Multi-Parton Contributions to $\bar{\mathrm{B}} \to \mathrm{X_s}\gamma$ at Next-To-Leading Order in QCD

Dissertation

zur Erlangung des akademischen Grades
eines Doktors der Naturwissenschaften

vorgelegt von
M.Sc. Lars-Thorben Moos

eingereicht bei der Naturwissenschaftlich-Technischen Fakultät
der Universität Siegen
Siegen 2024

Betreuer und erster Gutachter:

Prof. Dr. Tobias Huber
Universität Siegen

Zweiter Gutachter:

Prof. Dr. Guido Bell
Universität Siegen

Tag der mündlichen Prüfung:
25.10.2024

Abstract

The calculation of the branching ratio for the inclusive decay $\bar{B} \to X_s\gamma$ has been an active field of research for multiple decades, yielding results that work very well as a standard candle of the Standard Model of Particle Physics (SM). The large interest in this observable has already led to an almost complete expression for the next-to-leading order (NLO) part and a large number of next-to-next-to-leading order (NNLO) contributions to the theoretical prediction. With results from colliders becoming ever more precise, the need for higher precision of theoretical predictions arises.

In this work, we calculate the remaining pieces for the branching ratio of the four-body decay of a b quark into an s quark, a photon γ and two additional quarks $q\bar{q}$ at NLO in the strong coupling α_s. The calculation of this one-loop process $b \to s\gamma q\bar{q}$, which includes a virtual gluon, has to be supplemented by $b \to s\gamma q\bar{q}g$, since the loop calculation results in infrared divergences that have to be cancelled by the real-emission counterparts.

One focus of this thesis is the computation of the occurring four- and five-body phase space integrals and the calculational techniques that are crucial in obtaining them. These include the latest iterations of integration-by-parts (IBP) methods that we used to obtain our sets of master integrals and a description of the differential equations we used to solve them. We further lay out the different methods we used to determine the necessary boundary conditions for these, tailored to the different challenges we encountered.

Furthermore we describe our methods of renormalization and regularization. We are using dimensional regularization throughout this work and, in this framework, treating the final state quarks massless leads to residual collinear divergences. For these remaining divergent expressions, we employ splitting-function regularization to switch the regularization scheme from dimensional regularization to logarithms of the quark masses. We also give an overview over the ongoing effort in the calculation of the last missing piece for the future completion of the perturbative contributions at $\mathcal{O}(\alpha_s)$ which is tied to the next-to-leading order splitting function.

Using these techniques, we calculate the analytic results for our correction to the decay width of $\bar{B} \to X_s\gamma$ as one of the missing pieces for the NLO branching fraction. The expressions are made publicly available online in `Mathematica` format for further use. Finally, we discuss future steps for obtaining a numerical result.

Zusammenfassung

Das Verzweigungsverhältnis des inklusiven Zerfalls $\bar{B} \to X_s\gamma$ ist seit mehreren Jahrzehnten ein sehr aktives Feld der Teilchenphysik und funktioniert durch hohe theoretische und experimentelle Genauigkeit als eine Standardkerze für das Standardmodell der Teilchenphysik (SM). Durch das hohe Interesse an einer genauen theoretischen Vorhersage liegen heutzutage eine fast vollständige Berechnung auf nächst-zu-führender Ordnung und eine hohe Anzahl an Beiträgen auf nächst-zu-nächst-zu-führender Ordnung vor. Die stetig steigende Präzision von Messungen an Beschleunigern ruft auch nach einer steigenden Präzision der theoretischen Vorhersagen.

In der vorliegenden Arbeit berechnen wir die verbleibenden Beiträge von Vier-Körper-Zerfällen eines b-Quark in ein s-Quark, ein Photon γ und zwei zusätzliche Quarks $q\bar{q}$ zum Verzweigungsverhältnis in nächst-zu-führender Ordnung in der starken Kopplung α_s. Dieser Ein-Schleifen-Prozess $b \to s\gamma q\bar{q}$, der ein virtuelles Gluon g beinhaltet, muss durch den entsprechenden reellen Abstrahlungsprozess $b \to s\gamma q\bar{q}g$ ergänzt werden, um durch das Gluon hervorgerufenene Infrarot-Divergenzen zu eliminieren.

Ein Fokus dieser Arbeit liegt auf der Berechnung der auftretenden Vier- und Fünf-Teilchen Phasenraum-Integrale und den dafür benötigten Rechentechniken. Diese beinhalten aktuelle Methoden der Reduktion durch partielle Integration (IBP) mit denen wir jeweils unsere Basis an Masterintegralen identifiziert haben. Zudem beschreiben wir die Lösung dieser Masterintegrale mithilfe von Differentialgleichungen und die verschiedenen Arten, auf die wir die hierfür benötigten Randbedingungen bestimmt haben.

Weiterhin legen wir unsere Methodik für Renormierung und Regularisierung dar. Da wir die Quarks im Endzustand als masselos behandeln, führt unsere Nutzung von dimensionaler Regularisierung zu verbleibenden kollinearen Divergenzen. Für diese verbleibenden, divergenten Ausdrücke wechseln wir das Regularisierungsschema mittels Splitting-Funktionen und tauschen die verbleibenden Pole gegen einen natürlicheren Regulator, d.h. Logarithmen der Quark-Massen, ein. Dabei geben wir

einen Überblick über die momentan noch laufende Berechnung des letzten Teils der Splitting-Funktion auf nächst-zu-führender Ordnung, der für eine Vervollständigung der perturbativen Beiträge auf $\mathcal{O}(\alpha_s)$ in Zukunft benötigt wird.
Unter Nutzung dieser Techniken geben wir hier analytische Ergebnisse für unseren Beitrag zur Zerfallsbreite von $\bar{B} \to X_s\gamma$ als einen der letzten fehlenden Teile für einen vollständiges perturbatives Resultat für das NLO Verzweigungsverhältnis. Unsere finalen Ausdrücke werden online im `Mathematica`-Format öffentlich zugänglich gemacht. Als letztes diskutieren wir noch die nächsten Schritte, die nötig sind um eine numerische Vorhersage über das Verzweigungsverhältnis machen zu können.

Contents

Chapter 1

Introduction

1.1 General Motivation

The Standard Model of Particle Physics (SM) is one of the most successful theories in science. Since its formulation in the 1970s, it has been thoroughly tested and provides a very precise description of many phenomena in particle physics [1–3]. One of the most prominent among these is the anomalous magnetic moment of the electron, where theory and experiment agree up to 13 significant digits. Another great success of the theory is the proposition of the Higgs particle, which was finally found in 2012 at the Large Hadron Collider (LHC) [4, 5]. This was the last missing piece of the model, making it self-consistent.
Despite the success of the model, there are quite a few fields where it is not able to explain the full picture, as we will see in Sect. 2.1. This calls for very precise theoretical predictions complemented by the measuring of their experimental counterparts. Doing this makes it possible to see where exactly the results deviate, letting us introduce new models or extensions of the Standard Model to explain our observations.

1.2 Goal of this work

The inclusive radiative decay $\bar{B} \to X_s\gamma$ of the B-meson is one of the most precise tests of the Standard Model (SM) in the quark flavor sector. The dominant partonic

process $b \to s\gamma$ is a flavor changing neutral current (FCNC), which is forbidden at tree-level in the Standard Model. Because it is loop-induced, it is a rare process and very sensitive to new physics that modifies the particles running in the loop.
As this process has a very distinct signature, the value for the branching fraction has been measured very precisely up to now. The current value of the branching ratio of $\bar{B} \to X_s\gamma$ (with $E_\gamma > 1.6$ GeV) is measured with a precision of about 5% [6–9] and reads:

$$\mathcal{B}_{s\gamma}^{exp} = (3.32 \pm 0.15) \cdot 10^{-4} \,. \tag{1.1}$$

With uncertainties on the experimental side that are this small, the result needs to be supplemented accordingly by a theoretical value that is determined with comparable precision. The work on the theoretical prediction for this process has been carried out for the last twenty-five years [10–38]. This includes corrections up to next-to-next-to-leading order (NNLO) in the strong coupling α_s and the current value is at [11]:

$$\mathcal{B}_{s\gamma}^{SM} = (3.40 \pm 0.17) \cdot 10^{-4} \,, \tag{1.2}$$

which is in very good agreement with the experimental one.
With the upcoming run of Belle II [39] and the combination with data from the other B-factories [40], the uncertainties on the experimental side will decrease further, calling for even more precise predictions of the theoretical value.

1.3 Structure of this Thesis

In this work, we will focus on the remaining pieces that formally complete the next-to-leading order (NLO). First, the basic frameworks are discussed, giving an overview of the theoretical background in chapter 2 and of the computational methods in chapter 3. After these foundational chapters, the main parts of the calculation are discussed in chapters 4 and 5, which describe the four- and five-body calculation, respectively. After this, we discuss the renormalization of our results in chapter 6 and the regularization of remaining collinear expressions in chapter 7. Finally, the results are given in chapter 8.

Chapter 2

Theoretical Background

The goal of particle physics is to understand the nature of elementary particles and their interactions with each other. One of the most prominent and successful methods to probe those interactions is the analysis of collisions at particle colliders. In these experiments, a beam of accelerated particles (typically electrons, positrons or (anti-)protons) is either directed into an opposing beam or a fixed target. Detectors around the point of collision measure quantities of the particles emerging from the collision, such as energy distribution and charge. With those quantities determined, participating particles can be identified and scattering events reconstructed. Comparing the experimental outcomes with theoretical predictions allows for a check of the validity of the theoretical models and of the assumptions made about the interactions.
A prominent class of these observables are the decay widths for specific processes, which can then be related to the relative amount that the individual processes occur, resulting in branching fractions. The calculation of contributions to the theoretical prediction for such a decay width will be the main topic of this thesis.
After giving an introduction to the theoretical framework in the next sections, the exact quantity we are calculating will be discussed in Sect. 2.3.

	1st gen.	2nd gen.	3rd gen.
up-type quarks	u (2.2 MeV)	c (1.3 GeV)	t (173.1 GeV)
down-type quarks	d (4.7 MeV)	s (93 MeV)	b (4.18 GeV)
leptons	e (511 keV)	μ (106 MeV)	τ (1.78 GeV)
	ν_e ($<$ 1 eV)	ν_μ ($<$ 0.17 MeV)	ν_τ ($<$ 18.2 MeV)

Table 2.1: All fermions of the Standard Model and their respective masses. The numbers above are meant to illustrate the relative size of the values. The determination of the quark masses are scheme-dependent, which has to be taken into account when comparing results. For more information on how the above results were gathered, we refer to Ref. [41].

force	weak	electromagnetic	strong
gauge boson (Mass)	$W^\pm$ (80.39 GeV) Z^0 (91.19 GeV)	γ (0 GeV)	g (0 GeV)

Table 2.2: Gauge bosons of the Standard Model with their masses. For the summary of the latest measurements, we refer to Ref. [41].

2.1 Particles of the Standard Model

In the following, we want to give a brief overview of the Standard Model of Particle Physics with an emphasis on the parts that are relevant in this work. An in-depth description of the SM can be found in standard literature, e.g. Refs. [42, 43].

All matter that we encounter in everyday life is build from fermions, i.e. elementary particles with a half-integer spin. Protons and Neutrons, which form the nuclei of atoms, are formed by **quarks**, more precisely **up quarks** (or u quarks for short, with a charge of $2/3\,e$) and **down quarks** (or d quarks for short, with a charge of $-1/3\,e$). Around the nuclei we find **electrons** e with a charge of $-e$. The last type of elementary particles that we find are the very light **electron-neutrinos** ν_e, which have a charge of 0.

For all these particle types, we find two additional copies that only differ by mass. We call these different copies generations and the different types are often referred to as **flavor**. A collection of these can be found in Table 2.1. Note that the definitions of particle masses differ across the different types of fermions. For the quarks, one needs to define a mass scheme, as a single quark does not occur in nature outside of a bound state. More on these schemes and the current values can be found in Ref. [41].

For neutrinos, this task is even more difficult. As they only interact weakly, measuring their interactions needs very precise experimental setups. As of now, only upper limits for the respective masses of the generations can be determined [41].

All of the above mentioned particles have a corresponding anti-particle, that differs from its counterpart by having the opposing charge and is denoted with a bar over their symbol, e.g. the anti-up quark $\bar{u}$ with a charge of $-2/3\,e$.

One further important aspect is the mathematical property of the **handedness** of the particles [42]. A particle can be left- or right-handed, an exception for this are neutrinos, which only observed as left-handed in nature.

The particles we described interact via the four forces: gravity, electromagnetism, weak force and strong force. The latter three are combined in the **Standard Model**, in which the interactions are described by the exchange of spin-1 particles called **gauge bosons**. The gauge bosons and their masses can be found in Table 2.2. The strong interaction, which is, for example, responsible for holding together the nuclei of atoms, is mediated by the massless **gluons**. Electromagnetism, the force that is probably best know in everyday life, is mediated by the massless **photons**. The last of the three is the weak force. The most prominent observation of this force can be made when studying the β-decay of neutrons. It is mediated by the charged $\mathbf{W^{\pm}}$ and the neutral $\mathbf{Z^0}$ bosons. One special notion for the weak force is that only left-handed particles are partaking in the interactions mediated by the $W^{\pm}$ bosons. The last missing particle to complete the Standard Model is the **Higgs boson** with a mass of 124.97 GeV [4, 5]. Through the Higgs mechanism, we are able to ascribe masses to all particles in a gauge invariant way, especially giving the very large masses to the $W^{\pm}$ and Z^0 bosons. In this mechanism, the Higgs field gains a non-zero vacuum expectation value, breaking the original electroweak symmetry of the Standard Model. Spontaneously breaking this symmetry leaves us with two separate forces, electromagnetism and the weak force, and their respective exchange particles that we described above.

2.1.1 Indicators for New Physics

Despite the achievements of the SM, we know that the model does not describe the full picture. It works very well in many aspects, but there are phenomena that it is not able to describe sufficiently. In the following, I want to give a list of the most prominent fields that yield the potential for new physics:

- **Dark Matter:**
  Astrophysical experiments have established that there is a vast discrepancy between the amount of matter that we can see and the amount of matter that interacts gravitationally. 'Seeing', in this context, means an observation via strong, weak or electromagnetic interaction with other particles. To account for this observational gap, dark matter was introduced, but so far no definitive discovery has been made. This non-observation could, for example, stem from a very weak coupling between dark matter and SM particles or a new force. The search for dark matter is a very active field, experimentally as well as theoretically, and there are a lot of interesting approaches to shed light on the observations. An extensive collection of these can be found in Ref. [44] or chapter 27 of Ref. [41].

- **Neutrino Masses:**
  In the Standard Model, neutrinos do not have any mass. One reason for this is the non-observation of right-handed neutrinos, which would be needed for a Dirac mass term analogous to that of the other leptons.
  This stands in direct contrast to observations from neutrino oscillation experiments. In these, one compares the expected rate of a certain type of neutrino (electron, muon or tau) from e.g. the sun, a reactor or a particle accelerator to the one that is actually measured. For all these sources, the theoretically expected rate is understood very well. Experimental data shows that the relative content of the three generations fluctuates with distance from the source. This implies that neutrinos can change flavor and it can be shown that the fluctuation rate is tied to the mass differences between the generations. For a mass difference to exist, at least one of the three flavors needs to be massive.

There have been a number of ideas to implement massive neutrinos consistently within the Standard Model, e.g. Majorana neutrinos, although none of them can give a satisfatory explanation as of now. The Particle Data Group provides a comprehensive write-up, which can be found in chapter 14 of Ref. [45].

- **CP Asymmetry:**
  In principle, from the beginning of the universe, there should be the same amount of matter and anti-matter. This is vastly inconsistent with the observation that our universe today consists of matter. What this means is that there has to be a mechanism that creates a difference in the way that matter and anti-matter interact. The Standard Model technically does allow for such a mechanism, i.e. an asymmetry between particles and their corresponding anti-particles, which are related by a flip of their charge and parity, or CP transformation for short. Unfortunately, the amount of CP asymmetry that is measured is too low to account for the matter-antimatter-asymmetry that is observed in nature [41, 46].

- **Theory of all Four Forces:**
  The Standard Model incorporates a description of three of the four fundamental forces of nature (strong, weak and electromagnetic force). The fourth one, gravity, is described in General Relativity (GR). Since the Einstein equations that govern the behaviour of gravitational fields are non-linear, a quantization of gravitation similar to the other forces has not been achieved until now. The wish for unification of all forces is more of an aesthetical one, as there is no fundamental reason that all interactions have to be described by a single theory. Nevertheless, the success of unifying the first three sparked efforts to find a complete theory of all four forces, which would then (probably) use the Standard Model as a building block.

- **Flavor Anomalies:**
  In the last ten years, experiments have seen deviations from theory predictions in the flavor sector of the SM. Notable observables here are for example $R_{D^{(*)}}$, which compares the branching ratios of semi-leptonic B-meson decays to a D-meson that only differ by lepton flavor in the final state, or P_5', which describes

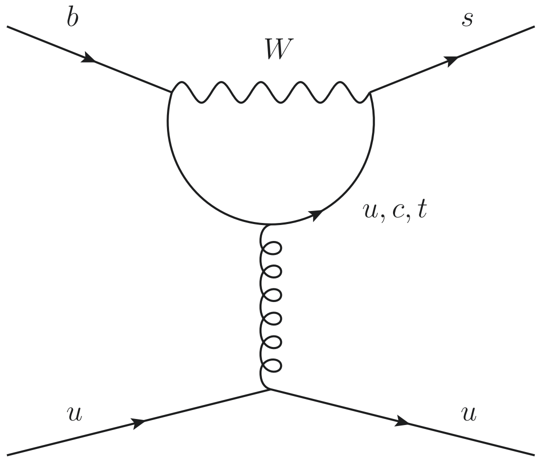


Figure 2.1: Sample diagram for the decay of a b quark in the Standard Model.

an angular distribution of the $B \to K^{*0}\mu^+\mu^-$ decay. With the newest set of measurements, though, we see that the tension in, for example, the observable $R_{K^{(*)}}$ vanished [47], whereas, for $R_{D^{(*)}}$, it became smaller but is still deviating about $3\,\sigma$ from theory [48]. P_5' is also still in tension with the Standard Model prediction [49,50]. Future analysis and experimental results are eagerly awaited to see if the remaining tensions will prevail or newly measured observables such as $B^+ \to K^+\nu\bar{\nu}$ [51] will show interesting deviations.

We see that there is still a number of things we do not completely understand about nature. To test the validity of existing theories and the plausibility of newly proposed ones, we need very precise observables. In the next section, we want to introduce such a class of observables, called inclusive rare B-decays.

2.2 Inclusive Rare B-Decays

Quarks are never found as free particles in nature. The most common **bound states** we encounter are either called **baryons** when three quarks or three anti-quarks are involved or **mesons**, which are bound states of a quark and an anti-quark.
With the whole zoo of different quarks, there is a plethora of combinations, making

it important to define a naming scheme.

Meson states are ordered by their heaviest consituent: If this is an s quark they are labelled kaons, if it is a c quark they are called D-mesons and if a b quark is involved, we call them B-mesons. Note that for historical reasons, the B-meson includes a $\bar{b}$ quark and vice versa.

As an additional tag, they have their combined charge as a superscript and the other involved quark as a subscript. If the meson is comprised of a b and an $\bar{s}$, we call it $\bar{B}_s^0$, for example. Examining the available literature, it becomes clear that this is not always carried out thoroughly. In the following, when we talk about the $\bar{B}_s^0$, we will suppress the charge superscript, just calling it $\bar{B}_s$.

With the emergence of B-factories (i.e. accelerator experiments operating at the production threshold of the Υ(4S), which is an excited state that decays mostly into two B-mesons), such as BaBar [40], Belle [40] and Belle II [39], and the growing precision of data, quark flavor physics is growing to be one of the most promising fields for the detection of new physics. The term flavor physics denotes that we are concerned with energy scales that are low enough to resolve the differences between the light quarks, i.e. all five active flavors.

Being sensitive to these differences in the interactions makes it possible to get precise determinations of the elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix [52], which governs the strength of the mixing between the generations. Furthermore, analysis of the flavor sector can yield complementary results for the CP violation in the Standard Model [53].

One of the most suitable subgroups to study flavor physics are the B-meson systems. Here, we can calculate predictions for our observables in perturbation theory, because the couplings (i.e. the strong coupling α_s and the electromagnetic coupling α_e) between the particles are much smaller than unity. This lets us calculate the result order-by-order in these couplings and gives a good handle on the estimation of missing higher orders.

In general, we also have to take into account so-called non-perturbative effects, which originate from the b-quark not being free but part of a meson. These effects are described, for example, by the modelling of shape functions, and for $\bar{B} \to X_s\gamma$ they have been estimated in Refs. [32,54,55]. What we can see in these works is that the large mass of the b quark makes it less sensitive to these corrections than e.g.

D-mesons or kaons (as these corrections are suppressed by the mass of the heavy quark), leading to a much cleaner theoretical description. For the latter, lighter mesons, one has to consider the wide field of bound states in their vicinity, which makes it much more complicated to get precise results.

B-mesons decay to a wide array of different final states. Of particular interest are the so-called rare decays. These are mediated by flavor changing neutral currents (FCNC), which (in the Standard Model) are forbidden at tree-level[1]. This means that the leading order contributions are already loop-suppressed $\big($being proportional to the coupling strength $\alpha_{s/e}/(4\pi) \ll 1\big)$, making the decay rate comparatively small and leading to the decays being rare. An example of such a loop-induced decay can be seen in Fig. 2.1.
This suppression by a loop-factor is not the only aspect that makes the decays rare. When looking at the mediator particles in the Standard model, we see that the weak force is the only one with gauge bosons that have non-vanishing masses, which are, additionally, very large compared to fermion masses (one exception being the top quark). As we will see in Sect. 2.4, each exchange of a gauge boson comes with a factor $(p^2 - M^2)^{-1}$, where p is the momentum and M is the mass of the mediator particle. At low momentum exchange, this is purely dominated by the large mass, which suppresses the size of the resulting quantity.
This means that at low energies, the other interactions are much more likely to occur, as their mediators are massless and do in turn not suffer from this suppression. For the process $\bar{B} \to X_s\gamma$, two more mechanisms have a relevant impact on the decay rate. The first is the **CKM suppression** that stems from the fact that we describe a process between two different quark generations and the final result will be proportional to off-diagonal elements of the CKM matrix. We will see later, in Eq. (2.15), that the products of CKM prefactors in our effective Lagrangian are all much smaller than unity [41].
The other effect that we have to take into account is the **Glashow-Iliopoulos-Maiani (GIM) mechanism** [56]. Fig. 2.1 shows that we sum over all the up-type

[1] The statement that rare decays are only mediated by FCNC is only true when talking about $1 \to 2$ decays, such as $b \to s\gamma$. As we will see in the next sections, multi-parton decays, such as the process $b \to s\gamma u\bar{u}$, for example, get contributions from a direct exchange of a W-boson at tree level (but are heavily CKM suppressed). These tree-level contributions were calculated in Ref. [10].

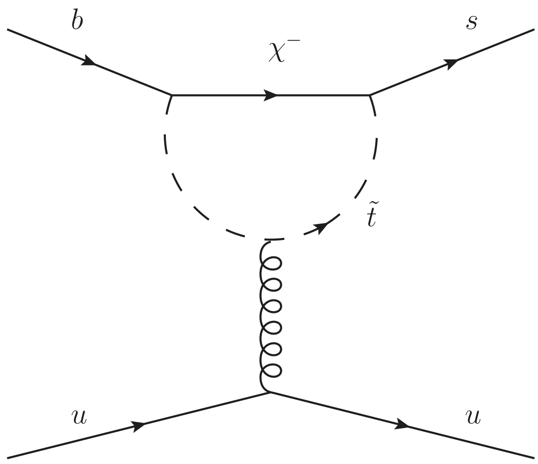


Figure 2.2: Sample diagram for an additional contribution to the process $b \to s\gamma$ in an extension of the Standard Model (here the Minimal Supersymmetric Standard Model [57]).

flavors in the loop. The single contributions of flavor i each come with a factor $f(m_i)V_{ib}V_{is}^*$, where the function $f(m_i)$ denotes the loop functions we encounter. If the masses were degenerate with $m = m_u = m_c = m_t$, this would lead to a contribution of

$$f(m)\,(V_{ub}V_{us}^* + V_{cb}V_{cs}^* + V_{tb}V_{ts}^*) = 0, \tag{2.1}$$

which vanishes from unitarity considerations. As the mass difference of the b and t quark is so large, this does not happen for the process at hand. For this reason, we call $\bar{B} \to X_s\gamma$ a **radiative decay** instead of a rare decay, giving us a semantic distinction to other FCNC processes, where the GIM cancellation takes effect.

Studying inclusive decays is driven in a large part by the prospect of very high precision, which makes them very sensitive to new particles beyond the SM. The usual method of indirect detection is to see new particles by the effect of them virtually participating via loop-corrections, an example for this is shown in Fig. 2.2. Since, for the radiative and rare decays, the SM effects are also starting at loop-level, the new contributions are potentially of the same size as the Standard Model ones, making the consistency with existing measurements a very important cross-check for newly proposed models as well as yielding discovery potential.

2.3 Branching Fractions

One of the most important observables for radiative and rare decays is the branching fraction $\mathcal{B}(X \to Y)$. This quantity denotes the probability of the initial state X decaying into the final state Y.

In this work, the main goal is a more precise theoretical prediction of the value for $\mathcal{B}(\bar{B} \to X_s\gamma)_{E_\gamma > 1.6\ \mathrm{GeV}}$, which describes the fraction with which a B meson decays into a final state with a total strangeness of -1[2] that contains a photon with an energy larger than 1.6 GeV. As a formula, we can write it as:

$$\mathcal{B}(\bar{B} \to X_s\gamma)_{E_\gamma > E_0} = \frac{\Gamma(\bar{B} \to X_s\gamma)_{E_\gamma > E_0}}{\Gamma_{\mathrm{total}}(\bar{B})}, \tag{2.2}$$

i.e. the inclusive partonic decay rate of the process normalized to the total decay rate.

The value that is chosen for the cut E_0 in Eq. (2.2) is a consequence of both theoretical and experimental constraints. For the theoretical prediction on the one hand, the constraints stem from the uncertainties on the shape function of the $\bar{B}$-meson, which describes the non-perturbative processes inside the meson that become more important the less inclusive the process gets when raising the cut-off. These shape functions have been studied in Refs. [32, 54, 55] and, considering the current state of the art, their uncertainties are irreducible. It has been found that the optimal value for E_0 for controlling these non-perturbative effects on the theoretical side is around 1.6 GeV, which leads to an uncertainty of around 5%.

For the experimental results, on the other hand, the cut-off has to be chosen as high as possible, as the detection of the photon becomes more robust, the higher the energy cut-off is set. Furthermore, as we exclude charm final states by definition, the background signals of intermediate decays of ψ and ψ' mesons have to be subtracted (and thus modelled and understood) correctly. Their relative impact becomes smaller for higher values of the cut-off. For an E_0 of 1.8 GeV, this effect can be as large as 5%, while it becomes negligible at 2.1 GeV [58].

These constraints are opposing each other diametrically and so for current predictions, the method described in Ref. [58] is used. They propose a lowering of the cut

[2] s and $\bar{s}$ quarks are assigned a strangeness of -1 and $+1$, respectively.

on the experimental side as much as possible below 2.1 GeV and using extrapolation to the theoretical prediction at 1.6 GeV. The extrapolation that was put forward in Ref. [58] has a precision of 0.3% up to 2.0 GeV and breaks down above that value.

To link the experimentally measured quantity $\mathcal{B}(\bar{B} \to X_s\gamma)_{E_\gamma > E_0}$ to the theoretical prediction, we rewrite it as [58]:

$$\mathcal{B}(\bar{B} \to X_s\gamma)_{E_\gamma > E_0} = \mathcal{B}(\bar{B} \to X_c e\bar{\nu}) \left| \frac{V_{ts}^* V_{tb}}{V_{cb}} \right|^2 \frac{6\,\alpha_{em}}{\pi\, C} \left[P(E_0) + N(E_0) \right], \tag{2.3}$$

normalized with the semileptonic phase space factor

$$C = \left| \frac{V_{ub}}{V_{cb}} \right|^2 \frac{\Gamma(\bar{B} \to X_c e\bar{\nu})}{\Gamma(\bar{B} \to X_u e\bar{\nu})}, \tag{2.4}$$

where V_{ij} are the entries of the CKM matrix, which governs the strength of mixing between the quark flavors.
This factor C can be determined from independent experimental channels by fitting the moments of $\bar{B} \to X_c l\bar{\nu}$ spectra. Rewriting it in this way has proven to reduce higher-order effects and scale dependence [58]. Furthermore, relating $\mathcal{B}(\bar{B} \to X_s\gamma)_{E_\gamma > E_0}$ to the semileptonic process $\mathcal{B}(\bar{B} \to X_c e\bar{\nu})$ in Eq. (2.3) gets rid of the strong dependence on the mass of the b-quark to the fifth power.
The two terms $P(E_0)$ and $N(E_0)$ in Eq. (2.3) denote the perturbative and non-perturbative contributions to the branching ratio, respectively. One advantage of inclusive decays is that, with a low enough cut on the photon energy, the size of non-perturbative effects is minimized. For the rate we want to calculate, we can write

$$\Gamma(\bar{B} \to X_s\gamma)_{E_\gamma > E_0} = \Gamma(b \to X_s^{\text{parton}}\gamma)_{E_\gamma > E_0} + \mathcal{O}(1/m_b), \tag{2.5}$$

which indicates that the contributions from non-perturbative effects are suppressed by m_b, i.e. the mass of the b quark. This is one of the reasons that makes this process a standard candle, since this suppression gives a very good handle on the estimation of these corrections. As stated above, for a cut on the photon energy of $E_0 = 1.6$ GeV, the size of the non-perturbative effects is estimated to be around 5% [32].

