *Let $\mathfrak{T}$ be a triangulated category. Then any morphism $\alpha\colon A \to B$ in $\mathfrak{T}$ can be extended, up to isomorphism*[3]*, to only one exact triangle.*

Proof. Any exact triangle of the form

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A).$$

is, by Lemma B.11, isomorphic to the standard exact triangle

$$A \xrightarrow{\alpha} B \xrightarrow{\iota(\alpha)} \mathrm{Cone}(\alpha) \xrightarrow{\pi(\alpha)} \Sigma(A)$$

associated to α, simply by setting u resp. v to id_A resp. id_B. $\square$

Remark B.20. By Corollary B.19, exact triangles with equal domains define an equivalence relation on the set of all exact triangles and each equivalence class can be represented by a standard exact triangle.

Definition B.21. A **witness of exactness** of a triangle

$$A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$$

is defined by an isomorphism $\lambda\colon C \xrightarrow{\sim} \mathrm{Cone}(\alpha)$ which satisfies $\iota \bullet \lambda = \iota(\alpha)$ and $\lambda \bullet \pi(\alpha) = \pi$.

Lemma B.22. *Let $\mathfrak{T}$ be a triangulated category. Then a morphism $\alpha\colon A \to B$ in $\mathfrak{T}$ is an isomorphism if and only if* $\mathrm{Cone}(\alpha)$ *is zero.*

Proof. If α is an isomorphism, then the assertion follows by **TR 1**, **TR 3** and Lemma B.11:

$$\begin{array}{ccccccc}
A & \xrightarrow{\alpha} & B & \xrightarrow{\iota(\alpha)} & \mathrm{Cone}(\alpha) & \xrightarrow{\pi(\alpha)} & \Sigma(A) \\
{\scriptstyle\alpha}\downarrow{\scriptstyle\sim} & \circlearrowleft & {\scriptstyle\mathrm{id}_B}\downarrow{\scriptstyle\sim} & \circlearrowleft & {\scriptstyle\exists w}\,\vdots\,{\scriptstyle\sim} & \circlearrowleft & \downarrow{\scriptstyle\Sigma(\alpha)} \\
B & \xrightarrow[\mathrm{id}_B]{} & B & \longrightarrow & 0 & \longrightarrow & \Sigma(B)
\end{array}$$

On the other hand, if $\mathrm{Cone}(\alpha)$ is zero, then the triangle $A \xrightarrow{\alpha} B \to 0 \to \Sigma(A)$ is exact and so is then its inverse rotation $0 \to A \xrightarrow{\alpha} B \to 0$. Hence, by **TR 1**, **TR 3** and Lemma B.11 there exists an isomorphism of exact triangles

$$\begin{array}{ccccccc}
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \longrightarrow & 0 \\
\downarrow{\scriptstyle\sim} & \circlearrowleft & {\scriptstyle\mathrm{id}_A}\downarrow{\scriptstyle\sim} & \circlearrowleft & {\scriptstyle\exists w}\,\vdots\,{\scriptstyle\sim} & \circlearrowleft & \downarrow \\
0 & \longrightarrow & A & \xrightarrow[\mathrm{id}_A]{} & A & \longrightarrow & 0
\end{array}$$

$\square$

Furthermore, we find in [**LH09**, Exercise 1.4.2.1], [**Nee01**, pp. 42-45] or [**Hap88**, Lemma 1.4] the following equivalent statements.

[3]Not necessarily unique.

Lemma B.23. *Let $\mathfrak{T}$ be a triangulated category and $\alpha\colon A \to B$ a morphism in $\mathfrak{T}$ and*

$$A \xrightarrow{\alpha} B \xrightarrow{\iota(\alpha)} \operatorname{Cone}(\alpha) \xrightarrow{\pi(\alpha)} \Sigma(A)$$

the standard exact triangle associated to α; then the following statements are equivalent:

(1) α is a monomorphism,
(2) $\iota(\alpha)$ is an epimorphism,
(3) $\pi(\alpha)\colon \operatorname{Cone}(\alpha) \to \Sigma(A)$ is zero,
(4) there exist morphisms $A \xleftarrow{s} B \xleftarrow{t} \operatorname{Cone}(\alpha)$ such that

$$\alpha \bullet s = \mathrm{id}_A, \qquad s \bullet \alpha + \iota(\alpha) \bullet t = \mathrm{id}_B, \qquad t \bullet \iota(\alpha) = \mathrm{id}_{\operatorname{Cone}(\alpha)},$$

i.e., $B \cong A \oplus \operatorname{Cone}(\alpha)$.

In other words, every monomorphism in a triangulated category is split and every epimorphism is also split.

Definition B.24. Let $F\colon \mathfrak{T}_1 \to \mathfrak{T}_2$ be an additive functor between triangulated categories. Then F is called **exact** if

(1) There exists a natural isomorphism $\mu\colon \Sigma_1 \bullet F \Rightarrow F \bullet \Sigma_2$.
(2) If $A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma_1 A$ is an exact triangle, then the triangle

$$F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\iota)} F(C) \xrightarrow{F(\pi)\bullet \mu_A} \Sigma_2 F(A)$$

is as well.

Remark B.25. Let $\mathfrak{T}_1, \mathfrak{T}_2$ be two triangulated categories. For any exact functor $F\colon \mathfrak{T}_1 \to \mathfrak{T}_2$, there is a natural isomorphism $\Sigma_1^{-1} \bullet F \Rightarrow F \bullet \Sigma_2^{-1}$. This is a direct consequence of Lemma 2.86 and the following isomorphisms for any object A in $\mathfrak{T}_1$ and U in $\mathfrak{T}_2$

$$\begin{aligned}
\operatorname{Hom}_{\mathfrak{T}_2}\Big(U, F(\Sigma_1^{-1}(A))\Big) &\cong \operatorname{Hom}_{\mathfrak{T}_2}\Big(\Sigma_2(U), \Sigma_2(F(\Sigma_1^{-1}(A)))\Big)\\
&\cong \operatorname{Hom}_{\mathfrak{T}_2}\Big(\Sigma_2(U), F(\Sigma_1(\Sigma_1^{-1}(A)))\Big)\\
&\cong \operatorname{Hom}_{\mathfrak{T}_2}(\Sigma_2(U), F(A))\\
&\cong \operatorname{Hom}_{\mathfrak{T}_2}\Big(U, \Sigma_2^{-1}(F(A))\Big).
\end{aligned}$$

Definition B.26. Let $\mathfrak{T}$ be a triangulated category. A full replete[4] additive subcategory $\mathfrak{D}$ of $\mathfrak{T}$ is called **triangulated subcategory** of $\mathfrak{T}$ if

(1) $\mathfrak{D}$ is closed under the functors Σ and Σ^{-1}.
(2) For any exact triangle $A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$ in $\mathfrak{T}$, if A and B are in $\mathfrak{D}$, then C is also in $\mathfrak{D}$.

A triangulated subcategory $\mathfrak{D}$ is called **thick** if for any direct sum $A \oplus B$ in $\mathfrak{D}$, both A and B are also in $\mathfrak{D}$. In other words, $\mathfrak{D}$ is closed under taking direct summands.

Remark B.27. Since Σ is an autoequivalence and the triangulated subcategory $\mathfrak{D}$ is closed under Σ and Σ^{-1}, we can restrict Σ to an autoequivalence on $\mathfrak{D}$. A triangle in $\mathfrak{D}$ will be declared as exact if it is exact in $\mathfrak{T}$. In this way, $\mathfrak{D}$ is a triangulated category and the inclusion functor $\mathfrak{D} \hookrightarrow \mathfrak{T}$ is exact.

[4]A subcategory $\mathscr{D} \subseteq \mathscr{C}$ is called **replete** if for any object A in $\mathscr{D}$ and any isomorphism $\alpha\colon A \xrightarrow{\sim} B$ in $\mathscr{C}$, both B and α are also in $\mathscr{D}$.

Remark B.28. For any triangulated subcategory $\mathscr{D} \subseteq \mathfrak{T}$, the rotation axiom **TR 2** implies that for any exact triangle $A \xrightarrow{\alpha} B \xrightarrow{\iota} C \xrightarrow{\pi} \Sigma(A)$ in $\mathfrak{T}$, if two out of the objects A, B or C are in $\mathfrak{D}$, then so is the third.

Example B.29. Let $F\colon \mathfrak{T}_1 \to \mathfrak{T}_2$ be an exact functor. Then the full subcategory generated by $\ker(F) \coloneqq \{A \in \mathfrak{T}_1 | F(A) = 0\}$ is a triangulated subcategory of $\mathfrak{T}_1$.

Lemma B.30. *Let $F\colon \mathfrak{T}_1 \to \mathfrak{T}_2$ be an exact functor between triangulated categories. If F admits a right adjoint $G\colon \mathfrak{T}_2 \to \mathfrak{T}_1$, then G is also exact.*

Proof. In the following we will only construct the natural isomorphism $\mu\colon \Sigma_2 \bullet G \to G \bullet \Sigma_1$ that turns G into an exact functor. For a complete proof we refer the reader to [**Nee01**, Lemma 5.3.6] or [**Huy06**, Proposition 1.41]. Let A be an object in $\mathfrak{T}_2$. Since F is exact we can construct a natural isomorphism $\xi\colon \operatorname{Hom}_{\mathfrak{T}_1}(-, G(\Sigma_2(A))) \Rightarrow \operatorname{Hom}_{\mathfrak{T}_1}(-, \Sigma_1(G(A)))$, whose component at an object U in $\mathfrak{T}_1$ is given by the isomorphism $\xi_U\colon \operatorname{Hom}_{\mathfrak{T}_1}(U, G(\Sigma_2(A))) \to \operatorname{Hom}_{\mathfrak{T}_1}(U, \Sigma_1(G(A)))$ defined by the following equalities:

$$\begin{aligned}
\operatorname{Hom}_{\mathfrak{T}_1}(U, G(\Sigma_2(A))) &\cong \operatorname{Hom}_{\mathfrak{T}_2}(F(U), \Sigma_2(A)) \\
&\cong \operatorname{Hom}_{\mathfrak{T}_2}\Big(\Sigma_2^{-1}(F(U)), A\Big) \\
&\cong \operatorname{Hom}_{\mathfrak{T}_2}\Big(F(\Sigma_1^{-1}(U)), A\Big) \\
&\cong \operatorname{Hom}_{\mathfrak{T}_1}\Big(\Sigma_1^{-1}(U), G(A)\Big) \\
&\cong \operatorname{Hom}_{\mathfrak{T}_1}(U, \Sigma_1(G(A))).
\end{aligned}$$

By Lemma 2.86, the natural isomorphism ξ corresponds to *the* canonical isomorphism $\mu_A \coloneqq \xi_{G(\Sigma_2(A))}\Big(\mathrm{id}_{G(\Sigma_2(A))}\Big)\colon G(\Sigma_2(A)) \to \Sigma_1(G(A))$. This defines the natural isomorphism $\mu\colon \Sigma_2 \bullet G \Rightarrow G \bullet \Sigma_1,\ A \mapsto \mu_A$. □

APPENDIX C

A Demo for Computing $\mathrm{Ext}^n(A, B)$ as $\mathrm{Hom}(A, \Sigma^n(B))$ in $\mathcal{D}^b(\mathbb{Q}[x, y]\text{-}\mathbf{fpmod})$

Let $\mathbb{Q}$ be the field of rationals and R the polynomial ring $\mathbb{Q}[x, y]$. The category R-**fpmod** of finitely presented left R-modules can be modeled by the FREYD category $\mathcal{A}(R\text{-}\mathbf{rows})$ where R-**rows** is the category of rows over R. Since R is computable and commutative, the category $\mathcal{A}(R\text{-}\mathbf{rows})$ is Abelian with enough projectives and is equipped with an $\mathcal{A}(R\text{-}\mathbf{rows})$-homomorphism structure that is equivalent to the external Hom bifunctor (cf. Section 2.2.3).

The R-module $\mathbb{Q}$ is presented by the matrix $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. In the following we construct $\mathbb{Q}$ as an object in $\mathcal{A}(R\text{-}\mathbf{rows})$ and compute $\mathrm{Ext}^1(\mathbb{Q}, \mathbb{Q})$ using two approaches.

We start by loading the JULIA package `CapAndHomalg` [**CAP21a**] and the `GAP` packages `FreydCategoriesForCAP` [**BP19a**] and `DerivedCategories` [**Sal21c**]:

```
julia> using CapAndHomalg
CapAndHomalg v1.1.8
Imported OSCAR's components GAP and Singular_jll
Type: ?CapAndHomalg for more information
julia> LoadPackage( "FreydCategoriesForCAP" )
julia> LoadPackage( "DerivedCategories" )
```

Next, we construct the ring R and the categories R-**rows** and $\mathcal{A}(R\text{-}\mathbf{rows})$:

```
julia> ℚ = HomalgFieldOfRationalsInDefaultCAS( )
GAP: Q

julia> R = ℚ["x,y"]
GAP: Q[x,y]

julia> Rrows = CategoryOfRows( R )
GAP: Rows( Q[x,y] )

julia> InfoOfInstalledOperationsOfCategory( Rrows )
59 primitive operations were used to derive 238 operations for this category which
* IsEquippedWithHomomorphismStructure
* IsLinearCategoryOverCommutativeRing
* IsRigidSymmetricClosedMonoidalCategory
* IsClosedMonoidalCategory
* IsAdditiveCategory

julia> Rmod = FreydCategory( Rrows )
GAP: Freyd( Rows( Q[x,y] ) )
```

```
julia> InfoOfInstalledOperationsOfCategory( Rmod )
57 primitive operations were used to derive 324 operations for this category which
* IsEquippedWithHomomorphismStructure
* IsLinearCategoryOverCommutativeRing
* IsSymmetricClosedMonoidalCategory
* IsClosedMonoidalCategory
* IsAbelianCategoryWithEnoughProjectives

julia> m = HomalgMatrix( "[ [ x ], [ y ] ]", 2, 1, R )
GAP: <A 2 x 1 matrix over an external ring>

julia> m = m / Rrows
GAP: <A morphism in Rows( Q[x,y] )>

julia> Q = m / Rmod
GAP: <An object in Freyd( Rows( Q[x,y] ) )>

julia> Show( Q )
```

$$\left(R^{1\times 2} \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} R^{1\times 1}\right)_{\mathcal{A}}$$

Next, we create the categories $\mathcal{C}^b(\mathcal{A}(R\text{-}\mathbf{rows})), \mathcal{K}^b(\mathcal{A}(R\text{-}\mathbf{rows}))$ and $\mathcal{D}^b(\mathcal{A}(R\text{-}\mathbf{rows}))$; then interpret $\mathbb{Q}$ as an object in $\mathcal{D}^b(\mathcal{A}(R\text{-}\mathbf{rows}))$:

```
julia> C_R = ComplexCategoryByCochains( Rmod )
GAP: Cochain complexes( Freyd( Rows( Q[x,y] ) ) )

julia> K_R = HomotopyCategoryByCochains( Rmod )
GAP: Homotopy category( Freyd( Rows( Q[x,y] ) ) )

julia> D_R = DerivedCategoryByCochains( Rmod )
GAP: Derived category( Freyd( Rows( Q[x,y] ) ) )

julia> Q = Q / C_R / K_R / D_R
GAP: <An object in Derived category( Freyd( Rows( Q[x,y] ) ) ) with active lower bound
    0 and active upper bound 0>
```

The *first approach* toward computing $\mathrm{Ext}^n(\mathbb{Q},\mathbb{Q})$ depends on computing it as the n'th derived functor of the external Hom functor

$$\mathrm{Hom}_{\mathcal{A}(R\text{-}\mathbf{rows})}(-,\mathbb{Q})\colon \mathcal{A}(R\text{-}\mathbf{rows})^{\mathrm{op}} \to \mathcal{A}(R\text{-}\mathbf{rows}) \subset \mathbf{Ab}$$

i.e., we apply $\mathrm{Hom}_{\mathcal{A}(R\text{-}\mathbf{rows})}(-,\mathbb{Q})$ "degreewise" on a projective resolution of $\mathbb{Q}$ and then we compute the n'th cohomology object. The *second approach* depends on Definition 3.52, i.e., on computing a generating set for the external Hom group

$$\mathrm{Ext}^n(\mathbb{Q},\mathbb{Q}) \coloneqq \mathrm{Hom}_{\mathcal{D}^b(\mathcal{A}(R\text{-}\mathbf{rows}))}(\mathbb{Q},\Sigma^n(\mathbb{Q})).$$

The first approach yields only a presentation matrix of $\mathrm{Ext}^n(\mathbb{Q},\mathbb{Q})$ as an R-finitely presented module, while the second approach yields an explicit generating set for $\mathrm{Ext}^n(\mathbb{Q},\mathbb{Q})$ as a group of morphisms in the bounded derived category.

In the following we perform the first approach:

```
julia> Q = HomalgMatrix( "[ [ x ], [ y ] ]", 2, 1, R ) / Rrows / Rmod
GAP: <An object in Freyd( Rows( Q[x,y] ) )>

julia> PQ = ProjectiveCochainResolution( Q, true )
GAP: <An object in Cochain complexes( Freyd( Rows( Q[x,y] ) ) ) with active lower bound
     -2 and active upper bound 0>

julia> Show( PQ )
```

$$\begin{array}{c}
\left(R^{1\times 0} \xrightarrow{()_{0\times 1}} R^{1\times 1}\right)_{\mathcal{A}} \\
\uparrow \\
\begin{pmatrix} -x \\ -y \end{pmatrix} \\
\vert_{-1} \\
\left(R^{1\times 0} \xrightarrow{()_{0\times 2}} R^{1\times 2}\right)_{\mathcal{A}} \\
\uparrow \\
\begin{pmatrix} y & -x \end{pmatrix} \\
\vert_{-2} \\
\left(R^{1\times 0} \xrightarrow{()_{0\times 1}} R^{1\times 1}\right)_{\mathcal{A}}
\end{array}$$

```
julia> Hom_PQ_Q = CochainComplex(
                        [ HomStructure( DifferentialAt( PQ, -1 ), Q ),
                          HomStructure( DifferentialAt( PQ, -2 ), Q ) ],
                        0
                    )
GAP: <An object in Cochain complexes( Freyd( Rows( Q[x,y] ) ) ) with active lower bound
     0 and active upper bound 2>

julia> Show( Hom_PQ_Q )
```

$$\left(R^{1\times 2} \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} R^{1\times 1}\right)_{\mathcal{A}}$$
$$\uparrow$$
$$\begin{pmatrix} y \\ -x \end{pmatrix}$$
$$|_1$$
$$\left(R^{1\times 4} \xrightarrow{\begin{pmatrix} x & \cdot \\ y & \cdot \\ \cdot & x \\ \cdot & y \end{pmatrix}} R^{1\times 2}\right)_{\mathcal{A}}$$
$$\uparrow$$
$$\begin{pmatrix} -x & -y \end{pmatrix}$$
$$|_0$$
$$\left(R^{1\times 2} \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} R^{1\times 1}\right)_{\mathcal{A}}$$

```
julia> ext1 = CohomologyAt( Hom_PQ_Q, 1 )
GAP: <An object in Freyd( Rows( Q[x,y] ) )>

julia> Show( ext1 )
```

$$\left(R^{1\times 8} \xrightarrow{\begin{pmatrix} \cdot & y \\ y & \cdot \\ \cdot & x \\ x & \cdot \\ \cdot & x \\ \cdot & y \\ x & \cdot \\ y & \cdot \end{pmatrix}} R^{1\times 2}\right)_{\mathcal{A}}$$

```
julia> ext1 = SimplifyObject( ext1, infinity )
GAP: <An object in Freyd( Rows( Q[x,y] ) )>

julia> Show( ext1 )
```

$$\left(R^{1\times 4} \xrightarrow{\begin{pmatrix} \cdot & y \\ y & \cdot \\ \cdot & x \\ x & \cdot \end{pmatrix}} R^{1\times 2}\right)_{\mathcal{A}}$$

This says that $\mathrm{Ext}^1(\mathbb{Q},\mathbb{Q})$ is presented by the above matrix, i.e., it is generated by two elements subject to 4 relations.