In this thesis we will be concerned with corrections to the perturbative quantity $P(E_0)$, more precisely the decay width $\Gamma(\bar{B} \to X_s^{\text{partonic}}\gamma)$. These two are related by:

$$\frac{\Gamma(b \to X_s^{\text{parton}}\gamma)_{E_\gamma > E_0}}{|V_{cb}/V_{ub}|^2\, \Gamma(b \to X_u^{\text{parton}} e\bar{\nu})} = \left| \frac{V_{ts}^* V_{tb}}{V_{cb}} \right|^2 \frac{6\, \alpha_{em}}{\pi} P(E_0)\,. \tag{2.6}$$

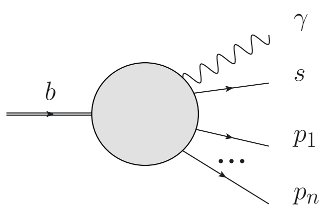
In Eq. (2.5), the $b \to X_s^{\text{parton}}\gamma$ denotes a b quark decaying into a hard photon (with an energy above E_0) and any number of additional particles with a combined strangeness of S = −1 (i.e. the sum of strange quarks has to be bigger by 1 than the sum of anti-strange quarks). The only additional constraint is that **every final state involving charm quarks is excluded** from the definition. This keeps the observable clean, as we avoid problems with the landscape of resonances around the D-mesons.

Using this, the rate can be written as:

$$\Gamma(b \to X_s^{\text{parton}}\gamma) = \Gamma(b \to s\gamma) + \Gamma(b \to sg\gamma) + \Gamma(b \to sq\bar{q}\gamma) + \Gamma(b \to sq\bar{q}g\gamma) + \ldots\,, \tag{2.7}$$

where $q = u, d, s$ (since c is excluded by definition and b as well as t are excluded by kinematics). This series can be written diagrammatically as

$$\Gamma(b \to X_s^{\text{parton}}\gamma) \propto \sum_n \int \mathrm{dPS}_{n+2} \sum_{\text{Diagrams}} \left| \vphantom{\int} \right.$$



$$\left. \vphantom{\int} \right|^2 . \tag{2.8}$$

The contributions we are calculating in this work are the cases for $p_1 = q, p_2 = \bar{q}$ at NLO in α_s and $p_1 = q, p_2 = \bar{q}, p_3 = g$ at tree-level.

We can see from Eq. (2.8) that, in the end, we need to solve integrals over the four- and five-body phase space, respectively.

In the next section, we go over the framework we are using to calculate these contributions, making clear what the grey blob in Eq. (2.8) represents. For this, a general introduction into the effective weak theory is given, which is then specialized to our case.

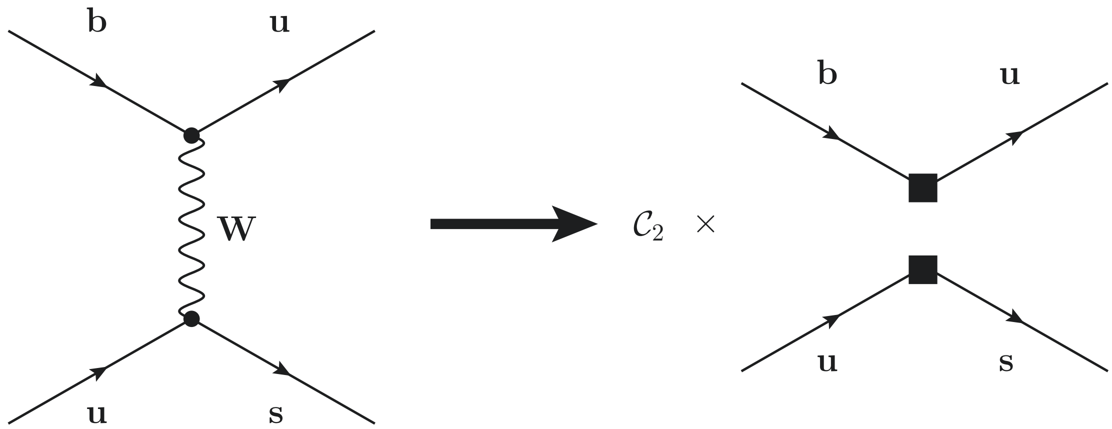


Figure 2.3: Example diagram for 4-fermion process mediated by the W boson.

2.4 The Effective Weak Theory

Calculating the rates of inclusive B decay modes in the Standard Model leads to conceptual problems at higher order. Short distance QCD effects yield **large logarithms** of the form $\alpha_s^n(m_b)\log^m(m_b/M)$, with M being either m_t or m_W. These logarithms lead to huge rate enhancements, making it difficult to get meaningful results.

Fortunately, there is a solution to this problem. The logarithmically enhanced terms can be resummed with renormalization group techniques [59]. The most suitable framework is the one of the so-called **effective field theories**. In the following section, we give a brief overview of the concept, introducing the core ideas and tools of the method.

2.4.1 Concept of Effective Theories

The primary goal of an effective theory is the separation of short- and long-distance contributions to physical processes. This is achieved by **integrating out** all high-energy degrees of freedom, which in the effective weak theory means all particles with a mass above a certain energy.

We want to illustrate this method, called **operator product expansion**, with one of the contributions that is important for our calculation, shown in Fig. 2.3: the

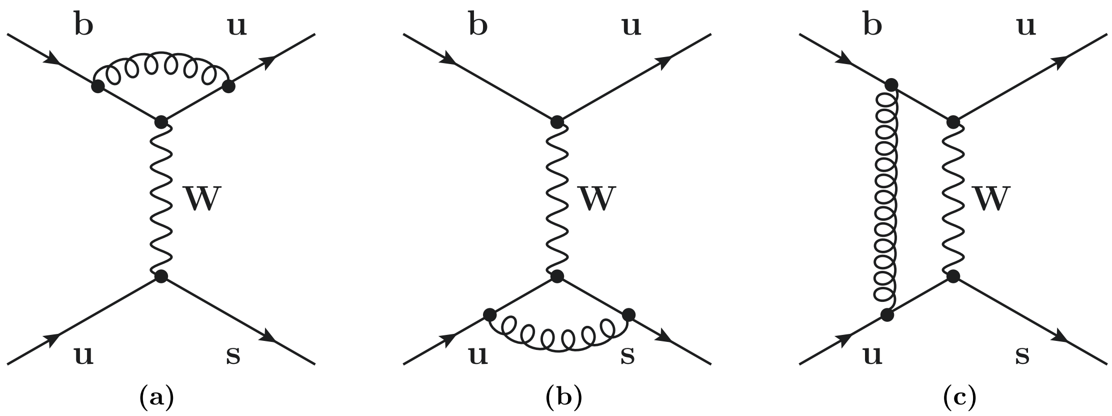


Figure 2.4: One-Loop QCD corrections to the current-current exchange in the Standard Model.

decay $b \to s u \bar{u}$.

In the Standard Model, using Feynman rules, the amplitude for the tree-level W-exchange is:

$$\mathcal{A} = -\frac{G_F}{\sqrt{2}} V_{ub}^* V_{us} (\bar{u}\gamma^\mu(1-\gamma_5)b) \; \frac{M_W^2}{p^2 - M_W^2} \; (\bar{s}\gamma^\mu(1-\gamma_5)u)\,, \tag{2.9}$$

where G_F is the Fermi constant, which includes the weak coupling g_2 as well as the mass of the W-boson M_W and is defined as

$$\frac{G_F}{\sqrt{2}} = \frac{g_2^2}{8M_W^2}\,. \tag{2.10}$$

For weak meson decays, the momentum transfer p of the W propagator satisfies the relation $p^2 \ll M_W^2$. This allows us to expand in the ratio p^2/M_W^2, leading to the amplitude

$$\mathcal{A} = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{us} (\bar{u}\gamma^\mu(1-\gamma_5)b) \; (\bar{s}\gamma^\mu(1-\gamma_5)u) \; + \mathcal{O}(p^2/M_W^2)\,. \tag{2.11}$$

What we did here is expressing the amplitude in terms of **effective operators**, where the first term is an operator of mass dimension six and the neglected terms of higher orders in the expansion correspond to higher dimensional ones.

We see that the result of (2.11) can also be obtained from an effective Hamiltonian

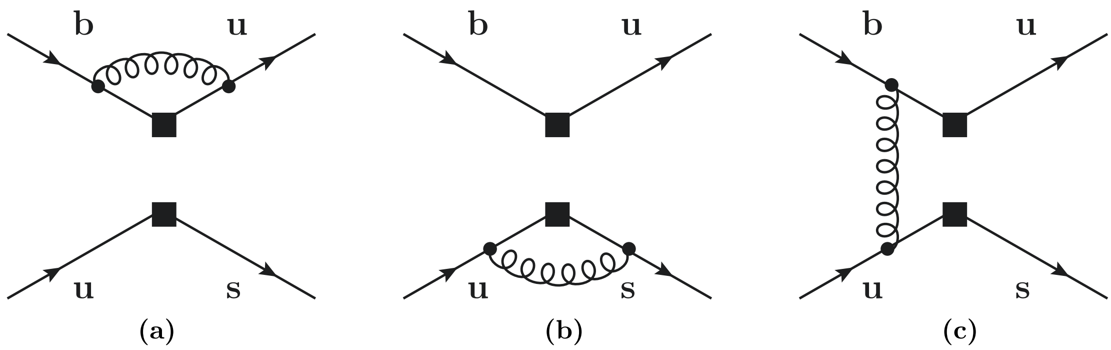


Figure 2.5: One-Loop QCD corrections to the current-current exchange in the Effective Theory.

of the form

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{us}\, \mathcal{C}_2 P_2 \ + \ \text{higher dimensional operators}\,, \tag{2.12}$$

where we introduce the notation

$$P_2 = (\bar{u}\gamma^\mu(1-\gamma_5)b)\ (\bar{s}\gamma^\mu(1-\gamma_5)u) \tag{2.13}$$

for our effective operator and $\mathcal{C}_2$ as its corresponding effective coupling, called **Wilson coefficient**. At tree-level, the Wilson coefficient is just 1, this can be read off when comparing the full amplitude to the one calculated via the effective Hamiltonian. This process is called **matching**.
The matching, although trivial at tree-level, gets more involved as QCD corrections are taken into account. Some of the one-loop corrections to the process above can be seen in Fig. 2.4 for the Standard Model and in Fig. 2.5 for the effective theory. Calculating all contributions in both theories and requiring them to be equal then allows for a determination of the Wilson coefficient order by order in α_s.

We encounter two additional subtleties in the calculation: First, one can see in Figs. 2.4 and 2.5 a) and b) that both models do not change the color of the respective currents (since the W does not carry color charge). But when calculating diagram c), the gluon connects the currents, allowing for color exchange. As our operator P_2 (called the color-singlet operator) does not allow for this, we need to introduce a second one (called the color-exchange operator), defined as

$$P_1 = (\bar{u}\gamma^\mu(1-\gamma_5)T^a b)\ (\bar{s}\gamma^\mu(1-\gamma_5)T^a u)\,. \tag{2.14}$$

The process at hand, i.e. the decay of a B-meson, does not only get contributions from these so-called current-current diagrams. As we have already seen in Fig. 2.1, there are more possible interactions that lead to the same final states. They are also incorporated and lead to four more effective operators that we need to consider, the so-called penguin-operators.

The second of the aforementioned subtleties is that we do not only encounter these so-called physical operators in our calculation. As the physical operators only span a complete basis in four dimensions, we also have to incorporate a set of evanescent operators that give finite contributions when shifting away from four dimensions in dimensional regularization. In the effective weak theory, they are constructed in such a way that we can disregard them for the calculation of bare amplitudes as long as we compensate their effect during the calculation of counterterms in the renormalization process.

In principle, this framework, called the **effective weak theory** [60, 61], contains more operators, e.g. including semi-leptonic, electro- and chromo-magnetic contributions. At next-to-leading order in α_s, considering the four- and five-body contributions, however, they are either not contributing to the process or their insertions have already been calculated in Ref. [62].

2.4.2 Operator Basis

As stated before, we want to determine additional contributions to the perturbative part of $\mathcal{B}(\bar{B} \to X_s\gamma)_{E_\gamma > E_0}$.

The relevant four-body (five-body) process $b \to s\bar{q}q\gamma(g)$ we are calculating are (up to $\mathcal{O}(\alpha_s)$) described by the following Lagrangian:

$$\mathcal{L}_{eff} = \mathcal{L}_{QED+QCD} + \frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \Big[\sum_{i=1}^{2} \mathcal{C}_i^u P_i^u + \sum_{i=3}^{6} \mathcal{C}_i\, P_i \Big] + h.c.\,, \tag{2.15}$$

where $\mathcal{L}_{QED+QCD}$ is the QCD and QED part of the Standard Model Lagrangian. This part is then complemented by six additional effective operators of dimension six:

$$
\begin{aligned}
P_1^u &= (\bar{s}_L\gamma_\mu T^a u_L)(\bar{u}_L\gamma^\mu T^a b_L)\,, & P_2^u &= (\bar{s}_L\gamma_\mu u_L)(\bar{u}_L\gamma^\mu b_L)\,, \\
P_3 &= (\bar{s}_L\gamma_\mu b_L)\sum_q(\bar{q}\gamma^\mu q)\,, & P_4 &= (\bar{s}_L\gamma_\mu T^a b_L)\sum_q(\bar{q}\gamma^\mu T^a q)\,, \\
P_5 &= (\bar{s}_L\gamma_\mu\gamma_\nu\gamma_\rho b_L)\sum_q(\bar{q}\gamma^\mu\gamma^\nu\gamma^\rho q)\,, & P_6 &= (\bar{s}_L\gamma_\mu\gamma_\nu\gamma_\rho T^a b_L)\sum_q(\bar{q}\gamma^\mu\gamma^\nu\gamma^\rho T^a q)\,.
\end{aligned}
\tag{2.16}
$$

Note that in this definition of the Lagrangian, $\mathcal{C}_{1,2}^u$ contain CKM phases, i.e. $\mathcal{C}_{1,2}^u = -V_{us}^*V_{ub}/V_{ts}^*V_{tb}\ C_{1,2}$, where $C_{1,2}$ are the coefficients as defined in Ref. [61]. For the other coefficients, the relation $\mathcal{C}_{3,..,6} = C_{3,..,6}$ applies, where $C_{3,..,6}$ are also defined as in Ref. [61].

The first two operators, P_1^u and P_2^u, are called **current-current operators**. They arise from diagrams similar to those in Fig. 2.5. Both external currents couple to the W-boson, leading to them both being left-handed.

The other four operators, P_3 to P_6, are **penguin operators**. As their name suggests, they originate from penguin diagrams, such as the one in Fig. 2.1. Here, only one of the currents is exclusively left-handed, while the $\bar{q}q$-part of the operator (the lower current in Fig. 2.1) is a vector current.

This will play an important role in the discussion of our calculational framework later, as the left-handed projector is defined as $P_L = \frac{1-\gamma_5}{2}$, introducing γ_5 into the calculation, leading to some subtleties that need to be addressed.

As stated in the last section, we also need evanescent operators. The ones that are relevant for our calculation are [17]:

$$
\begin{aligned}
E_1 &= (\bar{s}_L\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}T^a u_L)(\bar{u}_L\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}T^a b_L) - 16P_1^u\,, \\
E_2 &= (\bar{s}_L\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}u_L)(\bar{u}_L\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}b_L) - 16P_2^u\,, \\
E_3 &= (\bar{s}_L\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}b_L)\sum_q(\bar{q}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^{\mu_5}q) + 64P_3 - 20P_5\,,
\end{aligned}
$$

$$E_4 = (\bar{s}_L \gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5} T^a b_L) \sum_q (\bar{q}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^{\mu_5} T^a q) + 64 P_4 - 20 P_6 \,. \quad (2.17)$$

A more detailed discussion of their properties and the calculation of their contributions can be found in Sect. 6.2.

With this base of operators, following the notation of Ref. [62], we can write the perturbative contribution to the rate as:

$$\Gamma(b \to X_s^{\text{parton}}\gamma)_{E_\gamma > E_0} = \Gamma_0 \sum_{i,j=1}^{6} \mathcal{C}_i^*(\mu)\, \mathcal{C}_j(\mu)\, \widetilde{G}_{ij}(\mu, \delta) \,, \quad (2.18)$$

with a normalization factor

$$\Gamma_0 = \frac{G_F^2 m_b^5 \alpha_e |V_{ts}^* V_{tb}|^2}{32\pi^4} \,. \quad (2.19)$$

The entries of the matrix $\widetilde{G}_{ij}(\mu, \delta)$ are the interference terms of the operators P_i and P_j, integrated over the phase space. Their dependence on the photon energy cut is denoted by $\delta = 1 - 2E_0/m_b$.

In this work, we focus on the next-to-leading order part of the four-body contributions, which have to be supplemented by five-body contributions to render the expression free of infrared divergences. For an in-depth discussion of the latter, we refer to Sect. 5.
The four- and five-body contributions to $\widetilde{G}_{ij}(\mu, \delta)$, are defined by $\hat{G}_{ij}(\mu, \delta)$:

$$\Gamma(b \to s q\bar{q}\gamma)_{E_\gamma > E_0} + \Gamma(b \to s q\bar{q} g\gamma)_{E_\gamma > E_0} = \Gamma_0 \sum_{i,j} \mathcal{C}_i^*(\mu)\, \mathcal{C}_j(\mu)\, \hat{G}_{ij}(\mu, \delta) \,. \quad (2.20)$$

Furthermore, the $\hat{G}_{ij}$ can be written as an expansion in α_s:

$$\hat{G}_{ij}(\mu, \delta) = \hat{G}_{ij}^{(0)}(\delta) + \frac{\alpha_s(\mu)}{4\pi} \hat{G}_{ij}^{(1)}(\mu, \delta) + \mathcal{O}(\alpha_s^2) \,. \quad (2.21)$$

Fig. 2.6 shows the different contributions: Panel (a) shows the tree-level contributions, $\hat{G}_{ij}^{(0)}(\delta)$, calculated in Ref. [10] and panel (b) shows the NLO contributions

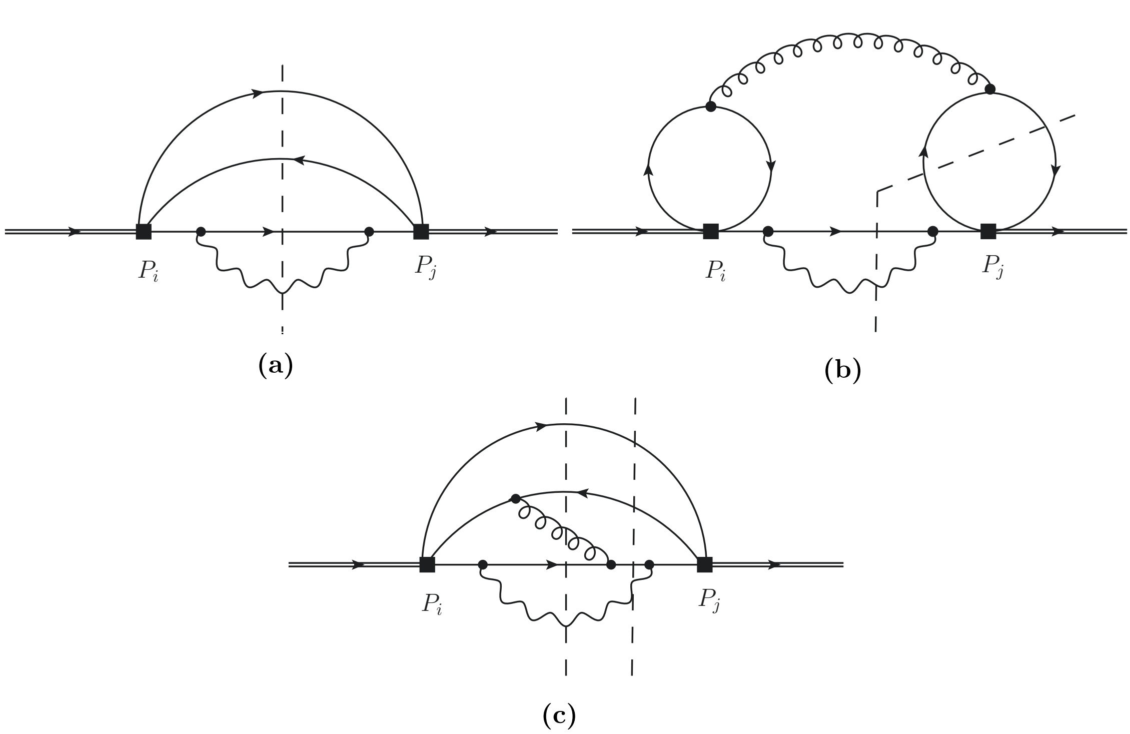


Figure 2.6: Sample four-body cut-diagrams contributing to the branching ratio. The ones from (c) were not calculated up until now and are subject of this work.

that were calculated in Ref. [62]. The last panel, (c), shows the up to now uncalculated contributions. The latter not only involve four-particle cuts, but to cancel all infrared divergences, the corresponding five-particle cuts also have to be considered. Calculating multi-parton contributions to the matrix $\widetilde{G}_{ij}(\mu, \delta)$ of Eq. (2.18) is a goal of this work. Our resulting matrix, which, together with the result from Ref. [62] constitutes the matrix $\hat{G}_{ij}^{(1)}(\mu, \delta)$, will be called $G_{ij}(\mu, \delta)$.

Before we come to the actual calculation of the decay width, the next chapter will first introduce methods, tools and notation that will be helpful in later chapters.

Chapter 3

Methods for Computing Integrals in Quantum Field Theory

In this work, we are interested in calculating branching fractions, i.e. the probability of finding a certain final state normalized to all possible final states. The size of such a branching fraction depends, at first order, on the strength of the involved couplings.

Another important factor are the possible momentum configurations of the involved particles. To get a result including all configurations, one integrates over the n-particle phase space PS_n, where n is the number of particles in the final state.

Beyond tree-level, when calculating higher-order corrections, the quantities start to also depend on the intermediate (virtual) states that occur at loop-level.

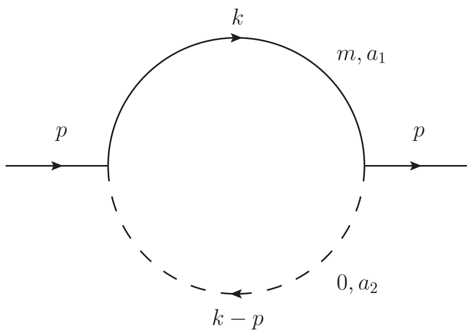
To include all possible momentum configurations of those virtual particles, their corresponding propagator factors are integrated with respect to the so-called loop momenta.

In the following, we want to introduce the basic concepts of the methods we used and the standard nomenclature, including Feynman integrals, phase space integration, integration-by-parts relations, integral families and differential equations.

3.1 Feynman Integrals

In this section we will discuss Feynman integrals and introduce basic notation. When calculating loop integrals, we integrate propagator factors, such as $[(p-k)^2 - m^2 + i0]^{-1}$ over the D-dimensional momentum space measure $d^D k$. Here, $D = 4 - 2\epsilon$, i.e. we shift away from the four-dimensional space-time by 2ϵ. This is done since, in four dimensions, the integrals are often divergent. By shifting the dimension away from four, one can **regulate** these divergences, leading to the result being a Laurent series in ϵ. With the method of renormalization, we can then cancel the factors of ϵ^{-n}, set ϵ to zero and obtain a finite result[1].
The $i0$ in the propagator is called the Feynman prescription and is needed to determine the sign when doing a Wick rotation from Minkowski to Euclidean space. It is usually dropped for readability.

A very simple example of a Feynman integral is the one-loop semi-massive bubble:



$$= \mathbf{I}[a_1, a_2; m, 0; p]\,, \tag{3.1}$$

where the notation m_i, a_i on a line denotes a propagator raised to the power a_i with a mass m_i and the incoming momentum is p.
We can write $\mathbf{I}[a_1, a_2; m, 0; p]$ as an integral over the loop momentum k, yielding

$$\mathbf{I}[a_1, a_2; m, 0; p] = \int \frac{d^D k}{(2\pi)^D} \, \frac{1}{[k^2 - m^2 + i0]^{a_1}[(k-p)^2 + i0]^{a_2}} \tag{3.2}$$

For $a_1 = a_2 = 1$, this integral is divergent in four dimensions. As stated above, this is avoided by a shift of the dimension, which makes the divergence explicit as a pole

[1] This power of the pole is not directly related to the number of particles in the phase space, although both are denoted by n here.

in ϵ. This method, called **dimensional regularization**, will be discussed briefly in the next section.

3.1.1 Dimensional Regularization

When calculating Feynman integrals in four dimensions, we encounter two types of divergences. The first ones are **ultraviolet (UV) divergences**, connected to the loop momentum k tending to infinity.

In Eq. (3.2) (from here on omitting the dependence on m_i and p), we observe this behaviour for $a_1 = a_2 = 1$ and very large k:

$$\mathbf{I}[1,1] \sim \int \frac{d^4k}{(2\pi)^4} \, \frac{1}{[k^2]^2} \sim \infty \,. \tag{3.3}$$

The second type of singularity that we encounter is rooted in the denominators approaching zero. These **infrared (IR) divergences** are subdivided into **soft** and **collinear divergences**. Taking the propagator of a massive particle with four-momentum p_1 and a massless particle with four-momentum p_2, we can derive:

$$\frac{1}{(p_1+p_2)^2 - m_1^2} = \frac{1}{2\,p_1 p_2} = \frac{1}{E_1 E_2 - \vec{p_1}\cdot\vec{p_2}} = \frac{1}{E_1 E_2 \left(1 - \sqrt{1 - \frac{m_1^2}{E_1^2}}\cos(\theta_{12})\right)}\,, \tag{3.4}$$

where E_i and $\vec{p_i}$ denote energy and three-momentum of the corresponding particle. We can distinguish two limits for the denominator in Eq. (3.4). The first is $E_2 \to 0$, the energy of the massless particle going to zero, which corresponds to a soft divergence (even though the emitting particle is massive). The second type of divergence only occurs if both particles are massless, i.e. $m_1 = 0$. In this limit, the denominator vanishes for $\theta_{12} \to 0$, the angle between the two particles going to zero, which is called a collinear divergence.

When calculating Feynman integrals, these divergences are a problem, since we know that, as real-world observations, our results, e.g. branching ratios or cross-sections, can not be infinite.

The problem is solved by shifting the dimension away from four using $D = 4 - 2\epsilon$ instead. This allows us to make the divergences explicit as poles in ϵ. In the end of the calculation, when the UV poles are renormalized and the IR poles cancel or are regularized in a different scheme, terms proportional to ϵ^{-n} vanish and we can go back to four dimensions by taking the limit $\epsilon \to 0$.

3.1.2 Feynman Parametrization

After introducing the dimensional regulator, we can start solving the integral without encountering (explicitly) infinite results. One of the most fundamental methods for this is the use of **Feynman parameters**. The general formula for two denominators reads:

$$\frac{1}{A^m B^n} = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \int_0^1 dx \, \frac{x^{m-1}\bar{x}^{n-1}}{[Ax + B\bar{x}]^{m+n}} \,, \tag{3.5}$$

where the standard notation $\bar{x} = 1 - x$ is used.
This method essentially trades the introduction of an additional integral for the benefit of changing a product of two denominators into a sum, which often simplifies computation significantly.
Applying this to our example yields:

$$\mathbf{I}[a_1, a_2] = \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \, \frac{x^{a_1-1}\bar{x}^{a_2-1}}{[(k^2 - m^2)x + (k-p)^2\bar{x}]^{a_1+a_2}} \,. \tag{3.6}$$

For simplicity, we now take $a_1 = a_2 = 1$ and again, impose the condition $p^2 = m^2$:

$$\mathbf{I}[1, 1] = \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \, \frac{1}{[(k^2 - m^2)x + (k-p)^2\bar{x}]^2} \,, \tag{3.7}$$

and, after completing the square and shifting k, we arrive at:

$$\mathbf{I}[1, 1] = \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \, \frac{1}{[(k^2 - x^2 m^2)]^2} \,. \tag{3.8}$$

For one-loop integrals in this form, we know the general solution for performing the loop momentum integration. The formula can e.g. be found in Ref. [42] or Ref. [43]. It reads:

$$\int \frac{d^D k}{(2\pi)^D} \, \frac{k^{2a}}{[(k^2 - \Delta)]^b} = i(-1)^{a-b} \frac{1}{(4\pi)^{D/2}} \frac{1}{\Delta^{b-a-D/2}} \frac{\Gamma(a + D/2)\Gamma[(b - a - D/2)}{\Gamma(b)\Gamma(D/2)} \,. \tag{3.9}$$

Using Eq. (3.9) then leads to:

$$\begin{aligned}\mathbf{I}[1,1] &= \int_0^1 dx\ (iS_\Gamma)(-1)\Gamma(1-\epsilon)\Gamma(\epsilon)(x^2m^2)^{-\epsilon} \\ &= (iS_\Gamma)\frac{(-1)\Gamma(1-\epsilon)\Gamma(\epsilon)\Gamma(1-2\epsilon)}{\Gamma(2-2\epsilon)}\left(m^2\right)^{-\epsilon} \\ &= -(iS_\Gamma)\left(m^2\right)^{-\epsilon}\left[\frac{1}{\epsilon}+2\right]+\mathcal{O}(\epsilon)\,. \end{aligned} \tag{3.10}$$

Here, we introduce the notation

$$S_\Gamma = \frac{1}{\Gamma(1-\epsilon)\,(4\pi)^{2-\epsilon}}\,,$$

collecting the factors that regularly occur in Feynman and phase space integrals. As we can see, the integral is divergent in four dimensions, i.e. it has a pole in ϵ. In the end, these poles have to cancel to allow the limit $\epsilon \to 0$. The method by which this is achieved, called **renormalization**, will be discussed in a later chapter.

3.2 Integral Families

In the last section we explicitly calculated a diagram for linear propagator powers. In general, this does not have to be the case, propagators can occur for any integer power, even negative ones.
If integrals only differ by the power of their denominators, they can be assigned to a so-called **family of integrals**.
Additionally to the denominator powers being different, the integrals occurring in calculations often have numerators that contain products of all the occurring momenta. At next-to-leading order, this can already lead to thousands of integrals that need to be computed, which gets impractical, if not impossible, very quickly.
For this reason, several methods of simplifying the integrals and reducing their number have been found. In this section, we want to briefly introduce the concept of two of them, namely Passarino-Veltman reduction and integration-by-parts relations.

3.2.1 Passarino-Veltman Reduction

The integrals discussed up to this point have been scalar ones, meaning they did not have open Lorentz indices. This is only the simplest case, as we see that in real calculations we encounter vector (one open index) or even tensor (more than one open index) integrals. A vector integral, for example, can look as follows:

$$I^\mu = \int \frac{d^D k}{(2\pi)^D} \, \frac{k^\mu}{[(k^2-m^2)][(k-p)^2]} \,. \tag{3.11}$$

We now use the knowledge that this integral only depends on one external momentum, i.e. p^μ. This means that after the loop integration is done, this can be the only object with an open index, leading to the ansatz:

$$I^\mu = p^\mu A \,. \tag{3.12}$$

With this ansatz, we can contract both sides with $2p_\mu$ (again, we work in the case $p^2 = m^2$ for simplicity):

$$\begin{aligned}
2p_\mu I^\mu &= \int \frac{d^D k}{(2\pi)^D} \, \frac{2p_\mu k^\mu}{[(k^2-m^2)][(k-p)^2]} \\
&= \int \frac{d^D k}{(2\pi)^D} \, \frac{(k-p)^2 - k^2 - p^2}{[(k^2-m^2)][(k-p)^2]} \\
&= \int \frac{d^D k}{(2\pi)^D} \left(\frac{1}{k^2-m^2} + \frac{1}{(k-p)^2} - \frac{2m^2}{[(k^2-m^2)][(k-p)^2]} \right) \qquad (3.13) \\
&= \mathbf{I}[1,0] - 2m^2 \mathbf{I}[1,1] \\
&= 2p_\mu \, p^\mu A = 2m^2 A \,. \qquad (3.14)
\end{aligned}$$

In Eq. (3.13) we have used that **scaleless integrals are set to zero** when using dimensional regularization. After the shift $k \to k' + p$ in the second term, there is no external scale present anymore and the contribution vanishes.
From Eqs. (3.14) we can now deduce that:

$$I^\mu = p^\mu \left(\frac{\mathbf{I}[1,0]}{2m^2} - \mathbf{I}[1,1] \right) , \tag{3.15}$$

and we have reduced the calculation of the vector integral to the computation of scalar integrals.

This method can be extended to multiple external vectors as well as more than one open Lorentz index. In general, the ansatz has to include all possible Lorentz structures that can occur in the result. These then have to be contracted with the different momenta and metric tensors. Doing these contractions leads to a matrix $A_{i,j}$ (instead of just A in the example above), that is then inverted, yielding the result in terms of scalar integrals [63].

3.2.2 Integration-by-Parts Reduction

In this section we will discuss a method that is even more powerful when it comes to reducing integrals. This method, called Integration-by-parts (IBP) [64], interconnects integrals of a family, e.g. integrals with high denominator and numerator powers to ones with lower powers.
This means that we can effectively narrow down the problem of calculating a large number of integrals to the problem of reducing them to a subset that is significantly smaller. The IBP reduction expresses the original integrals as a linear combination of this small subset, the so-called **master integrals**, which are then left to be computed.
Here, we want to introduce the concept and notation of this method, which is extensively used in this work.

We first consider a general family F_m of multi-loop integrals (the index m will later run from one to the maximum number of families to label them, for now it is kept general):

$$F_m[a_1, ..., a_n] = \int \frac{d^D k_1}{(2\pi)^D} ... \int \frac{d^D k_L}{(2\pi)^D} \frac{1}{{D_1}^{a_1} {D_2}^{a_2} \cdots {D_n}^{a_n}} \,. \tag{3.16}$$

The family F_m is defined by the set of denominators D_i, which are Feynman propagators, e.g. $D_1 = [(k_1 + p_2)^2 - m_2^2]$. The a_i denote the power of the propagators, which can take any integer value.

With this definition, we can now consider carrying out integration-by-parts on the integrals. The general form of such a relation can be written as

$$\int \frac{d^D k_1}{(2\pi)^D} \cdots \int \frac{d^D k_L}{(2\pi)^D} \frac{\partial}{\partial k_i^\mu} \frac{v^\mu}{D_1{}^{a_1} D_2{}^{a_2} \cdots D_n{}^{a_n}} = 0\,. \tag{3.17}$$

The derivatives are taken with respect to any of the loop-momenta and the numerator v^μ stands for any of the loop- or external momenta. With L the number of loops and E the number of independent external momenta, this means that for any given set of values for the a_i we can get $L(L+E)$ relations.

To illustrate the procedure of an IBP reduction, we can look at our example diagram from Eq. (3.1) again, leading to the general relations:

$$\int \frac{d^D k}{(2\pi)^D} \frac{\partial}{\partial k^\mu} \frac{v^\mu}{[(k^2-m^2)]^{a_1}[(k-p)^2]^{a_2}} = 0\,, \qquad v = k, p\,. \tag{3.18}$$

Explicitly calculating both equations then yields ($v^\mu = k^\mu$):

$$\begin{aligned} &(D-2a_1-a_2)\mathbf{I}[a_1,a_2] - a_2\mathbf{I}[a_1-1,a_2+1]- \\ &m^2 a_1 \mathbf{I}[a_1+1,a_2] - a_2(m^2-p^2)\mathbf{I}[a_1,a_2+1] = 0\,, \end{aligned} \tag{3.19}$$

and ($v^\mu = p^\mu$):

$$\begin{aligned} &(-a_1+a_2)\mathbf{I}[a_1,a_2] + a_1\mathbf{I}[a_1+1,a_2-1] - a_2\mathbf{I}[a_1-1,a_2+1]- \\ &(m^2+p^2)a_1\mathbf{I}[a_1+1,a_2] - a_2(m^2-p^2)\mathbf{I}[a_1,a_2+1] = 0\,. \end{aligned} \tag{3.20}$$

We can now see explicitly how the integrals with different values for a_1 and a_2 are related.
We now start at low a_i and increase them, the results can then be solved for:

$$\mathbf{I}[a_1+1,a_2] = \frac{(-D+a_1+2\,a_2)}{a1} \frac{\mathbf{I}[a_1,a_2]}{(p^2-m^2)} + \frac{\mathbf{I}[a_1+1,a_2-1]}{p^2-m^2}\,, \tag{3.21}$$

and

$$\mathbf{I}[a_1, a_2+1] = \frac{D(m^2+p^2) - 3\,a_2\,m^2 - (2\,a_1+a_2)p^2}{a_2} \frac{\mathbf{I}[a_1,a_2]}{(p^2-m^2)} + \frac{\mathbf{I}[a_1-1,a_2+1]}{p^2-m^2} - \frac{2\,a_1 m^2}{a_2} \frac{\mathbf{I}[a_1+1,a_2-1]}{(p^2-m^2)} \,. \tag{3.22}$$

Both of these equations now lower $A = a_1 + a_2$ and by repeatedly applying them we can reduce all occurring integrals to master integrals. For this problem, these are $\mathbf{I}[1,1]$ and $\mathbf{I}[1,0]$. In principle, $\mathbf{I}[0,1]$ does also occur, but it is a scaleless integral and thus set to zero.
This process becomes quite cumbersome very quickly when dealing with a large number of integrals and/or denominators. To automate the procedure of IBP reductions, the **Laporta algorithm** was introduced [65]. To illustrate the basics of the algorithm, we need to define two quantities: The first one is Q, which, for a given integral, denotes the number of lines of a respective integral, i.e. the number of positive a_i. The second one is A which is defined as $A = \sum_{i=1}^{N} |a_i|$.
The Laporta algorithm starts at the highest appearing values of Q, using Gaussian elimination to express the relations in terms of a subset of the occurring integrals, the so-called master integrals. It then decreases Q stepwise to include **lower sectors**, i.e. integrals with fewer distinct denominators. In each of the respective sectors, A is then increased stepwise also. This leads to a number of master integrals for each sector, relating integrals with high A to integrals with lower A from the same sector and to integrals from lower sectors.
It has been shown that with growing Q and A, the number of equations grows faster than the number of unknowns, and with that the ensuing reduction always leads to a finite number of master integrals [66].

The method of IBP reductions is very powerful and opened up the field for calculations at four and even five loops and beyond, applications at this level can be found, for example, in Refs. [67–70].
Increasing the number of loops and particles in the final state increases the possible combinations of momenta. As the denominators in the family have to form a linearly independent basis to make it possible to express every occurring scalar product as

a combination of them, the choice of these bases can have a significant impact on the reduction process.
For example, if we need more than one family, there can be multiple choices in how to group the original set of integrals, which can in turn result in a different number of master integrals. As the minimal number of master integrals cannot depend on the external choices, we know that in such a case, some of the master integrals have to be expressible as a linear combination of the others.
If the supplemented integrals have too low values of Q and A and it is implemented in a way that only goes over the minimally needed values, the Laporta algorithm can miss some relations. To circumvent this, one can on the one hand cross-check the results with a different set of families or on the other hand give the reduction extra integrals with higher powers of numerators and denominators.

3.3 Phase Space Integration

When calculating processes with more than one particle in the initial and final state, one has to take all their possible momentum configurations into account. This is taken care of by integrating over the whole **phase space** that the particles can occupy.
In this work we are concerned with the decay of a b quark, the corresponding structure being $1 \to n$ with n **massless** particles in the final state.
The phase space integral for such a decay process is defined as:

$$\int dPS_n = \left(\prod_{f=1}^{n} \int \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) (2\pi)^4 \delta^{(4)}(p_b - \sum p_f)\,, \tag{3.23}$$

where p_b is the momentum of the incoming b quark and the p_f (E_f) are the momenta (energies) of the particles in the final state. To make the above formula applicable in a real calculation, it is often rewritten in terms of the (dimensionless) **momentum invariants**, defined as $s_{ij} = 2\, p_i \cdot p_j / M^2$ (in our case $M^2 = m_b^2$).
Doing this eliminates trivial dependences on the angular configurations of the particles. This essentially means that the momentum integrations (over s_{ij}) are disentangled from the integrations over the angular volumes ($d\Omega_D$). The latter then

simply result in:

$$V(D) = \int d\Omega_D = \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})} \,. \tag{3.24}$$

After doing this change of variables, one arrives at the following expressions for the different number of particles (for the derivation of the two-, three- and four-particle expressions, we refer to Ref. [71]):

$$dPS_2 = (2\pi)^{2-D}(m_b^2)^{\frac{D-4}{2}}(s_{12})^{\frac{D-4}{2}}\frac{d\Omega_{D-1}}{2^{D-1}}\; ds_{12}\; \delta(1-s_{12})\,, \tag{3.25}$$

for the two-particle case and

$$\begin{aligned} dPS_3 =& (2\pi)^{3-2D}2^{-1-D}(m_b^2)^{D-3}(s_{12}s_{13}s_{23})^{\frac{D-4}{2}}\; \delta(1-s_{12}-s_{13}-s_{23}) \\ & d\Omega_{D-1}\; d\Omega_{D-2}\; ds_{12}\; ds_{13}\; ds_{23}\,, \end{aligned} \tag{3.26}$$

for the three-particle case.
Most integrations in this work are done for four and five particles in the final state, requiring the expressions for dPS_4 and dPS_5. Since they become rather lengthy, only dPS_4 is shown here, whereas dPS_5 (which was originally calculated in Ref. [72]) can be found in Appendix H:

$$\begin{aligned} dPS_4 =& (2\pi)^{4-3D}2^{-1-2D}(m_b^2)^{\frac{3D-8}{2}}\delta(1-s_{12}-s_{13}-s_{23}-s_{14}-s_{24}-s_{34}) \\ & (-\Delta_4)^{\frac{D-5}{2}}\Theta(-\Delta_4)\; d\Omega_{D-1}\; d\Omega_{D-2}\; d\Omega_{D-3}\; ds_{12}\; ds_{13}\; ds_{23}\; ds_{14}\; ds_{24}\; ds_{34}\,. \end{aligned} \tag{3.27}$$

Δ_n is the Gram determinant, for $n = 4$ it is:

$$\Delta_4 = \lambda(s_{12}s_{34}, s_{13}s_{24}, s_{14}s_{23}), \tag{3.28}$$

with

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2(xy + xz + yz). \tag{3.29}$$

In this section we only considered integrations over the full phase space to explain the concept and notation. In our calculation the phase space is restricted, with a lower cut on the energy of the photon. There are several subtleties tied to this, which will be explained when doing the actual computations in Sect. 4.3.

3.4 Mellin-Barnes Representation

With the tools introduced up to this point, we are able to generate integral expressions from Feynman diagrams and further reduce these expressions to arrive at a set of master integrals. These master integrals then need to be computed to yield analytical results. We have already introduced the method of Feynman parameters, which is very useful. When handling integrals that contain more loops and a higher number of lines (i.e. propagators with positive power), this method alone becomes inefficient (if not impossible) to use very quickly though, as a lot of parameters have to be introduced. This makes the resulting integrals much more complicated than in our example, calling for additional methods of computation.
One of these methods is the so-called **Mellin-Barnes representation**, which factorizes sums in a denominator at the cost of an additional integration:

$$\frac{1}{(A_1+A_2)^c} = \frac{1}{\Gamma(c)} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{d\sigma}{2\pi i}\, \Gamma(-\sigma)\, \Gamma(c+\sigma)\, A_1^{\sigma}\, A_2^{-c-\sigma}\,. \tag{3.30}$$

The contour of the integrations is chosen in such a way as to separate the poles in the $\Gamma(-\sigma)$ and $\Gamma(c+\sigma)$. What this essentially means is that our choice makes the real part of all the arguments in the Γ functions positive.
By applying the formula multiple times, it can be generalized to more than two terms in the sum:

$$\begin{aligned} \frac{1}{(A_1+A_2+...+A_n)^c} &= \frac{1}{\Gamma(c)} \frac{1}{(2\pi i)^{n-1}} \int_{\gamma_1-i\infty}^{\gamma_1+i\infty} d\sigma_1 \;\cdots \int_{\gamma_n-i\infty}^{\gamma_n+i\infty} d\sigma_n \\ &\Gamma(-\sigma_1)\,\cdots\,\Gamma(-\sigma_{n-1})\,\Gamma(c+\sigma_1+\cdots+\sigma_{n-1})\,A_1^{\sigma_1}\,...\,A_{n-1}^{\sigma_{n-1}}\,A_n^{-c-\sigma_1-...-\sigma_{n-1}}\,. \end{aligned} \tag{3.31}$$

Here we must again chose the contours to separate the poles, which becomes a tedious effort for larger n and a growing number of Γ-functions. There are algorithms to find appropriate values for the γ_i, e.g. implemented in Refs. [73] and [74].
The above procedure may sound counterintuitive at first, since the first step in

most of the calculations is often the introduction of Feynman parameters, changing products into sums, whereas this method works in the opposite direction. But, using both methods in succession can make it possible to carry out Feynman integrations analytically that were not doable before.
For example, factorizing an expression can reduce the dependence of variables to very basic structures that can be resolved via the help of e.g. the Beta-function B:

$$B(x,y) = \int_0^1 dt\; t^{x-1}\, \bar{t}^{\,y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,, \tag{3.32}$$

where we defined $\bar{t} = 1-t$. Carrying out the Feynman parameter integrations often introduces additional Γ-functions that depend on the Mellin-Barnes variables. Low-dimensional representations can be solved by summing their residues on either side of the contour, after using analytic continuation to $\epsilon \to 0$ (The regularization of the integrals used in Ref. [73] also uses $\epsilon \neq 0$ to define the contour, which then has to be continued to zero).
For higher-dimensional representations, this is often not possible. For these, we rely on the help of additional auxiliary functions to simplify the expressions and lower the dimension of the representation. The functions and relations that were most helpful for this work are collected and discussed in Sect. 3.6.

There are two lemmata connected to the Mellin-Barnes representation [75]. These prove to be very powerful, as we often encounter expressions that are of the structure we see on the left sides of Eqs. (3.33) and (3.34).
The first lemma reads:

$$\int_{c-i\infty}^{c+i\infty} \frac{d\sigma}{2\pi i}\Gamma(\lambda_1+\sigma)\Gamma(\lambda_2+\sigma)\Gamma(\lambda_3-\sigma)\Gamma(\lambda_4-\sigma) = \frac{\Gamma(\lambda_1+\lambda_3)\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_2+\lambda_3)\Gamma(\lambda_2+\lambda_4)}{\Gamma(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}\,, \tag{3.33}$$

and the second one is:

$$\int_{c-i\infty}^{c+i\infty} \frac{d\sigma}{2\pi i} \frac{\Gamma(\lambda_1+\sigma)\Gamma(\lambda_2+\sigma)\Gamma(\lambda_3+\sigma)\Gamma(\lambda_4-\sigma)\Gamma(\lambda_5-\sigma)}{\Gamma(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\sigma)} = \frac{\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_2+\lambda_5)}{\Gamma(\lambda_1+\lambda_2+\lambda_4+\lambda_5)} \frac{\Gamma(\lambda_1+\lambda_5)\Gamma(\lambda_3+\lambda_4)}{\Gamma(\lambda_1+\lambda_3+\lambda_4+\lambda_5)} \frac{\Gamma(\lambda_2+\lambda_4)\Gamma(\lambda_3+\lambda_5)}{\Gamma(\lambda_2+\lambda_3+\lambda_4+\lambda_5)}. \tag{3.34}$$

When encountering not only Γ-functions but also their derivatives (such as Polygamma-functions $\Psi^{(n)}$), additional functional relations can be obtained from the above formulas by taking derivatives with respect to the arguments λ_i on both sides. An automatic application of these lemmata is implemented in the package `barnesroutines.m` [76].

3.5 Differential Equations

3.5.1 The Principal Method

In this section we introduce the method of differential equations [77–79]. It has become one of the most important methods to compute master integrals and huge efforts are channeled into the research of this field. The main appeal of the methodology is the fact that the solution of the integrals is achieved without direct evaluations of the integrands (in the following, these are often called the 'integration kernels'). This is achieved with the following steps: We start with a set of master integrals that the integration-by-parts reduction yields. We can then utilize their functional dependence on kinematical invariants to build a system of differential equations for this basis of master integrals. After applying derivatives with respect to the kinematical quantities, we can reexpress them in terms of the propagators of the basis, the powers of which have changed compared to the original set of masters. These results can then be IBP reduced again to get back to our original basis, leaving us with a **system of first order differential equations**.
To solve this system, one **boundary condition** for each master integral is required, which can be calculated at a fixed point. Usually at one of the invariants being zero or one, or from the asymptotic behaviour in a certain region (we will see later that

the choice of this point can not be completely arbitrary, because this swaps two infinitesimal operations and we have to preserve the analytical behaviour of these limits after swapping). Eliminating the dependence on one variable (and choosing the point properly) makes the explicit calculation of the remaining function much easier, reducing the complexity of the original problem significantly.

To illustrate the method, let us come back to the family of integrals we have already analyzed in the sections before:

$$\mathbf{I}[a_1, a_2] = \int \frac{d^D k}{(2\pi)^D} \, \frac{1}{[k^2 - m^2]^{a_1}[(k-p)^2]^{a_2}} \tag{3.35}$$

In the section about integration-by-parts relations we have seen that our base consists of two integrals and we can write down the basis vector

$$\vec{f} = \begin{pmatrix} \mathbf{I}[1,0] \\ \mathbf{I}[1,1] \end{pmatrix} . \tag{3.36}$$

For the next step we want to find a dimensionless quantity that our base depends on and we choose $x = p^2/m^2$, which reduces the dimensionality of the problem from two to one. In the process of the variable transformation we need to translate the derivative with respect to x into a derivative with respect to the quantities of the kernel, which in our case is p^μ. Using the chain rule and multiplying by p^μ, we arrive at

$$\frac{\partial \vec{f}}{\partial x} = \frac{p^\mu}{2x} \frac{\partial \vec{f}}{\partial p^\mu} . \tag{3.37}$$

This can now be used in Eq. (3.35), where, after rewriting the result in terms of our basis, it yields

$$\frac{\partial \vec{f}}{\partial x} = \frac{\partial}{\partial x} \begin{pmatrix} \mathbf{I}[1,0] \\ \mathbf{I}[1,1] \end{pmatrix} = \frac{1}{2x} \begin{pmatrix} 0 \\ (m^2 - p^2)\mathbf{I}[1,2] + \mathbf{I}[0,2] - \mathbf{I}[1,1] \end{pmatrix} . \tag{3.38}$$

This system is not yet in its desired form, as we want to have a closed system of equations, making it necessary to use the integration-by-parts relations we derived earlier. This brings it into the form

$$\begin{aligned}\frac{\partial \vec{f}}{\partial x} = \frac{\partial}{\partial x}\begin{pmatrix}\mathbf{I}[1,0]\\ \mathbf{I}[1,1]\end{pmatrix} &= \begin{pmatrix}0\\ \frac{D-2}{2m^2}(\frac{1}{1-x}+\frac{1}{x})\mathbf{I}[1,0]-(\frac{D-3}{1-x}+\frac{D-2}{2x})\mathbf{I}[1,1]\end{pmatrix}\\ &= \begin{pmatrix}0 & 0\\ \frac{1-\epsilon}{m^2}(\frac{1}{1-x}+\frac{1}{x}) & -(\frac{1-2\epsilon}{1-x}+\frac{1-\epsilon}{x})\end{pmatrix}\begin{pmatrix}\mathbf{I}[1,0]\\ \mathbf{I}[1,1]\end{pmatrix} \mathrel{\hat{=}} \hat{A}(x,\epsilon)\vec{f}\,.\end{aligned} \tag{3.39}$$

Eq. (3.39) now defines the matrix $\hat{A}(x,\epsilon)$ that connects our basis vector $\vec{f}$ to its derivative.

A few observations on this can be made: First, the equation for $\mathbf{I}[1,0]$ is trivial, since it does not depend on p^μ and therefore the system does not produce additional information about it.

Secondly, the dependence on x of the matrix follows a clear pattern, and we can further decompose the matrix into

$$\hat{A}(x,\epsilon) = \sum_k \frac{\hat{a}_k(\epsilon)}{x-x_k}\,, \tag{3.40}$$

where the singularity structure of the equation is manifest through the given **singular points** x_k. This is called the **Fuchsian form**, where the $\hat{a}_k$ are matrices that contain functions of ϵ [80].

In general, the matrix does not have to take this simple form directly: It can have an additional term for a (spurious) singularity at infinity

$$\hat{A}(x,\epsilon) = \sum_k \frac{\hat{a}_k(\epsilon)}{x-x_k} + p(x,\epsilon)\,. \tag{3.41}$$

It has been shown by Ref. [81] that one can algorithmically introduce transformations on the system to eliminate these spurious poles.

In our case, only $x_k = \{0,1\}$ appear and we can make an ansatz for $\mathbf{I}[1,1]$ as a Laurent series in ϵ. When using our knowledge of $\mathbf{I}[1,0]$ and calculating $\mathbf{I}[1,1]$ for $x=1$ as boundary conditions, we can solve the equation order by order in ϵ using standard methods for the solution of linear differential equations of first order.

The boundary conditions up to $\mathcal{O}(\epsilon)$ read

$$\mathbf{I}[1,0] = (iS_\Gamma)(m^2)^{1-\epsilon}\left(\frac{1}{\epsilon} + 1 + \epsilon\left(1 + \frac{\pi^2}{6}\right) + \mathcal{O}(\epsilon^2)\right) \tag{3.42}$$

and

$$\mathbf{I}[1,1]\Big|_{x=0} = (iS_\Gamma)(m^2)^{-\epsilon}\left(\frac{1}{\epsilon} + 2 + \left(4 + \frac{\pi^2}{6}\right)\epsilon + \mathcal{O}(\epsilon^2)\right), \tag{3.43}$$

leading to the solution for $\mathbf{I}[1,1]$:

$$\begin{aligned}\mathbf{I}[1,1] = (iS_\Gamma)(m^2)^{-\epsilon}\Bigg(&\frac{1}{\epsilon} + \left(2 + \frac{1-x}{x}\log(1-x)\right) \\ &+ \epsilon\left(4 + \frac{\pi^2}{6} + \frac{x-1}{x}\left(\mathrm{Li}_2(x) + \log^2(1-x) - 2\log(1-x)\right)\right) + \mathcal{O}(\epsilon^2)\Bigg).\end{aligned} \tag{3.44}$$

The process described above is very powerful and led to huge advancements in multi-loop calculations, as the complexity of the problem gets lowered quite significantly. Thus, integrals that were deemed too complex before become accessible to computation, opening up the field for higher and higher precision.

3.5.2 The Canonical Form

What we arrived at in the last section was a system of equations that relates the vector of basis integrals to its derivative with respect to certain kinematical variables. This matrix depends on the kinematical variables and the dimensional regulator ϵ. In this section, we want to introduce the **canonical form** of the system of equations, which decouples the dependence on the kinematics from the dependence on ϵ.

First, we consider a general transformation of our basis vector $\vec{f}$ via the transformation T with

$$\vec{g} = T^{-1}\vec{f}, \tag{3.45}$$

changing the form of our system as follows:

$$\partial_x\vec{f} = \hat{A}\vec{f} \quad \Rightarrow \quad \partial_x(T\vec{g}) = \hat{A}(T\vec{g}). \tag{3.46}$$

Making the differential equation in $\vec{g}$ explicit, we arrive at:

$$\partial_x \vec{g} = \left(T^{-1}\hat{A}T - T^{-1}\partial_x T\right)\vec{g}\,. \tag{3.47}$$

As utilizing the method of differential equations in calculations gained traction, it was observed that it is favourable to have the dependence on ϵ via just a prefactor of ϵ, since this decouples the different orders of the solution [82].
An algorithmic approach to the construction of the matrix T was proposed in Ref. [83], making it possible to systematically bring differential equations into a canonical form.
Utilizing this, several tools emerged that could find such an ϵ-base for a given system of equations automatically, for example `epsilon` [84], `fuchsia` [85] and `CANONICA` [86].
Making use of these algorithms, i.e. finding a suitable transformation T, the differential equations can be brought into the form

$$\partial_x \vec{g} = \epsilon \hat{A}_\epsilon \vec{g} = \epsilon \left[\sum_k \frac{\hat{a}_k}{x - x_k}\right]\vec{g}\,, \tag{3.48}$$

where the $\hat{a}_k$ are matrices that contain only rational numbers.
The members of the set of x_k, which is also called the **alphabet**, depend on the problem itself. The constituents encountered most commonly are $x_k = \{0, \pm 1\}$, but, depending on the kinematics, the alphabet can also contain rational functions or square roots of the invariants that appear in the integrals.
It is one of the challenges of this method to choose the variables of the equations in such a manner as to arrive at the simplest alphabet.

Let us now illustrate how this transformation works, using the set of master integrals from the last sections:

$$\vec{f} = \begin{pmatrix} \mathbf{I}[1,0] \\ \mathbf{I}[1,1] \end{pmatrix}, \qquad \hat{A} = \begin{pmatrix} 0 & 0 \\ \frac{1-\epsilon}{m^2}\left(\frac{1}{1-x} + \frac{1}{x}\right) & -\left(\frac{1-2\epsilon}{1-x} + \frac{1-\epsilon}{x}\right) \end{pmatrix}. \tag{3.49}$$

Using one of the aforementioned codes, we find the transformation

$$T = \begin{pmatrix} 1 & 0 \\ \frac{(\epsilon-1)(x-1)}{2(2\epsilon-1)x} & \frac{3(\epsilon-1)(x-1)}{(2\epsilon-1)x} \end{pmatrix}, \tag{3.50}$$

leading us to the system of equations in our new basis $\vec{g}$:

$$\partial_x \vec{g} = \epsilon \begin{pmatrix} 0 & 0 \\ -\frac{1}{6x} & \frac{1}{x} + \frac{2}{1-x} \end{pmatrix} \vec{g}\,. \tag{3.51}$$

The two integrals spanning the new basis $\vec{g}$ are linear combinations of the ones from the original basis, following the transformation from Eq. (3.45):

$$\vec{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \mathbf{I}[1,0] \\ \left(\frac{1}{3(1-x)} - \frac{1}{6}\right)\mathbf{I}[1,0] + \left(-\frac{1}{3(1-x)} + \frac{2\epsilon-1}{3(\epsilon-1)}\right)\mathbf{I}[1,1] \end{pmatrix}. \tag{3.52}$$

We can solve these equations systematically in terms of **iterated integrals** and we see in Eq. (3.51) that, for a given order in ϵ, only solutions from lower orders in ϵ partake in the equation.

Another helpful quantity is the concept of a **weight** that can be attached to these iterated integrals, corresponding to the number of iterations that have been carried out. This means that the appearing functions and transcendental constants also get a weight attached to them, for weight n these include π^n, Li_n, $\log^n(x)$ and $\zeta(n)$.

Having this notion of weight then leads to an interesting observation: If one factors out the appropriate (ϵ-dependent) prefactor and attaches a weight of $-n$ to ϵ^n, one is able to arrive at **uniformly transcendental** expressions (or UT expressions, for short). An expression being uniformly transcendental means that all terms in the result have the same weight.