In the following we perform the second approach:

$$\mathrm{Ext}^1(\mathbb{Q},\mathbb{Q}) := \mathrm{Hom}_{\mathcal{D}^b(\mathcal{A}(R\text{-}\mathbf{rows}))}\Big(\mathbb{Q},\Sigma^1(\mathbb{Q})\Big).$$

By Corollary 4.35, $\mathcal{D}^b(\mathcal{A}(R\text{-}\mathbf{rows}))$ can be equipped with an $\mathcal{A}(R\text{-}\mathbf{rows})$-homomorphism structure. The generators of $\mathrm{Ext}^1(\mathbb{Q},\mathbb{Q})$ can be computed via the isomorphism

$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{A}(R\text{-}\mathbf{rows}))}\Big(\mathbb{Q},\Sigma^1(\mathbb{Q})\Big) \xrightarrow{\nu_{\mathbb{Q},\Sigma^1(\mathbb{Q})}} \mathrm{Hom}_{\mathcal{A}(R\text{-}\mathbf{rows})}\Big(\mathbb{1},H\Big(\mathbb{Q},\Sigma^1(\mathbb{Q})\Big)\Big).$$

by first computing a generating set for

$$\mathrm{Hom}_{\mathcal{A}(R\text{-}\mathbf{rows})}\Big(\mathbb{1},H\Big(\mathbb{Q},\Sigma^1(\mathbb{Q})\Big)\Big);$$

and then computing their pre-images under the isomorphism $\nu_{\mathbb{Q},\Sigma^1(\mathbb{Q})}$.

We start by computing the distinguished object $\mathbb{1}$ of the $\mathcal{A}(R\text{-}\mathbf{rows})$-homomorphism structure of $\mathcal{D}^b(\mathcal{A}(R\text{-}\mathbf{rows}))$:

```
julia> 𝟙 = DistinguishedObjectOfHomomorphismStructure( D_R )
GAP: <A projective object in Freyd( Rows( Q[x,y] ) )>

julia> Show( 𝟙 )
```

$$\left(R^{1\times 0} \xrightarrow{()_{0\times 1}} R^{1\times 1}\right)_{\mathcal{A}}$$

I.e., the distinguished object $\mathbb{1}$ corresponds under the equivalence $\mathcal{A}(R\text{-}\mathbf{rows}) \cong R\text{-}\mathbf{fpmod}$ to the finitely presented row R-module $R^{1\times 1}$.

Next, we compute $H(\mathbb{Q},\Sigma^1(\mathbb{Q}))$ in $\mathcal{A}(R\text{-}\mathbf{rows})$:

```
julia> H_Q_shiftQ = HomomorphismStructureOnObjects( Q, Shift( Q, 1 ) )
GAP: <An object in Freyd( Rows( Q[x,y] ) )>

julia> Show( H_Q_shiftQ )
```

$$\left(R^{1\times 5} \xrightarrow{\begin{pmatrix} -y & x & 1 \\ \cdot & x & 1 \\ x & \cdot & \cdot \\ \cdot & -y & \cdot \\ -y & \cdot & 1 \end{pmatrix}} R^{1\times 3}\right)_{\mathcal{A}}$$

$\mathrm{Hom}_{\mathcal{A}(R\text{-}\mathbf{rows})}(\mathbb{1},H(\mathbb{Q},\Sigma^1(\mathbb{Q})))$ is generated as an R-module by the following three morphisms:

$$g_1 := \quad \left(R^{1\times 0} \xrightarrow{()_{0\times 1}} R^{1\times 1}\right)_{\mathcal{A}} \xrightarrow{\begin{pmatrix} 1 & \cdot & \cdot \end{pmatrix}} \left(R^{1\times 5} \xrightarrow{\begin{pmatrix} -y & x & 1 \\ \cdot & x & 1 \\ x & \cdot & \cdot \\ \cdot & -y & \cdot \\ -y & \cdot & 1 \end{pmatrix}} R^{1\times 3}\right)_{\mathcal{A}},$$

$$g_2 := \qquad \left(R^{1\times 0} \xrightarrow{()_{0\times 1}} R^{1\times 1}\right)_{\mathcal{A}} \xrightarrow{\begin{pmatrix} \cdot & 1 & \cdot \end{pmatrix}} \left(R^{1\times 5} \xrightarrow{\begin{pmatrix} -y & x & 1 \\ \cdot & x & 1 \\ x & \cdot & \cdot \\ \cdot & -y & \cdot \\ -y & \cdot & 1 \end{pmatrix}} R^{1\times 3}\right)_{\mathcal{A}},$$

$$g_3 := \qquad \left(R^{1\times 0} \xrightarrow{()_{0\times 1}} R^{1\times 1}\right)_{\mathcal{A}} \xrightarrow{\begin{pmatrix} \cdot & \cdot & 1 \end{pmatrix}} \left(R^{1\times 5} \xrightarrow{\begin{pmatrix} -y & x & 1 \\ \cdot & x & 1 \\ x & \cdot & \cdot \\ \cdot & -y & \cdot \\ -y & \cdot & 1 \end{pmatrix}} R^{1\times 3}\right)_{\mathcal{A}}.$$

That is, $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{A}(R\text{-}\mathbf{rows}))}(\mathbb{Q},\Sigma^1(\mathbb{Q}))$ is generated (over R) by $\varphi_i := \nu^{-1}_{\mathbb{Q},\Sigma^1(\mathbb{Q})}(g_i)$ for $i = 1,2,3$.

```
julia> m1 = HomalgMatrix( "[ [ 1, 0, 0 ] ]", 1, 3, R ) / Rows
GAP: <A morphism in Rows( Q[x,y] )>

julia> g1 = FreydCategoryMorphism( 𝟙, m1, H_Q_shiftQ )
GAP: <A morphism in Freyd( Rows( Q[x,y] ) )>

julia> φ1 = InterpretMorphismFromDistinguishedObjectToHomomorphismStructureAsMorphism(
    Q, Shift( Q, 1 ), g1 )
GAP: <A morphism in Derived category( Freyd( Rows( Q[x,y] ) ) )>

julia> Show( φ1 ) # morphisms in D^b(A(R-rows)) are defined by roofs over K^b(A(R-rows))
```

$$\begin{array}{ccccccccc}
\left(R^{1\times 2} \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} R^{1\times 1}\right)_{\mathcal{A}} & \leftarrow & \begin{pmatrix} -1 \end{pmatrix} & - & \left(R^{1\times 0} \xrightarrow{()_{0\times 1}} R^{1\times 1}\right)_{\mathcal{A}} & - & ()_{1\times 0} & \rightarrow & \left(R^{1\times 0} \xrightarrow{()_{0\times 0}} R^{1\times 0}\right)_{\mathcal{A}} \\
\uparrow & & & & \uparrow & & & & \uparrow \\
()_{0\times 1} & & & & \begin{pmatrix} y \\ x \end{pmatrix} & & & & ()_{1\times 0} \\
|_{-1} & & & & |_{-1} & & & & |_{-1} \\
\left(R^{1\times 0} \xrightarrow{()_{0\times 0}} R^{1\times 0}\right)_{\mathcal{A}} & \leftarrow & ()_{2\times 0} & - & \left(R^{1\times 0} \xrightarrow{()_{0\times 2}} R^{1\times 2}\right)_{\mathcal{A}} & - & \begin{pmatrix} -1 \\ \cdot \end{pmatrix} & \rightarrow & \left(R^{1\times 2} \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} R^{1\times 1}\right)_{\mathcal{A}} \\
\uparrow & & & & \uparrow & & & & \uparrow \\
()_{0\times 0} & & & & \begin{pmatrix} x & -y \end{pmatrix} & & & & ()_{0\times 1} \\
|_{-2} & & & & |_{-2} & & & & |_{-2} \\
\left(R^{1\times 0} \xrightarrow{()_{0\times 0}} R^{1\times 0}\right)_{\mathcal{A}} & \leftarrow & ()_{1\times 0} & - & \left(R^{1\times 0} \xrightarrow{()_{0\times 1}} R^{1\times 1}\right)_{\mathcal{A}} & - & ()_{1\times 0} & \rightarrow & \left(R^{1\times 0} \xrightarrow{()_{0\times 0}} R^{1\times 0}\right)_{\mathcal{A}}
\end{array}$$

```
julia> m2 = HomalgMatrix( "[ [ 0, 1, 0 ] ]", 1, 3, R ) / Rows
GAP: <A morphism in Rows( Q[x,y] )>

julia> g2 = FreydCategoryMorphism( 𝟙, m2, H_Q_shiftQ )
GAP: <A morphism in Freyd( Rows( Q[x,y] ) )>

julia> φ2 = InterpretMorphismFromDistinguishedObjectToHomomorphismStructureAsMorphism(
    Q, Shift( Q, 1 ), g2 )
GAP: <A morphism in Derived category( Freyd( Rows( Q[x,y] ) ) )>
```

```
julia> Show( φ2 )
```

$$
\begin{array}{ccccccccc}
\left(R^{1\times 2} \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} R^{1\times 1}\right)_{\mathcal{A}} & \leftarrow & \begin{pmatrix} -1 \end{pmatrix} & - & \left(R^{1\times 0} \xrightarrow{()_{0\times 1}} R^{1\times 1}\right)_{\mathcal{A}} & - & ()_{1\times 0} & \rightarrow & \left(R^{1\times 0} \xrightarrow{()_{0\times 0}} R^{1\times 0}\right)_{\mathcal{A}} \\
\uparrow & & & & \uparrow & & & & \uparrow \\
()_{0\times 1} & & & & \begin{pmatrix} y \\ x \end{pmatrix} & & & & ()_{1\times 0} \\
|_{-1} & & & & |_{-1} & & & & |_{-1} \\
\left(R^{1\times 0} \xrightarrow{()_{0\times 0}} R^{1\times 0}\right)_{\mathcal{A}} & \leftarrow & ()_{2\times 0} & - & \left(R^{1\times 0} \xrightarrow{()_{0\times 2}} R^{1\times 2}\right)_{\mathcal{A}} & - & \begin{pmatrix} \cdot \\ 1 \end{pmatrix} & \rightarrow & \left(R^{1\times 2} \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} R^{1\times 1}\right)_{\mathcal{A}} \\
\uparrow & & & & \uparrow & & & & \uparrow \\
()_{0\times 0} & & & & \begin{pmatrix} x & -y \end{pmatrix} & & & & ()_{0\times 1} \\
|_{-2} & & & & |_{-2} & & & & |_{-2} \\
\left(R^{1\times 0} \xrightarrow{()_{0\times 0}} R^{1\times 0}\right)_{\mathcal{A}} & \leftarrow & ()_{1\times 0} & - & \left(R^{1\times 0} \xrightarrow{()_{0\times 1}} R^{1\times 1}\right)_{\mathcal{A}} & - & ()_{1\times 0} & \rightarrow & \left(R^{1\times 0} \xrightarrow{()_{0\times 0}} R^{1\times 0}\right)_{\mathcal{A}}
\end{array}
$$

```
julia> m3 = HomalgMatrix( "[ [ 0, 0, 1 ] ]", 1, 3, R ) / Rows
GAP: <A morphism in Rows( Q[x,y] )>

julia> g3 = FreydCategoryMorphism( 𝟙, m3, H_Q_shiftQ )
GAP: <A morphism in Freyd( Rows( Q[x,y] ) )>

julia> φ3 = InterpretMorphismFromDistinguishedObjectToHomomorphismStructureAsMorphism(
    Q, Shift( Q, 1 ), g3 )
GAP: <A morphism in Derived category( Freyd( Rows( Q[x,y] ) ) )>

julia> Show( φ3 )
```

$$
\begin{array}{ccccccccc}
\left(R^{1\times 2} \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} R^{1\times 1}\right)_{\mathcal{A}} & \leftarrow & \begin{pmatrix} -1 \end{pmatrix} & - & \left(R^{1\times 0} \xrightarrow{()_{0\times 1}} R^{1\times 1}\right)_{\mathcal{A}} & - & ()_{1\times 0} & \rightarrow & \left(R^{1\times 0} \xrightarrow{()_{0\times 0}} R^{1\times 0}\right)_{\mathcal{A}} \\
\uparrow & & & & \uparrow & & & & \uparrow \\
()_{0\times 1} & & & & \begin{pmatrix} y \\ x \end{pmatrix} & & & & ()_{1\times 0} \\
|_{-1} & & & & |_{-1} & & & & |_{-1} \\
\left(R^{1\times 0} \xrightarrow{()_{0\times 0}} R^{1\times 0}\right)_{\mathcal{A}} & \leftarrow & ()_{2\times 0} & - & \left(R^{1\times 0} \xrightarrow{()_{0\times 2}} R^{1\times 2}\right)_{\mathcal{A}} & - & \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} & \rightarrow & \left(R^{1\times 2} \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} R^{1\times 1}\right)_{\mathcal{A}} \\
\uparrow & & & & \uparrow & & & & \uparrow \\
()_{0\times 0} & & & & \begin{pmatrix} x & -y \end{pmatrix} & & & & ()_{0\times 1} \\
|_{-2} & & & & |_{-2} & & & & |_{-2} \\
\left(R^{1\times 0} \xrightarrow{()_{0\times 0}} R^{1\times 0}\right)_{\mathcal{A}} & \leftarrow & ()_{1\times 0} & - & \left(R^{1\times 0} \xrightarrow{()_{0\times 1}} R^{1\times 1}\right)_{\mathcal{A}} & - & ()_{1\times 0} & \rightarrow & \left(R^{1\times 0} \xrightarrow{()_{0\times 0}} R^{1\times 0}\right)_{\mathcal{A}}
\end{array}
$$

We notice that $\varphi_3 := \nu^{-1}(g_3) = 0$, hence g_3 should also be zero:

```
julia> IsZero( g_3 )
true
```

Hence, $\mathrm{Ext}^1(\mathbb{Q},\mathbb{Q})$ is generated (over R) by $\{\varphi_1,\varphi_2\}$. This could have been detected if we had first simplified $H(\mathbb{Q},\Sigma^1(\mathbb{Q}))$:

```
julia> sH_Q_shiftQ = SimplifyObject( H_Q_shiftQ, infinity )
<An object in Freyd( Rows( Q[x,y] ) )>

julia> Show( sH_Q_sigmaQ )
```

$$\left(R^{1\times 4} \xrightarrow{\begin{pmatrix} y & \cdot \\ x & \cdot \\ \cdot & -y \\ \cdot & -x \end{pmatrix}} R^{1\times 2}\right)_{\mathcal{A}}$$

```
julia> f = SimplifyObject_IsoToInputObject( H_Q_shiftQ, infinity )
GAP: <A morphism in Freyd( Rows( Q[x,y] ) )>

julia> IsIsomorphism( f )
true

julia> Show( f )
```

$$\left(R^{1\times 4} \xrightarrow{\begin{pmatrix} y & \cdot \\ x & \cdot \\ \cdot & -y \\ \cdot & -x \end{pmatrix}} R^{1\times 2}\right)_{\mathcal{A}} \xrightarrow{\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix}} \left(R^{1\times 5} \xrightarrow{\begin{pmatrix} -y & x & 1 \\ \cdot & x & 1 \\ x & \cdot & \cdot \\ \cdot & -y & \cdot \\ -y & \cdot & 1 \end{pmatrix}} R^{1\times 3}\right)_{\mathcal{A}}$$

APPENDIX D

A Demo for the Stable Category of a Frobenius Category

Let $E := \mathbb{Q}[e_0, e_1, e_2]$ be the $\mathbb{Z}$-graded exterior algebra with $\deg e_i = -1, i = 0, 1, 2$. In the following, we will equip the category E-**fpgrmod** of finitely presented graded E-modules with the class of lifting objects introduced in Example 2.60; then construct its associated stable category.

We start by constructing the $\mathbb{Z}$-graded exterior $\mathbb{Q}$-algebra E:

```
gap> LoadPackage( "FreydCategoriesForCAP" );
true
gap> LoadPackage( "StableCategories" );
true
gap> Q := HomalgFieldOfRationalsInDefaultCAS( );
Q
gap> S := GradedRing( Q["x,y,z"] );
Q[x,y,z]
(weights: yet unset)
gap> SetWeightsOfIndeterminates( S, [ 1, 1, 1 ] );
gap> E := KoszulDualRing( S );
Q{e0,e1,e2}
(weights: [ -1, -1, -1 ])
```

Let E-**grrows** be the category of graded rows over E and $\mathcal{A}(E\text{-}\mathbf{grrows})$ its Freyd category. Then E-**fpgrmod** $\cong \mathcal{A}(E\text{-}\mathbf{grrows})$ (cf. Example 2.37). It follows from Remark 4.14 and Example 4.9 and Corollary 4.28 that E-**grrows** and $\mathcal{A}(E\text{-}\mathbf{grrows})$ are equipped with a $\mathbb{Q}$-**mat**-homomorphism structures.

```
gap> E_GRows := CategoryOfGradedRows( E );
Graded rows( Q{e0,e1,e2} (with weights [ -1, -1, -1 ]) )
gap> InfoOfInstalledOperationsOfCategory( E_GRows );
45 primitive operations were used to derive 161 operations for this category which
* IsEquippedWithHomomorphismStructure
* IsLinearCategoryOverCommutativeRing
* IsAdditiveCategory
gap> RangeCategoryOfHomomorphismStructure( E_GRows );
Category of matrices over Q
gap> E_fpgrmod := FreydCategory( E_GRows : FinalizeCategory := false );
Category of f.p. graded left modules over Q{e0,e1,e2} (with weights [ -1, -1, -1 ])
gap> InfoOfInstalledOperationsOfCategory( E_fpgrmod );
40 primitive operations were used to derive 112 operations for this category which
* IsEquippedWithHomomorphismStructure
* IsLinearCategoryOverCommutativeRing
* IsAbelianCategoryWithEnoughProjectives
```

```
gap> RangeCategoryOfHomomorphismStructure( E_fpgrmod )
Category of matrices over Q
```

Next, we equip $\mathcal{A}(E\text{-}\mathbf{grrows})$ with the class of lifting objects introduced in Example 2.60. In order to do so, we need to "add" the corresponding categorical "primitive methods"[1] to $\mathcal{A}(E\text{-}\mathbf{grrows})$ (cf. [**Sal21e**]).

```
gap> AddIsLiftingObject( E_fpgrmod,
        {cat, obj} -> IsProjective( obj )
);

gap> AddLiftingObject( E_fpgrmod,
        {cat, obj} -> SomeProjectiveObject( obj )
);

gap> AddMorphismFromLiftingObject( E_fpgrmod,
        {cat, obj} -> EpimorphismFromSomeProjectiveObject( obj )
);

gap> AddSectionOfMorphismFromLiftingObject( E_fpgrmod,
        {cat, obj} -> ProjectiveLift(
                            IdentityMorphism( obj ),
                            MorphismFromLiftingObject( obj )
                      )
);

gap> AddLiftingMorphismWithGivenLiftingObjects( E_fpgrmod,
        function( cat, L_S, alpha, L_R )
            local S, R, ell_S, ell_R;
            S := Source( alpha );
            R := Range( alpha );
            ell_S := MorphismFromLiftingObject( S );
            ell_R := MorphismFromLiftingObject( R );
            return ProjectiveLift( PreCompose( ell_S, alpha ), ell_R );
        end
);

gap> Finalize( E_fpgrmod );
gap> InfoOfInstalledOperationsOfCategory( E_fpgrmod );
44 primitive operations were used to derive 283 operations for this category which
* IsEquippedWithHomomorphismStructure
* IsLinearCategoryOverCommutativeRing
* IsAbelianCategoryWithEnoughInjectives
* IsAbelianCategoryWithEnoughProjectives
```

[1] For the documentation of primitive methods we refer to [**GSP22**].

Since $\mathcal{A}(E\text{-}\mathbf{grrows})$ is equipped with $\mathbb{Q}$-$\mathbf{mat}$-homomorphism structure, it has decidable linear systems (cf. Theorem 4.17). In particular, it has decidable lifts. Hence, we can derive an algorithm to decide whether a morphism $\varphi\colon M \to N$ lifts along $\ell_N\colon L_N \to N$. That is, the associated stable category has decidable equality of morphisms (cf. Remark 2.56).

```
gap> CanCompute( E_fpgrmod, "IsLiftable" );
true
gap> CanCompute( E_fpgrmod, "IsLiftableAlongMorphismFromLiftingObject" );
true
gap> stable_E_fpgrmod := StableCategoryByClassOfLiftingObjects( E_fpgrmod );
Stable category( Category of f.p. graded left modules over Q{e0,e1,e2} (with weights [
    -1, -1, -1 ]) ) defined by a class of lifting objects
gap> InfoOfInstalledOperationsOfCategory( stable_E_fpgrmod );
40 primitive operations were used to derive 121 operations for this category which
* IsEquippedWithHomomorphismStructure
* IsLinearCategoryOverCommutativeRing
* IsAdditiveCategory
gap> RangeCategoryOfHomomorphismStructure( stable_E_fpgrmod )
Category of matrices over Q
```

Consider the following objects

$$M := \left(E(1) \xrightarrow{\begin{pmatrix} 2e_0 - 4e_1 & 3e_0e_1 & 5e_0e_1 \end{pmatrix}} E(0) \oplus E(-1)^{\oplus 2}\right)_{\mathcal{A}}$$

and

$$N := \left(E(0)^{\oplus 3} \xrightarrow{\begin{pmatrix} -e_0 & -2e_0 & -3e_0 \\ -3e_0 & e_0 + 2e_1 & 2e_0 + 2e_1 \\ 4e_0 + 6e_2 & -3e_0 + 3e_1 & e_2 \end{pmatrix}} E(-1)^{\oplus 3}\right)_{\mathcal{A}}$$

in $\mathcal{A}(E\text{-}\mathbf{grrows})$. In the following we construct the morphism

$$\varphi\colon M \xrightarrow{\begin{pmatrix} \frac{1}{3}e_0 & \cdot & \cdot \\ 1 & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix}} N$$

in $\mathcal{A}(E\text{-}\mathbf{grrows})$ and check whether $[\varphi] = 0$.

```
gap> sM := GradedRow( [ [ [ 1 ], 1 ] ], E );
<A graded row of rank 1>
gap> rM := GradedRow( [ [ [ 0 ], 1 ], [ [ -1 ], 2 ] ], E );
<A graded row of rank 3>
gap> m := HomalgMatrix( "[ [ 2*e0-4*e1, 3*e0*e1, 5*e0*e1 ] ]", 1, 3, E );
<A 1 x 3 matrix over a graded ring>
gap> m := GradedRowOrColumnMorphism( sM, m, rM );
<A morphism in Category of graded rows over Q{e0,e1,e2} (with weights [ -1, -1, -1 ])>
gap> M := m / E_fpgrmod;
<An object in Category of f.p. graded left modules over Q{e0,e1,e2} (with weights [ -1,
     -1, -1 ])>
gap> sN := GradedRow( [ [ [ 0 ], 3 ] ], E );
<A graded row of rank 3>
```

```
gap> rN := GradedRow( [ [ [ -1 ], 3 ] ], E );
<A graded row of rank 3>
gap> n := HomalgMatrix(  "[ [ -e0, -2*e0, -3*e0 ], \
                            [ -3*e0, e0+2*e1, 2*e0+2*e1 ], \
                            [ 4*e0+6*e2, -3*e0+3*e1, e2 ] ]", 3, 3, E );
<A 3 x 3 matrix over a graded ring>
gap> n := GradedRowOrColumnMorphism( sN, n, rN );
<A morphism in Category of graded rows over Q{e0,e1,e2} (with weights [ -1, -1, -1 ])>
gap> N := n / E_fpgrmod;
<An object in Category of f.p. graded left modules over Q{e0,e1,e2} (with weights [ -1,
     -1, -1 ])>
gap> IsWellDefined( M ) and IsWellDefined( N );
true
gap> phi := HomalgMatrix(  "[ [ 1/3*e0, 0, 0 ], \
                              [ 1,      0, 0 ], \
                              [ 1,      0, 0 ] ]", 3, 3, E );
<A 3 x 3 matrix over a graded ring>
gap> phi := GradedRowOrColumnMorphism( rM, phi, rN );
<A morphism in Category of graded rows over Q{e0,e1,e2} (with weights [ -1, -1, -1 ])>
gap> phi := FreydCategoryMorphism( M, phi, N );
<A morphism in Category of f.p. graded left modules over Q{e0,e1,e2} (with weights [
    -1, -1, -1 ])>
gap> IsWellDefined( phi );
true
gap> IsZeroForMorphisms( phi );
false
gap> P := ProjectionFunctor( stable_E_fpgrmod );
Canonical projection onto stable category
gap> Display( P );
Canonical projection onto stable category:
Category of f.p. graded left modules over Q{e0,e1,e2} (with weights [ -1, -1, -1 ])
  |
  V
Stable category( Category of f.p. graded left modules over Q{e0,e1,e2} (with weights [
    -1, -1, -1 ]) )
gap> P_phi := P( phi );
<A morphism in Stable category( Category of f.p. graded left modules over Q{e0,e1,e2} (
    with weights [ -1, -1, -1 ]) ) defined by IsLiftableThroughLiftingObject>
gap> IsZeroForMorphisms( P_phi );
true
```

Since $[\varphi] = 0$, there exists a lift morphism $E\colon M \to L_N$ of φ along $\ell_N\colon L_N \to N$.

```
gap> L_N := LiftingObject( N );
<A projective object in Category of f.p. graded left modules over Q{e0,e1,e2} (with
    weights [ -1, -1, -1 ])>
gap> Show( L_N );
```

$$\left(0 \xrightarrow{()_{0\times 3}} E(-1)^{\oplus 3}\right)_{\mathcal{A}}$$

```
gap> ell_N := MorphismFromLiftingObject( N );
<A morphism in Category of f.p. graded left modules over Q{e0,e1,e2} (with weights [
    -1, -1, -1 ])>
gap> Show( ell_N );
```

$$\left(0 \xrightarrow{()_{0\times 3}} E(-1)^{\oplus 3}\right)_{\mathcal{A}} \xrightarrow{\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}} \left(E(0)^{\oplus 3} \xrightarrow{\begin{pmatrix} -e_0 & -2e_0 & -3e_0 \\ -3e_0 & e_0+2e_1 & 2e_0+2e_1 \\ 4e_0+6e_2 & -3e_0+3e_1 & e_2 \end{pmatrix}} E(-1)^{\oplus 3}\right)_{\mathcal{A}}$$

```
gap> lambda := WitnessForBeingLiftableAlongMorphismFromLiftingObject( phi );
<A morphism in Category of f.p. graded left modules over Q{e0,e1,e2} (with weights [
    -1, -1, -1 ])>
gap> Show( lambda );
```

$$\left(E(1) \xrightarrow{\begin{pmatrix} 2e_0-4e_1 & 3e_0e_1 & 5e_0e_1 \end{pmatrix}} E(0) \oplus E(-1)^{\oplus 2}\right)_{\mathcal{A}} \xrightarrow{\begin{pmatrix} -2e_0 & -\frac{7}{12}e_0+\frac{7}{6}e_1 & -\frac{7}{12}e_0+\frac{7}{6}e_1 \\ 1 & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix}} \left(0 \xrightarrow{()_{0\times 3}} E(-1)^{\oplus 3}\right)_{\mathcal{A}}$$

```
gap> IsCongruentForMorphisms( PreCompose( lambda, ell_N ), phi );
true
```

Using Remark 4.12 we can compute bases of the $\mathbb{Q}$-vector spaces $\mathrm{Hom}_{\mathcal{A}(E\text{-}\mathbf{grrows})}(M,N)$ and $\mathrm{Hom}_{\mathcal{A}(E\text{-}\mathbf{grrows})/\mathcal{L}}(P(M),P(N))$:

```
gap> HomomorphismStructureOnObjects( M, N );
<A vector space object over Q of dimension 11>
gap> Hom_MN := BasisOfExternalHom( M, N );;
gap> Length( Hom_MN );
11
```

The morphism-datum matrices (cf. Definition 2.28) of the above 11 morphisms are given by:

$$\left\{ \begin{pmatrix} e_0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} e_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} -\frac{1}{12}e_0 & \cdot & -\frac{1}{4}e_0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} -2e_2 & \frac{11}{3}e_0 & 4e_0-\frac{1}{3}e_2 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} -210e_2 & 11e_2 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} \right\}.$$

```
gap> PM := P( M );;
gap> PN := P( N );;
gap> HomomorphismStructureOnObjects( PM, PN );
<A vector space object over Q of dimension 2>
gap> Hom_PM_PN := BasisOfExternalHom( PM, PN );;
gap> Length( Hom_PM_PN );
```

2

The morphism-datum matrices of the above two morphisms are given by:

$$\{\begin{pmatrix} -210e_2 & 11e_2 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}\}.$$

The category $\mathcal{A}(E\text{-}\mathbf{grrows})$ is Abelian with enough projectives and injectives and since E is a Frobenius algebra, the category $\mathcal{A}(E\text{-}\mathbf{grrows})$ is Frobenius (cf. Example 5.37).

Next, we compute the natural isomorphism $\nu(M)\colon M \xrightarrow{\sim} M^{**}$.

```
gap> nu_M := IsomorphismOntoDoubleDualOfFpModuleByFreyd( M );
<A morphism in Category of f.p. graded left modules over Q{e0,e1,e2} (with weights [
    -1, -1, -1 ])>
gap> Show( nu_M );
```

$$(R(1) \xrightarrow{\begin{pmatrix} 2e_0-4e_1 & 3e_0e_1 & 5e_0e_1 \end{pmatrix}} R(0) \oplus R(-1)^{\oplus 2})_{\mathcal{A}} \xrightarrow{\begin{pmatrix} 5e_1 & \cdot & 5 \\ \cdot & 5 & \cdot \\ -2 & -3 & \cdot \end{pmatrix}} (R(1) \xrightarrow{\begin{pmatrix} \cdot & \cdot & e_0-2e_1 \end{pmatrix}} R(-1)^{\oplus 2} \oplus R(0))_{\mathcal{A}}$$

In the following, we compute a monomorphism q_M from M into some injective object Q_M.

```
gap> q_M := MonomorphismIntoSomeInjectiveObject( M );
<A morphism in Category of f.p. graded left modules over Q{e0,e1,e2} (with weights [
    -1, -1, -1 ])>
gap> IsMonomorphism( q_M );
true
gap> IsInjective( Range( q_M ) );
true
gap> Show( q_M );
```

$$(R(1) \xrightarrow{\begin{pmatrix} 2e_0-4e_1 & 3e_0e_1 & 5e_0e_1 \end{pmatrix}} R(0) \oplus R(-1)^{\oplus 2})_{\mathcal{A}} \xrightarrow{\begin{pmatrix} \cdot & 5e_1 & 5e_0 \\ 5 & \cdot & \cdot \\ -3 & -2 & -4 \end{pmatrix}} (0 \xrightarrow{()_{0\times 3}} R(-1)^{\oplus 3})_{\mathcal{A}}$$

Since $[\varphi] = 0$, there exists a colift morphism of φ along q_M:

```
gap> lambda := Colift( q_M, phi );
<A morphism in Category of f.p. graded left modules over Q{e0,e1,e2} (with weights [
    -1, -1, -1 ])>
gap> Show( lambda );
```

$$(0 \xrightarrow{()_{0\times 3}} R(-1)^{\oplus 3})_{\mathcal{A}} \xrightarrow{\begin{pmatrix} \frac{1}{5} & \cdot & \cdot \\ \cdot & \frac{7}{30} & \frac{7}{30} \\ -\frac{2}{5} & -\frac{7}{60} & -\frac{7}{60} \end{pmatrix}} (R(0)^{\oplus 3} \xrightarrow{\begin{pmatrix} -e_0 & -2e_0 & -3e_0 \\ -3e_0 & e_0+2e_1 & 2e_0+2e_1 \\ 4e_0+6e_2 & -3e_0+3e_1 & e_2 \end{pmatrix}} R(-1)^{\oplus 3})_{\mathcal{A}}$$

```
gap> IsCongruentForMorphisms( PreCompose( q_M, lambda ), phi );
true
```

APPENDIX E

A Demo for the Happel Theorem

Let k be a field, $\mathfrak{q}$ be a finite right quiver. In Section 2.2.5 we defined the free category $\mathcal{F}_{\mathfrak{q}}$, its linear closure $k\mathcal{F}_{\mathfrak{q}}$ and the k-linear finitely presented category $\mathbf{A} := k\mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle$ defined by $\mathfrak{q}$ subject to a set of relations $\rho \subseteq k\mathcal{F}_{\mathfrak{q}}$.

The category of k-linear functors $\mathrm{Hom}_k(\mathbf{A}, k\text{-}\mathbf{mat})$ is denoted[1] by **mod-A** and is called the category of **A**-modules[2]. That is

(1) an object F in **mod-A** is a functor $F : \mathbf{A} \to k\text{-vec}$ and its data structure is a pair of lists. Namely, a list of vector spaces (represents the images of the objects of **A** under F) and a list of k-linear maps (represents the images of the generating morphisms of **A** under F);
(2) a morphism $\psi : F \to G$ is a natural transformation and its data structure is a list of morphisms (represents the images of the objects of **A** under ψ).

In this appendix we use the Julia package `CapAndHomalg` [**CAP21a**] to demonstrate the following computations:

(1) Create a quiver $\mathfrak{q}$, the free category $\mathcal{F}_{\mathfrak{q}}$, the k-linear closure $k\mathcal{F}_{\mathfrak{q}}$ and the k-linear finitely presented category defined by $\mathfrak{q}$ subject to an admissible set of relations $\rho \subset k\mathcal{F}_{\mathfrak{q}}$.
(2) Construct the Abelian category **mod-A** of **A**-modules.
(3) Construct the Yoneda embedding $Y : \mathbf{A}^{\mathrm{op}} \hookrightarrow \mathbf{mod\text{-}A}$ and the Yoneda equivalence $Y : \mathbf{A}^{\mathrm{op},\oplus} \xrightarrow{\sim} \mathbf{proj\text{-}A}$ and use it to compute the isomorphism classes of the indecomposable projective objects in **mod-A** (cf. Corollary 2.90).
(4) Construct the categories $\mathcal{C}^b(\mathbf{mod\text{-}A})$, $\mathcal{K}^b(\mathbf{mod\text{-}A})$ and $\mathcal{D}^b(\mathbf{mod\text{-}A})$ and use the Yoneda equivalence to construct equivalences
$$\mathcal{K}^b(\mathbf{A}^{\mathrm{op},\oplus}) \simeq \mathcal{K}^b(\mathbf{proj\text{-}A}) \simeq \mathcal{D}^b(\mathbf{mod\text{-}A}).$$
(5) Create an object C in $\mathcal{K}^b(\mathbf{A}^{\mathrm{op},\oplus})$ and compute its image in $\mathcal{D}^b(\mathbf{mod\text{-}A})$.
(6) Use C to construct a complete strong exceptional sequence $\mathscr{E} = (E_1, E_2, E_3, E_4)$ in **mod-A** where $T_{\mathscr{E}} = \bigoplus_1^4 E_i$ is a generalized tilting object.
(7) Compute the quiver $\mathfrak{q}_{\mathscr{E}}$ and the abstraction k-algebroid $\mathbf{A}_{\mathscr{E}}$ of $\mathscr{E}$.
(8) Compute the isomorphism $\mathscr{E} \simeq \mathbf{A}_{\mathscr{E}}$ and the equivalences
$$\mathcal{K}^b(\mathscr{E}^{\oplus}) \simeq \mathcal{K}^b\left(\mathbf{A}_{\mathscr{E}}^{\oplus}\right) \simeq \mathcal{K}^b(\mathbf{A}_{\mathscr{E}}\text{-}\mathbf{proj}) \simeq \mathcal{D}^b(\mathbf{A}_{\mathscr{E}}).$$

[1] Another widely used notation for **mod-A** is $\mathrm{reps}_k(\mathfrak{q}, \rho)$ which stands for the ρ-bounded finite-dimensional quiver k-representations of $\mathfrak{q}$.

[2] The notation is justified by the equivalence $\mathbf{mod\text{-}A} \simeq \mathbf{fdmod}\text{-}\mathbb{A}$ where $\mathbb{A}$ is the quotient k-algebra of the path algebra $k\mathfrak{q}$ subject to $\langle\rho\rangle$ (cf. Theorem 2.70) and **fdmod**-$\mathbb{A}$ is the category of finite-dimensional right $\mathbb{A}$-modules.

where $\mathbf{A}_{\mathscr{E}}$**-proj** is the full subcategory generated by projective objects of $\mathbf{A}_{\mathscr{E}}$**-mod** and $\mathcal{D}^b(\mathbf{A}_{\mathscr{E}}) := \mathcal{D}^b(\mathbf{A}_{\mathscr{E}}\text{-}\mathbf{mod})$.

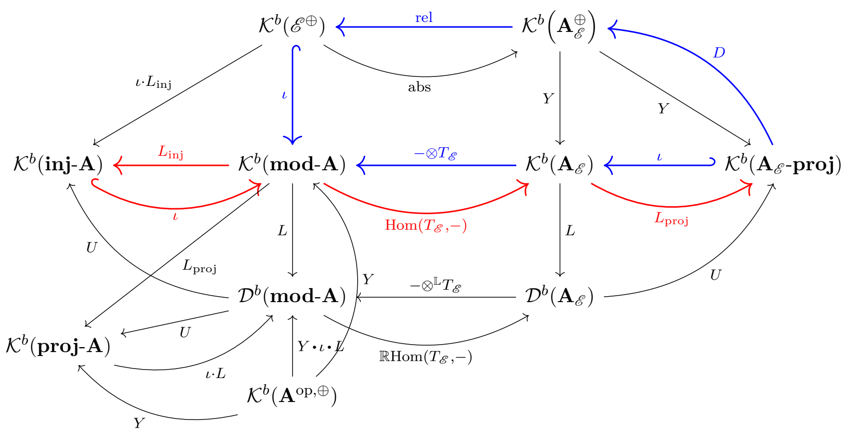
(9) Construct the adjoint functors

$$- \otimes T_{\mathscr{E}} \; : \mathbf{A}_{\mathscr{E}}\text{-}\mathbf{mod} \to \mathbf{mod}\text{-}\mathbf{A} \; : \mathrm{Hom}(T_{\mathscr{E}}, -)$$

(10) Construct the adjoint derived equivalences

$$- \otimes^{\mathbb{L}} T_{\mathscr{E}} \; : \mathcal{D}^b(\mathbf{A}_{\mathscr{E}}) \xrightarrow{\sim} \mathcal{D}^b(\mathbf{mod}\text{-}\mathbf{A}) \; : \mathbb{R}\mathrm{Hom}(T_{\mathscr{E}}, -)$$

and use it to compute an $\mathscr{E}$-replacement of an object $\mathcal{D}^b(\mathbf{mod}\text{-}\mathbf{A})$ (cf. Corollary 6.7).



The JULIA package `CapAndHomalg` mentioned above provides an interface to various `GAP` packages most of which are based on

- `homalg` project [**hom22**],
- CAP project [**GSP22**] and
- `HigherHomologicalAlgebra` `GAP` meta-package [**Sal21a**].

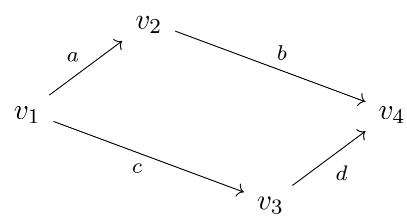
The `GAP` package `QPA` [**Qt21**] provides the data structure of quivers and their associated algebras and representations. In particular, it can be used to check equality of morphisms in $\mathbf{A}$ via performing non-commutative Gröbner bases computations.

We start by loading `CapAndHomalg` and the `GAP` package `DerivedCategories` [**Sal21c**]:

```
julia> using CapAndHomalg
CapAndHomalg v1.1.8
Imported OSCAR's components GAP and Singular_jll
Type: ?CapAndHomalg for more information
julia> LoadPackage( "DerivedCategories" )
```

(1) Create a quiver $\mathtt{q}$, the free category $\mathcal{F}_{\mathtt{q}}$, the k-linear closure $k\mathcal{F}_{\mathtt{q}}$ and a k-linear finitely presented category defined by $\mathtt{q}$ subject to an admissible set of relations $\rho \subset k\mathcal{F}_{\mathtt{q}}$.

Let q be the right quiver:



and let $\mathbf{A} := k\mathcal{F}_{\mathfrak{q}}/\langle\rho\rangle$ be the k-linear finitely presented category defined by q subject to the set of relations $\rho = \{ab - cd\}$. The set ρ is admissible because $\mathfrak{q}$ is acyclic and every relation in ρ is a linear combination of paths of length at least 2.