This can be used as a cross-check for results, as all solutions belonging to a single system should be brought into this form by factoring out the same ϵ-dependent prefactor.

We can determine the necessary factor for example at the boundaries of $\vec{g}$, plugging in Eqs. (3.42) and (3.43):

$$\vec{g}|_{x=0} = 2\pi i S_\Gamma \frac{1}{(1-\epsilon)} \begin{pmatrix} \frac{1}{2\pi\epsilon} + \frac{\pi\epsilon}{12} + \mathcal{O}(\epsilon^2) \\ \frac{1}{12\pi\epsilon} + \frac{\pi\epsilon}{72} + \mathcal{O}(\epsilon^2) \end{pmatrix} , \tag{3.53}$$

which, with the prescriptions from above, has a uniform transcendental weight of zero.

Before we now solve the equation, it will be useful to first discuss the types of special functions that are often used in Feynman integral calculations. For the above problem, we will see that the solution can be very elegantly expressed in terms of harmonic polylogarithms, but we will also talk about hypergeometric functions and the Meijer-G function, since they are all related and used throughout this work.

3.6 Useful Classes of Functions

3.6.1 Hypergeometric Functions

When calculating Feynman integrals, one of the most basic tools used in their computation are **hypergeometric functions**. The first hypergeometric function one encounters in the calculation of master integrals, called ${}_2F_1(a, b; c; z)$, is defined as

$$\int_0^1 dt\ t^a\ (1-t)^b\ (1-ut)^c = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}\ {}_2F_1(-c, a+1; a+b+2; u) . \tag{3.54}$$

Integrating further leads to the ${}_3F_2$ function:

$$\int_0^1 dt\ t^a\ (1-t)^b\ {}_2F_1(a_1, a_2; b_1; ut) = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}\ {}_3F_2(a_1, a_2, a+1; b_1, a+b+2; u) , \tag{3.55}$$

or, in general:

$$\int_0^1 dt\ t^a\ (1-t)^b\ {}_{p-1}F_{q-1}(a_1, ..., a_{p-1}; b_1, ..., b_{q-1}; ut) \tag{3.56}$$

$$= \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}\ {}_pF_q(a_1, ..., a_{p-1}, a+1; b_1, ..., b_{q-1}, a+b+2; u) . \tag{3.57}$$

These functions are of great use, since their analytic properties are very well studied. Especially at low p and q, they allow for a wide variety of argument transformations and reshuffling of the indices, leading to large simplification of integral expressions. An extensive amount of helpful relations is collected in Ref. [87].
They can also be expanded around a small parameter, in our case ϵ, to get the explicit coefficients of the Laurent series in the end. For this, the package `HypExp` [88] can be used. As this expansion swaps two limits, one has to be careful, as long as there are still integrations to be done on the functions, as these can introduce additional poles in ϵ. If that is the case, a premature expansion leads to incorrect results.

3.6.2 Iterated Integrals and Harmonic Polylogarithms

One structure that we encounter regularly, especially when solving differential equations in the ϵ-form, are iterated integrals. In Eq. (3.51), we see that we have a very clear structural dependence on x, with either x or $1-x$ in the denominator. As we solve these equations iteratively order by order in ϵ, we can make use of a very powerful class of functions called **harmonic polylogarithms (HPLs)**.
They are defined in the following way [89]:

$$H_{a_1,\dots a_n}(z) = \int_0^z dt\; f_{a_1}(t)\; H_{a_2,\dots,a_n}(t)\,, \quad \text{(at least one } a_i \neq 0)\,, \tag{3.58}$$

and

$$H_{a_1,\dots a_n}(z) = \log^n(z)\,, \quad a_i = \dots = a_n = 0\,. \tag{3.59}$$

with the starting point of the iteration $H_{\{\}}(z) = 1$.
The $f_{a_1}(t)$ are **weight functions**, given by:

$$f_0(t) = \frac{1}{t}\,, \quad f_1(t) = \frac{1}{1-t}\,, \quad f_{-1}(t) = \frac{1}{1+t}\,. \tag{3.60}$$

Using the subscripts $a_i = \{0, \pm 1\}$ is called **A-notation** and it allows us to assign the notion of weight to an HPL: We say that an HPL with n indices (in A-notation) is of weight n.

From Eq. (3.58) we can read off the basic HPLs of weight one:

$$H_0 = \log(z)\,, \quad H_1 = -\log(1-z)\,, \quad H_{-1} = \log(1+z)\,. \tag{3.61}$$

At weight two, we see products of these, but also start to encounter Polylogarithms, such as $\mathrm{Li}_2(z)$, and constants of corresponding weight, e.g. π^2.

A shorthand notation that is useful for HPLs of higher weight is the **M-notation** (opposed to the A-notation from before). Here, the number of zeroes to the left of ± 1 is replaced by $\pm(m+1)$:

$$H_{0,0,0,1,0,0,-1} = H_{4,-3}\,. \tag{3.62}$$

Besides their behaviour concerning iterated integrals, the harmonic polylogarithms have a lot of useful properties, which are all implemented in the `Mathematica` package `HPL` [90], which is included in `HypExp` [88].
They fulfil a **Hopf algebra**, meaning that products of HPLs can be rewritten as sums thereof. Denoting indices with a vector, the algebra looks as follows:

$$H_{\vec{a}}(z)\, H_{\vec{b}}(z) = \sum_{\vec{c}\,\in\,\vec{a}\,\sqcup\!\sqcup\,\vec{b}} H_{\vec{c}}(z), \tag{3.63}$$

where $\vec{a}\sqcup\!\sqcup\vec{b}$ is the set of index vectors that contain all entries from $\vec{a}$ and $\vec{b}$ while preserving the internal ordering, respectively. Take as an example the integral

$$F_1(z) = \int_0^z dt\ f_1(t)\ \frac{1}{6}\log(t)^3(\mathrm{Li}_2(t)\log(t) - 2\mathrm{Li}_3(t))\,, \tag{3.64}$$

where $f_1(t)$ is one of the weight functions from Eq. (3.60).
We can now express them as HPLs and make use of the Hopf algebra:

$$\begin{aligned} F_1(z) &= \int_0^z dt\ f_1(t)\,(H_{0,0,0}(t)H_{0,1,0}(t)) \\ &= \int_0^z dt\ f_1(t)\,(H_{5,0}(t) + H_{4,0,0}(t) + H_{3,0,0,0}(t) + H_{2,0,0,0,0}(t)) \\ &= H_{1,5,0}(z) + H_{1,4,0,0}(z) + H_{1,3,0,0,0}(z) + H_{1,2,0,0,0,0}(z)\,. \end{aligned} \tag{3.65}$$

With the rewriting in terms of HPLs, we were able to solve the integral algorithmically in terms of well understood functions. These can then also (for the lower weights at least) be reexpressed in terms of polylogarithms, logarithms and constants again.

Another property that proved very useful in our calculation was the use of the Hopf algebra to extract the logarithmic dependence of a function. This is used, for example, in the study of the asymptotic behaviour in a singular point.
Consider that we have integrated an expression and arrived at the result

$$F_2(z) = H_{1,0,1,0}(z)\,, \tag{3.66}$$

and as a boundary condition for the integration we want to compare the behaviour in $z \to 0$ with a given function. These logarithmic divergences in zero are indicated by zeroes at the end of the index array ('trailing zeroes'). The Hopf algebra (implemented in `HypExp`) gives us a prescription to extract these explicitly in the form of logarithms:

$$F_2(z) = H_0(z)H_1(z)H_2(z) - 2H_1(z)H_3(z) - 2H_0(z)H_{2,1}(z) + H_{2,2}(z) + 4H_{3,1}(z)\,, \tag{3.67}$$

where we can read of the coefficients of $H_0(z) = \log(z)$.

Explicit Solution of an Integral through Iteration

Let us now again consider the two integrals from Sect. 3.5.2. With the differential equation in Eq. (3.51) and the boundaries in Eq. (3.53), we have all the ingredients to solve the system.
We start at the lowest order, which in this case is ϵ^{-1}. As is the nature of the differential equation in ϵ-form, at the starting order, we only have to determine the constant by reading it off from the boundary condiction, because the integrand from

the previous order is zero. The result from this is

$$\begin{pmatrix} g_1^{(-1)} \\ g_2^{(-1)} \end{pmatrix} = 2\pi i S_\Gamma \begin{pmatrix} \frac{1}{2\pi} \\ \frac{1}{12\pi} \end{pmatrix} . \tag{3.68}$$

For the next order, we now plug this result as $\vec{g}$ into the right side of Eq. (3.51) and use the harmonic polylogarithms to express the iterated integrals:

$$\begin{pmatrix} g_1^{(0)} \\ g_2^{(0)} \end{pmatrix} = 2\pi i S_\Gamma \begin{pmatrix} c_1 \\ \frac{1}{6\pi} H_1(x) + c_2 \end{pmatrix} . \tag{3.69}$$

We can now take the limit $x \to 0$ and by comparing with the boundary condition in the same limit, we conclude

$$\begin{pmatrix} g_1^{(0)} \\ g_2^{(0)} \end{pmatrix} = 2\pi i S_\Gamma \begin{pmatrix} 0 \\ \frac{1}{6\pi} H_1(x) \end{pmatrix} . \tag{3.70}$$

The same procedure is done for the next order, leading to the result:

$$\begin{pmatrix} g_1^{(1)} \\ g_2^{(1)} \end{pmatrix} = 2\pi i S_\Gamma \begin{pmatrix} \frac{\pi}{12} \\ \frac{1}{6\pi}(H_{0,1}(x) + 2H_{1,1}(x)) + \frac{\pi}{72} \end{pmatrix} . \tag{3.71}$$

Putting together the previous findings, we get the result in the UT-basis:

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = 2\pi i S_\Gamma \begin{pmatrix} \frac{1}{2\pi\epsilon} + \frac{\pi\epsilon}{12} \\ \frac{1}{12\pi\epsilon} + \frac{1}{6\pi} H_1(x) + \epsilon \left(\frac{1}{6\pi}(H_{0,1}(x) + 2H_{1,1}(x)) + \frac{\pi}{72} \right) \end{pmatrix} . \tag{3.72}$$

This can now be turned back to basic functions and, using the transformation matrix from Eq. (3.50), returned to the original basis of $\mathbf{I}[1,0]$ and $\mathbf{I}[1,1]$ to compare the results. Doing this correctly yields the exact result from Eq. (3.44) again, but arguably in a much more compact form.

The only limiting factor for the iteration here is the highest known order of the boundary condition, because, as we saw in the last section, the integration and the limit $x \to 0$ can be carried out algorithmically to (in principle) arbitrary weight.

3.6.2.1 Generalized Polylogarithms

As of their primary definition, harmonic polylogarithms are only defined for the letters $\{0, \pm 1\}$. For sets of master integrals that contain a high number of (massive) lines, it is often not possible to find a basis that avoids letters differing from these basic three. To include these cases, the definition of HPLs can be expanded, resulting in so-called **Generalized Polylogarithms** [91]. Their basic definition looks very similar to that in Eq. (3.58):

$$G_{a_1,\dots a_n}(z) = \int_0^z dt \, \frac{1}{t - a_1} \, G_{a_2,\dots,a_n}(t) \,. \tag{3.73}$$

In contrast to the definition of HPLs though, a_1 is not restricted to $\{0, \pm 1\}$ any more.

The generalization of the definition now lets us include a larger class of iterated integrals that we can now also apply the Hopf algebra to, e.g. extracting logarithmic dependencies or simplifiying the product of two functions, as for the HPLs.

An implementation of this can e.g. be found in the package `PolyLogTools` [92].

3.6.3 The MeijerG Function

Another tool that proved very helpful in the computation of loop integrals is the **MeijerG** function [93], which was constructed to include most special functions as particular limits. As we will see in this section, it is (by design) very closely related to hypergeometric functions as well as Mellin-Barnes integrals, making it a very potent link between the methods.

One of the standard ways to define the functions is:

$$G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} \{a_1, ..., a_n\} \, , \, \{a_{n+1}, ..., a_p\} \\ \{b_1, ..., b_m\} \, , \, \{b_{m+1}, ..., b_q\} \end{matrix} \right) = \int_L \frac{ds}{2\pi i} \frac{\prod\limits_{j=1}^{m} \Gamma(b_j - s) \prod\limits_{k=1}^{n} \Gamma(1 - a_k + s)}{\prod\limits_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod\limits_{k=n+1}^{p} \Gamma(a_k - s)} z^s \,. \tag{3.74}$$

After transforming an integral to its Mellin-Barnes representation, we can then convert the corresponding integrals over Γ-functions into a MeijerG function.
As mentioned before, we also can relate hypergeometric functions to MeijerG functions, for example via the following formula:

$$
{}_pF_{p-1}(a_1, ..., a_p; b_1, ..., b_{p-1}; -z) = \\
\frac{\Gamma(b_1)...\Gamma(b_{p-1})}{\Gamma(a_1)...\Gamma(a_p)} G^{1,p}_{p,p}\left(z \,\middle|\, \begin{array}{c} \{1-a_1, ..., 1-a_p\}\,,\,\{\} \\ \{0\}\,,\,\{1-b_1, ..., 1-b_{p+1}\} \end{array} \right) . \tag{3.75}
$$

These types of conversions prove very helpful when encountering complicated arguments in the ${}_pF_q$ that do not immediately allow for a transformation (e.g. because the transformation only allows positive z). Turning the expression into a MeijerG with the help of Eq. (3.75) and then via Eq. (3.74) into a Mellin-Barnes integral can then enable further analytic progress.
For a collection of the analytic properties and transformations, we refer to Ref. [87].

Chapter 4

Calculation of the Four-Body Virtual Contributions

As we discussed before in Sect. 2.3, the main goal of this work is the calculation of **multi-parton corrections** to the branching fraction $\mathcal{B}(\bar{B} \to X_s\gamma)_{E_\gamma > E_0}$. To be more precise, our focus is on the pieces that are missing at next-to-leading order in QCD, complementing the four-body contributions that were calculated previously [10, 62].

The actual calculation of this involves multiple steps: In this chapter, we calculate the **one-loop four-body contributions** to the process. To cancel out infrared divergences occurring in these diagrams, we then proceed in computing the corresponding **five-body contributions** in Chapter 5, where the gluon is emitted as an on-shell parton. As we have seen in Fig. 2.6, these diagrams are very similar, as most of them can be obtained by appropriately shifting the cut. Adding these up eliminates the infrared divergences associated with the gluon, leaving only those from collinear photons and ultraviolet (UV) ones.
To get rid of the UV divergences, we **renormalize** the expressions by adding the appropriate counterterms. This procedure will be discussed in Chapter 6.
Finally, Chapter 7 will then address the remaining infinite pieces, i.e. the collinear divergences associated with the photon. We will explain the change of scheme we employ to trade the explicit poles in ϵ for logarithms of the light quark masses,

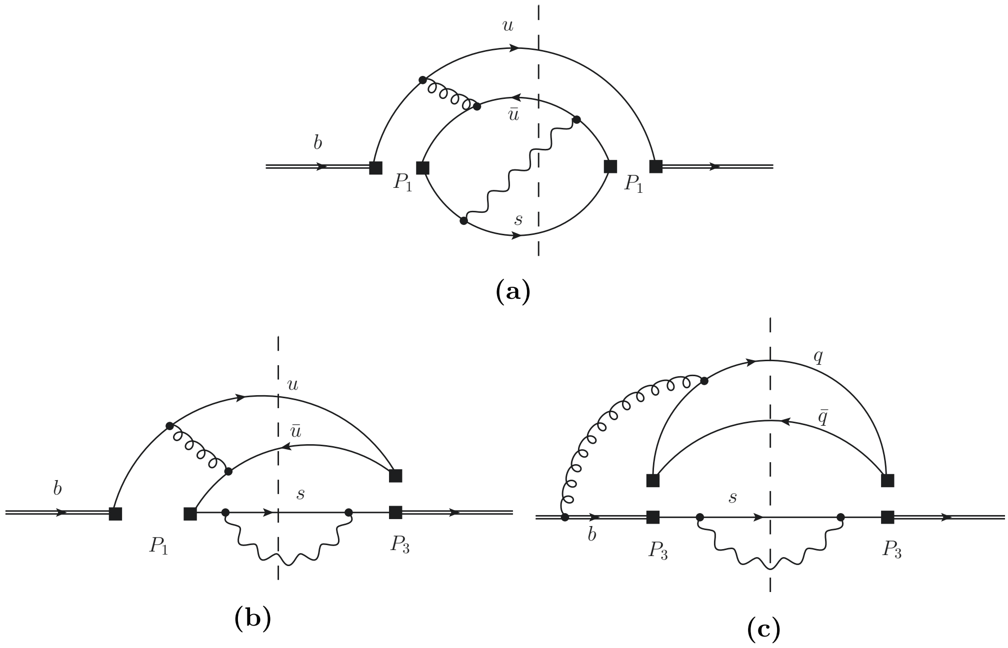


Figure 4.1: Examples for the different insertions that are encountered in the calculation. We see that the cases **a)** and **b)** only allow the additional quark-antiquark pair to be $u\bar{u}$, while the insertion of a penguin (cf. **c)**) on both sides makes it possible to have $q = u, d, s$ in the final state.

so-called collinear logarithms, via **DGLAP splitting functions**.

With our procedure outlined in this way, we can now start with the first part of our calculation, the four-body one-loop bare diagrams.

4.1 Operator Insertions

The operators that are relevant for our computation have already been listed in Eq. (2.16). We have to compute every possible insertion of these on the left- and on the right-hand side of the cut. We will denote these combinations by $(P_i \times P_j)$. What we can read off from the operator basis is that each of the operators P_2^u, P_3 and P_5 is related to one of the others (P_1^u, P_4 and P_6, respectively) by a change of occurring color matrices. This makes the calculation a lot simpler as it means that

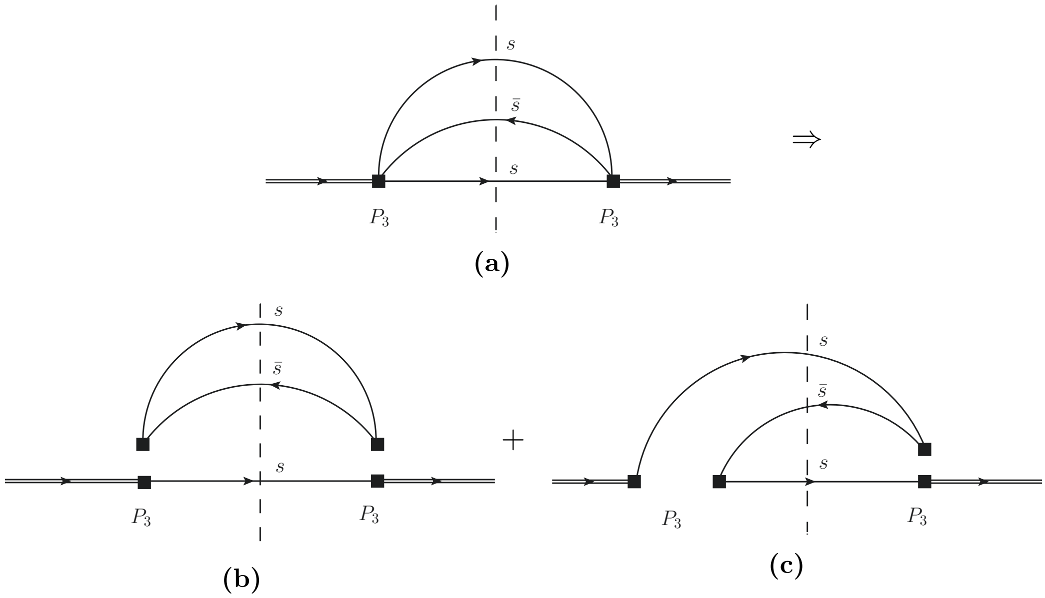


Figure 4.2: In the case of all final state quarks being s quarks, the operator insertions **a)** allow for two cases: **b)** The 'straight insertion', parallel to the case of $q = u, d$ and **c)** the 'crossed' insertions, where there is only one fermion line.

these insertions only differ by color factors, which we can add after the calculation of the contributions.
For example, we compute the insertion ($P_3 \times P_3$) without carrying out color traces. In the final result we can then supplement the correct color factors for P_4 and thus get the results ($P_3 \times P_4$), ($P_4 \times P_3$) and ($P_4 \times P_4$) without doing the Dirac algebra and the loop calculations again. These color factors are given in Sect. 5.8.

Our list of operators can be divided into two subparts: The current-current operators P_1^u and P_2^u, denoted by (C), and the four penguin operators P_3-P_6, which we will denote by (P). From this we can see that, diagrammatically, we encounter three types of insertions: ($C \times C$), ($P \times P$) and ($C \times P$). Examples for these three types of diagrams are shown in Fig. 4.1.
One subtlety arises when considering the case of penguin operators with $q = s$. In that case, it is possible to form more than one split of the fermion lines leading to two different contributions, called the 'straight' and the 'crossed' insertion. These two different types can be seen in Fig. 4.2 and have to be computed separately.

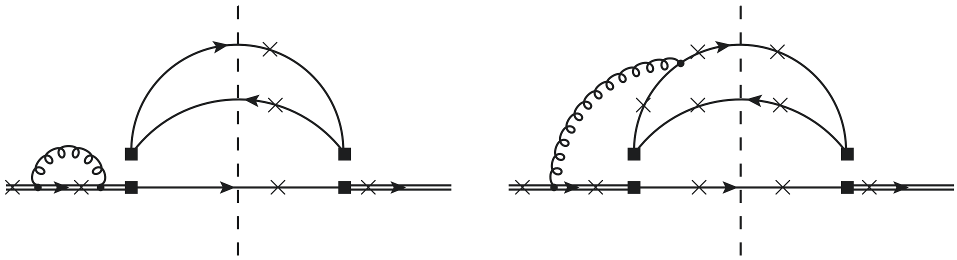


Figure 4.3: This figure shows the different ways that the gluon and photon can attach to the quark legs. A cross denotes a possible spot for the photon to attach, it always has to choose one on the left and one on the right of the cut. When counting the number of diagrams, we can distinguish between two different groups of gluon behaviour:
Left: The gluon attaches to the same leg it originated from, giving the photon two spots it can attach to the left of the cut. Note that if the photon attaches elsewhere, the diagram would be counted towards the wave function renormalization procedure.
Right: The gluon changes the quark line. There are six of these configurations with six possibilities for the photon to attach, respectively.

4.2 Gluons and Photons

For a given insertion of operators, we now have to accommodate for the photon and gluon, accounting for all possible attachments of the two. In our case, there are two different topologies for the gluon, as we can see in Fig. 4.3. As there are four of the first one and six of the second one, and counting all the possibilities for the photon, we arrive at $4 \times 2 \times 4 + 6 \times 6 \times 4 = 176$ diagrams for each operator insertion.

4.3 Generation and Evaluation of the Diagrams

Now that it is clear which diagrams are neccessary for the calculation, all needed parts are generated with `QGRAF` [94]. What is created are the left and right side of the cut independently, which are fused in the next step, when output of `QGRAF` is fed into `FORM` [95]. Here, the amplitudes are combined by carrying out spin sums, the symbolic vertices and propagators are replaced by their Feynman rules and the resulting fermion traces are evaluated.
In general, this step is very straightforward, but in this calculation one major subtlety was encountered: The operators of Eq. (2.16) contain the matrix γ_5. The problems that arise from this and our solution will be topic of the next section.

4.3.1 The treatment of γ_5

The Dirac matrix γ_5 is introduced in weak processes as a part of the axial current $\gamma_\mu\gamma_5$. It is defined in the following way:

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3\,. \tag{4.1}$$

From this relation, one can infer the anticommutation property

$$\{\gamma_5, \gamma_\mu\} = 0\,, \tag{4.2}$$

and that the matrix is its own inverse:

$$\gamma_5^2 = 1\,. \tag{4.3}$$

We also can define the behaviour of traces containing γ_5:

$$\mathrm{Tr}(\gamma_5\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}) = i\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\mathrm{Tr}(1)\,. \tag{4.4}$$

It gets clear from the definition in Eq. (4.1) that γ_5 is an inherently four dimensional object. This leads to problems when calculating quantities in dimensional regularization that involve traces over γ_5, as there is no unique way to analytically continue the above properties from four to D dimensions. For example, one can show that retaining the cyclicity of the trace and anticommutation of γ_5 leads to different results for different starting points of the trace. A very comprehensive analysis of the ambiguities that can arise in calculations with weak axial currents is given in Ref. [96].

In order to treat the occurring expressions containing γ_5 consistently in D dimensions, we need a scheme that deals with these emerging ambiguities. Many schemes have been proposed over the years, such as the 't Hooft-Veltmann scheme [97] that splits γ_5 into a four dimensional and an ϵ dimensional part, eliminating the ambiguity in the anticommutation relation but introducing a lot of new terms and bookkeeping. Other common schemes are the Larin scheme [98] and naive dimensional regularisation (NDR) [99].

For our calculation, we adapted the so called KKS scheme, which was developed by Kreimer, Körner and Schilcher [100, 101] .
It is a 'reading-point scheme' and can be broken down to the following rules that one has to implement for traces containing γ_5:

1. Loss of cyclicity of traces. From this follows the need for a consistent reading point prescription.
2. Start the traces at the axial vertex $\gamma_\mu\gamma_5$
3. If there is more than one axial vertex, average over all possible starting points ('Bosonization'):
$$tr(\gamma_{\mu_1}\gamma_{\mu_2}\gamma_5\gamma_{\mu_3}\gamma_{\mu_4}\gamma_5) = \frac{1}{2}\left(tr(\gamma_{\mu_2}\gamma_5\gamma_{\mu_3}\gamma_{\mu_4}\gamma_5\gamma_{\mu_1}) + tr(\gamma_{\mu_4}\gamma_5\gamma_{\mu_1}\gamma_{\mu_2}\gamma_5\gamma_{\mu_3})\right)$$
4. Anticommute all occurring γ_5 to the end of the trace
5. Use $\gamma_5^2 = 1$ until a maximum of one γ_5 remains
6. If a single γ_5 remains, replace it by $\frac{i}{4!}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$

As Fig. 4.1 shows, we encounter two different types of trace topologies. In the case of one trace, we are either left with a trace free of γ_5, which can be treated normally, or with only one remaining γ_5. The treatment of the latter expressions, which result in a single ε-tensor, will be topic of the next section.
For the case of two traces, there is an additional case that we have to consider: If two traces that are multiplied contain a single γ_5, we end up with a product of two Levi-Civita tensors after step 6 of our procedure. This product has to be treated consistently and we chose the following procedure, linking their product to metric tensors:

$$\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\varepsilon^{\nu_1\nu_2\nu_3\nu_4} = \det\begin{pmatrix} g^{\mu_1\nu_1} & ... & g^{\mu_1\nu_4} \\ ... & & ... \\ g^{\mu_4\nu_1} & ... & g^{\mu_4\nu_4} \end{pmatrix}. \tag{4.5}$$

This prescription respects the symmetric properties and restores the correct dependence on the dimension. The resulting metric tensors are then handled as D-dimensional objects in the usual way, which is discussed, for example, in Ref. [102].

Applying this scheme allowed us to replicate the tree-level results from Ref. [10].
After going through the above procedure, we are now only left with either traces without γ_5 or with expressions containing a single $\varepsilon^{\mu_1\mu_2\mu_3\mu_4}$.

4.3.2 Cancellation of terms containing $\varepsilon^{\mu_1\mu_2\mu_3\mu_4}$

One can show that the pieces proportional to the antisymmetric tensor cancel out in the end by the following argument, similar to the one made in Ref. [62]:
Introducing the fully antisymmetric tensor as we did above leads to terms in the calculation that contain pieces such as $\varepsilon_{\mu_1\mu_2\mu_3\mu_4}p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_4^{\mu_4}$ in the final result. As it is not straightforward how to correctly interpret these terms, it is fortunate that they cancel out when fully carrying out the angular integrations occurring in the phase space.
Let us go to the rest frame of the b quark and fix all momentum invariants $p_i \cdot p_j$. We can derive from momentum conservation that all the energies $p_b \cdot p_i$ are fixed. From this, we can infer that the $\vec{p}_i \cdot \vec{p}_j$ are fixed, still leaving us with free choice of the coordinate system. If we define a plane with $\vec{p}_1$ and $\vec{p}_2$, this fixes $\vec{p}_3$ up to its orientation relative to the plane. Choosing this sign then also fixes $\vec{p}_4$.
With this argument we can see that the expression $\varepsilon_{\mu_1\mu_2\mu_3\mu_4}p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_4^{\mu_4}$ is fixed by the invariants $p_i \cdot p_j$ up to the sign we chose. Now, when carrying out the phase space integration, we encounter terms that have the form $F(p_i \cdot p_j)\ \varepsilon_{\mu_1\mu_2\mu_3\mu_4}p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_4^{\mu_4}$. Since the function $F(p_i \cdot p_j)$ is parity-even, i.e. does not depend on the orientation of the momenta, when encountering it alone, we can trivially carry out the angular integrations. But, as we have shown above, this does not hold true for the second part, $\varepsilon^{\mu_1\mu_2\mu_3\mu_4}$, as it changes the sign under change of orientation of e.g. $\vec{p}_3$.
What is integrated then is the product of a parity-even and a parity-odd expressiong (yielding a parity-odd one) that vanishes after integrating over the symmetric angular variables. Logically, this should also be the case, as our final observable is parity-even and there should be no parity-odd contributions.
The logic of the above statements also holds true when imposing cut on the phase space via a lower bound on the photon energy. We will see in Eqs. (4.13) and (4.14) that we can relate the cut parameter to momentum invariants, which then simply contribute to $F(p_i \cdot p_j)$.