We start by creating the quiver $\mathfrak{q}$:

```
julia> q = RightQuiver( "q(v1,v2,v3,v4)[a:v1->v2,b:v2->v4,c:v1->v3,d:v3->v4]" )
GAP: q(v1,v2,v3,v4)[a:v1->v2,b:v2->v4,c:v1->v3,d:v3->v4]
```

Next, we assign LaTeX strings to the vertices and arrows of $\mathfrak{q}$ and $\mathfrak{q}^{\mathrm{op}}$:

```
julia> SetLabelsAsLaTeXStrings( q,
                        [ "v_1", "v_2", "v_3", "v_4" ],
                        [ "a", "b", "c", "d" ]
                    );
julia> q_op = OppositeQuiver( q )
GAP: q_op(v1,v2,v3,v4)[a:v2->v1,b:v4->v2,c:v3->v1,d:v4->v3]
julia> SetLabelsAsLaTeXStrings( q_op,
                        [ "v_1", "v_2", "v_3", "v_4" ],
                        [ "a", "b", "c", "d" ]
                    );

julia> F_q = FreeCategory( q )
GAP: Category freely generated by the right quiver q(v1,v2,v3,v4)[a:v1->v2,b:v2->v4,c:
    v1->v3,d:v3->v4]

julia> ℚ = HomalgFieldOfRationals( )
GAP: Q
julia> k = ℚ
GAP: Q

julia> kF_q = k[ F_q ]
GAP: Algebroid( Q * q )

julia> ρ = [ PreCompose( kF_q."a", kF_q."b" ) - PreCompose( kF_q."c", kF_q."d" ) ]
1-element Array{GapObj,1}:
 GAP: (v1)-[-1*(c*d) + 1*(a*b)]->(v4)

julia> A =  kF_q / ρ
GAP: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ] )

julia> InfoOfInstalledOperationsOfCategory( A )
23 primitive operations were used to derive 63 operations for this category which
```

```
* IsEquippedWithHomomorphismStructure
* IsLinearCategoryOverCommutativeRing
```

We can construct the objects of **A** by using their labels as vertices in `q`:

```
julia> v1 = A."v1"
GAP: <(v1)>
julia> v2 = A."v2"
GAP: <(v2)>
julia> v3 = A."v3"
GAP: <(v3)>
julia> v4 = A."v4"
GAP: <(v4)>
```

The list of all objects of **A**:

```
julia> SetOfObjects( A )
GAP: [ <(v1)>, <(v2)>, <(v3)>, <(v4)> ]
```

The category **A** is equipped with a homomorphism structure over k-**mat** (cp. Example 4.13):

```
julia> RangeCategoryOfHomomorphismStructure( A )
GAP: Category of matrices over Q
julia> HomomorphismStructureOnObjects( v1, v4 )
GAP: <A vector space object over Q of dimension 1>
```

So, $\mathrm{Hom}_{\mathbf{A}}(v_1, v_4)$ is a 1 dimensional k-vector space. Its basis is given by:

```
julia> B_v1_v4 = BasisOfExternalHom( v1, v4 )
GAP: [ (v1)-[{ 1*(a*b) }]->(v4) ]
```

Morphisms can also be created by their labels as arrows in `q`:

```
julia> b = A."b"
GAP: (v2)-[{ 1*(b) }]->(v4)
julia> Show( b )
```

$$v_2 - (b) \to v_4$$

The list of all generating morphisms[3] of **A**:

```
julia> SetOfGeneratingMorphisms( A )
GAP: [ (v1)-[{ 1*(a) }]->(v2), (v2)-[{ 1*(b) }]->(v4), (v1)-[{ 1*(c) }]->(v3), (v3)-[{
    1*(d) }]->(v4) ]
```

[3] I.e., the morphisms that are represented by the arrows of `q`.

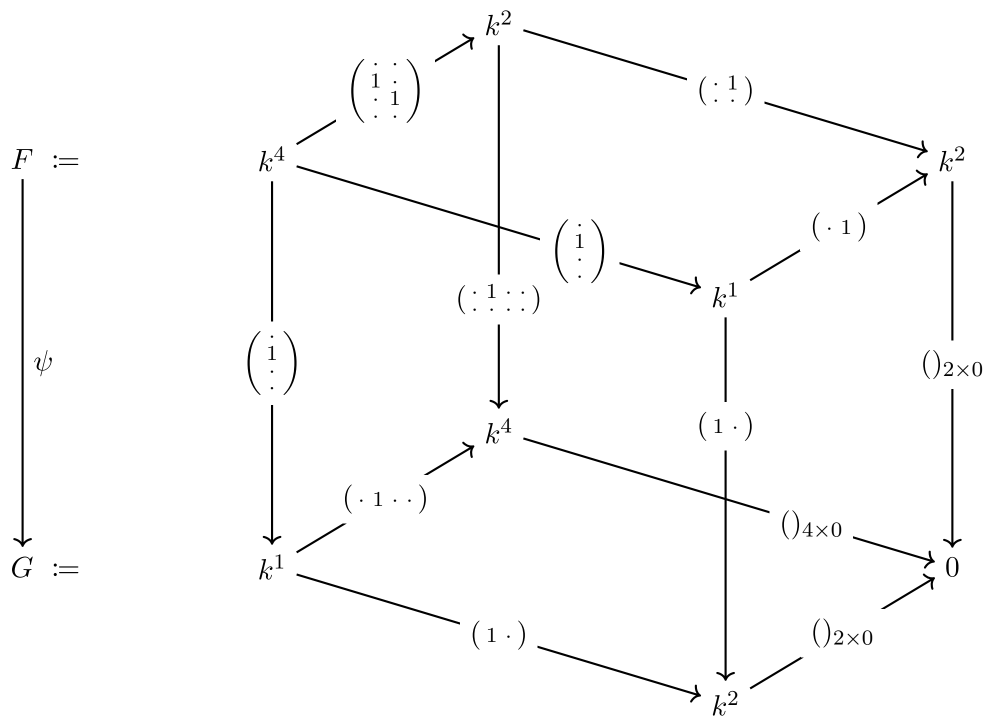
(2) Construct the category mod-A

```
julia> k_vec = MatrixCategory( k )
GAP: Category of matrices over Q
```

The category **mod-A** can be constructed by using the `GAP` package `FunctorCategories` [**BS21a**]:

```
julia> mod_A = Hom( A, k_vec )
GAP: The category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ] ) ->
    Category of matrices over Q
julia> InfoOfInstalledOperationsOfCategory( mod_A )
120 primitive operations were used to derive 312 operations for this category which
* IsEquippedWithHomomorphismStructure
* IsLinearCategoryOverCommutativeRing
* IsAbelianCategoryWithEnoughInjectives
* IsAbelianCategoryWithEnoughProjectives
```

Let us create the following morphism $\psi\colon F \to G$:



As mentioned above, the data structure of an object in **mod-A** is a pair of lists: list of k-vectors spaces and a list of k-linear maps:

```
julia> Fv1 = 4 / k_vec;
```

```
julia> Fv2 = 2 / k_vec;
julia> Fv3 = 1 / k_vec;
julia> Fv4 = 2 / k_vec
GAP: <A vector space object over Q of dimension 2>
julia> Fa = HomalgMatrix(
                "[ [ 0, 0 ], [ 1, 0 ], [ 0, 1 ], [ 0, 0 ] ]", 4, 2, k) / k_vec;
julia> Fb = HomalgMatrix( "[ [  0,  1 ], [  0,  0 ] ]", 2, 2, k ) / k_vec;
julia> Fc = HomalgMatrix( "[ [ 0 ], [ 1 ], [ 0 ], [ 0 ] ]", 4, 1, k ) / k_vec;
julia> Fd = HomalgMatrix( "[ [ 0, 1 ] ]", 1, 2, k ) / k_vec
GAP: <A morphism in Category of matrices over Q>
julia> F = AsObjectInFunctorCategory( A, [ Fv1, Fv2, Fv3, Fv4 ], [ Fa, Fb, Fc, Fd ] )
GAP: <(v1)->4, (v2)->2, (v3)->1, (v4)->2; (a)->4x2, (b)->2x2, (c)->4x1, (d)->1x2>
julia> Show( F )
```

$$\begin{array}{lll} v_1 & \mapsto & k^{1\times 4} \\ v_2 & \mapsto & k^{1\times 2} \\ v_3 & \mapsto & k^{1\times 1} \\ v_4 & \mapsto & k^{1\times 2} \\ \hline \\ a & \mapsto & \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \\ \cdot & 1 \\ \cdot & \cdot \end{pmatrix} \\ \\ b & \mapsto & \begin{pmatrix} \cdot & 1 \\ \cdot & \cdot \end{pmatrix} \\ \\ c & \mapsto & \begin{pmatrix} \cdot \\ 1 \\ \cdot \\ \cdot \end{pmatrix} \\ \\ d & \mapsto & \begin{pmatrix} \cdot & 1 \end{pmatrix} \end{array}$$

```
julia> IsWellDefined( F )
true
```

The object F is a functor, so we can apply it to morphisms of $\mathbf{A}$:

```
julia> m = PreCompose( A."a", A."b" )
GAP: (v1)-[{ 1*(a*b) }]->(v4)
julia> Show( m )
```

$$v_1 - (ab) \to v_4$$

```
julia> F_m = F( m )
GAP: <A morphism in Category of matrices over Q>
julia> Show( F_m )
```

$$k^{1\times 4} \xrightarrow{\begin{pmatrix} \cdot & \cdot \\ \cdot & 1 \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}} k^{1\times 2}$$

```
julia> Gv1 = 1 / k_vec;
julia> Gv2 = 4 / k_vec;
julia> Gv3 = 2 / k_vec;
julia> Gv4 = 0 / k_vec
GAP: <A vector space object over Q of dimension 0>
julia> Ga = HomalgMatrix( "[ [ 0, 1, 0, 0 ] ]", 1, 4, k ) / k_vec;
julia> Gb = HomalgZeroMatrix( 4, 0, k ) / k_vec;
julia> Gc = HomalgMatrix( "[ [ 1, 0 ] ]", 1, 2, k ) / k_vec;
julia> Gd = HomalgZeroMatrix( 2, 0, k ) / k_vec
GAP: <A morphism in Category of matrices over Q>
julia> G = AsObjectInFunctorCategory( A, [ Gv1, Gv2, Gv3, Gv4 ], [ Ga, Gb, Gc, Gd ] )
GAP: <(v1)->1, (v2)->4, (v3)->2, (v4)->0; (a)->1x4, (b)->4x0, (c)->1x2, (d)->2x0>
julia> Show( G )
```

$$\begin{array}{lcc} v_1 & \mapsto & k^{1\times 1} \\ v_2 & \mapsto & k^{1\times 4} \\ v_3 & \mapsto & k^{1\times 2} \\ v_4 & \mapsto & k^{1\times 0} \\ \hline a & \mapsto & \begin{pmatrix} \cdot & 1 & \cdot & \cdot \end{pmatrix} \\ b & \mapsto & ()_{4\times 0} \\ c & \mapsto & \begin{pmatrix} 1 & \cdot \end{pmatrix} \\ d & \mapsto & ()_{2\times 0} \end{array}$$

The data structure of a morphism in **mod-A** is a list of k-linear maps:

```
julia> ψ_v1 = HomalgMatrix( "[ [ 0 ], [ 1 ], [ 0 ], [ 0 ] ]", 4, 1, k ) / k_vec;
julia> ψ_v2 = HomalgMatrix( "[ [ 0, 1, 0, 0 ], [ 0, 0, 0, 0 ] ]", 2, 4, k ) / k_vec;
julia> ψ_v3 = HomalgMatrix( "[ [ 1, 0 ] ]", 1, 2, k ) / k_vec;
julia> ψ_v4 = HomalgZeroMatrix( 2, 0, k ) / k_vec
GAP: <A morphism in Category of matrices over Q>
julia> ψ = AsMorphismInFunctorCategory(
                F,
                [ ψ_v1, ψ_v2, ψ_v3, ψ_v4 ],
                G
           )
GAP: <(v1)->4x1, (v2)->2x4, (v3)->1x2, (v4)->2x0>
```

```
julia> Show( ψ )
```

$$\begin{array}{rcl} v_1 & \mapsto & \begin{pmatrix} \cdot \\ 1 \\ \cdot \\ \cdot \end{pmatrix} \\[1em] v_2 & \mapsto & \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \\[1em] v_3 & \mapsto & \begin{pmatrix} 1 & \cdot \end{pmatrix} \\[1em] v_4 & \mapsto & ()_{2\times 0} \end{array}$$

```
julia> IsMonomorphism( ψ )
false
julia> IsEpimorphism( ψ )
false
```

Since $\mathfrak{q}$ is acyclic and **A** is admissible, the category **mod-A** is Abelian with enough injectives and projectives and its global dimension is bounded by the number of vertices of $\mathfrak{q}$. Let us compute the kernel object and kernel embedding of ψ:

```
julia> K_ψ = KernelObject( ψ )
GAP: <(v1)->3, (v2)->1, (v3)->0, (v4)->2; (a)->3x1, (b)->1x2, (c)->3x0, (d)->0x2>
julia> Show( K_ψ )
```

$$\begin{array}{rcl} v_1 & \mapsto & \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \\[1em] v_2 & \mapsto & \begin{pmatrix} \cdot & 1 \end{pmatrix} \\[1em] v_3 & \mapsto & ()_{0\times 1} \\[1em] v_4 & \mapsto & \begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix} \end{array}$$

Furthermore, **mod-A** is has homomorphism structure over k-**mat**:

```
julia> RangeCategoryOfHomomorphismStructure( mod_A )
GAP: Category of matrices over Q
julia> HomStructure( F, G )
GAP: <A vector space object over Q of dimension 1>
julia> HomStructure( G, F )
```

```
GAP: <A vector space object over Q of dimension 6>
julia> Hom_GF = BasisOfExternalHom( G, F );
julia> τ = -5 * Hom_GF[3] + 2 * Hom_GF[5] + 15 * Hom_GF[6]
GAP: <(v1)->1x4, (v2)->4x2, (v3)->2x1, (v4)->0x2>
julia> Show( τ )
```

$$v_1 \mapsto \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$v_2 \mapsto \begin{pmatrix} \cdot & -5 \\ \cdot & \cdot \\ \cdot & 2 \\ \cdot & 15 \end{pmatrix}$$

$$v_3 \mapsto \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$$

$$v_4 \mapsto ()_{0\times 2}$$

```
julia> P_F = SomeProjectiveObject( F )
GAP: <(v1)->4, (v2)->4, (v3)->4, (v4)->5; (a)->4x4, (b)->4x5, (c)->4x4, (d)->4x5>
julia> IsProjective( P_F )
true
julia> Show( P_F )
```

$$\begin{array}{ccc} v_1 & \mapsto & k^{1\times 4} \\ v_2 & \mapsto & k^{1\times 4} \\ v_3 & \mapsto & k^{1\times 4} \\ v_4 & \mapsto & k^{1\times 5} \\ \hline \end{array}$$

$$a \mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

$$b \mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix}$$

$$c \mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

$$d \mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix}$$

```
julia> π_F = EpimorphismFromSomeProjectiveObject( F )
GAP: <(v1)->4x4, (v2)->4x2, (v3)->4x1, (v4)->5x2>
julia> Show( π_F )
```

$$v_1 \mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

$$v_2 \mapsto \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \\ \cdot & 1 \\ \cdot & \cdot \end{pmatrix}$$

$$v_3 \mapsto \begin{pmatrix} \cdot \\ 1 \\ \cdot \\ \cdot \end{pmatrix}$$

$$v_4 \mapsto \begin{pmatrix} \cdot & \cdot \\ \cdot & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & \cdot \end{pmatrix}$$

```
julia> I_F = SomeInjectiveObject( F )
GAP: <(v1)->5, (v2)->3, (v3)->2, (v4)->2; (a)->5x3, (b)->3x2, (c)->5x2, (d)->2x2>
julia> IsInjective( I_F )
true
julia> Show( I_F )
```

$$\begin{array}{lcc} v_1 & \mapsto & k^{1\times 5} \\ v_2 & \mapsto & k^{1\times 3} \\ v_3 & \mapsto & k^{1\times 2} \\ v_4 & \mapsto & k^{1\times 2} \\ \hline \end{array}$$

$$a \mapsto \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}$$

$$b \mapsto \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \\ \cdot & 1 \end{pmatrix}$$

$$c \mapsto \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & \cdot \\ \cdot & 1 \end{pmatrix}$$

$$d \mapsto \begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix}$$

```
julia> ι_F = MonomorphismIntoSomeInjectiveObject( F )
GAP: <(v1)->4x5, (v2)->2x3, (v3)->1x2, (v4)->2x2>
julia> Show( ι_F )
```

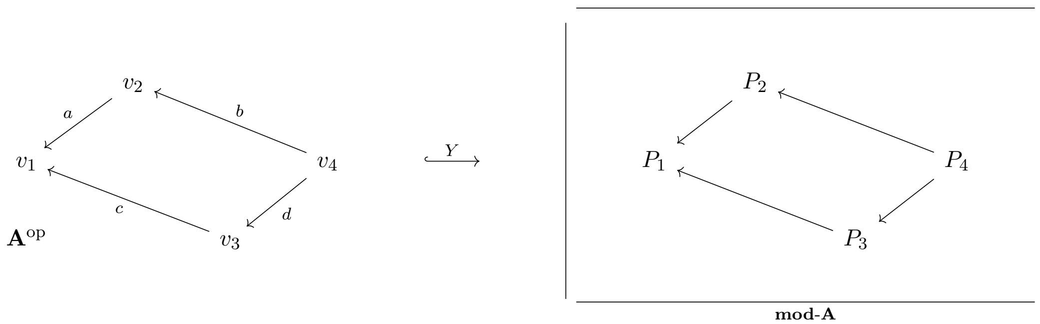
$$
\begin{array}{lcc}
v_1 & \mapsto & k^{1\times 5} \\
v_2 & \mapsto & k^{1\times 3} \\
v_3 & \mapsto & k^{1\times 2} \\
v_4 & \mapsto & k^{1\times 2} \\
\hline
a & \mapsto & \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} \\
b & \mapsto & \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \\ \cdot & 1 \end{pmatrix} \\
c & \mapsto & \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & \cdot \\ \cdot & 1 \end{pmatrix} \\
d & \mapsto & \begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix}
\end{array}
$$

(3) Construct the Yoneda embedding $Y : \mathbf{A}^{\mathrm{op}} \hookrightarrow$ mod-A and the induced Yoneda equivalence $Y : \mathbf{A}^{\mathrm{op},\oplus} \xrightarrow{\sim}$ proj-A.

The Yoneda embedding $Y : \mathbf{A}^{\mathrm{op}} \hookrightarrow \mathbf{mod\text{-}A}$ maps an object $v \in \mathbf{A}^{\mathrm{op}}$ to the functor $P_v \coloneqq Y(v) \coloneqq \mathrm{Hom}_{\mathbf{A}}(v,\text{-}) : \mathbf{A} \to k\text{-}\mathbf{mat}$. It is well known that the images of the Yoneda embedding are projective objects in **mod-A**. We start by creating the opposite algebroid $\mathbf{A}^{\mathrm{op}}$:

```
julia> A_op = OppositeAlgebroid( A )
GAP: Algebroid( (Q * q_op) / [ -1*(d*c) + 1*(b*a) ] )
julia> Y = YonedaEmbedding( A_op )
GAP: Yoneda embedding functor
julia> Display( Y )
Yoneda embedding functor:

Algebroid( (Q * q_op) / [ -1*(d*c) + 1*(b*a) ] )
  |
  V
The category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of
    matrices over Q
julia> IsIdenticalObj( RangeOfFunctor( Y ), mod_A )
true
```

Since **A** is admissible, the images of the Y are, up to isomorphism, *the* indecomposable projective objects of **mod-A**.




```
julia> P1 = Y( A_op."v1" )
GAP: <(v1)->1, (v2)->1, (v3)->1, (v4)->1; (a)->1x1, (b)->1x1, (c)->1x1, (d)->1x1>
julia> Show( P1 )
```

$$\begin{array}{rcl} v_1 & \mapsto & k^{1\times 1} \\ v_2 & \mapsto & k^{1\times 1} \\ v_3 & \mapsto & k^{1\times 1} \\ v_4 & \mapsto & k^{1\times 1} \\ \hline a & \mapsto & \left(1 \right) \\ b & \mapsto & \left(1 \right) \\ c & \mapsto & \left(1 \right) \\ d & \mapsto & \left(1 \right) \end{array}$$

```
julia> P2 = Y( A_op."v2" )
GAP: <(v1)->0, (v2)->1, (v3)->0, (v4)->1; (a)->0x1, (b)->1x1, (c)->0x0, (d)->0x1>
julia> Show( P2 )
```

$$\begin{array}{rcl} v_1 & \mapsto & k^{1\times 0} \\ v_2 & \mapsto & k^{1\times 1} \\ v_3 & \mapsto & k^{1\times 0} \\ v_4 & \mapsto & k^{1\times 1} \\ \hline a & \mapsto & ()_{0\times 1} \\ b & \mapsto & \left(1 \right) \\ c & \mapsto & ()_{0\times 0} \\ d & \mapsto & ()_{0\times 1} \end{array}$$

```
julia> P3 = Y( A_op."v3" )
julia> Show( P3 )
```

$$\begin{array}{ccc} v_1 & \mapsto & k^{1\times 0} \\ v_2 & \mapsto & k^{1\times 0} \\ v_3 & \mapsto & k^{1\times 1} \\ v_4 & \mapsto & k^{1\times 1} \\ \hline a & \mapsto & ()_{0\times 0} \\ b & \mapsto & ()_{0\times 1} \\ c & \mapsto & ()_{0\times 1} \\ d & \mapsto & (\ 1\ ) \end{array}$$

```
julia> P4 = Y( A_op."v4" )
julia> Show( P4 )
```

$$\begin{array}{ccc} v_1 & \mapsto & k^{1\times 0} \\ v_2 & \mapsto & k^{1\times 0} \\ v_3 & \mapsto & k^{1\times 0} \\ v_4 & \mapsto & k^{1\times 1} \\ \hline a & \mapsto & ()_{0\times 0} \\ b & \mapsto & ()_{0\times 1} \\ c & \mapsto & ()_{0\times 0} \\ d & \mapsto & ()_{0\times 1} \end{array}$$

In the following we apply Y to the morphism $\mathbf{A}^{\mathrm{op}} \ni \alpha = ba : v_4 \to v_1$

```
julia> α =  PreCompose( A_op."b", A_op."a" )
GAP: (v4)-[{ 1*(b*a) }]->(v1)
julia> Show( α )
```

$$v_4 - (ba) \to v_1$$

```
julia> P_α = Y( α )
GAP: <(v1)->0x1, (v2)->0x1, (v3)->0x1, (v4)->1x1>
julia> Show( P_α )
```

$$v_1 \mapsto ()_{0\times 1}$$

$$v_2 \mapsto ()_{0\times 1}$$

$$v_3 \mapsto ()_{0\times 1}$$

$$v_4 \mapsto \left(1 \right)$$

If we restrict the Yoneda embedding $Y\colon \mathbf{A}^{\mathrm{op}} \hookrightarrow \textbf{mod-A}$ to its image, we get an isomorphism

$$Y\colon \mathbf{A}^{\mathrm{op}} \xrightarrow{\sim} \mathbf{ind}_0(\textbf{proj-A})$$

where $\mathbf{ind}_0(\textbf{proj-A})$ is the skeletal of the full subcategory generated by the indecomposable projective objects in **mod-A**. In the following we construct this isomorphism:

```
julia> ind0_proj_A = FullSubcategoryGeneratedByIndecProjectiveObjects( mod_A )
GAP: Full subcategory generated by the 4 indecomposable projective objects( The
    category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of
    matrices over Q )
julia> ind0_proj_A[ 1 ]
GAP: An object in full subcategory given by: <(v1)->1, (v2)->1, (v3)->1, (v4)->1; (a
    )->1x1, (b)->1x1, (c)->1x1, (d)->1x1>
julia> IsEqualForObjects( P1, UnderlyingCell( ind0_proj_A[ 1 ] ) )
true
julia> KnownFunctors( A_op, ind0_proj_A )
1: Yoneda isomorphism
julia> Y = Functor( A_op, ind0_proj_A, 1 )
GAP: Isomorphism functor from Algebroid onto full subcategory generated by
    indecomposable projective objects
julia> Display( Y )
Isomorphism functor from Algebroid onto full subcategory generated by indecomposable
    projective objects:

Algebroid( (Q * q_op) / [ -1*(d*c) + 1*(b*a) ] )
  |
  V
Full subcategory generated by the 4 indecomposable projective objects( The category of
    functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of matrices
    over Q )
julia> Y( A_op."v1" )
GAP: An object in full subcategory given by: <(v1)->1, (v2)->1, (v3)->1, (v4)->1; (a
    )->1x1, (b)->1x1, (c)->1x1, (d)->1x1>
```

If we extend the functor Y to the additive closures, we get an equivalence

$$\mathbf{A}^{\mathrm{op},\oplus} \simeq \mathbf{ind}_0^{\oplus}(\textbf{proj-A}) \simeq \textbf{proj-A}.$$

The forward equivalence is the extension of Yoneda isomorphism to additive closures and the backward equivalence is the direct sum decomposition functor of projective objects.

```
julia> A_op_plus = AdditiveClosure( A_op )
GAP: Additive closure( Algebroid( (Q * q_op) / [ -1*(d*c) + 1*(b*a) ] ) )
julia> InfoOfInstalledOperationsOfCategory( A_op_plus )
23 primitive operations were used to derive 113 operations for this category which
* IsEquippedWithHomomorphismStructure
* IsLinearCategoryOverCommutativeRing
* IsAdditiveCategory
julia> proj_A = FullSubcategoryGeneratedByProjectiveObjects( mod_A )
GAP: Full additive subcategory generated by projective objects( The category of
    functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of matrices
    over Q )
julia> InfoOfInstalledOperationsOfCategory( proj_A )
53 primitive operations were used to derive 119 operations for this category which
* IsEquippedWithHomomorphismStructure
* IsLinearCategoryOverCommutativeRing
* IsAdditiveCategory
```

The above categories are also equipped with k-**mat**-homomorphism structures:

```
julia> RangeCategoryOfHomomorphismStructure( A_op_plus )
GAP: Category of matrices over Q
julia> RangeCategoryOfHomomorphismStructure( proj_A )
GAP: Category of matrices over Q
```

In the following we create the equivalences between $\mathbf{A}^{\mathrm{op},\oplus} \simeq$ **proj-A**

```
julia> KnownFunctors( A_op_plus, proj_A )
1: Yoneda embedding
julia> Y = Functor( A_op_plus, proj_A, 1 )
GAP: Yoneda embedding
julia> Display( Y )
Yoneda embedding:

Additive closure( Algebroid( (Q * q_op) / [ -1*(d*c) + 1*(b*a) ] ) )
  |
  V
Full additive subcategory generated by projective objects( The category of functors:
    Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q )
julia> KnownFunctors( proj_A, A_op_plus )
1: Decomposition of projective objects
julia> D = Functor( proj_A, A_op_plus, 1 )
GAP: Decomposition of projective objects
julia> Display( D )
Decomposition of projective objects:

Full additive subcategory generated by projective objects( The category of functors:
    Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q )
  |
```

```
  V
Additive closure( Algebroid( (Q * q_op) / [ -1*(d*c) + 1*(b*a) ] ) )
```

In the following we use $D : \textbf{proj-A} \xrightarrow{\sim} \mathbf{A}^{\mathrm{op},\oplus}$ to decompose a projective object $P \in \textbf{proj-A}$:

```
julia> K = DirectSum( KernelObject( ψ ), CokernelObject( ψ ) )
GAP: <(v1)->3, (v2)->4, (v3)->1, (v4)->2; (a)->3x4, (b)->4x2, (c)->3x1, (d)->1x2>
julia> IsProjective( K )
false
julia> P = SomeProjectiveObject( K )
GAP: <(v1)->3, (v2)->6, (v3)->4, (v4)->9; (a)->3x6, (b)->6x9, (c)->3x4, (d)->4x9>
julia> Show( P )
```

$$\begin{array}{lll} v_1 & \mapsto & k^{1\times 3} \\ v_2 & \mapsto & k^{1\times 6} \\ v_3 & \mapsto & k^{1\times 4} \\ v_4 & \mapsto & k^{1\times 9} \\ \hline a & \mapsto & \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix} \\ b & \mapsto & \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix} \\ c & \mapsto & \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \\ d & \mapsto & \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix} \end{array}$$

```
julia> P =  P / proj_A
GAP: An object in full subcategory given by: <(v1)->3, (v2)->6, (v3)->4, (v4)->9; (a
    )->3x6, (b)->6x9, (c)->3x4, (d)->4x9>
julia> DP = D( P )
GAP: <An object in Additive closure( Algebroid( (Q * q_op) / [ -1*(d*c) + 1*(b*a) ] ) )
     defined by 9 underlying objects>
julia> Show( DP )
```

$${v_1}^{\oplus 3} \oplus {v_2}^{\oplus 3} \oplus v_3 \oplus {v_4}^{\oplus 2}$$

In the following, we apply the Yoneda isomorphism to a morphism $\varphi : D(P) \to D(P)$:

```
julia> HomStructure( DP, DP )
GAP: <A vector space object over Q of dimension 49>
julia> φ = Sum( BasisOfExternalHom( DP, DP ) )
GAP: <A morphism in Additive closure( Algebroid( (Q * q_op) / [ -1*(d*c) + 1*(b*a) ] )
    ) defined by a 9 x 9 matrix of underlying morphisms>
julia> Show( φ )
```

$$v_1{}^{\oplus 3} \oplus v_2{}^{\oplus 3} \oplus v_3 \oplus v_4{}^{\oplus 2} \xrightarrow{\begin{pmatrix} v_1 & v_1 & v_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_1 & v_1 & v_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_1 & v_1 & v_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & a & a & v_2 & v_2 & v_2 & 0 & 0 & 0 \\ a & a & a & v_2 & v_2 & v_2 & 0 & 0 & 0 \\ a & a & a & v_2 & v_2 & v_2 & 0 & 0 & 0 \\ c & c & c & 0 & 0 & 0 & v_3 & 0 & 0 \\ ba & ba & ba & b & b & b & d & v_4 & v_4 \\ ba & ba & ba & b & b & b & d & v_4 & v_4 \end{pmatrix}} v_1{}^{\oplus 3} \oplus v_2{}^{\oplus 3} \oplus v_3 \oplus v_4{}^{\oplus 2}$$

```
julia> Yφ = Y( φ )
GAP: A morphism in full subcategory given by: <(v1)->3x3, (v2)->6x6, (v3)->4x4, (v4)->9
    x9>
julia> Show( Yφ )
```

$$v_1 \mapsto \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$v_2 \mapsto \begin{pmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$v_3 \mapsto \begin{pmatrix} 1 & 1 & 1 & \cdot \\ 1 & 1 & 1 & \cdot \\ 1 & 1 & 1 & \cdot \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$v_4 \mapsto \begin{pmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

```
julia> D( Yφ ) == φ
true
```

(4) Construct the equivalences $\mathcal{K}^b(\mathbf{A}^{\mathrm{op},\oplus}) \simeq \mathcal{K}^b(\textbf{proj-A}) \simeq \mathcal{D}^b(\textbf{mod-A})$

The equivalence $\mathbf{A}^{\mathrm{op},\oplus} \simeq \textbf{proj-A}$ can be lifted to an equivalence between the (bounded) complexes categories: $\mathrm{Ch}^b(\mathbf{A}^{\mathrm{op},\oplus}) \simeq \mathrm{Ch}^b(\textbf{proj-A})$ and the (bounded) homotopy categories: $\mathcal{K}^b(\mathbf{A}^{\mathrm{op},\oplus}) \simeq \mathcal{K}^b(\textbf{proj-A})$.

Since `q` is acyclic, the global dimension of **mod-A** is finite and bounded above by the number of vertices in `q`. In this example the global dimension of **A** is 2. Hence, we obtain an equivalence: $\mathcal{K}^b(\textbf{proj-A}) \simeq \mathcal{D}^b(\textbf{mod-A})$.

To sum up, we have the following equivalences:

$$\mathcal{K}^b(\mathbf{A}^{\mathrm{op},\oplus}) \simeq \mathcal{K}^b(\textbf{proj-A}) \simeq \mathcal{D}^b(\textbf{mod-A}).$$

The `GAP` package `QPA` can be used to compute the global dimension of **mod-A**. We compute the endomorphism k-algebra $\mathbb{A} := \mathrm{End}_k\, \mathbf{A} \cong \mathrm{End}_k\, \mathscr{E} \cong \mathrm{End}_k\, T_{\mathscr{E}}$ as a quotient of the path algebra k`q` and compute the global dimension of $\mathbb{A}$:

```
julia> 𝔸 = UnderlyingQuiverAlgebra( A )
```

```
GAP: (Q * q) / [ -1*(c*d) + 1*(a*b) ]
julia> Dimension( 𝔸 )
9
julia> GlobalDimensionOfAlgebra( 𝔸, 1 )
false
julia> GlobalDimensionOfAlgebra( 𝔸, 2 )
2
```

We start by creating the homotopy categories $\mathcal{K}^b(\mathbf{A}^{\mathrm{op},\oplus})$ and $\mathcal{K}^b(\mathbf{proj}\text{-}\mathbf{A})$:

```
julia> K𝐀_op_plus = HomotopyCategoryByCochains( 𝐀_op_plus )
GAP: Homotopy category( Additive closure( Algebroid( (Q * q_op) / [ -1*(d*c) + 1*(b*a)
    ] ) ) )
julia> Kproj_𝐀 = HomotopyCategoryByCochains( proj_𝐀 )
GAP: Homotopy category( Full additive subcategory generated by projective objects( The
    category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of
    matrices over Q ) )
```

Of course both categories are equipped with $\mathbb{Q}$-**mat**-equipped with a homomorphism structures:

```
julia> RangeCategoryOfHomomorphismStructure( K𝐀_op_plus )
GAP: Category of matrices over Q
julia> RangeCategoryOfHomomorphismStructure( Kproj_𝐀 )
GAP: Category of matrices over Q
julia> 𝑌 = ExtendFunctorToHomotopyCategoriesByCochains( Y )
GAP: Extension of ( Yoneda embedding ) to homotopy categories
julia> Display( 𝑌 )
GAP: Extension of ( Yoneda embedding ) to homotopy categories:

Homotopy category( Additive closure( Algebroid( (Q * q_op) / [ -1*(d*c) + 1*(b*a) ] ) )
     )
  |
  V
Homotopy category( Full additive subcategory generated by projective objects( The
    category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of
    matrices over Q ) )
julia> IsIdenticalObj( SourceOfFunctor( 𝑌 ), K𝐀_op_plus )
                && IsIdenticalObj( Kproj_𝐀, RangeOfFunctor( 𝑌 ) )
true
julia> 𝐷 = ExtendFunctorToHomotopyCategoriesByCochains( D )
GAP: Extension of ( Decomposition of projective objects ) to homotopy categories
julia> Display( 𝐷 )
GAP: Extension of ( Decomposition of projective objects ) to homotopy categories

Homotopy category( Full additive subcategory generated by projective objects( The
    category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of
    matrices over Q ) )
  |
```

```
  V
Homotopy category( Additive closure( Algebroid( (Q * q_op) / [ -1*(d*c) + 1*(b*a) ] ) )
     )
julia> IsIdenticalObj( SourceOfFunctor( D ), Kproj_A )
                && IsIdenticalObj( KA_op_plus, RangeOfFunctor( D ) )
true
```

The equivalence $\mathcal{K}^b(\textbf{proj-A}) \simeq \mathcal{D}^b(\textbf{mod-A})$ is the composition:

$$\mathcal{K}^b(\textbf{proj-A}) \hookrightarrow \mathcal{K}^b(\textbf{mod-A}) \xrightarrow{L} \mathcal{D}^b(\textbf{mod-A})$$

where L is the natural localization functor. That is, L maps a morphism $\beta : B \to C$ in $\mathcal{K}^b(\textbf{mod-A})$ to the morphism in $\mathcal{D}^b(\textbf{mod-A})$) represented by the roof[4]

$$(B \xleftarrow{\mathrm{id}_B} B \xrightarrow{\beta} C) \ : B \to C.$$

```
julia> Cmod_A = CochainComplexCategory( mod_A )
GAP: Cochain complexes( The category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(
    a*b) ] ) -> Category of matrices over Q )
julia> Kmod_A = HomotopyCategoryByCochains( mod_A )
GAP: Homotopy category( The category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(
    a*b) ] ) -> Category of matrices over Q )
julia> Dmod_A = DerivedCategoryByCochains( mod_A )
GAP: Derived category( The category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a
    *b) ] ) -> Category of matrices over Q )
julia> IsIdenticalObj( mod_A, AmbientCategory( proj_A ) )
true
julia> I = InclusionFunctor( proj_A );
julia> I = ExtendFunctorToHomotopyCategoriesByCochains( I )
GAP: Extension of a functor to homotopy categories
julia> Display( I )
Extension of a functor to homotopy categories:

Homotopy category( Full additive subcategory generated by projective objects( The
    category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of
    matrices over Q ) )
  |
  V
Homotopy category( The category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b)
    ] ) -> Category of matrices over Q )
julia> IsIdenticalObj( Kmod_A, RangeOfFunctor( I ) )
true
julia> L = LocalizationFunctor( Kmod_A )
GAP: Localization functor in derived category
julia> Display( L )
Localization functor in derived category:
```

[4]A roof in $\mathcal{K}^b(\textbf{mod-A})$) is by definition a pair of morphisms $(A \xleftarrow{\alpha} B \xrightarrow{\beta} C)$ where α is a quasi-isomorphism. Morphisms in the derived categories are equivalence classes of roofs Definition 3.41.

```
Homotopy category( The category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b)
    ] ) -> Category of matrices over Q )
  |
  V
Derived category( The category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ]
     ) -> Category of matrices over Q )
julia> IsIdenticalObj( Dmod_A, RangeOfFunctor( L ) )
true
```

On the other hand, the equivalence

$$\mathcal{D}^b(\mathbf{mod}\text{-}\mathbf{A}) \xrightarrow{U} \mathcal{K}^b(\mathbf{proj}\text{-}\mathbf{A})$$

can be computed by the universal property of derived categories. More precisely, the functor

$$\mathcal{K}^b(\mathbf{mod}\text{-}\mathbf{A}) \xrightarrow{L_{\mathrm{proj}}} \mathcal{K}^b(\mathbf{proj}\text{-}\mathbf{A})$$

which maps cells in $\mathcal{K}^b(\mathbf{mod}\text{-}\mathbf{A})$ to their projective replacements in $\mathcal{K}^b(\mathbf{proj}\text{-}\mathbf{A})$ is a localization functor with respects to quasi-isomorphisms, hence factors uniquely along L via the desired functor U which maps a morphism in $\mathcal{D}^b(\mathbf{mod}\text{-}\mathbf{A})$ represented by a roof $A \xleftarrow{\alpha} B \xrightarrow{\beta} C$ to $(L_{\mathrm{proj}}(\alpha))^{-1} \cdot L_{\mathrm{proj}}(\beta) : L_{\mathrm{proj}}(A) \to L_{\mathrm{proj}}(C)$ in $\mathcal{K}^b(\mathbf{proj}\text{-}\mathbf{A})$. Note that $\alpha : B \to A$ is by definition a quasi-isomorphism in $\mathcal{K}^b(\mathbf{mod}\text{-}\mathbf{A})$, hence its projective replacement is an isomorphism in $\mathcal{K}^b(\mathbf{proj}\text{-}\mathbf{A})$.

```
julia> L_proj = LocalizationFunctorByProjectiveObjects( Kmod_A )
GAP: Localization functor by projective objects
julia> Display( L_proj )
Localization functor by projective objects:

Homotopy category( The category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b)
    ] ) -> Category of matrices over Q )
  |
  V
Homotopy category( Full additive subcategory generated by projective objects( The
    category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of
    matrices over Q ) )
julia> U = UniversalFunctorFromDerivedCategory( L_proj )
GAP: Universal functor from derived category onto a localization category
julia> Display( U )
Universal functor from derived category onto a localization category:

Derived category( The category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ]
     ) -> Category of matrices over Q )
  |
  V
Homotopy category( Full additive subcategory generated by projective objects( The
    category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of
    matrices over Q ) )
```

Now we can compute the composition

$$\mathcal{D}^b(\textbf{mod-A}) \xrightarrow{U} \mathcal{K}^b(\textbf{proj-A}) \xrightarrow{KD} \mathcal{K}^b(\mathbf{A}^{\mathrm{op},\oplus})$$

```
julia> UD = PreCompose( U, D );
julia> Display( UD )
Composition of Universal functor from derived category onto a localization category and
     Extension of a functor to homotopy categories:

Derived category( The category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ]
     ) -> Category of matrices over Q )
  |
  V
Homotopy category( Additive closure( Algebroid( (Q * q_op) / [ -1*(d*c) + 1*(b*a) ] ) )
     )
```

and the other way around

$$\mathcal{K}^b(\mathbf{A}^{\mathrm{op},\oplus}) \xrightarrow{Y} \mathcal{K}^b(\textbf{proj-A}) \hookrightarrow \mathcal{K}^b(\textbf{mod-A}) \xrightarrow{L} \mathcal{D}^b(\textbf{mod-A})$$

```
julia> YIL = PreCompose( [ Y, I, L ] );
julia> Display( YIL )
Composition of Composition of Extension of a functor to homotopy categories and
    Extension of a functor to homotopy categories and Localization functor in derived
    category:

Homotopy category( Additive closure( Algebroid( (Q * q_op) / [ -1*(d*c) + 1*(b*a) ] ) )
     )
  |
  V
Derived category( The category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ]
     ) -> Category of matrices over Q )
```

(5) Create an object in $\mathcal{K}^b(\mathbf{A}^{\mathrm{op},\oplus})$ and compute its image in $\mathcal{D}^b(\textbf{mod-A})$.