4.4 Phase Space Parametrization with Cut on the Photon Energy

In Sect. 3.3, we already listed dPS_4, the measure for the four-particle phase space. With the help of this, the unpolarized decay rate for a decaying b quark is given by

$$\Gamma = \frac{1}{2m_b}\frac{1}{2N_c}\int dPS_4 \sum_{\substack{\text{spin}\\ \text{color}}} |\mathcal{M}|^2 \,, \tag{4.6}$$

where we average over incoming and sum over outgoing spin and color configurations.

As we argued in the last section, the $\sum|\mathcal{M}|^2 = \mathcal{K}(s_{ij})$ only depends on the dimensionless momentum invariants, which we define as

$$s_{ij} = \frac{2k_i \cdot k_j}{m_b^2}\,. \tag{4.7}$$

From here on, introducing shorthand notation, we will also use

$$s_{ijk} = s_{ij} + s_{ik} + s_{jk} \qquad s_{ijkl} = s_{ij} + s_{ik} + s_{jk} + s_{il} + s_{jl} + s_{kl}\,. \tag{4.8}$$

As the kernels that are integrated over do not depend on the angular configurations, we can carry out the respective integrations separately:

$$\int d\Omega_{D-1} d\Omega_{D-2} d\Omega_{D-3} = \frac{8\pi^{\frac{3D-6}{2}}}{\Gamma(\frac{D-1}{2})\Gamma(\frac{D-2}{2})\Gamma(\frac{D-3}{2})}\,. \tag{4.9}$$

Plugging this into the decay rate and also using the explicit form of the phase space measure, we arrive at

$$\Gamma = N(D)\int [s_{ij}]\ \delta(1-\sum s_{ij})\ \mathcal{K}(s_{ij})\ (-\Delta_4)^{\frac{D-5}{2}}\Theta(-\Delta_4)\,, \tag{4.10}$$

with the prefactor defined as

$$N(D) = \frac{\tilde{\mu}^{6\epsilon} 2^{8-5D} \pi^{1-3D/2} m_b^{3D-9}}{4N_c\Gamma(\frac{D-1}{2})\Gamma(\frac{D-2}{2})\Gamma(\frac{D-3}{2})}\,. \tag{4.11}$$

4.4.1 Introducing the Cut on E_γ

In our calculation we are interested in the kinematics of the four-body decay

$$b(p_b) \to q(p_1)\bar{q}(p_2)s(p_3)\gamma(p_4)\,.$$

We additionally impose a restriction on the energy of the photon that we define in the restframe of the b quark (following e.g. Ref. [62]) as

$$E_\gamma = \frac{p_b \cdot p_4}{m_b} = \frac{m_b}{2}(s_{14} + s_{24} + s_{34}) > \frac{m_b}{2}(1-\delta)\,. \tag{4.12}$$

leading to:

$$(s_{14} + s_{24} + s_{34}) > (1-\delta)\,. \tag{4.13}$$

We now have to incorporate this cut on the phase space into our integrations. This is done by introducing an additional integration and a delta function, a method that was for example used in Ref. [62]. We define the additional cut variable z and multiply the integrand with the function $\delta(1 - z - s_{14} - s_{24} - s_{34})$. Now we can integrate over this newly introduced variable from 0 to δ, which enforces the restrictions on the phase space:

$$\int_0^\delta dz \int_0^1 [ds_{ij}]\, \delta(1-z-s_{14}-s_{24}-s_{34})\delta(z-s_{12}-s_{13}-s_{23})\mathcal{K}(s_{ij})\, (-\Delta_4)^{\frac{D-5}{2}}\Theta(-\Delta_4)\,, \tag{4.14}$$

where the first delta function fixes the cut and the other one originates from momentum conservation. We can now use these to fix two of the invariants, arriving at five integrations left to perform:

$$\Gamma_{E_\gamma > E_0} = N(D) \int_0^\delta dz \int_0^{\bar{z}} ds_{34} \int_0^{\bar{z}-s_{34}} ds_{14} \int_0^z ds_{12} \int_0^{z-s_{12}} ds_{23} \times \mathcal{K}(s_{ij})\, (-\Delta_4)^{\frac{D-5}{2}}\Theta(-\Delta_4)\Bigg|_{\substack{s_{13}=z-s_{12}-s_{23}\\ s_{24}=\bar{z}-s_{14}-s_{34}}}\,, \tag{4.15}$$

where we again used the 'bar'-notation $\bar{z} = 1 - z$.

We use a suitable parametrization that factorizes the Gram determinant Δ_4 as well as mapping the integrations to the unit hypercube. We adapt here the one introduced in Ref. [62], where a detailed derivation can be found.

The parametrization is as follows:

$$\begin{aligned} s_{12} &= zvw & s_{34} &= \bar{z}\bar{v} \\ s_{14} &= \bar{z}vx & s_{23} &= (a^+ - a^-)u + a^- \\ s_{13} &= z - s_{12} - s_{23} & s_{24} &= \bar{z} - s_{14} - s_{34}\,, \end{aligned} \tag{4.16}$$

with

$$a^\pm = z[\bar{v}wx + \bar{w}\bar{x} \pm 2(\bar{v}w\bar{w}x\bar{x})^{1/2}]\,. \tag{4.17}$$

With this the determinant now factorizes:

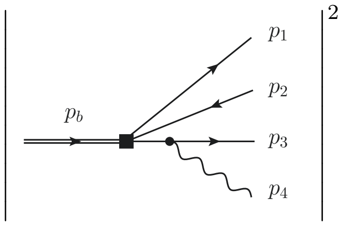
$$\Gamma_{E_\gamma > E_0} = N(D)\; 4^{D-4} \int_0^\delta dz\,(z\bar{z})^{D-3} \int_0^1 du\,dv\,dx\,dw\,(u\bar{u})^{\frac{D-5}{2}} v^{D-3} (\bar{v}x\bar{x}w\bar{w})^{\frac{D-4}{2}}\,\mathcal{K}\,. \tag{4.18}$$

These integrals now have to be carried out and, for integration kernels $\mathcal{K}(u, v, x, w, z)$ that are simple enough, this can be done (semi-)automatically in terms of hypergeometric functions.
Before we solve our integrals with this parametrization, we first reduce the number of integrals significantly by carrying out an integration-by-parts (IBP) reduction as our next step, the basics of which were discussed in Sect. 3.2.2.

4.4.2 Implementing the Cut on E_γ into the IBP Reduction

One minor point when employing IBP reductions is that the method is oblivious to phase space restrictions such as our cut on the photon energy. To solve this, Ref. [103] proposed the method of **reversed unitarity**, trading delta functions for propagators.
To illustrate the method, we look at a contribution occurring in the tree level case

of our calculation:

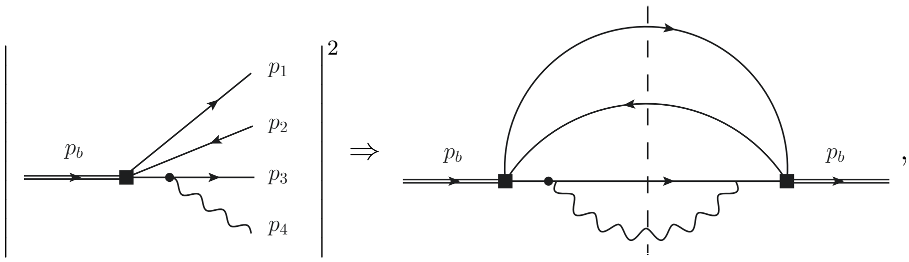


$$\sim \int \frac{d^D p_1\, d^D p_2\, d^D p_3\, \delta(p_1^2)\delta(p_2^2)\delta(p_3^2)\delta(p_4^2)\delta(z - s_{123})}{[(p_3+p_4)^2]^2}\,, \tag{4.19}$$

where the first four delta functions are enforcing the on-shell conditions for the final-state particles and the last delta functions implements the cut condition.
We now can use Cutkosky rules [104] to replace these by propagators:

$$2\pi i\ \delta(p^2 - m^2) \to \frac{1}{p^2 - m^2 + i0} - \frac{1}{p^2 - m^2 - i0}\,. \tag{4.20}$$

In diagrammatic form this can be depicted as



(4.21)

showing the principle of trading the square of a Born amplitude for a multiloop forward scattering diagram. A cut propagator in the diagram means that the Cutkosky rules have been used.
These replacements make the resulting expressions eligible for the processing by an integration-by-parts routine, letting us reduce the number of integrals by multiple orders of magnitude. After the reduction process, the relation from Eq. (4.20) is used backwards, turning the multiloop-integrals back into their original form.
Note that the IBP reduction yields the same result, irrespective of the sign of the imaginary part $\pm i0$ in Eq. (4.20) (if one reverts the expressions back after the reduction). To avoid unnecessary bookkeeping, we can thus use the relation

$$2\pi i\ \delta(p^2 - m^2) \to \frac{1}{p^2 - m^2}\,. \tag{4.22}$$

4.5 Specifics of the IBP Reduction

After applying the reverse unitarity relations, we can group the diagrams into three families that we call $\mathcal{F}_1 - \mathcal{F}_3$:

$$\mathcal{F}_1 = \{p_1^2,\, p_2^2,\, p_3^2\,, (p_b - p_{123})^2,\, p_{123}^2 - z\, m_b^2,\, k^2,\, (k + p_b - p_{13})^2,\, (k + p_2)^2,\\ (k - p_3)^2,\, (k - p_1)^2,\, (p_b - p_{13})^2,\, (p_b - p_{23})^2,\, (p_1 + p_3)^2,\, (p_2 + p_3)^2\}\,, \quad (4.23)$$

$$\mathcal{F}_2 = \{p_1^2,\, p_2^2,\, p_3^2\,, (p_b - p_{123})^2,\, p_{123}^2 - z\, m_b^2,\, k^2,\, (k - p_b + p_{12})^2,\, (k - p_2)^2,\\ (k - p_b)^2 - m_b^2,\, (k - p_3)^2,\, (p_b - p_{23})^2,\, (p_b - p_{12})^2,\, (p_1 + p_3)^2,\, (p_2 + p_3)^2\}\,, \quad (4.24)$$

$$\mathcal{F}_3 = \{p_1^2,\, p_2^2,\, p_3^2\,, (p_b - p_{123})^2,\, p_{123}^2 - z\, m_b^2,\, k^2,\, (k - p_{123})^2 - m_b^2,\, (k - p_2)^2,\\ (k - p_b)^2 - m_b^2,\, (k - p_3)^2,\, (p_b - p_{12})^2,\, (p_b - p_{13})^2,\, (p_1 + p_3)^2,\, (p_2 + p_3)^2\}\,. \quad (4.25)$$

To fit all the occurring integrals into these three families, we can use symmetries in the momenta to our advantage:
If there were no restrictions on the phase space, we could interchange p_1, p_2, p_3 and p_4 at will, but since we cut the energy of the photon, we have to be careful with the corresponding momentum p_4. Note that we have used the replacement rule $p_4 = p_b - p_{123}$ in the above families to avoid hidden linear dependencies.
The first five propagators are the same in each family and originate from reverse unitarity relations, which means that their occurrence is mandatory. In these, we see a **symmetry in the first three momenta** p_1, p_2 and p_3.
We can interchange these three while keeping the result unchanged, making it possible to fit the originally $\mathcal{O}(10000)$ integrals into the three families with $\mathcal{O}(2000)$ integrals each.
We have labelled the families such that, with increasing family number, the number of massive propagators also increases. In the first family, we have collected all diagrams that contain no massive propagators, while in the second one there can be a

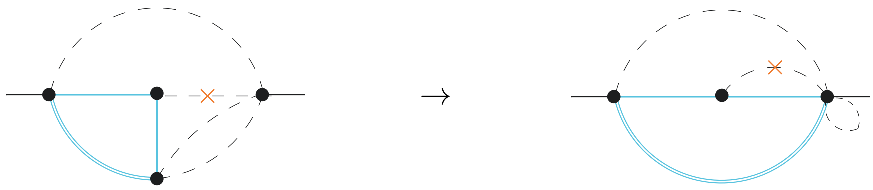


Figure 4.4: Left: Diagrammatic representation of a typical phase space integral occurring during the IBP reduction with the δ-functions turned into propagators.
Right: One of the propagators from reversed unitarity gets pinched in the reduction process, creating a massless tadpole. The integral thus becomes zero. Details on the diagrammatical representation of the integrals are found in Sect. 4.6.

maximum of one. In the third one, the maximum of massive propagators appears, which is two for the four-body one-loop part of the calculation.

4.5.1 Vanishing Integrals

As we stated above, the first five entries in each family are the cut propagators, which are the results of applying reverse unitarity. If, during the reduction process, the power of one of these becomes non-positive, we set the integral to zero (we adopt here the standard notation introduced in Sect. 3.2.2, i.e. denominators having positive and numerators having negative powers).
The reasoning behind this approach can be illustrated in multiple ways: First, let us assume that the propagator power of p_1^2 is lowered from one to zero by the term being multiplied by p_1^2:

$$\frac{1}{[p_1^2]^0} = \frac{p_1^2}{p_1^2} = \delta(p_1^2)\, p_1^2 \to 0\,. \tag{4.26}$$

When reverting the propagator back to a delta function, we see that, after integration, the expression vanishes.
The vanishing of the integral can also be made clear by a diagrammatic approach. Looking at Fig. 4.4, we see that "pinching" one of the cut propagators (i.e. lowering its power, thus eliminating it from the diagram) creates a massless tadpole in the diagram. As massless tadpoles create scaleless integrals, which are set to zero in dimensional regularization, we find in this another way to illustrate the validity of the approach.

4.6 The Four-Body Master Integrals

The reductions are carried out in `FIRE` [105] on the `OMNI` computing cluster [106]. Using the `C++` routine of `FIRE` (in comparison to its `Mathematica` interface) speeds the reduction up by a lot. The massless family only takes a few minutes while the other two can take up to a few days.

The three reductions yield 30 master integrals in total. All of them are given diagramatically in Fig. 4.5.

In this depiction, we use the **light blue solid lines** to denote denominators. If these denominators are massive, we use **double-lines** and if a denominator is squared, this is shown by an **additional dot** on it.

The **dashed black lines** indicate the propagators from reversed unitary that originate in the delta functions, the **orange cross** denoting the momentum of the photon that is subject to the energy cut condition. Numerators are included in the following way: If a **dotted red line** connects two propagators with momenta l_1 and l_2 flowing through them, respectively, the integral includes a numerator $(l_1 - l_2)^2 - m_1^2 - m_2^2$. We additionally list them here, in the notation of Eq. (3.16), as this makes a future comparison more convenient:

$$
\begin{aligned}
F_{4\mathrm{B}1} &= F_1[1,1,1,1,1;0,0,1,1,0,0,0,0,0]\,,\\
F_{4\mathrm{B}2} &= F_1[1,1,1,1,1;0,1,0,1,0,0,0,0,0]\,,\\
F_{4\mathrm{B}3} &= F_1[1,1,1,1,1;0,1,0,2,0,0,0,0,0]\,,\\
F_{4\mathrm{B}4} &= F_1[1,1,1,1,1;1,1,0,0,0,0,0,0,0]\,,\\
F_{4\mathrm{B}5} &= F_1[1,1,1,1,1;1,0,1,0,0,0,0,0,0]\,,\\
F_{4\mathrm{B}6} &= F_1[1,1,1,1,1;1,0,2,0,0,0,0,0,0]\,,\\
F_{4\mathrm{B}7} &= F_1[1,1,1,1,1;1,1,1,0,1,0,1,0,0]\,,
\end{aligned}
$$

$$
\begin{aligned}
F_{4\mathrm{B}8} &= F_2[1,1,1,1,1;0,0,0,1,0,0,0,0,0]\,,\\
F_{4\mathrm{B}9} &= F_2[1,1,1,1,1;0,0,0,1,1,0,0,0,0]\,,\\
F_{4\mathrm{B}10} &= F_2[1,1,1,1,1;0,0,0,1,2,0,0,0,0]\,,
\end{aligned}
$$

$$F_{4\mathrm{B}11} = F_2[1,1,1,1,1;0,0,0,2,1,0,0,0,0]\,,$$
$$F_{4\mathrm{B}12} = F_2[1,1,1,1,1;0,1,0,1,0,0,0,0,0]\,,$$
$$F_{4\mathrm{B}13} = F_2[1,1,1,1,1;0,1,0,2,0,0,0,0,0]\,,$$
$$F_{4\mathrm{B}14} = F_2[1,1,1,1,1;0,0,0,1,1,1,0,0,0]\,,$$
$$F_{4\mathrm{B}15} = F_2[1,1,1,1,1;0,1,0,1,0,1,0,0,0]\,,$$
$$F_{4\mathrm{B}16} = F_2[1,1,1,1,1;0,1,0,1,1,0,0,0,0]\,,$$
$$F_{4\mathrm{B}17} = F_2[1,1,1,1,1;1,1,0,1,0,0,0,0,0]\,,$$
$$F_{4\mathrm{B}18} = F_2[1,1,1,1,1;1,1,0,1,0,1,0,0,0]\,,$$
$$F_{4\mathrm{B}19} = F_2[1,1,1,1,1;1,1,0,1,1,0,0,0,0]\,,$$
$$F_{4\mathrm{B}20} = F_2[1,1,1,1,1;1,1,0,2,0,1,0,0,0]\,,$$
$$F_{4\mathrm{B}21} = F_2[1,1,1,1,1;1,1,0,1,1,1,0,0,0]\,,$$
$$F_{4\mathrm{B}22} = F_2[1,1,1,1,1;1,1,0,1,-1,0,0,0,0]\,,$$

$$F_{4\mathrm{B}23} = F_3[1,1,1,1,1;1,1,0,0,0,0,0,0,0]\,,$$
$$F_{4\mathrm{B}24} = F_3[1,1,1,1,1;0,1,0,0,1,-1,0,0,0]\,,$$
$$F_{4\mathrm{B}25} = F_3[1,1,1,1,1;0,1,0,1,1,0,0,0,0]\,,$$
$$F_{4\mathrm{B}26} = F_3[1,1,1,1,1;0,1,0,1,2,0,0,0,0]\,,$$
$$F_{4\mathrm{B}27} = F_3[1,1,1,1,1;1,1,0,1,0,0,0,0,0]\,,$$
$$F_{4\mathrm{B}28} = F_3[1,1,1,1,1;1,1,0,0,1,0,1,0,0]\,,$$
$$F_{4\mathrm{B}29} = F_3[1,1,1,1,1;1,1,0,1,1,0,1,0,0]\,,$$
$$F_{4\mathrm{B}30} = F_3[1,1,1,1,1;1,1,0,1,1,1,0,0,0]\,.$$

Let us make some remarks regarding the list of master integrals: First, note that there is not a complete congruence of the integrals listed here and the ones that our reduction in `FIRE` yielded as master integrals. If possible, the impementations of the Laporta algorthm avoid numerators and reduce their input to a list of denominator-only master integrals. Here, some of the original ones had to be replaced to make them suitable for the next step, i.e. the differential equations method. The search for new integrals was done by hand, for further reading on the topic of choosing the right integrals basis, cf. Refs. [107, 108].

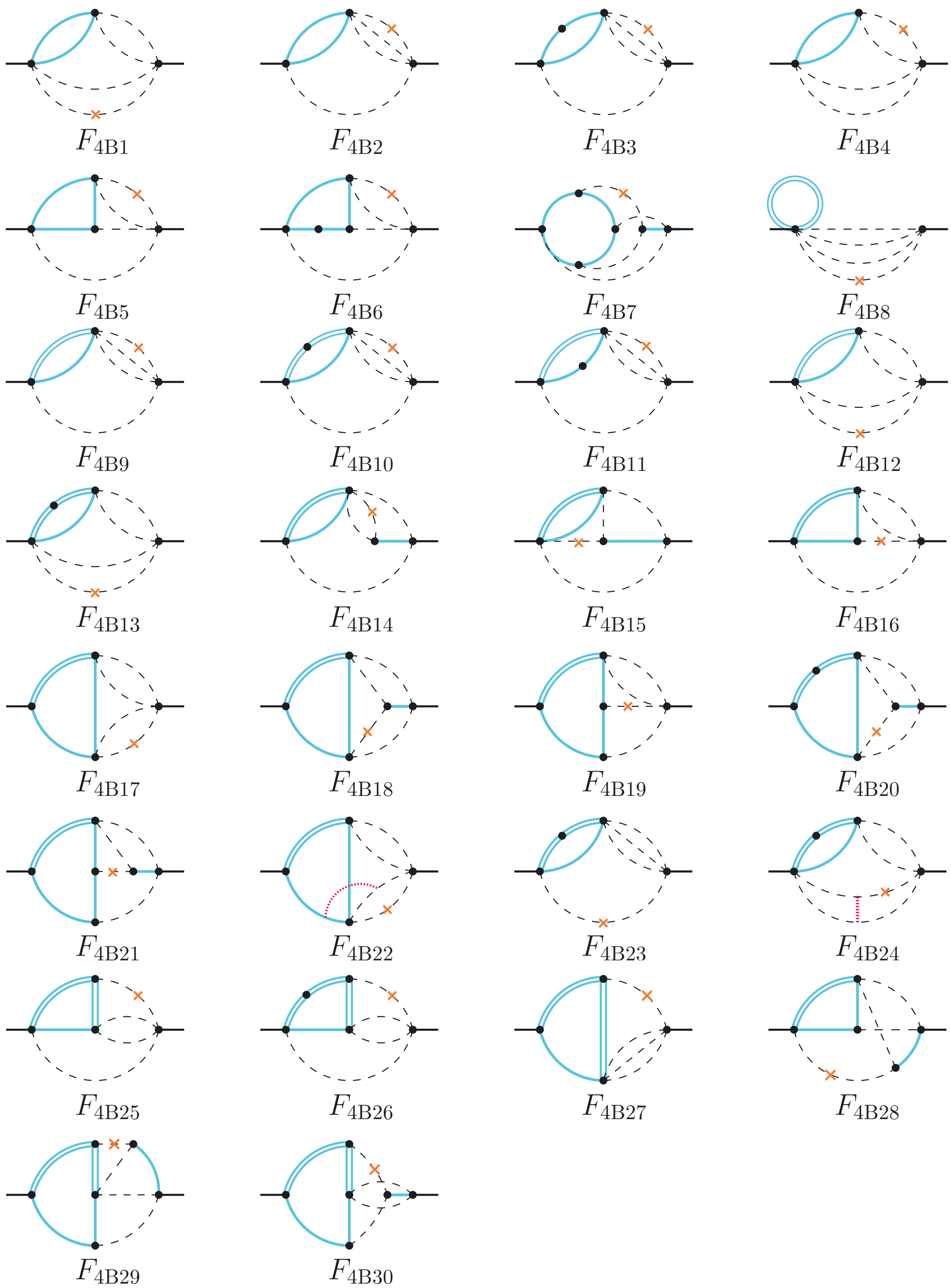


Figure 4.5: The full set of one-loop four-body master integrals that was calculated in the course of this work. The dashed lines indicated the cut propagators from reversed unitarity relations, the solid light blue (double-)lines indicate (massive) propagators. The dotted red lines show numerators in the following way: When connecting two lines with momenta l_1 and l_2, the corresponding numerator is $(l_1 - l_2)^2 - m_1^2 - m_2^2$. The orange cross denotes the cut propagator with momentum p_4 as the symmetry in the momenta is broken here through the cut on the photon energy.

Additionally, as the master integrals do not all stem from the same reduction, they are not necessarily forming a basis, i.e. they are not completely independent. One can see that e.g. $\mathcal{F}_2$ and $\mathcal{F}_3$ have a lot of denominators in common, meaning that if we go to a low enough sector there will be a significant overlap. There is an ambiguity in assigning the original integrals to the families, which can lead to hidden relations between the master integrals. A strong hint that this is the case is the need for unexpectedly high powers of ϵ in the solution of master integrals, when analyzing their prefactors.
These additional relations can be found by rewriting the integrals in terms of another family and applying the respective rules or by using another IBP reduction program, e.g. `Kira` [109] (For our calculation, we used these additional relations as cross-checks for the validity of the reduction). Another possible solution is the introduction of additional seed integrals, i.e. suggesting a master integral base to the program that is more fitting or adding higher line integrals at the start.
The relations we encountered in the calculation of the four-body integrals are given in Sect. 4.11.

4.7 Calculating the Master Integrals

To calculate the master integrals, we employ the method of differential equations, which was outlined in Sect. 3.5. For this, we are using the fact that our treatment of the cut on the photon energy E_γ via the delta function $\delta(z - s_{12} - s_{13} - s_{23})$ introduces a dependence on the dimensionless parameter z. After the phase space integrations we want to stay differential in this parameter, which means that the resulting expressions can only be a functions of z and the mass of the b quark, m_b. But, as the latter is the only variable with a mass dimension in our problem, we can set it to one (i.e. $m_b^2 = 1$) during the calculation and restore the correct power afterwards by dimensional considerations. We then use the dependence on z to build and solve our system of differential equations. Furthermore, as we see in Eqs. (4.23) – (4.25), we only have one denominator that directly depends on z, i.e. $(p_{123}^2 - z\, m_b^2)$.

Let us consider for example the integral

$$F_{4B5} = \int dPS_4 \frac{\delta(p_1^2)\delta(p_2^2)\delta(p_3^2)\delta(p_4^2)\delta(z - s_{123})}{[k^2]\,[(k + p_b - p_{13})^2]\,[(k - p_3)^2]}\,, \tag{4.27}$$

taken from the family $\mathcal{F}_1$. We use reverse unitarity, converting the delta functions into denominators and further make use of the shorthand notation introduced in Eq. (3.16). Note that, in this notation, negative numbers indicate numerators.
This leads to the following expression (note that we omit the prefactors from reversed unitarity relations, as we use them the other way again in the end):

$$F_{4B5} = \int dPS_4 \frac{1}{[p_1^2]\,[p_2^2]\,[p_3^2]\,[p_4^2]\,[(s_{123} - z)]\,[k^2]\,[(k + p_b - p_{13})^2]\,[(k - p_3)^2]} \tag{4.28}$$

$$= F_1[1, 1, 1, 1, 1; 1, 1, 0, 1, 0, 0, 0, 0, 0, 0]\,. \tag{4.29}$$

The semicolon is added for better readability to separate the (always present) first five denominators introduced by applying reverse unitarity from the rest.
We can now take the derivative with respect to z:

$$\begin{aligned}\partial_z F_{4B5} &= \partial_z \int dPS_4 \frac{1}{[p_1^2]\,[p_2^2]\,[p_3^2]\,[p_4^2]\,[(z - s_{123})]\,[k^2]\,[(k + p_b - p_{13})^2]\,[(k - p_3)^2]} \\ &= \int dPS_4 \frac{1}{[p_1^2]\,[p_2^2]\,[p_3^2]\,[p_4^2]\,[(z - s_{123})]^2\,[k^2]\,[(k + p_b - p_{13})^2]\,[(k - p_3)^2]} \\ &= F_1[1, 1, 1, 1, 2; 1, 1, 0, 1, 0, 0, 0, 0, 0, 0]\end{aligned} \tag{4.30}$$

and we see that the only effect is squaring the numerator stemming from the cut condition.
This is done for all the master integrals in Sect. 4.6, which are then given to the IBP reduction again to get a closed linear system of differential equations.
For illustration purpose, we give the explicit result for the example integral from above:

$$\begin{aligned}\partial_z F_{4B5} = &-\frac{2(1-2\epsilon)^3}{\epsilon(z-1)z(2z-1)} F_{4B2} - \frac{(1-2\epsilon)}{\epsilon(z-1)z(2z-1)} F_{4B3} \\ &+ \frac{2(1-2\epsilon)^3}{\epsilon(z-1)z(2z-1)} F_{4B4} + \frac{2(1-5\epsilon)}{2z-1} F_{4B5} + \frac{6(1-\epsilon)}{2(1-2\epsilon)(2z-1)} F_{4B6}\,.\end{aligned} \tag{4.31}$$

We see that the system is closed under the reduction which lets us bring it into the general form (cf. Sect. 3.5):

$$\partial_z \vec{F} = \hat{A}(z,\epsilon)\vec{F}\,, \tag{4.32}$$

where the vector $\vec{F}$ contains the master integrals and the matrix $\hat{A}(z,\epsilon)$ contains the reduction coefficients.

4.7.1 Solving the Differential Equations

The differential equations are generated for each family separately. For this, we define three sets of integrals that are closed under the IBP reductions. The condition that the sets have to be closed under the reduction leads to the fact that some of the lower-line integrals are contained in more than one family.
Using the labelling introduced in Fig. 4.5, the sets read:

$$\vec{F}_1 = \{F_{\rm 4B1}, F_{\rm 4B2}, F_{\rm 4B3}, F_{\rm 4B4}, F_{\rm 4B5}, F_{\rm 4B6}, F_{\rm 4B7}\}\,, \tag{4.33}$$

$$\begin{aligned}\vec{F}_2 = \{&F_{\rm 4B8}, F_{\rm 4B9}, F_{\rm 4B10}, F_{\rm 4B11}, F_{\rm 4B12}, F_{\rm 4B13}, F_{\rm 4B4}, F_{\rm 4B14},\\ &F_{\rm 4B15}, F_{\rm 4B16}, F_{\rm 4B17}, F_{\rm 4B18}, F_{\rm 4B19}, F_{\rm 4B20}, F_{\rm 4B21}, F_{\rm 4B22}\}\,,\end{aligned} \tag{4.34}$$

and

$$\begin{aligned}\vec{F}_3 = \{&F_{\rm 4B8}, F_{\rm 4B9}, F_{\rm 4B10}, F_{\rm 4B11}, F_{\rm 4B12}, F_{\rm 4B23}, F_{\rm 4B14}, F_{\rm 4B24},\\ &F_{\rm 4B15}, F_{\rm 4B25}, F_{\rm 4B26}, F_{\rm 4B27}, F_{\rm 4B28}, F_{\rm 4B29}, F_{\rm 4B30}\}\,.\end{aligned} \tag{4.35}$$

These sets are sorted by sectors, i.e. their number of positive entries in the F_i-notation. Choosing the right order brings the respective differential equation matrices into triangular form, which is favored in the following steps of obtaining the ϵ-form of the system.