In the following we want to apply the functor $Y \cdot I \cdot L$ to the object C in $\mathcal{K}^b(\mathbf{A}^{\mathrm{op},\oplus})$ defined by

$$C \;:=\; \quad 0 \longrightarrow v_4 \xrightarrow{(b \quad d)} v_2 \oplus v_3 \longrightarrow 0$$

where v_4 is concentrated in the cohomological index -1.

```
julia> C_m1 = [ A_op."v4" ] / A_op_plus
julia> C_0 =  [ A_op."v2", A_op."v3" ] / A_op_plus
GAP: <An object in Additive closure( Algebroid( (Q * q_op) / [ -1*(d*c) + 1*(b*a) ] ) )
     defined by 2 underlying objects>
julia> ∂_m1 = AdditiveClosureMorphism( C_m1,  [ [ A_op."b", A_op."d" ] ], C_0 )
GAP: <A morphism in Additive closure( Algebroid( (Q * q_op) / [ -1*(d*c) + 1*(b*a) ] )
    ) defined by a 1 x 2 matrix of underlying morphisms>
julia> Show( ∂_m1 )
```

$$\begin{array}{rcl} v_1 & \mapsto & ()_{0\times 0} \\ v_2 & \mapsto & ()_{0\times 1} \\ v_3 & \mapsto & ()_{0\times 1} \\ v_4 & \mapsto & \begin{pmatrix} 1 & 1 \end{pmatrix} \end{array}$$

```
julia> C = [ [ ∂_m1 ], -1 ] / KA_op_plus
GAP: <An object in Homotopy category( Additive closure( Algebroid( (Q * q_op) / [ -1*(d
    *c) + 1*(b*a) ] ) ) ) with active lower bound -1 and active upper bound 0>
julia> Show( C )
```

$$\begin{array}{c} v_2 \oplus v_3 \\ \uparrow \\ \begin{pmatrix} b & d \end{pmatrix} \\ |_{-1} \\ v_4 \end{array}$$

```
julia> IsWellDefined( C )
true
julia> W = YIL( C )
GAP: <An object in Derived category( The category of functors: Algebroid( (Q * q) / [
    -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q ) with active lower bound -1
    and active upper bound 0>
julia> IsWellDefined( W )
true
```

$W^{-1} = P_4$ and $W^0 = P_2 \oplus P_3$.

```
julia> ObjectAt( W, -1 )
GAP: <(v1)->0, (v2)->0, (v3)->0, (v4)->1; (a)->0x0, (b)->0x1, (c)->0x0, (d)->0x1>
julia> ObjectAt( W, 0 )
GAP: <(v1)->0, (v2)->1, (v3)->1, (v4)->2; (a)->0x1, (b)->1x2, (c)->0x1, (d)->1x2>
julia> ∂_m1 = DifferentialAt( W, -1 )
GAP: <(v1)->0x0, (v2)->0x1, (v3)->0x1, (v4)->1x2>
julia> Show( ∂_m1 )
```

$$\begin{array}{rcl} v_1 & \mapsto & ()_{0\times 0} \\ v_2 & \mapsto & ()_{0\times 1} \\ v_3 & \mapsto & ()_{0\times 1} \\ v_4 & \mapsto & \begin{pmatrix} 1 & 1 \end{pmatrix} \end{array}$$

```
julia> CohomologySupport( W )
GAP: [ 0 ]
```

Since 0 is an upper bound of W and its cohomology support[5] is [0], we can create the following acyclic complex

$$B := \qquad 0 \to W^{-1} \xrightarrow{\partial^{-1}} W^0 \xrightarrow{\text{CokernelProjection}(\partial^{-1})} \text{CokernelObject}(\partial^{-1}) \simeq H^0(W) \to 0$$

```
julia> H_0 = CohomologyAt( W, 0 )
GAP: <(v1)->0, (v2)->1, (v3)->1, (v4)->1; (a)->0x1, (b)->1x1, (c)->0x1, (d)->1x1>
julia> Show( H_0 )
```

$$\begin{array}{ccc} v_1 & \mapsto & k^{1\times 0} \\ v_2 & \mapsto & k^{1\times 1} \\ v_3 & \mapsto & k^{1\times 1} \\ v_4 & \mapsto & k^{1\times 1} \\ \hline \\ a & \mapsto & ()_{0\times 1} \\ \\ b & \mapsto & \begin{pmatrix} -1 \end{pmatrix} \\ \\ c & \mapsto & ()_{0\times 1} \\ \\ d & \mapsto & \begin{pmatrix} 1 \end{pmatrix} \end{array}$$

```
julia> ∂_0 = CokernelProjection( ∂_m1 )
GAP: <(v1)->0x0, (v2)->1x1, (v3)->1x1, (v4)->2x1>
julia> Show( ∂_0 )
```

$$\begin{array}{ccc} v_1 & \mapsto & ()_{0\times 0} \\ \\ v_2 & \mapsto & \begin{pmatrix} 1 \end{pmatrix} \\ \\ v_3 & \mapsto & \begin{pmatrix} 1 \end{pmatrix} \\ \\ v_4 & \mapsto & \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{array}$$

```
julia> IsEqualForObjects( H_0, Range( ∂_0 ) )
true
julia> B = DerivedCategoryObject( Dmod_A, [ ∂_m1, ∂_0 ], -1 )
```

[5] I.e., the cohomological indices where the cohomology object is not zero.

```
GAP: <An object in Derived category( The category of functors: Algebroid( (Q * q) / [
    -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q ) with active lower bound -1
    and active upper bound 1>
julia> IsWellDefined( B )
true
julia> CohomologySupport( B )
GAP: [  ]
```

Since B is an acyclic complex, it vanishes in the derived category. In the following, we check that applying the equivalence $U \bullet D$ on B returns an object which also vanishes in $\mathcal{K}^b(\mathbf{A}^{\mathrm{op},\oplus})$

```
julia> IsZero( B )
true
julia> UD_B = UD( B )
GAP: <An object in Homotopy category( Additive closure( Algebroid( (Q * q_op) / [ -1*(d
    *c) + 1*(b*a) ] ) ) ) with active lower bound -1 and active upper bound 1>
julia> Show( UD_B )
```

$$\begin{array}{c} v_2 \oplus v_3 \\ \uparrow \\ \begin{pmatrix} v_2 & 0 \\ 0 & v_3 \\ -b & -d \end{pmatrix} \\ |_0 \\ v_2 \oplus v_3 \oplus v_4 \\ \uparrow \\ \begin{pmatrix} b & d & v_4 \end{pmatrix} \\ |_{-1} \\ v_4 \end{array}$$

```
julia> IsZero( UD_B )
true
```

(6) Construct a complete strong exceptional sequence $\mathscr{E} = (E_1, E_2, E_3, E_4)$ in mod-A $\simeq$ mod-$\mathbb{A}$

Consider the following objects $E_1 := P_2,\ E_2 := P_3,\ E_3 := H^0(W),\ E_4 := P_1$ and let $T_{\mathscr{E}} := E_1 \oplus E_2 \oplus E_3 \oplus E_4$:

```
julia> E1 = P2;
julia> E2 = P3;
julia> E3 = CohomologyAt( W, 0 );
julia> E4 = P1
GAP: <(v1)->1, (v2)->1, (v3)->1, (v4)->1; (a)->1x1, (b)->1x1, (c)->1x1, (d)->1x1>
```

Using the new notation, we can rewrite the acyclic complex B as follows:

$$B := \qquad 0 \to P_4 \xrightarrow{\partial^{-1}} E_1 \oplus E_2 \xrightarrow{\text{CokernelProjection}(\partial^{-1})} E_3 \to 0$$

The above acyclic complex says that we can coresolve P_4 in terms of direct sums of E_1, E_2, E_3. That is, the object $P_1 \oplus P_2 \oplus P_3 \oplus P_4$ (which correspondes to $\mathbb{A}$ as an object in **mod**-$\mathbb{A}$) can also be coresolved by direct sums of E_1, E_2, E_3 and E_4.

```
julia> T = DirectSum( E1, E2, E3, E4 )
GAP: <(v1)->1, (v2)->3, (v3)->3, (v4)->4; (a)->1x3, (b)->3x4, (c)->1x3, (d)->3x4>
```

$$\begin{array}{ccc}
v_1 & \mapsto & k^{1\times 1} \\
v_2 & \mapsto & k^{1\times 3} \\
v_3 & \mapsto & k^{1\times 3} \\
v_4 & \mapsto & k^{1\times 4} \\
\hline
a & \mapsto & \begin{pmatrix} \cdot & \cdot & 1 \end{pmatrix} \\
b & \mapsto & \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \\
c & \mapsto & \begin{pmatrix} \cdot & \cdot & 1 \end{pmatrix} \\
d & \mapsto & \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}
\end{array}$$

```
julia> HomStructure( T, T )
GAP: <A vector space object over Q of dimension 9>
```

That is $\dim \operatorname{End}_k T_{\mathscr{E}} = 9$. Next, we want to prove that $\operatorname{Ext}^n(T_{\mathscr{E}}, T_{\mathscr{E}}) = 0$ for all $n \geq 1$. Since the global dimension of **mod-A** is 2, we have $\operatorname{Ext}^n(T_{\mathscr{E}}, T_{\mathscr{E}}) = 0$ for all $n \geq 3$. It remains to verify that $\operatorname{Ext}^1(T_{\mathscr{E}}, T_{\mathscr{E}}) = 0$ and $\operatorname{Ext}^2(T_{\mathscr{E}}, T_{\mathscr{E}}) = 0$.

It is well known that

$$\operatorname{Ext}^n(T_{\mathscr{E}}, T_{\mathscr{E}}) \simeq \operatorname{Hom}_{\mathcal{D}^b(\textbf{mod-A})}(T_{\mathscr{E}}, \Sigma^n T_{\mathscr{E}})$$

where $\Sigma : \mathcal{D}^b(\textbf{mod-A}) \xrightarrow{\sim} \mathcal{D}^b(\textbf{mod-A})$ is the shift autoequivalence on $\mathcal{D}^b(\textbf{mod-A})$.

```
julia> T = T / Cmod_A / Kmod_A / Dmod_A
GAP: <An object in Derived category( The category of functors: Algebroid( (Q * q) / [
    -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q ) with active lower bound 0
    and active upper bound 0>
julia> Shift( T, 1 )
GAP: <An object in Derived category( The category of functors: Algebroid( (Q * q) / [
    -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q ) with active lower bound -1
    and active upper bound -1>
julia> HomStructure( T, Shift( T, 0 ) )
GAP: <A vector space object over Q of dimension 9>
```

```
julia> HomStructure( T, Shift( T, 1 ) )
GAP: <A vector space object over Q of dimension 0>
julia> HomStructure( T, Shift( T, 2 ) )
GAP: <A vector space object over Q of dimension 0>
```

To sum up,

- $T_{\mathscr{E}}$ admits a finite projective resolution,
- $T_{\mathscr{E}}$ has no higher extensions, i.e., $\mathrm{Ext}^n(T_{\mathscr{E}}, T_{\mathscr{E}}) \simeq 0$ for all $n \geq 1$ and
- $P_1 \oplus P_2 \oplus P_3 \oplus P_4$ can be coresolved by direct summands of direct sums of $T_{\mathscr{E}}$.

Hence, the object $T_{\mathscr{E}} = E_1 \oplus E_2 \oplus E_3 \oplus E_4$ is a generalized tilting object in **mod-A** $\simeq$ **mod**-$\mathbb{A}$. Happel's theorem states that the derived functors

$$- \otimes^{\mathbb{L}} T_{\mathscr{E}} \ : \mathcal{D}^b(\mathbf{A}_{\mathscr{E}}) \xrightarrow{\sim} \mathcal{D}^b(\mathbf{mod\text{-}A}) : \ \mathbb{R}\mathrm{Hom}(T_{\mathscr{E}}, -)$$

induce an adjoint equivalences where $\mathbf{A}_{\mathscr{E}}$ is the abstraction k-algebroid of $\mathscr{E}$ and $\mathcal{D}^b(\mathbf{A}_{\mathscr{E}}) :=$ $\mathcal{D}^b(\mathbf{A}_{\mathscr{E}}) := \mathcal{D}^b(\mathbf{mod\text{-}A}_{\mathscr{E}}^{\mathrm{op}})$.

In the following we create $\mathscr{E}$. For a better readability, we label each object in $\mathscr{E}$ by its dimension vector:

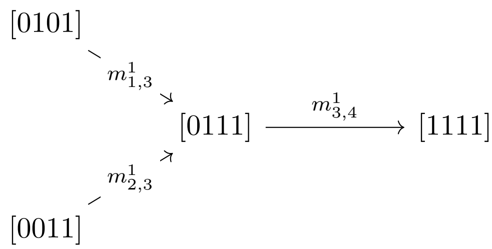
```
julia> ℰ =  CreateStrongExceptionalCollection(
                [ E1,       E2,       E3,       E4       ],
                [ "[0101]", "[0011]", "[0111]", "[1111]" ]
           )
GAP: <A strong exceptional sequence defined by the objects of the Full subcategory
    generated by 4 objects in The category of functors: Algebroid( (Q * q) / [ -1*(c*d)
     + 1*(a*b) ] ) -> Category of matrices over Q>
```

(7) Compute the quiver $\mathfrak{q}_{\mathscr{E}}$ and the abstraction k-algebroid $\mathbf{A}_{\mathscr{E}}$.

The abstraction k-algebroid $\mathbf{A}_{\mathscr{E}}$ of $\mathscr{E}$ can be computed as follows:

```
julia> A_ℰ = Algebroid( ℰ )
GAP: Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111] ) )
julia> q_ℰ = UnderlyingQuiver( A_ℰ )
GAP: quiver([0101],[0011],[0111],[1111])[m1_3_1:[0101]->[0111], m2_3_1:[0011]->[0111],
    m3_4_1:[0111]->[1111]]
julia> relationsOfAlgebroid( A_ℰ )
GAP: [  ]
julia> EndT = UnderlyingQuiverAlgebra( A_ℰ )
GAP: End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111] )
julia> Dimension( EndT )
9
julia> IsAdmissibleQuiverAlgebra( EndT )
true
```

That is, the quiver $\mathfrak{q}_{\mathscr{E}}$ of $\mathscr{E}$ consists of 4 vertices and 3 arrows:



The vertices are labeled by the strings we assigned to the objects of $\mathscr{E}$ and the arrows are labeled by $m^k_{i,j}$ which means that the arrow is the k'th arrow from the vertex indexed by i to the vertex indexed by j.

```
julia> u1 = Vertex( q_𝓔, 1 )
GAP: ([0101])
julia> u1 == q_𝓔."[0101]"
true
julia> m1_3_1 = q_𝓔."m1_3_1"
GAP: (m1_3_1)
```

(8) Compute the isomorphism $\mathscr{E} \simeq \mathbf{A}_{\mathscr{E}}$ and the equivalences
$\mathcal{K}^b(\mathscr{E}^{\oplus}) \simeq \mathcal{K}^b\left(\mathbf{A}_{\mathscr{E}}^{\oplus}\right) \simeq \mathcal{K}^b(\mathbf{A}_{\mathscr{E}}\text{-}\mathbf{proj}) \simeq \mathcal{D}^b(\mathbf{A}_{\mathscr{E}})$.

We call the isomorphism functors between $\mathscr{E}$ and $\mathbf{A}_{\mathscr{E}}$ *the abstraction functor* abs resp. *the realization functor* rel

$$\text{abs} \ : \mathscr{E} \xrightarrow{\sim} \mathbf{A}_{\mathscr{E}} : \ \text{rel}$$

```
julia> abs = IsomorphismOntoAlgebroid( 𝓔 )
GAP: Isomorphism functor from exceptional collection onto Algebroid
julia> abs( 𝓔[ 1 ] )
GAP: <([0101])>
julia> rel = IsomorphismFromAlgebroid( 𝓔 )
GAP: Isomorphism functor from Algebroid onto exceptional collection
julia> rel( A_𝓔."[1111]" )
GAP: An object in full subcategory given by: <(v1)->1, (v2)->1, (v3)->1, (v4)->1; (a
    )->1x1, (b)->1x1, (c)->1x1, (d)->1x1>
julia> rel( A_𝓔."[1111]" ) == 𝓔[ 4 ]
true
julia> m = rel( A_𝓔."m3_4_1" )
GAP: A morphism in full subcategory given by: <(v1)->0x1, (v2)->1x1, (v3)->1x1, (v4)->1
    x1>
julia> Source( m ) == 𝓔[ 3 ] && Range( m ) == 𝓔[ 4 ]
true
julia> Show( UnderlyingCell( m ) )
```

$$\begin{array}{lcl} v_1 & \mapsto & ()_{0\times 0} \\ v_2 & \mapsto & \left(\begin{array}{c} -1 \end{array} \right) \\ v_3 & \mapsto & ()_{0\times 1} \\ v_4 & \mapsto & \left(\begin{array}{c} 1 \end{array} \right) \end{array}$$

```
julia> m = rel( A_ℰ."m2_3_1" )
GAP: A morphism in full subcategory given by: <(v1)->0x0, (v2)->0x1, (v3)->1x1, (v4)->1
    x1>
julia> Source( m ) == ℰ[ 2 ] && Range( m ) == ℰ[ 3 ]
true
julia> Show( m )
```

$$\begin{array}{lcl} v_1 & \mapsto & ()_{0\times 0} \\ v_2 & \mapsto & ()_{0\times 1} \\ v_3 & \mapsto & \left(\begin{array}{c} 1 \end{array} \right) \\ v_4 & \mapsto & \left(\begin{array}{c} 1 \end{array} \right) \end{array}$$

```
julia> m = rel( A_ℰ."m1_3_1" )
GAP: A morphism in full subcategory given by: <(v1)->0x0, (v2)->1x1, (v3)->0x1, (v4)->1
    x1>
julia> Source( m ) == ℰ[ 1 ] && Range( m ) == ℰ[ 3 ]
true
julia> Show( m )
```

$$\begin{array}{lcl} v_1 & \mapsto & ()_{0\times 0} \\ v_2 & \mapsto & \left(\begin{array}{c} -1 \end{array} \right) \\ v_3 & \mapsto & ()_{0\times 1} \\ v_4 & \mapsto & \left(\begin{array}{c} 1 \end{array} \right) \end{array}$$

The above isomorphisms together with the Yoneda embedding induces equivalences:

$$\mathscr{E}^{\oplus} \simeq \mathbf{A}_{\mathscr{E}}^{\oplus} \simeq \mathbf{A}_{\mathscr{E}}\text{-}\mathbf{proj}.$$

```
julia> abs = ExtendFunctorToAdditiveClosures( abs )
GAP: Extension of Abstraction isomorphism to additive closures
julia> rel = ExtendFunctorToAdditiveClosures( rel )
```

```
GAP: Extension of Realization isomorphism to additive closures
julia> 𝐀_ℰ_op = OppositeAlgebroid( 𝐀_ℰ )
GAP: Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111] )^op )
julia> 𝐀_ℰ_mod = Hom( 𝐀_ℰ_op, k_vec )
GAP: The category of functors: Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111] )^op
    ) -> Category of matrices over Q
julia> InfoOfInstalledOperationsOfCategory( 𝐀_ℰ_mod )
120 primitive operations were used to derive 312 operations for this category which
* IsEquippedWithHomomorphismStructure
* IsLinearCategoryOverCommutativeRing
* IsAbelianCategoryWithEnoughInjectives
* IsAbelianCategoryWithEnoughProjectives
julia> 𝐀_ℰ_proj = FullSubcategoryGeneratedByProjectiveObjects( 𝐀_ℰ_mod )
GAP: Full additive subcategory generated by projective objects( The category of
    functors: Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111] )^op ) -> Category of
    matrices over Q )
julia> 𝐀_ℰ_plus = AdditiveClosure( 𝐀_ℰ )
GAP: Additive closure( Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111] ) ) )
julia> KnownFunctors( 𝐀_ℰ_plus, 𝐀_ℰ_proj )
1: Yoneda embedding
julia> KnownFunctors( 𝐀_ℰ_proj, 𝐀_ℰ_plus )
1: Decomposition of projective objects
```

The above isomorphisms can also be extended to equivalences of categories:

$$\mathcal{K}^b(\mathscr{E}^{\oplus}) \simeq \mathcal{K}^b\left(\mathbf{A}_{\mathscr{E}}^{\oplus}\right) \simeq \mathcal{K}^b(\mathbf{A}_{\mathscr{E}}\text{-}\mathbf{proj}) \simeq \mathcal{D}^b(\mathbf{A}_{\mathscr{E}}).$$

```
julia> abs = ExtendFunctorToHomotopyCategoriesByCochains( abs )
GAP: Extension of a functor to homotopy categories
julia> rel = ExtendFunctorToHomotopyCategoriesByCochains( rel )
GAP: Extension of a functor to homotopy categories
```