We then can define the equations in the original bases:

$$\partial_z \vec{F}_i = \hat{A}_i(z,\epsilon)\vec{F}_i \qquad i = 1,2,3\,. \tag{4.36}$$

The matrix $\hat{A}_1(z,\epsilon)$, for example, takes the form:

$$\hat{A}_1(z,\epsilon) = \begin{pmatrix} -\frac{5z\epsilon-2z-3\epsilon+1}{z\bar{z}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{5z\epsilon-2z-3\epsilon+1}{z\bar{z}} & \frac{\epsilon(3\epsilon-1)}{2z(2\epsilon-1)^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{2(2\epsilon-1)^2}{z\bar{z}} & -\frac{3z\epsilon-z+\epsilon}{z\bar{z}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{5z\epsilon-2z-2\epsilon+1}{z\bar{z}} & 0 & 0 & 0 \\ 0 & \frac{2(2\epsilon-1)^3}{z(2z-1)\bar{z}\epsilon^2} & -\frac{2\epsilon-1}{z(2z-1)\bar{z}\epsilon} & -\frac{2(2\epsilon-1)^3}{z(2z-1)\bar{z}\epsilon^2} & -\frac{2(5\epsilon-1)}{2z-1} & \frac{3(\epsilon+1)}{(2z-1)(4\epsilon-1)} & 0 \\ 0 & \frac{20(2\epsilon-1)^3(4\epsilon-1)}{3z(2z-1)\bar{z}\epsilon(\epsilon+1)} & \frac{4(z-3)(2\epsilon-1)(4\epsilon-1)}{3z(2z-1)\bar{z}(\epsilon+1)} & -\frac{20(2\epsilon-1)^3(4\epsilon-1)}{3z(2z-1)\bar{z}\epsilon(\epsilon+1)} & \frac{2(z+2)\epsilon(4\epsilon-1)(5\epsilon-1)}{3z(2z-1)\bar{z}(\epsilon+1)} & -\frac{\left(4z^2-3z+3\right)\epsilon}{z(2z-1)\bar{z}} & 0 \\ 0 & -\frac{8(2\epsilon-1)^3(4\epsilon-1)}{z(2z-1)\bar{z}^3\epsilon^2(3\epsilon+1)} & \frac{8(2\epsilon-1)(4\epsilon-1)}{(2z-1)\bar{z}^3\epsilon(3\epsilon+1)} & \frac{8(2\epsilon-1)^3(4\epsilon-1)}{z(2z-1)\bar{z}^3\epsilon^2(3\epsilon+1)} & \frac{4(4\epsilon-1)(5\epsilon-1)}{z(2z-1)\bar{z}^2(3\epsilon+1)} & -\frac{4(z+1)(\epsilon+1)}{z(2z-1)\bar{z}^2(3\epsilon+1)} & -\frac{5z\epsilon+z-3\epsilon}{z\bar{z}} \end{pmatrix}. \tag{4.37}$$

The matrices $\hat{A}_2(z,\epsilon)$ and $\hat{A}_3(z,\epsilon)$ are not explicitly given, but can be constructed by reversing the relation from Eq. (3.47), which yields:

$$A_i(z,\epsilon) = T_i A_{i,\epsilon} T_i^{-1} + (\partial_z T_i) T_i^{-1}\,,$$

where the results from Appendix C and Appendix F can be plugged in.

An interesting observation for the above matrix (which also holds true for the other two) is that it contains poles in $z = 1/2$. But, as we will see later on, these drop out after the change of basis. If they persisted, we would have to use generalized polylogarithms for the construction of our solution, but as only z and $\bar{z}$ survive, we can restrict ourselves to harmonic polylogarithms. This convenience does not fully carry over to the five-body case (as we will see in chapter 5), where the rotation to the new base does not get rid of all letters that lie outside of the realm of HPLs.
Another observation regarding $\vec{F}_1$ is the complete decoupling of F_{4B1}. This integral is nominally part of the family, but it does not mix with the other ones in the re-

duction. This will also hold true for the change into the new basis, where we also see that the corresponding integral is not brought into the form of uniform transcendentality by the same factor as the others.

The matrices $\hat{A}_i(z,\epsilon)$ can now be brought into ϵ-form. There are multiple implementations of the algorithmic approach by Lee [83], for our calculation we use the program `epsilon` [84].
Running the program gives us the necessary ingredients to change our basis, namely the transformation matrices T_i and the differential equation matrices $A_{i,\epsilon}$. Changing our base from $\vec{F}_i$ to $\vec{G}_i$, these matrices are defined in the following way:

$$\vec{G}_i = \hat{T}_i^{-1}\vec{F}_i \,, \tag{4.38}$$

relating the old and new base, resulting in a new system of equations for each family:

$$\partial_z \vec{G}_i = \epsilon \hat{A}_{i,\epsilon}(z)\vec{G}_i \,. \tag{4.39}$$

The transformation matrices T_i are given explicitly in Appendix F.
The matrices can be simplified even further, because of their simple dependence on only $1/z$ and $1/\bar{z}$, splitting them into

$$\hat{A}_{i,\epsilon}(z) = \frac{1}{z}\hat{A}_{i,z} + \frac{1}{\bar{z}}\hat{A}_{i,\bar{z}} \,, \tag{4.40}$$

where the entries of the matrices are now only rational constants. These matrices are given in Appendix C.
As for the nomenclature of these integrals, the three bases are:

$$\vec{G}_1 = \{G_{4\text{B}1}, G_{4\text{B}2}, G_{4\text{B}3}, G_{4\text{B}4}, G_{4\text{B}5}, G_{4\text{B}6}, G_{4\text{B}7}\} \,, \tag{4.41}$$

$$\begin{aligned}\vec{G}_2 = \{&G_{4\text{B}8}, G_{4\text{B}9}, G_{4\text{B}10}, G_{4\text{B}11}, G_{4\text{B}12}, G_{4\text{B}13}, G_{4\text{B}14}, G_{4\text{B}15}, \\ &G_{4\text{B}16}, G_{4\text{B}17}, G_{4\text{B}18}, G_{4\text{B}19}, G_{4\text{B}20}, G_{4\text{B}21}, G_{4\text{B}22}, G_{4\text{B}38}\} \,,\end{aligned} \tag{4.42}$$

and

$$\vec{G}_3 = \{G_{4\mathrm{B}23}, G_{4\mathrm{B}24}, G_{4\mathrm{B}25}, G_{4\mathrm{B}26}, G_{4\mathrm{B}27}, G_{4\mathrm{B}28}, G_{4\mathrm{B}29}, G_{4\mathrm{B}30}, G_{4\mathrm{B}31}, G_{4\mathrm{B}32}, G_{4\mathrm{B}33}, G_{4\mathrm{B}34}, G_{4\mathrm{B}35}, G_{4\mathrm{B}36}, G_{4\mathrm{B}37}\}\,. \tag{4.43}$$

We now have successfully brought our equations into the form introduced in Sect. 3.5.2. This enables us to solve them via iterated integrals, using harmonic polylogarithms, as discussed in Sect. 3.6.2.
The only piece that is missing at this point in the calculation of the integrals are the boundary conditions to fix the constant part at each order in ϵ.

4.8 Calculation of the Boundary Conditions

This section will discuss the different methods we used to acquire the boundary conditions needed for the calculation of the integrals from families F_1 to F_3.
Each determination of a boundary condition was done via one of the following three methods: **full analytic dependence**, **asymptotic behaviour without Mellin-barnes representation** and **asymptotic behaviour with Mellin-barnes representation**. To make clear what we mean by these three methods, we now give an example for each of them.

Case 1: Full Analytic Dependence

In the case of the simplest integrals (having a low number of denominators and being non-massive mostly), we were able to calculate them analytically. If we were not using the differential equations method, we would be done at this point, but because the system we want to solve is constructed via an IBP reduction, these integrals mix into the equations of the ones from higher sectors. This makes it necessary to compute an asymptotic expansion for them, too. As an example, let us look at $F_{4\mathrm{B}2/3}$, which only differ by the power of one of the denominators:

$$F_{4\mathrm{B}2/3} = \int dPS_4 \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2]^n [(k+p_{234})^2]}\,. \tag{4.44}$$

where $F_{4B2/3}$ corresponds to $n=1$ and $n=2$, respectively.
Now we introduce Feynman parameters and perform the loop integration (m_b^2 is set to 1):

$$\begin{aligned} F_{4B2/3} &= \int dPS_4\ (-iS_\Gamma)(-1)^n\Gamma(1-\epsilon)\Gamma(-1+n+\epsilon)\int dy\frac{y^{n-1}}{[-(p_{234}^2 y\bar{y})-i0]^{-1+n+\epsilon}} \\ &= \int dPS_4\ (iS_\Gamma)\frac{\Gamma(1-\epsilon)^2\Gamma(-1+n+\epsilon)\Gamma(2-n-\epsilon)}{\Gamma(3-n-2\epsilon)}(-1)^\epsilon(s_{234})^{1-n-\epsilon}\,. \end{aligned} \tag{4.45}$$

In these types of integrals, one has to be careful with the $i0$-prescription when pulling out the sign from the denominator, as the value of the imaginary part has to be consistent before and after the evaluation of the integral.
For the next step, we introduce our parametrization from Eq. (4.16), leading to:

$$\begin{aligned} F_{4B2/3} &= N(D)\ 4^{-2\epsilon}\ (iS_\Gamma)\ (-1)^\epsilon\frac{\Gamma(1-\epsilon)^2\Gamma(-1+n+\epsilon)\Gamma(2-n-\epsilon)}{\Gamma(3-n-2\epsilon)} \\ &\int_0^\delta dz\int_0^1 du\ dv\ dx\ dw\ (z\bar{z})^{1-2\epsilon}(u\bar{u})^{-1/2-\epsilon}v^{1-2\epsilon}(\bar{v}x\bar{x}w\bar{w})^{-\epsilon}[v(1-z\bar{w})]^{1-n-\epsilon}\,, \end{aligned} \tag{4.46}$$

which we can solve in terms of Γ- and hypergeometric functions:

$$\begin{aligned} F_{4B2/3} &= N(D)\ 4^{-2\epsilon}\ (iS_\Gamma)\ (-1)^\epsilon\frac{\Gamma(-1+n+\epsilon)\Gamma(2-n-\epsilon)\Gamma(3-n-3\epsilon)}{\Gamma(3-n-2\epsilon)\Gamma(4-n-4\epsilon)} \\ &\frac{\Gamma(1-\epsilon)^7\Gamma(1/2-\epsilon)^2}{\Gamma(2-2\epsilon)^2\Gamma(1-2\epsilon)}\int_0^\delta dz\ (z\bar{z})^{1-2\epsilon}\ {}_2F_1(n+\epsilon-1,1-\epsilon,2-2\epsilon;z)\,. \end{aligned} \tag{4.47}$$

With this, we can then calculate the boundary condition for G_{4B2}, which given by:

$$G_{4B2} = \frac{4(4\epsilon-1)(2\epsilon-1)^3}{3(z-1)z\epsilon^3(3\epsilon+1)}F_{4B2} + \frac{2(4\epsilon-1)(2\epsilon-1)}{3z\epsilon^2(3\epsilon+1)}F_{4B3}\,. \tag{4.48}$$

After plugging in the result from Eq. (4.47) and expanding with `HypExp`, we can take the limit of $z\to 0$ to get the asymptotic behaviour of the integral.
Note that we are interchanging the order of the remaining integrations and the ϵ-expansion. This is only possible because at this point, the integration in z cannot

generate further divergences here. If it did, changing the two procedures would not be allowed. One also has to be careful not to take the limit $z \to 0, 1$ too soon in the calculation for the same reason. This will become clearer for the other types of boundaries in the next section.

The result for the boundary up to $\mathcal{O}(\epsilon^2)$ reads[1]:

$$\widetilde{G}_{4\text{B}2} = \frac{i}{\epsilon^4}\frac{\pi S_\Gamma^4}{(1+3\epsilon)(1-2\epsilon)}\Bigg[-\frac{4}{3}+\epsilon\left(\frac{8}{3}H_0(z)-\frac{4}{3}i\pi\right)+ \\ +\epsilon^2\left(\frac{8}{3}i\pi H_0(z)-\frac{16}{3}H_{0,0}(z)+2\pi^2\right)+\mathcal{O}(\epsilon^3)\Bigg], \quad (4.49)$$

where the coefficients in front have been taken out to make the uniform transcendentality of the integral explicit.

Case 2: Asymptotic Behaviour Order-By-Order without Mellin-Barnes Representation

For our set of integrals, it is not always possible to determine the full analytic results, even for the boundary conditions. Two different methods were used for obtaining the asymptotic behaviour in terms of a series in ϵ, i.e. as an asymptotic expansion with and without the help of Mellin-Barnes representations. We first want to describe the latter in this section, the next section will then address the former method.
For illustration, we consider the integral $F_{4\text{B}9}$:

$$F_{4\text{B}9} = \int dPS_4 \int \frac{d^Dk}{(2\pi)^D}\frac{1}{[k^2]^n[(k+p_{124})^2-m_b{}^2]}\,. \quad (4.50)$$

We can use the same method as in the first case, up to a certain point:

$$F_{4\text{B}9} = \int dPS_4 (iS_\Gamma)\Gamma(1-\epsilon)\Gamma(\epsilon)\int dy\ \frac{1}{[\bar{y}-(p_{234}^2 y\bar{y})-i0]^\epsilon}$$

[1]Here we introduce the notation $\widetilde{G}$ and $\widetilde{F}$ for the boundary conditions.

$$
\begin{aligned}
&= \int dPS_4\ (iS_\Gamma)\Gamma(1-\epsilon)\Gamma(\epsilon) \int dy\ \frac{\bar{y}^{-\epsilon}}{[1-y\,s_{124}]^\epsilon} \\
&= \int dPS_4\ (iS_\Gamma)\frac{\Gamma(1-\epsilon)^2\Gamma(\epsilon)}{\Gamma(2-\epsilon)} \int dy\ {}_2F_1(\epsilon,1;2-\epsilon;s_{124}) \\
&= N(D)\ 4^{-2\epsilon}\ (iS_\Gamma)\ \frac{\Gamma(1-\epsilon)^2\Gamma(\epsilon)}{\Gamma(2-\epsilon)} \int_0^\delta dz \int_0^1 du\ dv\ dx\ dw\ (z\bar{z})^{1-2\epsilon} \\
&\quad (u\bar{u})^{-1/2-\epsilon} v^{1-2\epsilon} (\bar{v}x\bar{x}w\bar{w})^{-\epsilon}\ {}_2F_1(\epsilon,1;2-\epsilon;v(1-z\bar{w})) \\
&= N(D)\ 4^{-2\epsilon}\ (iS_\Gamma)\ \frac{\Gamma(1-\epsilon)^5\Gamma(\epsilon)}{\Gamma(2-\epsilon)\Gamma(3-3\epsilon)} \frac{\Gamma(1/2-\epsilon)^2}{\Gamma(1-2\epsilon)} \\
&\quad \int_0^\delta dz\ (z\bar{z})^{1-2\epsilon} \int_0^1 dw\ (w\bar{w})^{-\epsilon}\ {}_3F_2(\epsilon,1,2-2\epsilon;2-\epsilon,3-3\epsilon;1-z\bar{w})
\end{aligned} \tag{4.51}
$$

We see that by adding a mass, we introduce hypergeometric functions into the calculation, increasing its complexity, as one would suspect. The integration over w that remains in Eq. (4.51) cannot be done analytically.

But, as we only need the asymptotic behaviour in one point of z, we can set it to a definite value in the argument of the ${}_3F_2$ and compute the expansion in ϵ.

One has to be careful though with this choice of z, as the integration over w can still introduce new poles in ϵ. As this procedure interchanges the two limits of expansion in a small parameter and integration, we have to make sure our treatment is consistent.

For this reason, we cannot choose $z \to 0$ as our limit, which makes the argument of the hypergeometric function independent of w, subsequently leading to inconsistent results. Instead, we choose $z \to 1$, simplifiying the argument to w, which enables the computation of the remaining integral over w in terms of a hypergeometric function:

$$
\begin{aligned}
\widetilde{F}_{4B9}|_{z\to 1} &= N(D)\ 4^{-2\epsilon}\ (iS_\Gamma)\ \frac{\Gamma(1-\epsilon)^5\,\Gamma(\epsilon)}{\Gamma(2-\epsilon)\Gamma(3-3\epsilon)} \frac{\Gamma(1/2-\epsilon)^2}{\Gamma(1-2\epsilon)} \\
&\quad \int_0^\delta dz\ (z\bar{z})^{1-2\epsilon} \int_0^1 dw\ (w\bar{w})^{-\epsilon}\ {}_3F_2(\epsilon,1,2-2\epsilon;2-\epsilon,3-3\epsilon;w)
\end{aligned} \tag{4.52}
$$

$$
\begin{aligned}
&= N(D)\ 4^{-2\epsilon}\ (iS_\Gamma)\ \frac{\Gamma(1-\epsilon)^7\,\Gamma(\epsilon)}{\Gamma(2-\epsilon)\Gamma(2-2\epsilon)\Gamma(3-3\epsilon)} \frac{\Gamma(1/2-\epsilon)^2}{\Gamma(1-2\epsilon)} \\
&\quad \int_0^\delta dz\ (z\bar{z})^{1-2\epsilon}\ {}_3F_2(\epsilon,1,-\epsilon;2-2\epsilon,3-3\epsilon;1)\,.
\end{aligned} \tag{4.53}
$$

This result can now be expanded in ϵ, additionally setting terms $\sim \log(z)$ to zero as they vanish in the limit $z \to 1$:

$$\widetilde{F}_{4B9}|_{z\to 1} = z\bar{z}\frac{i\pi S_\Gamma^4}{\epsilon}\Bigg[1 + \epsilon\left(-2\log(\bar{z}) - \frac{\pi^2}{3} + 13\right) + \epsilon^2\Big(2\log^2(\bar{z}) + \frac{2}{3}\left(\pi^2 - 39\right)\log(\bar{z}) - 16\zeta(3) - \frac{10\pi^2}{3} + 103\Big) + \mathcal{O}(\epsilon^3)\Bigg]. \tag{4.54}$$

Note that the prefactor still contains a $\bar{z}$, which would vanish in our limit. But, as we can read off from the transformation matrices in Appendix F, some of the entries introduce powers of $\bar{z}$ that negate the ones from the old (F-)base. This means that, strictly, we can only take the limit $z \to 1$ **after the transformation** to the new (G-)base. This was taken care of in our calculation, the 'premature' limit was taken here only for illustration.
This method of setting $z \to 1$ works for a lot of integrals in the medium sectors, e.g. F_{4B10},F_{4B11} and F_{4B12}. These are, amongst others, related to G_{4B24}. Expanding them each to the appropriate order, transforming to the new base and taking the asymptotic limit $z \to 1$ yields the boundary condition:

$$\widetilde{G}_{4B24} = \frac{i}{\epsilon^4}\frac{2\pi S_\Gamma^4}{(1-2\epsilon)^2}\Bigg[-\frac{11}{120} - \frac{11}{60}\epsilon H_1(z) + \epsilon^2\left(-\frac{11}{30}H_{1,1}(z) - \frac{11}{720}\pi^2\right) + \epsilon^3\left(-\frac{11}{360}\pi^2 H_1(z) - \frac{11}{15}H_{1,1,1}(z) - \frac{77}{30}\zeta(3)\right) + \mathcal{O}(\epsilon^4)\Bigg]. \tag{4.55}$$

Case 3: Asymptotic Behaviour Order-By-Order with Mellin-Barnes Representation

For the last case of integrals, calculating the asymptotic behaviour is not as straightforward as before: For more than two propagators, we generally need to introduce more than one Feynman parameter, leading to more complicated structures in the denominator, which can often not be expressed in terms of hypergeometric func-

tions.
To give an example for this type of integral, we now want to look at F_{4B29}:

$$F_{4B29} = \int dPS_4 \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2][(k-p_3)^2][(k-p_{123})^2 - m_b{}^2][(k-p_b)^2 - m_b{}^2]s_{24}} \, . \tag{4.56}$$

As there are four propagators, we introduce three Feynman parameter integrations to combine them and resolve the loop integration afterwards:

$$F_{4B29} = \int dPS_4 \; (iS_\Gamma) \frac{\Gamma(1-\epsilon)\Gamma(2+\epsilon)}{s_{24}} \quad \times \\ \int dx_1 \, dx_2 \, dx_3 \, \frac{x_3 \bar{x}_3}{[\bar{x}_3 - (x_1 p_3 + x_2 p_4 - p_b)^2 x_3 \bar{x}_3 - i0]^{2+\epsilon}} \, . \tag{4.57}$$

Using our parametrization of the phase space, we then arrive at:

$$F_{4B29} = N(D) \; 4^{-2\epsilon} \; (iS_\Gamma) \; \Gamma(1-\epsilon)\Gamma(2+\epsilon) \int_0^\delta dz \int_0^1 du \; dv \; dx \; dw \; (z\bar{z})^{1-2\epsilon} \\ \int dx_1 \; dx_2 \; dx_3 \; \frac{(u\bar{u})^{-1/2-\epsilon} v^{1-2\epsilon} (\bar{v} x \bar{x} w \bar{w})^{-\epsilon} \; \bar{x}_3^{-1-\epsilon} x_3}{[x_1 x_3 \bar{v} + v x_1 x_3 z \bar{w} + \bar{x}_3 + v x_1 x_2 x_3 \bar{z} + x_2 y \bar{x}_1 \bar{z}]^{2+\epsilon}} (v \bar{x} \bar{z})^{-1} \, . \tag{4.58}$$

As we elaborated before, the structure in the denominator is highly complex and there is no straightforward analytical solution for the integrations. To disentangle the sum, we now recast the integral in terms of several Mellin-Barnes representations, iteratively using the formula from Eq. (3.30). This leads to the introduction of four additional integrations, which we trade for now factorized numerators:

$$F_{4B29} = N(D) \; 4^{-2\epsilon} \; (iS_\Gamma) \; \Gamma(1-\epsilon)\Gamma(2+\epsilon) \int_0^\delta dz \int_0^1 du \; dv \; dx \; dw \; (z\bar{z})^{1-2\epsilon} \\ \int \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \frac{dz_3}{2\pi i} \frac{dz_4}{2\pi i} \; \frac{\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(\epsilon + z_1 + z_2 + z_3 + z_4 + 2)}{\Gamma(\epsilon + 2)} \\ \int dx_1 \; dx_2 \; dx_3 (x_1 x_3 \bar{v})^{z_1} (x_2 x_3 \bar{x}_1 \bar{z})^{z_4} \bar{x}_3^{-\epsilon - z_1 - z_2 - z_3 - z_4 - 2} (v x_1 x_3 z \bar{w})^{z_2} (v x_1 x_2 x_3 \bar{z})^{z_3} \, . \tag{4.59}$$

At first glance this looks intimidating, but since all the terms are now factorized we can carry out all the phase space and Feynman parameter integrations:

$$\begin{aligned}
F_{4B29} =& N(D)\ (iS_\Gamma^4)\ 4^{6-5\epsilon}\ \pi^{6-3\epsilon} \int_0^\delta dz \int \frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i}\frac{dz_3}{2\pi i}\frac{dz_4}{2\pi i}\ z^{z_2-2\epsilon+1}\bar{z}^{z_3+z_4-2\epsilon} \\
& \frac{\Gamma\left(\frac{1}{2}-\epsilon\right)^2\Gamma(1-\epsilon)^6\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)^2\Gamma(-2\epsilon)}\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(z_4+1)\Gamma(z_1-\epsilon+1) \\
& \Gamma(z_2-\epsilon+1)\Gamma(z_3+z_4+1)\Gamma(z_1+z_2+z_3+1)\Gamma(z_2+z_3-2\epsilon+1) \\
& \frac{\Gamma(-z_1-z_2-z_3-z_4-2\epsilon-2)\Gamma(z_1+z_2+z_3+z_4+\epsilon+2)}{\Gamma(z_2-2\epsilon+2)\Gamma(z_3+z_4+2)\Gamma(z_1+z_2+z_3-3\epsilon+2)}\,.
\end{aligned} \tag{4.60}$$

After this, we are only left with the Mellin-Barnes integrations. Calculating such high-dimensional representations is very cumbersome, but as we are only interested in the asymptotic behaviour, we can make use of the expansion properties of these integrals. For this we are using the programs `MB` [73] and `MBasymptotics` [110], to first expand the expression as a series in ϵ and then determine the asymptotic behaviour in $z \to 0, 1$.
After using the aforementioned tools, the four-fold representation simplifies greatly, leaving us with a maximum of two-fold representations to be calculated.

Calculating the Mellin-Barnes integrals amounts to summing the residues of the expressions after closing the contour to the left or right side. The summations are either done with internal routines of `Mathematica` [111] or with the additional package `HarmonicSums` [112]. `HarmonicSums` is especially useful for sums over polylogarithms that are not included in `Mathematica`.
After completing all the above steps, we finally arrive at the boundary condition:

$$\begin{aligned}
\widetilde{F}_{4B29} = \frac{1}{\epsilon^3\bar{z}}\Bigg[& -\frac{\pi^2}{6}+\epsilon\left(\frac{2}{3}\pi^2\log(\bar{z})-7\zeta(3)-\frac{\pi^2}{3}\right) \\
& +\epsilon^2\Big(-\frac{2\pi^2}{3}\mathrm{Li}_2(\bar{z})+28\zeta(3)\log(\bar{z})-\frac{4}{3}\pi^2\log^2(\bar{z})+\frac{4}{3}\pi^2\log(\bar{z}) \\
& -14\zeta(3)-\frac{5\pi^4}{18}-\frac{2\pi^2}{3}\Big)+\mathcal{O}(\epsilon^3)\Bigg]\,,
\end{aligned} \tag{4.61}$$

which can be directly translated into the ϵ-basis via the matrix $\hat{T}_3$ (as there is only one entry for this part of the transformation), given in Appendix F.3, yielding the result for $\widetilde{G}_{4\mathrm{B}36}$:

$$\widetilde{G}_{4\mathrm{B}36} = \frac{i}{\epsilon^4}\frac{2\pi S_\Gamma^4}{(1-2\epsilon)^2}\Bigg[-\frac{1}{4}\pi^2\epsilon^2 + \epsilon^3\left(-\pi^2 H_1(z) - \frac{21}{2}\zeta(3)\right) + \epsilon^4\left(-42\zeta(3)H_1(z) - 4\pi^2 H_{1,1}(z) - \frac{5}{12}\pi^4\right) + \mathcal{O}(\epsilon^5)\Bigg]. \quad (4.62)$$

4.9 Results for the Boundary Conditions

With the three procedures from the previous sections, we were able to determine all necessary boundary conditions for the four-body integrals $F_{4\mathrm{B}1}$–$F_{4\mathrm{B}36}$. Most of the time, we determined them in the limit $z \to 1$, but for some the limit $z \to 0$ led to a simplification of the computation.

For the following integrals, we used $z \to 1$:

$$G_{4\mathrm{B}i} \xrightarrow{z\to 1} \widetilde{G}_{4\mathrm{B}i} \quad i \in \{8..34, 36, 37, 38\}\,, \quad (4.63)$$

whereas the remaining ones were computed for the limit $z \to 0$:

$$G_{4\mathrm{B}i} \xrightarrow{z\to 0} \widetilde{G}_{4\mathrm{B}i} \quad i \in \{1..7, 35\}\,. \quad (4.64)$$

Each family has a unique ϵ-dependent normalization factor, which read as follows:

$$\begin{aligned} n_0(\epsilon) &= 2\pi S_\Gamma^4\,, \\ n_1(\epsilon) &= \frac{i}{\epsilon^4}\frac{n_0(\epsilon)}{2(1+3\epsilon)(1-2\epsilon)}\,, \\ n_2(\epsilon) &= \frac{i}{\epsilon^5}\frac{n_0(\epsilon)}{(1-2\epsilon)}\,, \\ n_3(\epsilon) &= \frac{i}{\epsilon^4}\frac{n_0(\epsilon)}{(1-2\epsilon)^2}\,, \\ S_\Gamma &= \frac{1}{\Gamma(1-\epsilon)(4\pi)^{2-\epsilon}}\,. \end{aligned} \quad (4.65)$$

With these definitions, we can give the full array of boundaries that we calculated:

$$\widetilde{G}_{4\text{B}1} = \frac{n_0(\epsilon)}{\epsilon}\Bigg[1 + \epsilon\left(\gamma - 3H_0(z)\right) + \epsilon^2\left(-3\gamma H_0(z) + 9H_{0,0}(z) + \frac{1}{4}\pi^2 + \frac{\gamma^2}{2}\right)$$
$$+ \epsilon^3\Big(- \frac{3}{4}\pi^2 H_0(z) - \frac{3}{2}\gamma^2 H_0(z) + 9\gamma H_{0,0}(z) - 27H_{0,0,0}(z)$$
$$+ \frac{1}{3}\zeta(3) + \frac{1}{4}\gamma\pi^2 + \frac{\gamma^3}{6}\Big) + \mathcal{O}(\epsilon^4)\Bigg],$$

$$\widetilde{G}_{4\text{B}2} = n_1(\epsilon)\Bigg[- \frac{4}{3} + \epsilon\left(\frac{8}{3}H_0(z) - \frac{4}{3}i\pi\right) + \epsilon^2\left(\frac{8}{3}i\pi H_0(z) - \frac{16}{3}H_{0,0}(z) + 2\pi^2\right)$$
$$+ \epsilon^3\left(-4\pi^2 H_0(z) - \frac{16}{3}i\pi H_{0,0}(z) + \frac{32}{3}H_{0,0,0}(z) + \frac{80}{3}\zeta(3) + \frac{14}{9}i\pi^3\right)$$
$$+ \epsilon^4\Big(- \frac{160}{3}\zeta(3)H_0(z) - \frac{28}{9}i\pi^3 H_0(z) + 8\pi^2 H_{0,0}(z) + \frac{32}{3}i\pi H_{0,0,0}(z)$$
$$- \frac{64}{3}H_{0,0,0,0}(z) + \frac{80}{3}i\pi\zeta(3) - \frac{49}{90}\Big)\pi^4 + \mathcal{O}(\epsilon^5)\Bigg],$$

$$\widetilde{G}_{4\text{B}3} = n_1(\epsilon)\Bigg[0 + \mathcal{O}(\epsilon^5)\Bigg],$$

$$\widetilde{G}_{4\text{B}4} = n_1(\epsilon)\Bigg[- \frac{4}{3} + \epsilon\left(\frac{8}{3}H_0(z) - \frac{4}{3}i\pi\right) + \epsilon^2\left(\frac{8}{3}i\pi H_0(z) - \frac{16}{3}H_{0,0}(z) + \frac{20}{9}\pi^2\right)$$
$$+ \epsilon^3\left(-\frac{40}{9}\pi^2 H_0(z) - \frac{16}{3}i\pi H_{0,0}(z) + \frac{32}{3}H_{0,0,0}(z) + 32\zeta(3) + \frac{16}{9}i\pi^3\right)$$
$$+ \epsilon^4\Big(- 64\zeta(3)H_0(z) - \frac{32}{9}i\pi^3 H_0(z) + \frac{80}{9}\pi^2 H_{0,0}(z) + \frac{32}{3}i\pi H_{0,0,0}(z)$$
$$- \frac{64}{3}H_{0,0,0,0}(z) + 32i\pi\zeta(3) - \frac{32}{45}\pi^4\Big) + \mathcal{O}(\epsilon^5)\Bigg],$$

$$\widetilde{G}_{4\mathrm{B}5} = n_1(\epsilon)\Bigg[-\frac{2}{3}\pi^2\epsilon^2 + \epsilon^3\left(\frac{4}{3}\pi^2 H_0(z) - 16\zeta(3) - \frac{2}{3}i\pi^3\right) + \epsilon^4\left(32\zeta(3)H_0(z) + \frac{4}{3}i\pi^3 H_0(z) - \frac{8}{3}\pi^2 H_{0,0}(z) - 16i\pi\zeta(3) + \frac{1}{2}\pi^4\right) + \mathcal{O}(\epsilon^5)\Bigg],$$

$$\widetilde{G}_{4\mathrm{B}6} = n_1(\epsilon)\Big[0 + \mathcal{O}(\epsilon^5)\Big],$$

$$\widetilde{G}_{4\mathrm{B}7} = n_1(\epsilon)\Bigg[4 + \epsilon\left(4i\pi - 12H_0(z)\right) + \epsilon^2\left(-12i\pi H_0(z) + 36H_{0,0}(z) - \frac{64}{9}\pi^2\right) + \epsilon^3\left(\frac{64}{3}\pi^2 H_0(z) + 36i\pi H_{0,0}(z) - 108H_{0,0,0}(z) - \frac{328}{3}\zeta(3) - \frac{52}{9}i\pi^3\right) + \mathcal{O}(\epsilon^4)\Bigg],$$

$$\widetilde{G}_{4\mathrm{B}8} = n_2(\epsilon)\Bigg[-\frac{1}{120} - \epsilon\frac{1}{60}H_1(z) + \epsilon^2\left(\frac{\pi^2}{180} - \frac{1}{30}H_{1,1}(z)\right) + \epsilon^3\left(\frac{1}{90}\pi^2 H_1(z) - \frac{1}{15}H_{1,1,1}(z) + \frac{1}{10}\zeta(3)\right) + \epsilon^4\left(\frac{1}{5}\zeta(3)H_1(z) + \frac{1}{45}\pi^2 H_{1,1}(z) - \frac{2}{15}H_{1,1,1,1}(z) + \frac{\pi^4}{1800}\right) + \epsilon^5\Big(\frac{1}{900}\pi^4 H_1(z) + \frac{2}{5}\zeta(3)H_{1,1}(z) + \frac{2}{45}\pi^2 H_{1,1,1}(z) - \frac{4}{15}H_{1,1,1,1,1}(z) + \frac{1}{2}\zeta(5) - \frac{1}{15}\pi^2\zeta(3)\Big) + \mathcal{O}(\epsilon^6)\Bigg],$$

$$\widetilde{G}_{4\mathrm{B}9} = n_2(\epsilon)\Bigg[\frac{11}{360} + \epsilon\frac{11}{180}H_1(z) + \epsilon^2\left(\frac{11}{90}H_{1,1}(z) + \frac{11}{2160}\pi^2\right) + \epsilon^3\left(\frac{11}{1080}\pi^2 H_1(z) + \frac{11}{45}H_{1,1,1}(z) + \frac{77}{90}\zeta(3)\right)$$

$$+ \epsilon^4 \left(\frac{77}{45}\zeta(3)H_1(z) + \frac{11}{540}\pi^2 H_{1,1}(z) + \frac{22}{45}H_{1,1,1,1}(z) + \frac{847}{16200}\pi^4\right)$$
$$+ \mathcal{O}(\epsilon^5)\Bigg] ,$$

$$\widetilde{G}_{4\text{B}10} = n_2(\epsilon)\Bigg[\epsilon^2 \frac{11}{1080}\pi^2 + \epsilon^3 \left(\frac{11}{540}\pi^2 H_1(z) + \frac{22}{45}\zeta(3)\right)$$
$$+ \epsilon^4 \left(\frac{44}{45}\zeta(3)H_1(z) + \frac{11}{270}\pi^2 H_{1,1}(z) + \frac{44}{2025}\pi^4\right)$$
$$+ \epsilon^5 \Big(\frac{88}{2025}\pi^4 H_1(z) + \frac{88}{45}\zeta(3)H_{1,1}(z) + \frac{11}{135}\pi^2 H_{1,1,1}(z)$$
$$+ \frac{341}{30}\zeta(5) - \frac{11}{27}\pi^2\zeta(3)\Big) + \mathcal{O}(\epsilon^6)\Bigg] ,$$

$$\widetilde{G}_{4\text{B}11} = n_2(\epsilon)\Bigg[- \epsilon^2 \frac{11}{540}\pi^2 + \epsilon^3 \left(-\frac{11}{270}\pi^2 H_1(z) - \frac{44}{45}\zeta(3)\right)$$
$$+ \epsilon^4 \left(-\frac{88}{45}\zeta(3)H_1(z) - \frac{11}{135}\pi^2 H_{1,1}(z) - \frac{88}{2025}\pi^4\right)$$
$$+ \epsilon^5 \Big(- \frac{176}{2025}\pi^4 H_1(z) - \frac{176}{45}\zeta(3)H_{1,1}(z) - \frac{22}{135}\pi^2 H_{1,1,1}(z)$$
$$- \frac{341}{15}\zeta(5) + \frac{22}{27}\pi^2\zeta(3)\Big) + \mathcal{O}(\epsilon^6)\Bigg] ,$$

$$\widetilde{G}_{4\text{B}12} = n_2(\epsilon)\Bigg[\frac{1}{20} + \epsilon\frac{1}{5}H_1(z) + \epsilon^2 \left(H_{1,1}(z) + \frac{1}{20}\pi^2\right)$$
$$+ \epsilon^3 \left(\frac{7}{30}\pi^2 H_1(z) + \frac{28}{5}H_{1,1,1}(z) + \frac{7}{10}\zeta(3)\right)$$
$$+ \epsilon^4 \left(-\zeta(3)H_1(z) + \frac{19}{15}\pi^2 H_{1,1}(z) + \frac{164}{5}H_{1,1,1,1}(z) + \frac{97}{900}\pi^4\right)$$
$$+ \epsilon^5 \Big(\frac{83}{225}\pi^4 H_1(z) - \frac{82}{5}\zeta(3)H_{1,1}(z) + \frac{22}{3}\pi^2 H_{1,1,1}(z)$$
$$+ \frac{976}{5}H_{1,1,1,1,1}(z) + \frac{303}{10}\zeta(5) - \frac{29}{15}\pi^2\zeta(3)\Big) + \mathcal{O}(\epsilon^6)\Bigg] ,$$

$$
\begin{aligned}
\widetilde{G}_{4\text{B}13} = n_2(\epsilon)\bigg[&-\frac{1}{60} - \epsilon\frac{1}{10}H_1(z) + \epsilon^2\left(-\frac{3}{5}H_{1,1}(z) - \frac{1}{18}\pi^2\right) \\
&+ \epsilon^3\left(-\frac{1}{5}\pi^2 H_1(z) - \frac{18}{5}H_{1,1,1}(z) - \frac{6}{5}\zeta(3)\right) \\
&+ \epsilon^4\left(-\frac{4}{5}\zeta(3)H_1(z) - \frac{14}{15}\pi^2 H_{1,1}(z) - \frac{108}{5}H_{1,1,1,1}(z) - \frac{29}{300}\pi^4\right) \\
&+ \epsilon^5\Big(-\frac{133}{450}\pi^4 H_1(z) + 8\zeta(3)H_{1,1}(z) - \frac{76}{15}\pi^2 H_{1,1,1}(z) \\
&- \frac{648}{5}H_{1,1,1,1,1}(z) - \frac{168}{5}\zeta(5) + \frac{28}{15}\pi^2\zeta(3)\Big) + \mathcal{O}(\epsilon^6)\bigg],
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{G}_{4\text{B}14} = n_2(\epsilon)\bigg[&-\frac{1}{270} + \epsilon\left(-\frac{1}{90}H_1(z) - \frac{1}{270}i\pi\right) \\
&+ \epsilon^2\left(-\frac{1}{90}i\pi H_1(z) - \frac{1}{30}H_{1,1}(z) + \frac{\pi^2}{162}\right) \\
&+ \epsilon^3\left(\frac{1}{54}\pi^2 H_1(z) - \frac{1}{30}i\pi H_{1,1}(z) - \frac{1}{10}H_{1,1,1}(z) + \frac{4}{45}\zeta(3) + \frac{2}{405}i\pi^3\right) \\
&+ \epsilon^4\Big(\frac{4}{15}\zeta(3)H_1(z) + \frac{2}{135}i\pi^3 H_1(z) + \frac{1}{18}\pi^2 H_{1,1}(z) - \frac{1}{10}i\pi H_{1,1,1}(z) \\
&- \frac{3}{10}H_{1,1,1,1}(z) + \frac{4}{45}i\pi\zeta(3) - \frac{4}{2025}\pi^4\Big) + \epsilon^5\Big(\frac{4}{15}i\pi\zeta(3)H_1(z) \\
&- \frac{4}{675}\pi^4 H_1(z) + \frac{4}{5}\zeta(3)H_{1,1}(z) + \frac{2}{45}i\pi^3 H_{1,1}(z) + \frac{1}{6}\pi^2 H_{1,1,1}(z) \\
&- \frac{3}{10}i\pi H_{1,1,1,1}(z) - \frac{9}{10}H_{1,1,1,1,1}(z) + \frac{4}{5}\zeta(5) - \frac{4}{27}\pi^2\zeta(3) - \frac{i\pi^5}{2430}\Big) \\
&+ \mathcal{O}(\epsilon^6)\bigg],
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{G}_{4\text{B}15} = n_2(\epsilon)\bigg[&-\epsilon\frac{1}{60}H_1(z) + \epsilon^2\left(-\frac{2}{15}H_{1,1}(z) - \frac{\pi^2}{90}\right) \\
&+ \epsilon^3\left(-\frac{2}{45}\pi^2 H_1(z) - \frac{13}{15}H_{1,1,1}(z) - \frac{1}{12}\zeta(3)\right)
\end{aligned}
$$

$$+ \epsilon^4 \left(\frac{7}{30}\zeta(3)H_1(z) - \frac{2}{9}\pi^2 H_{1,1}(z) - \frac{16}{3}H_{1,1,1,1}(z) - \frac{17}{1350}\pi^4 \right)$$
$$+ \epsilon^5 \Big(- \frac{137}{2700}\pi^4 H_1(z) + \frac{43}{15}\zeta(3)H_{1,1}(z) - \frac{56}{45}\pi^2 H_{1,1,1}(z)$$
$$- \frac{484}{15}H_{1,1,1,1,1}(z) - \frac{49}{20}\zeta(5) + \frac{5}{18}\pi^2\zeta(3)\Big) + \mathcal{O}(\epsilon^6)\Bigg],$$

$$\widetilde{G}_{4B16} = n_2(\epsilon)\Bigg[- \epsilon\frac{1}{60}H_1(z) + \epsilon^2 \left(\frac{\pi^2}{180} - \frac{2}{15}H_{1,1}(z)\right)$$
$$+ \epsilon^3 \left(-\frac{1}{90}\pi^2 H_1(z) - \frac{13}{15}H_{1,1,1}(z) + \frac{7}{15}\zeta(3)\right)$$
$$+ \epsilon^4 \left(\frac{4}{3}\zeta(3)H_1(z) - \frac{7}{45}\pi^2 H_{1,1}(z) - \frac{16}{3}H_{1,1,1,1}(z) + \frac{19}{5400}\pi^4 \right)$$
$$+ \epsilon^5 \Big(- \frac{1}{54}\pi^4 H_1(z) + \frac{76}{15}\zeta(3)H_{1,1}(z) - \frac{10}{9}\pi^2 H_{1,1,1}(z)$$
$$- \frac{484}{15}H_{1,1,1,1,1}(z) + \frac{32}{5}\zeta(5) - \frac{1}{18}\pi^2\zeta(3)\Big) + \mathcal{O}(\epsilon^6)\Bigg],$$

$$\widetilde{G}_{4B17} = n_2(\epsilon)\Bigg[\epsilon^2\frac{\pi^2}{60} + \epsilon^3 \left(\frac{1}{30}\pi^2 H_1(z) + \frac{11}{20}\zeta(3)\right)$$
$$+ \epsilon^4 \left(\frac{11}{10}\zeta(3)H_1(z) + \frac{1}{15}\pi^2 H_{1,1}(z) + \frac{29}{1800}\pi^4 \right)$$
$$+ \epsilon^5 \Big(\frac{29}{900}\pi^4 H_1(z) + \frac{11}{5}\zeta(3)H_{1,1}(z) + \frac{2}{15}\pi^2 H_{1,1,1}(z)$$
$$+ \frac{177}{20}\zeta(5) - \frac{1}{3}\pi^2\zeta(3)\Big) + \mathcal{O}(\epsilon^6)\Bigg],$$

$$\widetilde{G}_{4B18} = n_2(\epsilon)\Bigg[\epsilon^2 \left(-\frac{1}{60}i\pi H_1(z) - \frac{1}{60}H_{1,1}(z) - \frac{\pi^2}{360}\right)$$
$$+ \epsilon^3 \left(\frac{1}{180}\pi^2 H_1(z) - \frac{7}{60}i\pi H_{1,1}(z) - \frac{3}{20}H_{1,1,1}(z) - \frac{7}{60}\zeta(3) - \frac{i\pi^3}{180}\right)$$
$$+ \epsilon^4 \Big(- \frac{4}{15}\zeta(3)H_1(z) + \frac{1}{12}\pi^2 H_{1,1}(z) - \frac{37}{60}i\pi H_{1,1,1}(z)$$

$$- \frac{11}{12}H_{1,1,1,1}(z) - \frac{1}{30}i\pi\zeta(3) + \frac{7}{2160}\pi^4\Big) + \mathcal{O}(\epsilon^5)\Bigg],$$

$$\begin{aligned}\widetilde{G}_{4B19} = n_2(\epsilon)\Bigg[&\epsilon^4\left(\frac{1}{6}i\pi^3 H_1(z) + \frac{1}{6}\pi^2 H_{1,1}(z) + \frac{7}{360}\pi^4\right)\\ &+ \epsilon^5\Big(9i\pi\zeta(3)H_1(z) - \frac{13}{180}\pi^4 H_1(z) + 9\zeta(3)H_{1,1}(z) + \frac{7}{6}i\pi^3 H_{1,1}(z)\\ &+ \frac{3}{2}\pi^2 H_{1,1,1}(z) + 9\zeta(5) + \frac{7}{6}\pi^2\zeta(3) + \frac{1}{18}i\pi^5\Big) + \mathcal{O}(\epsilon^6)\Bigg],\end{aligned}$$

$$\begin{aligned}\widetilde{G}_{4B20} = n_2(\epsilon)\Bigg[&\frac{5}{72} + \epsilon\left(\frac{5}{36}H_1(z) + \frac{1}{18}i\pi\right) + \epsilon^2\left(\frac{1}{9}i\pi H_1(z) + \frac{1}{18}H_{1,1}(z) - \frac{5}{36}\pi^2\right)\\ &+ \epsilon^3\left(\frac{11}{27}\pi^2 H_1(z) - \frac{1}{9}i\pi H_{1,1}(z) + \frac{16}{9}H_{1,1,1}(z) + \frac{31}{18}\zeta(3) + \frac{5}{54}i\pi^3\right)\\ &+ \mathcal{O}(\epsilon^4)\Bigg],\end{aligned}$$

$$\begin{aligned}\widetilde{G}_{4B21} = n_2(\epsilon)\Bigg[&-\frac{1}{6} - \frac{1}{3}\epsilon H_1(z) + \epsilon^2\left(\frac{2}{3}i\pi H_1(z) + \frac{2}{3}H_{1,1}(z)\right)\\ &- \epsilon^3\left(\frac{2}{9}\pi^2 H_1(z) - \frac{14}{3}i\pi H_{1,1}(z) - \frac{38}{3}H_{1,1,1}(z) + \frac{14}{3}\zeta(3) - \frac{2}{9}i\pi^3\right)\\ &+ \epsilon^4\Big(-\frac{44}{3}\zeta(3)H_1(z) - \frac{2}{3}i\pi^3 H_1(z) - \frac{20}{9}\pi^2 H_{1,1}(z) + \frac{74}{3}i\pi H_{1,1,1}(z)\\ &+ \frac{302}{3}H_{1,1,1,1}(z) + \frac{4}{3}i\pi\zeta(3) - \frac{74}{135}\pi^4\Big) + \mathcal{O}(\epsilon^5)\Bigg],\end{aligned}$$

$$\begin{aligned}\widetilde{G}_{4B22} = n_2(\epsilon)\Bigg[&-\frac{5}{24} + \epsilon\left(-\frac{5}{12}H_1(z) - \frac{1}{6}i\pi\right) + \epsilon^2\left(-\frac{1}{3}i\pi H_1(z) - \frac{1}{6}H_{1,1}(z) + \frac{1}{6}\pi^2\right)\\ &+ \epsilon^3\left(\frac{5}{9}\pi^2 H_1(z) - \frac{1}{3}i\pi H_{1,1}(z) + \frac{16}{3}H_{1,1,1}(z) - \frac{25\zeta(3)}{3} + \frac{i\pi^3}{9}\right) + \mathcal{O}(\epsilon^4)\Bigg],\end{aligned}$$

$$\begin{aligned}\widetilde{G}_{4B23} = n_3(\epsilon)\Bigg[&\frac{1}{40} + \frac{1}{20}\epsilon H_1(z) + \epsilon^2\left(\frac{1}{10}H_{1,1}(z) - \frac{\pi^2}{60}\right)\\
&+ \epsilon^3\left(-\frac{1}{30}\pi^2 H_1(z) + \frac{1}{5}H_{1,1,1}(z) - \frac{3\zeta(3)}{10}\right)\\
&+ \epsilon^4\left(-\frac{3}{5}\zeta(3)H_1(z) - \frac{1}{15}\pi^2 H_{1,1}(z) + \frac{2}{5}H_{1,1,1,1}(z) - \frac{\pi^4}{600}\right)\\
&+ \epsilon^5\Big(-\frac{1}{300}\pi^4 H_1(z) - \frac{6}{5}\zeta(3)H_{1,1}(z) - \frac{2}{15}\pi^2 H_{1,1,1}(z)\\
&+ \frac{4}{5}H_{1,1,1,1,1}(z) - \frac{3}{2}\zeta(5) + \frac{1}{5}\pi^2\zeta(3)\Big) + \mathcal{O}(\epsilon^6)\Bigg],\end{aligned}$$

$$\begin{aligned}\widetilde{G}_{4B24} = n_3(\epsilon)\Bigg[&-\frac{11}{120} - \frac{11}{60}\epsilon H_1(z) + \epsilon^2\left(-\frac{11}{30}H_{1,1}(z) - \frac{11}{720}\pi^2\right)\\
&+ \epsilon^3\left(-\frac{11}{360}\pi^2 H_1(z) - \frac{11}{15}H_{1,1,1}(z) - \frac{77}{30}\zeta(3)\right) + \mathcal{O}(\epsilon^4)\Bigg],\end{aligned}$$

$$\begin{aligned}\widetilde{G}_{4B25} = n_3(\epsilon)\Bigg[&-\pi^2\epsilon^2 + \epsilon^3\left(-\frac{11}{180}\pi^2 H_1(z) - \frac{22}{15}\zeta(3)\right)\\
&+ \epsilon^4\left(-\frac{44}{15}\zeta(3)H_1(z) - \frac{11}{90}\pi^2 H_{1,1}(z) - \frac{44}{675}\pi^4\right) + \mathcal{O}(\epsilon^5)\Bigg],\end{aligned}$$

$$\begin{aligned}\widetilde{G}_{4B26} = n_3(\epsilon)\Bigg[&\epsilon^2\frac{11}{180}\pi^2 + \epsilon^3\left(\frac{11}{90}\pi^2 H_1(z) + \frac{44}{15}\zeta(3)\right)\\
&+ \epsilon^4\left(\frac{88}{15}\zeta(3)H_1(z) + \frac{11}{45}\pi^2 H_{1,1}(z) + \frac{88}{675}\pi^4\right) + \mathcal{O}(\epsilon^5)\Bigg],\end{aligned}$$

$$\begin{aligned}\widetilde{G}_{4B27} = n_3(\epsilon)\Bigg[&\frac{1}{350} + \epsilon\frac{1}{1050}H_1(z) + \epsilon^2\left(-\frac{2}{75}H_{1,1}(z) - \frac{47}{6300}\pi^2\right)\\
&+ \epsilon^3\left(-\frac{67}{3150}\pi^2 H_1(z) - \frac{118}{525}H_{1,1,1}(z) - \frac{181}{1050}\zeta(3)\right)\end{aligned}$$

$$
+ \epsilon^4 \left(-\frac{121}{525}\zeta(3)H_1(z) - \frac{127}{1575}\pi^2 H_{1,1}(z) - \frac{776}{525}H_{1,1,1,1}(z) - \frac{419}{47250}\pi^4 \right)
+ \epsilon^5 \Big(-\frac{169}{6750}\pi^4 H_1(z) + \frac{118}{525}\zeta(3)H_{1,1}(z) - \frac{614}{1575}\pi^2 H_{1,1,1}(z)
- \frac{4792}{525}H_{1,1,1,1,1}(z) - \frac{247}{70}\zeta(5) + \frac{311}{1575}\pi^2\zeta(3)\Big)
+ \mathcal{O}(\epsilon^6) \Bigg],
$$

$$
\widetilde{G}_{4\mathrm{B}28} = n_3(\epsilon)\Bigg[\frac{1}{60} + \frac{1}{20}\epsilon H_1(z) + \epsilon^2\left(\frac{1}{6}H_{1,1}(z) - \frac{\pi^2}{120}\right)
+ \epsilon^3\left(-\frac{1}{45}\pi^2 H_1(z) + \frac{3}{5}H_{1,1,1}(z) - \frac{13}{60}\zeta(3)\right)
+ \epsilon^4\left(-\frac{2}{3}\zeta(3)H_1(z) - \frac{1}{15}\pi^2 H_{1,1}(z) + \frac{34}{15}H_{1,1,1,1}(z) - \frac{23}{10800}\pi^4\right)
+ \mathcal{O}(\epsilon^5)\Bigg],
$$

$$
\widetilde{G}_{4\mathrm{B}29} = n_3(\epsilon)\Bigg[-\frac{1}{20}\pi^2\epsilon^2 + \epsilon^3\left(-\frac{1}{10}\pi^2 H_1(z) - \frac{33}{20}\zeta(3)\right)
+ \epsilon^4\left(-\frac{33}{10}\zeta(3)H_1(z) - \frac{1}{5}\pi^2 H_{1,1}(z) - \frac{29}{600}\pi^4\right) + \mathcal{O}(\epsilon^5)\Bigg],
$$

$$
\widetilde{G}_{4\mathrm{B}30} = n_3(\epsilon)\Bigg[\frac{1}{3150} - \epsilon\frac{1}{1050}H_1(z) + \epsilon^2\left(-\frac{2}{175}H_{1,1}(z) - \frac{17}{9450}\pi^2\right)
+ \epsilon^3\left(-\frac{1}{175}\pi^2 H_1(z) - \frac{2}{25}H_{1,1,1}(z) - \frac{13}{350}\zeta(3)\right)
+ \epsilon^4\left(-\frac{19}{525}\zeta(3)H_1(z) - \frac{38}{1575}\pi^2 H_{1,1}(z) - \frac{88}{175}H_{1,1,1,1}(z) - \frac{37}{15750}\pi^4\right)
+ \epsilon^5\Big(-\frac{337}{47250}\pi^4 H_1(z) + \frac{82}{525}\zeta(3)H_{1,1}(z) - \frac{28}{225}\pi^2 H_{1,1,1}(z)
- \frac{536}{175}H_{1,1,1,1,1}(z) - \frac{59}{70}\zeta(5) + \frac{79}{1575}\pi^2\zeta(3)\Big) + \mathcal{O}(\epsilon^6)\Bigg],
$$

$$\begin{aligned}\widetilde{G}_{4\mathrm{B}31} = n_3(\epsilon)\Bigg[&\frac{1}{20}\epsilon H_1(z) + \epsilon^2\left(\frac{2}{5}H_{1,1}(z) - \frac{\pi^2}{60}\right)\\ &+ \epsilon^3\left(\frac{1}{30}\pi^2 H_1(z) + \frac{13}{5}H_{1,1,1}(z) - \frac{7}{5}\zeta(3)\right)\\ &+ \epsilon^4\left(-4\zeta(3)H_1(z) + \frac{7}{15}\pi^2 H_{1,1}(z) + 16H_{1,1,1,1}(z) - \frac{19}{1800}\pi^4\right)\\ &+ \mathcal{O}(\epsilon^5)\Bigg],\end{aligned}$$

$$\begin{aligned}\widetilde{G}_{4\mathrm{B}32} = n_3(\epsilon)\Bigg[&-\frac{1}{5}\epsilon H_1(z) + \epsilon^2\left(-\frac{8}{5}H_{1,1}(z) - \frac{13}{60}\pi^2\right)\\ &+ \epsilon^3\left(-\frac{7}{10}\pi^2 H_1(z) - \frac{52}{5}H_{1,1,1}(z) - 5\zeta(3)\right)\\ &+ \epsilon^4\left(-\frac{26}{5}\zeta(3)H_1(z) - 3\pi^2 H_{1,1}(z) - 64H_{1,1,1,1}(z) - \frac{74}{225}\pi^4\right)\\ &+ \mathcal{O}(\epsilon^5)\Bigg],\end{aligned}$$

$$\widetilde{G}_{4\mathrm{B}33} = n_3(\epsilon)\Big[0 + \mathcal{O}(\epsilon^5)\Big],$$

$$\widetilde{G}_{4\mathrm{B}34} = n_3(\epsilon)\Big[0 + \mathcal{O}(\epsilon^5)\Big],$$

$$\widetilde{G}_{4\mathrm{B}35} = n_3(\epsilon)\Big[0 + \mathcal{O}(\epsilon^6)\Big],$$

$$\widetilde{G}_{4\mathrm{B}36} = n_3(\epsilon)\Bigg[-\frac{1}{4}\pi^2\epsilon^2 + \epsilon^3\left(-\pi^2 H_1(z) - \frac{21}{2}\zeta(3)\right)$$

$$+\epsilon^4\left(-42\zeta(3)H_1(z)-4\pi^2H_{1,1}(z)-\frac{5}{12}\pi^4\right)+\mathcal{O}(\epsilon^5)\Bigg],$$

$$\begin{aligned}\widetilde{G}_{4\mathrm{B}37}=n_3(\epsilon)\Bigg[&-\frac{1}{8}-\frac{1}{4}\epsilon H_1(z)+\epsilon^2\left(\frac{1}{2}H_{1,1}(z)+\frac{1}{4}\pi^2\right)\\&+\epsilon^3\left(\frac{4}{3}\pi^2H_1(z)+11H_{1,1,1}(z)+\frac{1}{2}\zeta(3)\right)\\&+\epsilon^4\left(-4\zeta(3)H_1(z)+\frac{22}{3}\pi^2H_{1,1}(z)+98H_{1,1,1,1}(z)+\frac{43}{180}\pi^4\right)\\&+\mathcal{O}(\epsilon^5)\Bigg],\end{aligned}$$

$$\begin{aligned}\widetilde{G}_{4\mathrm{B}38}=n_2(\epsilon)\Bigg[&\frac{1}{225}+\epsilon\Big(\frac{37}{2700}H_1(z)+\frac{1}{180}i\pi\Big)\\&+\epsilon^2\Big(\frac{3}{100}i\pi H_1(z)+\frac{143}{2700}H_{1,1}(z)-\frac{26}{2025}\pi^2\Big)+\epsilon^3\Big(-\frac{121}{2700}\pi^2H_1(z)\\&+\frac{43}{300}i\pi H_{1,1}(z)+\frac{601}{2700}H_{1,1,1}(z)-\frac{433}{1350}\zeta(3)-\frac{2}{675}i\pi^3\Big)\\&+\epsilon^4\Big(-\frac{493}{675}\zeta(3)H_1(z)-\frac{1}{45}i\pi^3H_1(z)-\frac{163}{900}\pi^2H_{1,1}(z)\\&+\frac{193}{300}i\pi H_{1,1,1}(z)+\frac{2507}{2700}H_{1,1,1,1}(z)-\frac{8}{75}i\pi\zeta(3)-\frac{791}{60750}\pi^4\Big)\\&+\mathcal{O}(\epsilon^5)\Bigg].\end{aligned}$$

Except for the first boundary condition, $\widetilde{G}_{4\mathrm{B}1}$, we can bring every expression coming from the same family into UT form with the same factor $n_i(\epsilon)$. The first integral is an exception as it is not related to the others via the differential equations and is solved on its own. It can be made UT by a different factor, but this does not yield additional information, so it is left in its original form.