On the other hand, we have a natural embedding functor $\mathcal{K}^b(\mathscr{E}^{\oplus}) \hookrightarrow \mathcal{K}^b(\mathbf{mod}\text{-}\mathbf{A})$:

```
julia> I = EmbeddingFunctorFromAdditiveClosure( ℰ );
julia> I = ExtendFunctorToHomotopyCategoriesByCochains( I )
GAP: Extension of a functor to homotopy categories
julia> Display( I )
Embedding functor

Additive closure( Full subcategory generated by 4 objects in The category of functors:
    Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q )
  |
  V
The category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of
    matrices over Q
julia> N = RandomObject( SourceOfFunctor( rel ), ConvertJuliaToGAP( [ -1, 1, 2 ] ) )
GAP: <An object in Homotopy category( Additive closure( Algebroid( End( [0101] ⊕ [0011]
     ⊕ [0111] ⊕ [1111] ) ) ) ) with active lower bound -1 and active upper bound 1>
julia> Show( N )
```

$$
\begin{array}{c}
[0101] \oplus [1111] \\
\uparrow \\
\begin{pmatrix} 0 & 3[1111] \\ 0 & 3m^1_{3,4} \end{pmatrix} \\
|_0 \\
[1111] \oplus [0111] \\
\uparrow \\
\begin{pmatrix} -3m^1_{1,3}m^1_{3,4} & 3m^1_{1,3} \\ -3m^1_{3,4} & 3[0111] \end{pmatrix} \\
|_{-1} \\
[0101] \oplus [0111]
\end{array}
$$

```
julia> N = I( rel( N ) )
GAP: <An object in Homotopy category( The category of functors: Algebroid( (Q * q) / [
    -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q ) with active lower bound -1
    and active upper bound 1>
julia> N[-1]
GAP: <(v1)->0, (v2)->2, (v3)->1, (v4)->2; (a)->0x2, (b)->2x2, (c)->0x1, (d)->1x2>
julia> N[0]
GAP: <(v1)->1, (v2)->2, (v3)->2, (v4)->2; (a)->1x2, (b)->2x2, (c)->1x2, (d)->2x2>
julia> N[1]
GAP: <(v1)->1, (v2)->2, (v3)->1, (v4)->2; (a)->1x2, (b)->2x2, (c)->1x1, (d)->1x2>
```

(9) Construct the adjoint functors

$- \otimes T_{\mathscr{E}} \quad : \mathbf{A}_{\mathscr{E}}\textbf{-mod} \to \textbf{mod-A} \quad : \mathrm{Hom}(T_{\mathscr{E}}, -)$

For every object F in **mod-A** we have $\mathrm{Hom}_{\textbf{mod-A}}(T_{\mathscr{E}}, F) \simeq \bigoplus_{i=1}^{4} \mathrm{Hom}_{\textbf{mod-A}}(E_i, F)$. This enables us to interpret $\mathrm{Hom}_{\textbf{mod-A}}(T_{\mathscr{E}}, F)$ as an object in $\mathbf{A}_{\mathscr{E}}$**-mod**. The images of $E_i \in \mathscr{E}$ under $\mathrm{Hom}(T_{\mathscr{E}}, -)$ are, up to isomorphism, *the* indecomposable projective objects in $\mathbf{A}_{\mathscr{E}}$**-mod**. Its left adjoint $- \otimes T_{\mathscr{E}}$ is right exact and translates back the indecomposable projective objects to corresponding E_i's.

```
julia> mod_A
GAP: The category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ] ) ->
    Category of matrices over Q
julia> A_ℰ_mod
GAP: The category of functors: Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111] )^op
    ) -> Category of matrices over Q
julia> HomT = HomFunctorToCategoryOfFunctors( ℰ )
GAP: Hom(T,-) functor
julia> Display( HomT )
Hom(T,-) functor:

The category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of
    matrices over Q
  |
  V
```

```
The category of functors: Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111] )^op ) ->
    Category of matrices over Q
julia> tensorT = TensorFunctorFromCategoryOfFunctors( ℰ )
GAP: -⊗T functor
julia> Display( tensorT )
-⊗T functor:

The category of functors: Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111] )^op ) ->
    Category of matrices over Q
  |
  V
The category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of
    matrices over Q
julia> ϵ = CounitOfTensorHomAdjunction( E, tensorT, HomT )
GAP: Hom(T,-) ⊗ T ⇒ Id
julia> η = UnitOfTensorHomAdjunction( E, tensorT, HomT )
GAP: Id ⇒ Hom( T, -⊗T )
```

Let us compute the component of $\epsilon_F\colon \operatorname{Hom}(T_{\mathscr{E}}, F) \otimes T_{\mathscr{E}} \to F$:

```
julia> F
GAP: <(v1)->4, (v2)->2, (v3)->1, (v4)->2; (a)->4x2, (b)->2x2, (c)->4x1, (d)->1x2>
julia> tensorT_HomT_F = tensorT( HomT( F ) )
GAP: <(v1)->4, (v2)->2, (v3)->1, (v4)->1; (a)->4x2, (b)->2x1, (c)->4x1, (d)->1x1>
julia> Show( tensorT_HomT_F )
```

$$\begin{array}{lcl} v_1 & \mapsto & k^{1\times 4} \\ v_2 & \mapsto & k^{1\times 2} \\ v_3 & \mapsto & k^{1\times 1} \\ v_4 & \mapsto & k^{1\times 1} \\ \hline \\ a & \mapsto & \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ 1 & \cdot \\ \cdot & 1 \end{pmatrix} \\ \\ b & \mapsto & \begin{pmatrix} \cdot \\ 1 \end{pmatrix} \\ \\ c & \mapsto & \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \\ \\ d & \mapsto & \begin{pmatrix} 1 \end{pmatrix} \end{array}$$

```
julia> ϵ_F = ϵ( F )
GAP: <(v1)->4x4, (v2)->2x2, (v3)->1x1, (v4)->1x2>
julia> Source( ϵ_F ) == tensorT_HomT_F && Range( ϵ_F ) == F
```

```
true
julia> Show( ϵ_F )
```

$$
\begin{aligned}
v_1 &\mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix} \\
v_2 &\mapsto \begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} \\
v_3 &\mapsto \begin{pmatrix} 1 \end{pmatrix} \\
v_4 &\mapsto \begin{pmatrix} \cdot & 1 \end{pmatrix}
\end{aligned}
$$

Since ϵ is a natural transformation, the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Hom}(T_{\mathscr{E}}, F) \otimes T_{\mathscr{E}} & \xrightarrow{\mathrm{Hom}(T_{\mathscr{E}},\psi)\otimes T_{\mathscr{E}}} & \mathrm{Hom}(T_{\mathscr{E}}, G) \otimes T_{\mathscr{E}} \\
\downarrow{\scriptstyle \epsilon(F)} & & \downarrow{\scriptstyle \epsilon(G)} \\
F & \xrightarrow[\psi]{} & G
\end{array}
$$

```
julia> PreCompose( ϵ( F ), ψ ) == PreCompose( tensorT( HomT( ψ ) ), ϵ( G ) )
true
```

(10) Construct the adjoint derived equivalences

$$- \otimes^{\mathbb{L}} T_{\mathscr{E}} \; : \mathcal{D}^b(\mathbf{A}_{\mathscr{E}}) \xrightarrow{\sim} \mathcal{D}^b(\textbf{mod-A}) \; : \mathbb{R}\mathrm{Hom}(T_{\mathscr{E}}, -)$$

and use them to compute an $\mathscr{E}$-replacement of an object $\mathcal{D}^b(\textbf{mod-A})$

The right and left derived functors $\mathbb{R}\,\mathrm{Hom}(T_{\mathscr{E}}, -)$ and $-\otimes^{\mathbb{L}} T_{\mathscr{E}}$ can be computed by extending $\mathrm{Hom}(T_{\mathscr{E}}, -)$ and $- \otimes T_{\mathscr{E}}$ to the homotopy categories

$$- \otimes T_{\mathscr{E}} \; : \mathcal{K}^b(\mathbf{A}_{\mathscr{E}}) \xrightarrow{\sim} \mathcal{K}^b(\textbf{mod-A}) \; : \mathrm{Hom}(T_{\mathscr{E}}, -)$$

then applying them to injective resp. projective replacements (cf. Examples 3.67 and 3.71). To demonstrate this, we will do all computations in the homotopy categories. Let N be an object in $\mathcal{K}^b(\textbf{mod-A})$ and $\iota_N : N \to \mathcal{I}_N$ a quasi-isomorphism to the injective replacement $\mathcal{I}_N$ of N. Suppose $\pi_{\mathrm{Hom}(T_{\mathscr{E}},\mathcal{I}_N)} \colon \mathcal{P}_{\mathrm{Hom}(T_{\mathscr{E}},\mathcal{I}_N)} \to \mathrm{Hom}(T_{\mathscr{E}}, \mathcal{I}_N)$ is a quasi-isomorphism to $\mathrm{Hom}(T_{\mathscr{E}}, \mathcal{I}_N)$ from its projective replacement. The $\mathscr{E}$-replacement of N is defined by $\mathcal{P}_{\mathrm{Hom}(T_{\mathscr{E}},\mathcal{I}_N)} \otimes T_{\mathscr{E}}$ which lives in the image of the full embedding of $\mathcal{K}^b(\mathscr{E}^{\oplus}) \hookrightarrow \mathcal{K}^b(\textbf{mod-A})$. In particular, the cospan formed by the quasi-isomorphisms

$$\tau_N := \left(\pi_{\mathrm{Hom}(T_{\mathscr{E}},\mathcal{I}_N)} \otimes T_{\mathscr{E}}\right) \cdot \epsilon(\mathcal{I}_N) \colon \mathcal{P}_{\mathrm{Hom}(T_{\mathscr{E}},\mathcal{I}_N)} \otimes T_{\mathscr{E}} \to \mathcal{I}_N$$

and $\iota_N \colon N \to \mathcal{I}_N$ gives rise to an isomorphism $N \cong \mathcal{P}_{\mathrm{Hom}(T_{\mathscr{E}},\mathcal{I}_N)} \otimes T_{\mathscr{E}}$ in $\mathcal{D}^b(\textbf{mod-A})$.

```
julia> HomT = ExtendFunctorToHomotopyCategoriesByCochains( HomT )
GAP: Extension of a functor to homotopy categories
julia> tensorT = ExtendFunctorToHomotopyCategoriesByCochains( tensorT )
GAP: Extension of a functor to homotopy categories
julia> ϵ = ExtendNaturalTransformationToHomotopyCategoriesByCochains( ϵ )
GAP: Extention of the natural transformation ( Hom(T,-) ⊗ T ⇒ Id ) to homotopy
    categories
julia> N = P4 / Cmod_A / Kmod_A
GAP: <An object in Homotopy category( The category of functors: Algebroid( (Q * q) / [
    -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q ) with active lower bound 0
    and active upper bound 0>
julia> IN = InjectiveResolution( N, true )
GAP: <An object in Homotopy category( The category of functors: Algebroid( (Q * q) / [
    -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q ) with active lower bound 0
    and active upper bound 2>
julia> ForAll( ConvertJuliaToGAP( [ 0, 1, 2 ] ), i ->  IsInjective( IN[ i ] ) )
true
julia> HomT_IN = HomT( IN )
GAP: <An object in Homotopy category( The category of functors: Algebroid( End( [0101]
    ⊕ [0011] ⊕ [0111] ⊕ [1111] )^op ) -> Category of matrices over Q ) with active
    lower bound 0 and active upper bound 2>
julia> PHomT_IN = ProjectiveResolution( HomT_IN, true )
GAP: <An object in Homotopy category( The category of functors: Algebroid( End( [0101]
    ⊕ [0011] ⊕ [0111] ⊕ [1111] )^op ) -> Category of matrices over Q ) with active
    lower bound 0 and active upper bound 2>
julia> qHomT_IN = QuasiIsomorphismFromProjectiveResolution( HomT_IN, true )
GAP: <A morphism in Homotopy category( The category of functors: Algebroid( End( [0101]
     ⊕ [0011] ⊕ [0111] ⊕ [1111] )^op ) -> Category of matrices over Q ) with active
    lower bound 0 and active upper bound 2>
julia> IsWellDefined( qHomT_IN ) && IsQuasiIsomorphism( qHomT_IN )
true
julia> ℰ_rep_N = tensorT( PHomT_IN )
GAP: <An object in Homotopy category( The category of functors: Algebroid( (Q * q) / [
    -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q ) with active lower bound 0
    and active upper bound 2>
julia> ℰ_rep_N[0]
GAP: <(v1)->1, (v2)->2, (v3)->2, (v4)->3; (a)->1x2, (b)->2x3, (c)->1x2, (d)->2x3>
julia> ℰ_rep_N[0] == DirectSum( E1, E2, E4 )
true
julia> ℰ_rep_N[1]
GAP: <(v1)->2, (v2)->3, (v3)->3, (v4)->3; (a)->2x3, (b)->3x3, (c)->2x3, (d)->3x3>
julia> ℰ_rep_N[1] == DirectSum( E3, E4, E4 )
true
julia> ℰ_rep_N[2]
GAP: <(v1)->1, (v2)->1, (v3)->1, (v4)->1; (a)->1x1, (b)->1x1, (c)->1x1, (d)->1x1>
julia> ℰ_rep_N[2] == E4
true
julia> τN = PreCompose( tensorT( qHomT_IN ), ϵ( IN ) )
```

```
GAP: <A morphism in Homotopy category( The category of functors: Algebroid( (Q * q) / [
     -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q ) with active lower bound 0
    and active upper bound 2>
julia> ( Source( τN ) == ℰ_rep_N ) && ( Range( τN ) == IN )
true
julia> IsWellDefined( τN )
true
julia> IsQuasiIsomorphism( τN )
true
```

In the following we compute the $\mathscr{E}$-replacement of $N := P_4$ as an object in $\mathcal{K}^b\Big(\mathbf{A}_{\mathscr{E}}^{\oplus}\Big)$:

```
julia> A_ℰ_plus
GAP: Additive closure( Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111] ) ) )
julia> KA_ℰ_plus = HomotopyCategoryByCochains( A_ℰ_plus )
GAP: Homotopy category( Additive closure( Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕
    [1111] ) ) ) )
julia> A_ℰ_mod
GAP: The category of functors: Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111] )^op
    ) -> Category of matrices over Q
julia> KA_ℰ_mod = HomotopyCategoryByCochains( A_ℰ_mod )
GAP: Homotopy category( The category of functors: Algebroid( End( [0101] ⊕ [0011] ⊕
    [0111] ⊕ [1111] )^op ) -> Category of matrices over Q )
julia> L = LocalizationFunctorByProjectiveObjects( KA_ℰ_mod )
GAP: Localization functor by projective objects
julia> KA_ℰ_proj = RangeOfFunctor( L )
GAP: Homotopy category( Full additive subcategory generated by projective objects( The
    category of functors: Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111] )^op ) ->
    Category of matrices over Q ) )
julia> KnownFunctors( KA_ℰ_proj, KA_ℰ_plus )
1: Apply ExtendFunctorToHomotopyCategoriesByCochains on ( Decomposition of projective
    objects )
julia> D = Functor( KA_ℰ_proj, KA_ℰ_plus, 1 )
GAP: Extension of a functor to homotopy categories
julia> R = PreCompose( [ HomT, L, D ] );
julia> Display( R )
Composition of Composition of Extension of a functor to homotopy categories and
    Localization functor by projective objects and Extension of a functor to homotopy
    categories:

Homotopy category( The category of functors: Algebroid( (Q * q) / [ -1*(c*d) + 1*(a*b)
    ] ) -> Category of matrices over Q )
  |
  V
Homotopy category( Additive closure( Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111]
     ) ) ) )
julia> R_IN = R( IN )
GAP: <An object in Homotopy category( Additive closure( Algebroid( End( [0101] ⊕ [0011]
     ⊕ [0111] ⊕ [1111] ) ) ) ) with active lower bound 0 and active upper bound 2>
```

```
julia> Show( R_IN )
```

$$
\begin{array}{c}
[1111] \\
\uparrow \\
\begin{pmatrix} -m^{1}_{3,4} \\ -[1111] \\ [1111] \end{pmatrix} \\
|_{1} \\
[0111] \oplus [1111]^{\oplus 2} \\
\uparrow \\
\begin{pmatrix} -m^{1}_{1,3} & 0 & -m^{1}_{1,3} m^{1}_{3,4} \\ m^{1}_{2,3} & -m^{1}_{2,3} m^{1}_{3,4} & 0 \\ 0 & [1111] & [1111] \end{pmatrix} \\
|_{0} \\
[0101] \oplus [0011] \oplus [1111]
\end{array}
$$

```
julia> sR_IN = SimplifyObject( R_IN, infinity )
GAP: <An object in Homotopy category( Additive closure( Algebroid( End( [0101] ⊕ [0011]
    ⊕ [0111] ⊕ [1111] ) ) ) ) with active lower bound 0 and active upper bound 2>
julia> Show( sR_IN )
```

$$
\begin{array}{c}
0 \\
\uparrow \\
() \\
|_{1} \\
[0111] \\
\uparrow \\
\begin{pmatrix} m^{1}_{1,3} \\ -m^{1}_{2,3} \end{pmatrix} \\
|_{0} \\
[0101] \oplus [0011]
\end{array}
$$

```
julia> m = SimplifyObject_IsoToInputObject( R_IN, infinity )
GAP: <A morphism in Homotopy category( Additive closure( Algebroid( End( [0101] ⊕
    [0011] ⊕ [0111] ⊕ [1111] ) ) ) ) with active lower bound 0 and active upper bound
    2>
julia> IsWellDefined( m ) && IsIsomorphism( m )
julia> Show( m )
```

$$
\begin{array}{ccc}
0 & -\ 0 \ \rightarrow & [1111] \\
\uparrow & & \uparrow \\
() & & \begin{pmatrix} -m^1_{3,4} \\ -[1111] \\ [1111] \end{pmatrix} \\
|_1 & & |_1 \\
[0111] & -\ \begin{pmatrix} [0111] & -m^1_{3,4} & 0 \end{pmatrix} \ \rightarrow & [0111] \oplus [1111]^{\oplus 2} \\
\uparrow & & \uparrow \\
\begin{pmatrix} m^1_{1,3} \\ -m^1_{2,3} \end{pmatrix} & & \begin{pmatrix} -m^1_{1,3} & 0 & -m^1_{1,3}m^1_{3,4} \\ m^1_{2,3} & -m^1_{2,3}m^1_{3,4} & 0 \\ 0 & [1111] & [1111] \end{pmatrix} \\
|_0 & & |_0 \\
[0101] \oplus [0011] & -\ \begin{pmatrix} -[0101] & 0 & -m^1_{1,3}m^1_{3,4} \\ 0 & -[0011] & 0 \end{pmatrix} \ \rightarrow & [0101] \oplus [0011] \oplus [1111]
\end{array}
$$

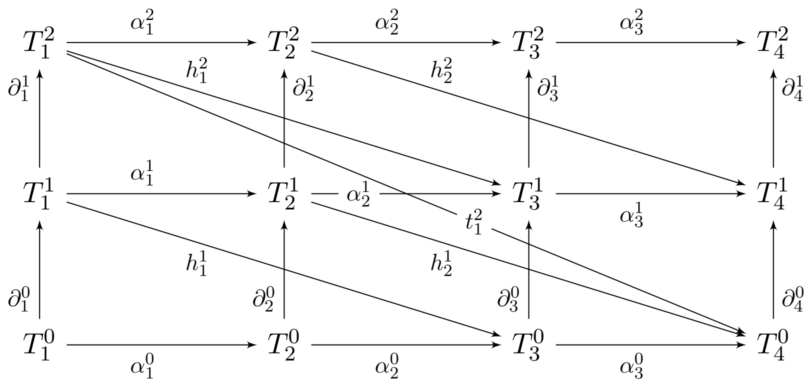
```
julia> I_rel_m = I( rel( m ) )
GAP: <A morphism in Homotopy category( The category of functors: Algebroid( (Q * q) / [
     -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q ) with active lower bound 0
    and active upper bound 2>
julia> IsIsomorphism( I_rel_m )
julia> Range( I_rel_m ) == ℰ_rep_N
true
julia> im = InverseForMorphisms( m )
julia> Show( im )
```

$$
\begin{array}{ccc}
[1111] & -\ () \ \rightarrow & 0 \\
\uparrow & & \uparrow \\
\begin{pmatrix} -m^1_{3,4} \\ -[1111] \\ [1111] \end{pmatrix} & & () \\
|_1 & & |_1 \\
[0111] \oplus [1111]^{\oplus 2} & -\ \begin{pmatrix} [0111] \\ 0 \\ 0 \end{pmatrix} \ \rightarrow & [0111] \\
\uparrow & & \uparrow \\
\begin{pmatrix} -m^1_{1,3} & 0 & -m^1_{1,3}m^1_{3,4} \\ m^1_{2,3} & -m^1_{2,3}m^1_{3,4} & 0 \\ 0 & [1111] & [1111] \end{pmatrix} & & \begin{pmatrix} m^1_{1,3} \\ -m^1_{2,3} \end{pmatrix} \\
|_0 & & |_0 \\
[0101] \oplus [0011] \oplus [1111] & -\ \begin{pmatrix} -[0101] & 0 \\ 0 & -[0011] \\ 0 & 0 \end{pmatrix} \ \rightarrow & [0101] \oplus [0011]
\end{array}
$$

APPENDIX F

A Demo for Computing a Standard Postnikov System

Let k be a field and $\mathfrak{q}$ be the following quiver



Let $\mathbf{A}_{\mathfrak{q}}$ the k-linear finitely presented category defined by $\mathfrak{q}$ subject to the relations

$$\{\partial_j^0 \cdot \partial_j^1 \mid 1 \leq j \leq 4\} \cup \{\partial_j^i \cdot \alpha_j^{i+1} - \alpha_j^i \cdot \partial_{j+1}^i \mid 0 \leq i \leq 1, 1 \leq j \leq 3\}$$

$$\cup\{\partial_j^0 \cdot h_j^1 - \alpha_j^0 \cdot \alpha_{j+1}^0 \mid 1 \leq j \leq 2\} \cup \{h_j^2 \cdot \partial_{j+2}^1 - \alpha_j^2 \cdot \alpha_{j+1}^2 \mid 1 \leq j \leq 2\}$$

$$\cup\{\partial_j^1 \cdot h_j^2 + h_j^1 \cdot \partial_{j+2}^0 - \alpha_j^1 \cdot \alpha_{j+1}^1 \mid 1 \leq j \leq 2\}$$

$$\cup\{\partial_1^1 \cdot t_1^2 + h_1^1 \cdot \alpha_3^0 - \alpha_1^1 \cdot h_2^1\} \cup \{\alpha_1^2 \cdot h_2^2 + t_1^2 \cdot \partial_4^0 - h_1^2 \cdot \alpha_3^1\}.$$

For every j with $1 \leq j \leq 4$, define T_j by the object of $\mathcal{K}^b\left(\mathbf{A}_{\mathfrak{q}}^{\oplus}\right)$ whose differential at index $0 \leq i \leq 1$ is ∂_j^i. For every j with $1 \leq j \leq 3$, define $\alpha_j \colon T_j \to T_{j+1}$ by the morphism whose component at index i is α_j^i.

Let T be the object in $\mathcal{C}^b\left(\mathcal{K}^b\left(\mathbf{A}_{\mathfrak{q}}^{\oplus}\right)\right)$ defined by the sequence

$$0 \to T_1 \xrightarrow{\alpha_1} T_2 \xrightarrow{\alpha_2} T_3 \xrightarrow{\alpha_3} T_4 \to 0$$

where T_1 is concentrated at index 1. In this demonstration, we want to use Algorithm 4 to compute an extension of T to a standard Postnikov system.

We start by loading the `CapAndHomalg` and loading `QPA2` and `HomotopyCategories`:

```
julia> using CapAndHomalg
CapAndHomalg v1.1.8
Imported OSCAR's components GAP and Singular_jll
Type: ?CapAndHomalg for more information
julia> LoadPackage( "QPA" )
```

```
julia> LoadPackage( "HomotopyCategories" )
```

Next, we define the right quiver q:

```
julia> q = RightQuiver(
    "q(T10,T11,T12,T20,T21,T22,T30,T31,T32,T40,T41,T42)[" *
    "d10:T10->T11,d11:T11->T12,d20:T20->T21,d21:T21->T22,d30:T30->T31,d31:T31->T32," *
    "d40:T40->T41,d41:T41->T42,alpha10:T10->T20,alpha11:T11->T21,alpha12:T12->T22," *
    "alpha20:T20->T30,alpha21:T21->T31,alpha22:T22->T32,alpha30:T30->T40," *
    "alpha31:T31->T41,alpha32:T32->T42,h11:T11->T30,h12:T12->T31,h21:T21->T40," *
    "h22:T22->T41,t12:T12->T40]" );

julia> SetLabelsAsLaTeXStrings( q,
    [ "T_1^0", "T_1^1", "T_1^2", "T_2^0", "T_2^1", "T_2^2",
      "T_3^0", "T_3^1", "T_3^2", "T_4^0", "T_4^1", "T_4^2" ],
    [ "\\partial_1^0", "\\partial_1^1", "\\partial_2^0", "\\partial_2^1",
      "\\partial_3^0", "\\partial_3^1", "\\partial_4^0", "\\partial_4^1",
      "\\alpha_1^0", "\\alpha_1^1", "\\alpha_1^2", "\\alpha_2^0", "\\alpha_2^1",
      "\\alpha_2^2", "\\alpha_3^0", "\\alpha_3^1", "\\alpha_3^2",
      "h_1^1", "h_1^2", "h_2^1", "h_2^2", "t_1^2" ] );
```

The next task is to construct the category of bounded complexes $\mathcal{C}^b\left(\mathcal{K}^b\left(\mathbf{A}_{\mathrm{q}}^{\oplus}\right)\right)$. This can be done in six steps:

(1) Construct the free category $\mathbf{F}_{\mathrm{q}}$ generated by quiver q.
(2) Construct the k-linear closure category $k[\mathbf{F}_{\mathrm{q}}]$ of $\mathbf{F}_{\mathrm{q}}$.
(3) Construct the quotient category $\mathbf{A}_{\mathrm{q}}$ of $k[\mathbf{F}_{\mathrm{q}}]$ modulo the two-sided ideal generated by the relations ρ.
(4) Construct the additive closure category $\mathbf{A}_{\mathrm{q}}^{\oplus}$ of $\mathbf{A}_{\mathrm{q}}$.
(5) Construct the bounded homotopy category $\mathcal{K}^b\left(\mathbf{A}_{\mathrm{q}}^{\oplus}\right)$ as a quotient category of $\mathcal{C}^b\left(\mathbf{A}_{\mathrm{q}}^{\oplus}\right)$ modulo the two-sided ideal generated by all null-homotopic morphisms.
(6) Construct the category of bounded complexes $\mathcal{C}^b\left(\mathcal{K}^b\left(\mathbf{A}_{\mathrm{q}}^{\oplus}\right)\right)$.

```
julia> Fq = FreeCategory( q )
GAP: Category freely generated by the right quiver ...

julia> ℚ = HomalgFieldOfRationals( )
GAP: Q
julia> k = ℚ
GAP: Q

julia> kFq = k[ Fq ]
GAP: Algebroid( Q * q )

julia> ρ =
[ PreCompose( kFq."d10", kFq."d11" ),  PreCompose( kFq."d20", kFq."d21" ),
  PreCompose( kFq."d30", kFq."d31" ),  PreCompose( kFq."d40", kFq."d41" ),
  PreCompose( kFq."d10", kFq."alpha11" ) - PreCompose( kFq."alpha10", kFq."d20" ),
```

```
  PreCompose( kFq."d11", kFq."alpha12" ) - PreCompose( kFq."alpha11", kFq."d21" ),
  PreCompose( kFq."d20", kFq."alpha21" ) - PreCompose( kFq."alpha20", kFq."d30" ),
  PreCompose( kFq."d21", kFq."alpha22" ) - PreCompose( kFq."alpha21", kFq."d31" ),
  PreCompose( kFq."d30", kFq."alpha31" ) - PreCompose( kFq."alpha30", kFq."d40" ),
  PreCompose( kFq."d31", kFq."alpha32" ) - PreCompose( kFq."alpha31", kFq."d41" ),
  PreCompose( kFq."d10", kFq."h11" ) - PreCompose( kFq."alpha10", kFq."alpha20" ),
  PreCompose( kFq."d20", kFq."h21" ) - PreCompose( kFq."alpha20", kFq."alpha30" ),
  PreCompose( kFq."h12", kFq."d31" ) - PreCompose( kFq."alpha12", kFq."alpha22" ),
  PreCompose( kFq."h22", kFq."d41" ) - PreCompose( kFq."alpha22", kFq."alpha32" ),
  PreCompose( kFq."d11", kFq."h12" ) + PreCompose( kFq."h11", kFq."d30" )
    - PreCompose( kFq."alpha11", kFq."alpha21" ),
  PreCompose( kFq."d21", kFq."h22" ) + PreCompose( kFq."h21", kFq."d40" )
    - PreCompose( kFq."alpha21", kFq."alpha31" ),
  PreCompose( kFq."d11", kFq."t12" ) + PreCompose( kFq."h11", kFq."alpha30" )
    - PreCompose( kFq."alpha11", kFq."h21" ),
  PreCompose( kFq."alpha12", kFq."h22" ) + PreCompose( kFq."t12", kFq."d40" )
    - PreCompose( kFq."h12", kFq."alpha31" ) ];


julia> Aq =  kFq / ρ;
GAP: Algebroid generated by the right quiver q(T10,T11,T12,T20,T21,T22,T30,T31,T32,T40,
    T41,T42)[d10:T10->T11,d11:T11->T12,d20:T20->T21,d21:T21->T22,d30:T30->T31,d31:T31->
    T32,d40:T40->T41,d41:T41->T42,alpha10:T10->T20,alpha11:T11->T21,alpha12:T12->T22,
    alpha20:T20->T30,alpha21:T21->T31,alpha22:T22->T32,alpha30:T30->T40,alpha31:T31->
    T41,alpha32:T32->T42,h11:T11->T30,h12:T12->T31,h21:T21->T40,h22:T22->T41,t12:T12->
    T40]

julia> SetUnderlyingNameForCapCategory( Aq,
  g"Algebroid defined by the right quiver q subject to 18 relations" )

julia> Aq
GAP: Algebroid defined by the right quiver q subject to 18 relations

julia> Aq_add = AdditiveClosure( Aq )
GAP: Additive closure( Algebroid defined by the right quiver q subject to 18 relations
    )

julia> KAq_add = HomotopyCategoryByCochains( Aq_add )
GAP: Homotopy category( Additive closure( Algebroid defined by the right quiver q
    subject to 18 relations ) )

julia> InfoOfInstalledOperationsOfCategory( KAq_add )
61 primitive operations were used to derive 152 operations for this category which
    * IsEquippedWithHomomorphismStructure
    * IsLinearCategoryOverCommutativeRing
    * IsAdditiveCategory
    * IsTriangulatedCategory

julia> CKAq_add = CochainComplexCategory( KAq_add )
```

```
GAP: Cochain complexes( Homotopy category( Additive closure( Algebroid defined by the
    right quiver q subject to 18 relations ) ) )
```

Now we can construct the objects T_1, T_2, T_3 and T_4 in $\mathcal{K}^b\left(\mathbf{A}_q^{\oplus}\right)$:

```
julia> T1 = HomotopyCategoryObject(
                KAq_add, [ Aq."d10" / Aq_add, Aq."d11" / Aq_add ], 0 );

julia> T2 = HomotopyCategoryObject(
                KAq_add, [ Aq."d20" / Aq_add, Aq."d21" / Aq_add ], 0 );

julia> T3 = HomotopyCategoryObject(
                KAq_add, [ Aq."d30" / Aq_add, Aq."d31" / Aq_add ], 0 );

julia> T4 = HomotopyCategoryObject(
                KAq_add, [ Aq."d40" / Aq_add, Aq."d41" / Aq_add ], 0 )
GAP: <An object in Homotopy category( Additive closure( Algebroid defined by the right
    quiver q subject to 18 relations ) ) with active lower bound 0 and active upper
    bound 2>
```

Similarly, we can construct the morphisms $\alpha_1 : T_1 \to T_2$, $\alpha_2 : T_2 \to T_3$ and $\alpha_3 : T_3 \to T_4$:

```
julia> α1 = HomotopyCategoryMorphism(
                T1,
                [ Aq."alpha10" / Aq_add,
                  Aq."alpha11" / Aq_add,
                  Aq."alpha12" / Aq_add ],
                0,
                T2 )

julia> α2 = HomotopyCategoryMorphism(
                T2,
                [ Aq."alpha20" / Aq_add,
                  Aq."alpha21" / Aq_add,
                  Aq."alpha22" / Aq_add ],
                0,
                T3 )

julia> α2 = HomotopyCategoryMorphism(
                T3,
                [ Aq."alpha30" / Aq_add,
                  Aq."alpha31" / Aq_add,
                  Aq."alpha32" / Aq_add ],
                0,
                T4 )
GAP: <A morphism in Homotopy category( Additive closure( Algebroid defined by the right
     quiver q subject to 18 relations ) ) with active lower bound 0 and active upper
    bound 2>
```

```
julia> T = CochainComplex( [ α₁, α₂, α₃ ], 1 )
GAP: <An object in Cochain complexes( Homotopy category( Additive closure( Algebroid
    defined by the right quiver q subject to 18 relations ) ) ) with active lower bound
     1 and active upper bound 4>

julia> IsEqualForMorphisms( DifferentialAt( T, 1 ), α₁ )
true
```

The object T is bounded above by 4. Hence, in any extension P_T of T to a standard Postnikov system, we should have $C_T^4 = T^4$ and $\mu_T^3 : T_3 \to C_T^4$ is equal to ∂_T^3. In other words the truncation $P_T^{\leq 4} = T$.

```
julia> P4 = PostnikovSystemAt( T, 4 )
GAP: <An object in Cochain complexes( Homotopy category( Additive closure( Algebroid
    defined by the right quiver q subject to 18 relations ) ) ) with active lower bound
     1 and active upper bound 4>

julia> IsEqualForObjects( P4, T )
true
```

The next iteration computes $C_T^3 = \text{Cocone}^{st}(\mu_T^3)$ and $\mu_T^2 : T_2 \to C_T^3$. We get the truncation

$$P_T^{\leq 3} := \quad 0 \to T_1 \xrightarrow{\alpha_1} T_2 \xrightarrow{\mu_T^2} C_T^3 \to 0$$

```
julia> P3 = PostnikovSystemAt( T, 3 )
GAP: <An object in Cochain complexes( Homotopy category( Additive closure( Algebroid
    defined by the right quiver q subject to 18 relations ) ) ) with active lower bound
     1 and active upper bound 3>

julia> C3 = ObjectAt( P3, 3 )
GAP: <An object in Homotopy category( Additive closure( Algebroid defined by the right
    quiver q subject to 18 relations ) ) with active lower bound 0 and active upper
    bound 3>

julia> μ2 = DifferentialAt( P3, 2 );

julia> Show( μ2 )
```

$$
\begin{array}{ccc}
0 & \xrightarrow{\quad 0 \quad} & T_4^2 \\
\uparrow & & \uparrow \\
() & & \begin{pmatrix} -\alpha_3^2 \\ -\partial_4^1 \end{pmatrix} \\
|_2 & & |_2 \\
T_2^2 & \xrightarrow{\quad (\alpha_2^2 \quad -h_2^2) \quad} & T_3^2 \oplus T_4^1 \\
\uparrow & & \uparrow \\
(\partial_2^1) & & \begin{pmatrix} \partial_3^1 & -\alpha_3^1 \\ 0 & -\partial_4^0 \end{pmatrix} \\
|_1 & & |_1 \\
T_2^1 & \xrightarrow{\quad (\alpha_2^1 \quad -h_2^1) \quad} & T_3^1 \oplus T_4^0 \\
\uparrow & & \uparrow \\
(\partial_2^0) & & (\partial_3^0 \quad -\alpha_3^0) \\
|_0 & & |_0 \\
T_2^0 & \xrightarrow{\quad (\alpha_2^0) \quad} & T_3^0
\end{array}
$$

The next iteration computes $C_T^2 = \mathrm{Cocone}^{st}(\mu_T^2)$ and $\mu_T^1 : T_1 \to C_T^2$. We get the truncation:

$$P_T^{\leq 2} := \quad 0 \to T_1 \xrightarrow{\mu_T^1} C_T^2 \to 0$$

```
julia> P2 = PostnikovSystemAt( T, 2 )
GAP: <An object in Cochain complexes( Homotopy category( Additive closure( Algebroid
    defined by the right quiver q subject to 18 relations ) ) ) with active lower bound
     1 and active upper bound 2>

julia> C2 = ObjectAt( P2, 2 )
GAP: <An object in Homotopy category( Additive closure( Algebroid defined by the right
    quiver q subject to 18 relations ) ) with active lower bound 0 and active upper
    bound 4>

julia> μ1 = DifferentialAt( P2, 1 );

julia> Show( μ1 )
```

$$
\begin{array}{ccccc}
0 & - & 0 & \rightarrow & T_4^2 \\
\uparrow & & & & \uparrow \\
0 & & & & \begin{pmatrix} \alpha_3^2 \\ \partial_4^1 \end{pmatrix} \\
\vert_3 & & & & \vert_3 \\
0 & - & 0 & \rightarrow & T_3^2 \oplus T_4^1 \\
\uparrow & & & & \uparrow \\
() & & & & \begin{pmatrix} -\alpha_2^2 & h_2^2 \\ -\partial_3^1 & \alpha_3^1 \\ 0 & \partial_4^0 \end{pmatrix} \\
\vert_2 & & & & \vert_2 \\
T_1^2 & - & \begin{pmatrix} \alpha_1^2 & -h_1^2 & t_1^2 \end{pmatrix} & \rightarrow & T_2^2 \oplus T_3^1 \oplus T_4^0 \\
\uparrow & & & & \uparrow \\
\begin{pmatrix} \partial_1^1 \end{pmatrix} & & & & \begin{pmatrix} \partial_2^1 & -\alpha_2^1 & h_2^1 \\ 0 & -\partial_3^0 & \alpha_3^0 \end{pmatrix} \\
\vert_1 & & & & \vert_1 \\
T_1^1 & - & \begin{pmatrix} \alpha_1^1 & -h_1^1 \end{pmatrix} & \rightarrow & T_2^1 \oplus T_3^0 \\
\uparrow & & & & \uparrow \\
\begin{pmatrix} \partial_1^0 \end{pmatrix} & & & & \begin{pmatrix} \partial_2^0 & -\alpha_2^0 \end{pmatrix} \\
\vert_0 & & & & \vert_0 \\
T_1^0 & - & \begin{pmatrix} \alpha_1^0 \end{pmatrix} & \rightarrow & T_2^0
\end{array}
$$

The next iteration computes $C_T^1 = \mathrm{Cocone}^{st}(\mu_T^1)$ and $\mu_T^0 : 0 \to C_T^1$ and the algorithm returns the truncation

$$P_T^{\leq 1} := \quad 0 \to C_T^1 \to 0$$

and terminates.

```
julia> P1 = PostnikovSystemAt( T, 1 )
GAP: <An object in Cochain complexes( Homotopy category( Additive closure( Algebroid
    defined by the right quiver q subject to 18 relations ) ) ) with active lower bound
     1 and active upper bound 1>

julia> C1 = ObjectAt( P1, 1 )
```

$$
\begin{array}{c}
T_4^2 \\
\uparrow \\
\begin{pmatrix} -\alpha_3^2 \\ -\partial_4^1 \end{pmatrix} \\
\vert_4 \\
T_3^2 \oplus T_4^1 \\
\uparrow \\
\begin{pmatrix} \alpha_2^2 & -h_2^2 \\ \partial_3^1 & -\alpha_3^1 \\ 0 & -\partial_4^0 \end{pmatrix} \\
\vert_3 \\
T_2^2 \oplus T_3^1 \oplus T_4^0 \\
\uparrow \\
\begin{pmatrix} -\alpha_1^2 & h_1^2 & -t_1^2 \\ -\partial_2^1 & \alpha_2^1 & -h_2^1 \\ 0 & \partial_3^0 & -\alpha_3^0 \end{pmatrix} \\
\vert_2 \\
T_1^2 \oplus T_2^1 \oplus T_3^0 \\
\uparrow \\
\begin{pmatrix} \partial_1^1 & -\alpha_1^1 & h_1^1 \\ 0 & -\partial_2^0 & \alpha_2^0 \end{pmatrix} \\
\vert_1 \\
T_1^1 \oplus T_2^0 \\
\uparrow \\
\begin{pmatrix} \partial_1^0 & -\alpha_1^0 \end{pmatrix} \\
\vert_0 \\
T_1^0
\end{array}
$$

The convolution of T is given by $\Sigma^{-1}(C_T^1)$:

```
julia> conv_T = Shift( C1, -1 )
GAP: <An object in Homotopy category( Additive closure( Algebroid defined by the right
    quiver q subject to 18 relations ) ) with active lower bound 1 and active upper
    bound 6>

julia> Show( conv_T )
```

$$
\begin{array}{c}
T_4^2 \\
\uparrow \\
\begin{pmatrix} \alpha_3^2 \\ \partial_4^1 \end{pmatrix} \\
|_5 \\
T_3^2 \oplus T_4^1 \\
\uparrow \\
\begin{pmatrix} -\alpha_2^2 & h_2^2 \\ -\partial_3^1 & \alpha_3^1 \\ 0 & \partial_4^0 \end{pmatrix} \\
|_4 \\
T_2^2 \oplus T_3^1 \oplus T_4^0 \\
\uparrow \\
\begin{pmatrix} \alpha_1^2 & -h_1^2 & t_1^2 \\ \partial_2^1 & -\alpha_2^1 & h_2^1 \\ 0 & -\partial_3^0 & \alpha_3^0 \end{pmatrix} \\
|_3 \\
T_1^2 \oplus T_2^1 \oplus T_3^0 \\
\uparrow \\
\begin{pmatrix} -\partial_1^1 & \alpha_1^1 & -h_1^1 \\ 0 & \partial_2^0 & -\alpha_2^0 \end{pmatrix} \\
|_2 \\
T_1^1 \oplus T_2^0 \\
\uparrow \\
\begin{pmatrix} -\partial_1^0 & \alpha_1^0 \end{pmatrix} \\
|_1 \\
T_1^0
\end{array}
$$

```
julia> IsEqualForObjects( conv_T, Convolution( T ) )
true
```

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