4.10 Results for the Four-Body Master Integrals

With the boundary conditions at hand, we can now finally calculate the integrals themselves. The z dependent part is fully determined by iterative application of the differential equations, where the ϵ-form allows us to calculate the results order by order. The constant terms are then fixed at each order separately by comparing the asymptotic behaviour in the same limit as the corresponding boundary condition.
Taking the same integral as in case three of the boundary calculation, $G_{4\mathrm{B}36}$, as an example, we arrive at the result:

$$\begin{aligned} G_{4\mathrm{B}36} = n_3(\epsilon)\Bigg[&\epsilon^2\frac{3}{2}H_{1,0}(z) + \epsilon^3\Big(-\frac{1}{2}\pi^2 H_1(z) + 3H_{1,2}(z) - \frac{21}{2}H_{1,0,0}(z) + 6H_{1,1,0}(z)\Big) \\ &+ \epsilon^4\Big(24S_{2,2}(z) + 9\zeta(3)H_1(z) + \pi^2 H_{1,0}(z) - 2\pi^2 H_{1,1}(z) - 33H_{1,3}(z) \\ &- 24H_{3,1}(z) + 6H_{1,1,2}(z) - 9H_{1,2,0}(z) + \frac{117}{2}H_{1,0,0,0}(z) - 42H_{1,1,0,0}(z) \\ &+ 18H_{1,1,1,0}(z)\Big) + \mathcal{O}(\epsilon^5)\Bigg] . \end{aligned} \tag{4.66}$$

The complete results for all 38 integrals $G_{4\mathrm{B}i}$ are collected in Appendix A. These can then be related back to the F_{4Bi} base via the inverse of the transformation defined in Eq. (4.38). They are also collected in electronic form (in both bases) on `GitHub` [113].
Looking at our results, we notice that the members of each family are brought into the form of uniform transcendentality by the same factor, which is a first sanity check for the results. An exception for this is $G_{4\mathrm{B}1}$, but this is to be expected since its determination is completely decoupled from that of the others.

4.11 Relations Between the Master Integrals

As we mentioned before, there are some additional relations between some of the integrals across the families. These were used as an additional cross-check for our results.

For completeness, we give them here:

$$F_{4B13} = \frac{(1-\epsilon)}{z-1} F_{4B8} + \frac{(2z(\epsilon-1)+2\epsilon-1)}{z-1} F_{4B12} + \frac{2z(4\epsilon-3)}{(z-1)^2} F_{4B24}\,, \tag{4.67}$$

$$\begin{aligned} F_{4B16} = {} & \frac{(\epsilon-1)}{\bar{z}^2\epsilon} F_{4B8} + \frac{(2-4\epsilon)}{\bar{z}\epsilon} F_{4B9} + \frac{\epsilon}{2\epsilon-1} F_{4B14} - \frac{1}{\bar{z}\epsilon} F_{4B11} \\ & + \frac{(6\bar{z}\epsilon - 4\bar{z} - 4\epsilon + 3)}{\bar{z}^2\epsilon} F_{4B12} - \frac{2(\bar{z}-1)(4\epsilon-3)}{\bar{z}^3\epsilon} F_{4B24} + \frac{\epsilon}{1-2\epsilon} F_{4B15}\,, \end{aligned} \tag{4.68}$$

$$\begin{aligned} F_{4B26} = {} & \frac{(\epsilon-1)(z(4\epsilon-3)+1)}{(z-1)^3\epsilon} F_{4B8} + \frac{(2-4\epsilon)}{z-1} F_{4B9} + \frac{(\epsilon-3\epsilon^2)}{2\epsilon-1} F_{4B14} \\ & + \frac{1}{z-1} F_{4B10} + \frac{(1-2\epsilon)}{\epsilon - z\epsilon} F_{4B11} + \frac{\epsilon(3\epsilon-1)}{2\epsilon-1} F_{4B15} \\ & + \frac{(2\left(z^2-12z+3\right)\epsilon^2 + \left(2z^2+23z-5\right)\epsilon - 2z^2 - 5z + 1)}{(z-1)^3\epsilon} F_{4B12} \\ & - \frac{2z(4\epsilon-3)((z+3)\epsilon - z - 1)}{(z-1)^4\epsilon} F_{4B24}\,. \end{aligned} \tag{4.69}$$

Chapter 5

Calculation of the Five-Body Gluon Bremsstrahlung Diagrams

After the calculation of the four-body one-loop diagrams we now turn to the real-emission counterpart, which in our case are the five-body gluon bremsstrahlung diagrams. These need to be added to the scattering amplitude to cancel the infrared divergences caused by the gluon in our calculation. This cancellation is described by the Kinoshita-Lee-Nauenberg theorem, which is discussed in-depth in the standard literature, for example in Refs. [42, 43].
If we integrated over the entire phase space, the same statement would hold for the infrared diveregences generated by the photon. But, because of the cut on E_γ this does not hold true and we have to use splitting functions to regulate these later in chapter 7.

5.1 Setup of the Calculation Compared to the Four-Body One-Loop Case

The general setup of the calculation is reused from the case of the four-body one-loop diagrams. This means that we have the same operator insertions as discussed in Sect. 4.1 and that we treat the occurrence of γ_5 in the same way we described in Sect. 4.3.

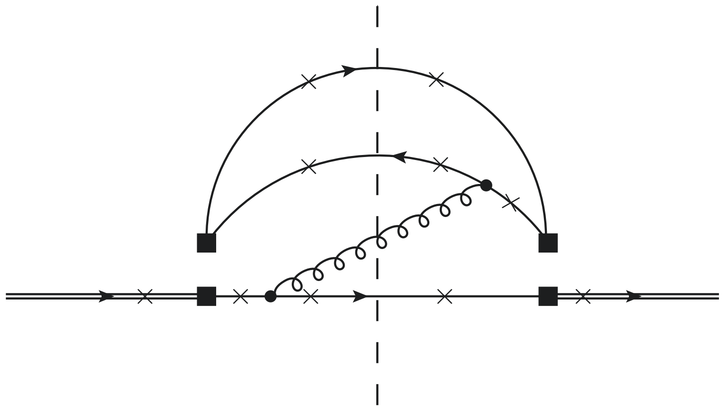


Figure 5.1: This figure shows the different ways that the photon can attach to the quark legs for a given gluon configuration. A cross denotes a possible spot for the photon to attach. It always has to choose one on the left and one on the right side of the cut.

Instead of 176 diagrams in the four-body case, we now encounter $16 \times 25 = 400$ diagrams. This number is put together from the 4×4 ways to attach the gluon on each side of the cut multiplied by the then 5×5 ways to add a photon to this. The possible attachments are shown in Fig. 5.1 for one of the 16 gluon configurations.
Note that there is no one-to-one correspondence between the four- and five-body diagrams as, for example, gluon insertions that are absorbed by wave-function renormalization in the four-particle case have corresponding five-particle cuts that are counted towards the unrenormalized squared amplitude.
We generate and process the diagrams using the same setup as before, i.e. utilizing `QGRAF` and `Form`. The resulting integrals are then reduced in `FIRE`.

5.2 Phase Space Parametrization for the Five-Body Diagrams

For the diagrams at hand, we have an extra particle, the gluon, in the final state and thus need to integrate them over the five-particle phase space dPS_5. The momenta of the process in question are assigned as follows:

$$b(p_b) \to q(p_1)\bar{q}(p_2)s(p_3)\gamma(p_4)g(p_5)\,.$$

Since there are more independent momentum invariants, this involves more integrations than in the four-particle case, making the direct calculation very cumbersome. The complete derivation of the parametrization we are using can be found in Ref. [72], a brief overview is given in Appendix H. The final result we are using for the invariants is the following:

$$\begin{aligned}
s_{1345}/q^2 &= t_7\,, & s_{34}/q^2 &= t_2t_6t_7\bar{t}_4\,,\\
s_{134}/q^2 &= t_6t_7\,, & s_{15}/q^2 &= t_7\bar{t}_6[1-t_9(1-t_2t_2)]-y_{10}\,,\\
s_{13}/q^2 &= t_6t_7\bar{t}_2\,, & s_{25}/q^2 &= y_8^- + (y_8^+ - y_8^-)t_8\\
s_{23}/q^2 &= t_3\bar{t}_7(1-t_2t_4)(t_6\bar{t}_9+t_9)\,, & s_{35}/q^2 &= t_7t_9\bar{t}_6(1-t_2t_4)\,,\\
s_{14}/q^2 &= t_2t_4t_6t_7 & s_{45}/q^2\,, &= y_{10}^- + (y_{10}^+ - y_{10}^-)t_{10}\,,\\
s_{24}/q^2 &= y_5^- + (y_5^+ - y_5^-)t_5)\,, & & \qquad (5.1)
\end{aligned}$$

with the phase space integrations reading:

$$\int d\Phi^D_{1\to5} = \mathcal{K}^{(5)}_\Gamma (q^2)^{2D-5}\int\limits_0^1 \prod_{j=2}^{10} dt_j\,[t_5\bar{t}_5]^{-1-\epsilon}[t_8\bar{t}_8t_{10}\bar{t}_{10}]^{-\frac{1}{2}-\epsilon}$$
$$\times\,[t_2t_6\bar{t}_6\bar{t}_7]^{1-2\epsilon}[(\bar{t}_2t_3\bar{t}_3t_4\bar{t}_4t_9\bar{t}_9)]^{-\epsilon}t_7^{2-3\epsilon}\,. \qquad (5.2)$$

The prefactor $\mathcal{K}^{(5)}_\Gamma$ is defined as:

$$\begin{aligned}
\mathcal{K}^{(5)}_\Gamma &= (2\pi)^{5-4D}2^{-2-2D}2^{-8\epsilon}V(D-1)V(D-2)V(D-3)V(D-4)\\
&= \frac{4^{4\epsilon}}{2^{17}\pi^9\Gamma(-2\epsilon)\Gamma(2-2\epsilon)}\,,
\end{aligned} \qquad (5.3)$$

with the angular volumes V defined as

$$V(D) = \frac{2\pi^{D/2}}{\Gamma(D/2)}\,. \qquad (5.4)$$

In the four-body case, the parametrization of the invariants included the cut variable z explicitly. For the five-body integrals, we chose a different approach for obtaining the boundary conditions that removes the dependence on the cut, allowing us to use the above parametrization. This will be discussed in detail in Sect. 5.6.

Not including the cut explicitly for the calculation of boundary conditions later gives us the freedom to relabel all the invariants in Eq. (5.1), but during the reduction step we still have to exclude p_4, the photon momentum, from this symmetry (as we did in the last chapter).

5.3 Specifics of the IBP Reduction

Similar to the four-body case, we are also conducting an integration-by-parts reduction on the integrals of the five-body diagrams. We can sort them into a total of six different families, of which the first three contain only massless propagators, the fourth containts a single massive one and the last two contain up to two massive propagators. They are defined as follows:

$$
\begin{aligned}
\mathcal{F}_{101} = \{& p_1^2,\, p_2^2,\, p_3^2\,, (p_b - p_{123})^2,\, p_5^2,\, p_{1235}^2 - z\, m_b^2,\, (p_b - p_{13})^2,\, (p_b - p_{235})^2, \\
& (p_b - p_{135})^2,\, (p_1 + p_5)^2,\, (p_2 + p_5)^2,\, (p_3 + p_5)^2,\, (p_1 + p_3)^2,\, (p_2 + p_3)^2\}\,, \\
\mathcal{F}_{102} = \{& p_1^2,\, p_2^2,\, p_3^2\,, (p_b - p_{123})^2,\, p_5^2,\, p_{1235}^2 - z\, m_b^2,\, (p_b - p_{23})^2,\, (p_b - p_{13})^2, \\
& (p_b - p_{235})^2,\, (p_b - p_{135})^2,\, (p_1 + p_5)^2,\, (p_3 + p_5)^2,\, (p_1 + p_3)^2,\, (p_2 + p_3)^2\}\,, \\
\mathcal{F}_{103} = \{& p_1^2,\, p_2^2,\, p_3^2\,, (p_b - p_{123})^2,\, p_5^2,\, p_{1235}^2 - z\, m_b^2,\, (p_b - p_{23})^2,\, (p_b - p_{13})^2, \\
& (p_b - p_{135})^2,\, (p_1 + p_5)^2,\, (p_2 + p_5)^2,\, (p_3 + p_5)^2,\, (p_1 + p_3)^2,\, (p_2 + p_3)^2\}\,, \\
\mathcal{F}_{104} = \{& p_1^2,\, p_2^2,\, p_3^2\,, (p_b - p_{123})^2,\, p_5^2,\, p_{1235}^2 - z\, m_b^2,\, (p_b - p_5)^2 - m_b^2,\, (p_b - p_{12})^2, \\
& (p_b - p_{235})^2,\, (p_b - p_{125})^2,\, (p_2 + p_5)^2,\, (p_3 + p_5)^2,\, (p_1 + p_3)^2,\, (p_2 + p_3)^2\}\,, \\
\mathcal{F}_{105} = \{& p_1^2,\, p_2^2,\, p_3^2\,, (p_b - p_{123})^2,\, p_5^2,\, p_{1235}^2 - z\, m_b^2,\, (p_{123})^2 - m_b^2,\, (p_b - p_5)^2 - m_b^2, \\
& (p_b - p_{12})^2,\, (p_b - p_{135})^2,\, (p_2 + p_5)^2,\, (p_3 + p_5)^2,\, (p_1 + p_3)^2,\, (p_2 + p_3)^2\}\,, \\
\mathcal{F}_{106} = \{& p_1^2,\, p_2^2,\, p_3^2\,, (p_b - p_{123})^2,\, p_5^2,\, p_{1235}^2 - z\, m_b^2,\, (p_{123})^2 - m_b^2,\, (p_b - p_5)^2 - m_b^2, \\
& (p_b - p_{13})^2,\, (p_b - p_{135})^2,\, (p_b - p_{125})^2,\, (p_1 + p_5)^2,\, (p_1 + p_3)^2,\, (p_2 + p_3)^2\}\,.
\end{aligned}
\tag{5.5}
$$

Here, as before, we eliminated p_4 in favor of the other momenta to make all the denominators linearly independent. When sorting the integrals into the six families, we can use symmetries in the momenta of all final-state particles except the photon, i.e. p_1, p_2, p_3 and p_5.
Using reversed unitarity in this case leads to an additional propagator, p_5^2, compared to the four-body reduction, so now the first 6 denominators are fixed and their corresponding diagrams are set to zero if their power becomes non-positive.

5.3.1 Relations Between the Four- and Five-Body Reductions

One interesting observation we made in the course of the calculation is that there is a correspondence between the reductions of the four- and five-body case. When applying reversed unitarity, we treat the involved momenta as loop momenta. This leads to the interesting fact that, regarding the reduction algorithm, the behaviour of the p_i does not conceptually differ from that of k anymore.
Observing this, one can establish a relation between the reductions by relabelling p_5 as k or the other way round. This makes it in priciple possible to reduce the overall number of required families, because under relabeling they can be mapped onto each other.
The downside to this is that the aforementioned vanishing of the diagrams for negative powers of the denominators is not as straightforward to implement, as we do not have an on-shell condition for k. Hence, this has to be done by hand diagram-by-diagram when turning k back into p_5. For our case, the non-vanishing of certain integrals leads to a larger number of terms during the reduction process, making the combined reductions slower than the separate ones.
This could be optimized for future projects similar to this one, while for us it serves as a semi-independent cross-check for the reduction results.

5.4 The Five-Body Master Integrals

The reductions, as in the four-body case, are done in `FIRE` while utilising the `OMNI` cluster of the Siegen University, yielding a total of 36 master integrals across the six families. We collect them here, using the families defined in Sect. 5.5 and the notation introduced in Sect. 4.6. The integrals are additionally given diagrammatically in Fig. 5.2.

5.4.1 List of the Five-Body Master Integrals

$$
\begin{aligned}
F_{5\text{B}1} &= F_{101}[1,1,1,1,1,1;0,0,0,0,0,0,0,0]\,,\\
F_{5\text{B}2} &= F_{101}[1,1,1,1,1,1;1,0,0,0,0,0,0,0]\,,\\
F_{5\text{B}3} &= F_{101}[1,1,1,1,1,1;1,0,0,1,0,0,0,0]\,,\\
F_{5\text{B}4} &= F_{101}[1,1,1,1,1,1;1,0,0,2,0,0,0,0]\,,\\
F_{5\text{B}5} &= F_{101}[1,1,1,1,1,1;0,1,1,1,1,0,0,0]\,,
\end{aligned}
$$

$$
\begin{aligned}
F_{5\text{B}6} &= F_{102}[1,1,1,1,1,1;1,1,0,0,0,0,0,0]\,,\\
F_{5\text{B}7} &= F_{102}[1,1,1,1,1,1;1,2,0,0,0,0,0,0]\,,\\
F_{5\text{B}8} &= F_{102}[1,1,1,1,1,1;1,1,-1,0,0,0,0,0]\,,\\
F_{5\text{B}9} &= F_{102}[1,1,1,1,1,1;1,1,0,0,0,-1,0,0]\,,\\
F_{5\text{B}10} &= F_{102}[1,1,1,1,1,1;1,1,1,1,0,0,0,0]\,,
\end{aligned}
$$

$$
\begin{aligned}
F_{5\text{B}11} &= F_{103}[1,1,1,1,1,1;1,1,0,0,1,0,0,0]\,,\\
F_{5\text{B}12} &= F_{103}[1,1,1,1,1,1;1,1,0,1,1,0,0,0]\,,
\end{aligned}
$$

$$
\begin{aligned}
F_{5\text{B}13} &= F_{104}[1,1,1,1,1,1;1,0,0,0,0,0,0,0]\,,\\
F_{5\text{B}14} &= F_{104}[1,1,1,1,1,1;1,0,0,1,0,0,0,0]\,,\\
F_{5\text{B}15} &= F_{104}[1,1,1,1,1,1;1,1,0,0,0,0,0,0]\,,\\
F_{5\text{B}16} &= F_{104}[1,1,1,1,1,1;1,1,0,0,-1,1,0,0]\,,\\
F_{5\text{B}17} &= F_{104}[1,1,1,1,1,1;1,0,0,1,0,1,0,0]\,,\\
F_{5\text{B}18} &= F_{104}[1,1,1,1,1,1;1,1,1,0,0,0,0,0]\,,\\
F_{5\text{B}19} &= F_{104}[1,1,1,1,1,1;1,1,1,-1,0,-1,0,0]\,,\\
F_{5\text{B}20} &= F_{104}[1,1,1,1,1,1;1,1,1,1,0,0,0,0]\,,
\end{aligned}
$$

$$F_{5\text{B}21} = F_{104}[1,1,1,1,1,1;1,0,1,1,0,1,0,0]\,,$$

$$\begin{aligned}
F_{5\text{B}22} &= F_{105}[1,1,1,1,1,1;1,0,0,0,0,0,0,0]\,,\\
F_{5\text{B}23} &= F_{105}[1,1,1,1,1,1;1,1,0,0,0,0,0,0]\,,\\
F_{5\text{B}24} &= F_{105}[1,1,1,1,1,1;1,0,0,1,0,0,0,0]\,,\\
F_{5\text{B}25} &= F_{105}[1,1,1,1,1,1;1,0,1,0,0,0,0,0]\,,\\
F_{5\text{B}26} &= F_{105}[1,1,1,1,1,1;1,0,2,0,0,0,0,0]\,,\\
F_{5\text{B}27} &= F_{105}[1,1,1,1,1,1;1,0,0,1,0,1,0,0]\,,\\
F_{5\text{B}28} &= F_{105}[1,1,1,1,1,1;1,1,0,1,0,1,0,0]\,,\\
F_{5\text{B}29} &= F_{105}[1,1,1,1,1,1;1,-1,1,0,0,0,0,0]\,,\\
F_{5\text{B}30} &= F_{105}[1,1,1,1,1,1;1,1,1,0,0,0,0,0]\,,\\
F_{5\text{B}31} &= F_{105}[1,1,1,1,1,1;1,1,2,0,0,0,0,0]\,,\\
F_{5\text{B}32} &= F_{105}[1,1,1,1,1,1;1,1,1,0,0,1,0,0]\,,
\end{aligned}$$

$$\begin{aligned}
F_{5\text{B}33} &= F_{106}[1,1,1,1,1,1;0,1,1,0,0,-2,0,0]\,,\\
F_{5\text{B}34} &= F_{106}[1,1,1,1,1,1;1,-1,1,0,0,0,0,-1]\,,\\
F_{5\text{B}35} &= F_{106}[1,1,1,1,1,1;1,0,1,1,0,0,0,0]\,,\\
F_{5\text{B}36} &= F_{106}[1,1,1,1,1,1;1,1,1,1,0,0,0,0]\,.
\end{aligned}$$

As we discussed in the four-body case, some of the integrals that `FIRE` suggest as master integrals have to be replaced after this step by manually interchaning them with ones that are more suitable. This pertains to either integrals that already have a dot on one of the propagators from reversed unitarity (i.e. one of the first six entries is ≥ 2), as it is unclear how to treat these properly, or to integrals that result in a non-decoupling of z and ϵ in the differential equation later on.

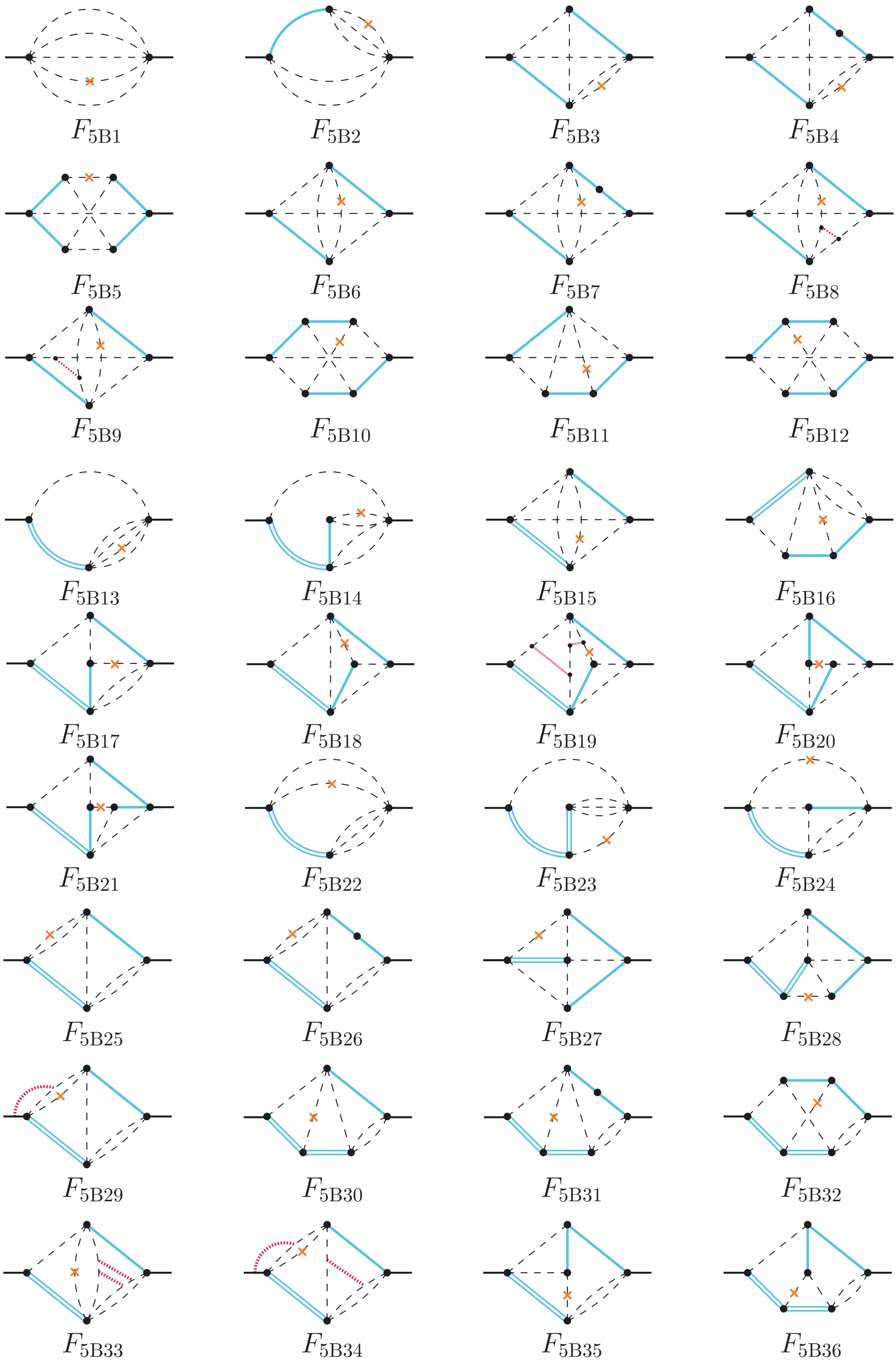


Figure 5.2: The full set of master integrals for the five-body bremsstrahlung contributions that was calculated in the course of this work. The dashed lines indicated the cut propagators from reversed unitarity relations, the solid light blue (double-)lines indicate (massive) propagators. The dotted red lines show numerators in the following way: When connecting two lines with momenta l_1 and l_2, the corresponding numerator is $(l_1 - l_2)^2 - m_1^2 - m_2^2$. The orange cross denotes the cut propagator with momentum p_4 as the symmetry in the momenta is broken here through the cut on the photon energy.

5.5 Calculating the Master Integrals

The method we use to construct the general solutions for the five-body master integrals is the same that is described in Sect. 4.7 for the four-body case.
The derivative in z acting on a given integral squares the denominator originating from the δ-function implementing the cut, i.e.

$$\partial_z[s_{1235}-z]^{-1} = [s_{1235}-z]^{-2}\,.$$

These new integrals are reduced again and lead to six systems of equations, one for each family. They all are of the form:

$$\partial_z\vec{F} = \hat{A}(z,\epsilon)\vec{F}\,. \tag{5.6}$$

5.5.1 Solving the differential equations

After the IBP reduction gives us our basis of master integrals and we have made the neccessary changes discussed above, we arrive at the following sets of integrals:

$$\vec{F}_{101} = \{F_{5\mathrm{B}1}, F_{5\mathrm{B}2}, F_{5\mathrm{B}3}, F_{5\mathrm{B}4}, F_{5\mathrm{B}5}\}\,, \tag{5.7}$$

$$\vec{F}_{102} = \{F_{5\mathrm{B}1}, F_{5\mathrm{B}2}, F_{5\mathrm{B}6}, F_{5\mathrm{B}7}, F_{5\mathrm{B}8}, F_{5\mathrm{B}3}, F_{5\mathrm{B}4}, F_{5\mathrm{B}9}, F_{5\mathrm{B}10}\}\,, \tag{5.8}$$

$$\vec{F}_{103} = \{F_{5\mathrm{B}1}, F_{5\mathrm{B}2}, F_{5\mathrm{B}3}, F_{5\mathrm{B}4}, F_{5\mathrm{B}6}, F_{5\mathrm{B}7}, F_{5\mathrm{B}9}, F_{5\mathrm{B}10}, F_{5\mathrm{B}11}\}\,, \tag{5.9}$$

$$\begin{aligned}\vec{F}_{104} = \{&F_{5\mathrm{B}1}, F_{5\mathrm{B}2}, F_{5\mathrm{B}3}, F_{5\mathrm{B}4}, F_{5\mathrm{B}13}, F_{5\mathrm{B}14}, F_{5\mathrm{B}15},\\ &F_{5\mathrm{B}16}, F_{5\mathrm{B}17}, F_{5\mathrm{B}18}, F_{5\mathrm{B}19}, F_{5\mathrm{B}20}, F_{5\mathrm{B}21}\}\,,\end{aligned} \tag{5.10}$$

$$\begin{aligned}\vec{F}'_{105} = \{&F_{5\mathrm{B}1}, F_{5\mathrm{B}2}, F_{5\mathrm{B}13}, F_{5\mathrm{B}22}, F_{5\mathrm{B}23}, F_{5\mathrm{B}15}, F_{5\mathrm{B}16}, F_{5\mathrm{B}24},\\ &F_{5\mathrm{B}25}, F_{5\mathrm{B}26}, F_{5\mathrm{B}27}, F_{5\mathrm{B}28}, F_{5\mathrm{B}29}, F_{5\mathrm{B}30}, F_{5\mathrm{B}31}, F_{5\mathrm{B}32}\}\,,\end{aligned} \tag{5.11}$$

and

$$\vec{F}'_{106} = \{F_{5\text{B}1}, F_{5\text{B}2}, F_{5\text{B}13}, F_{5\text{B}22}, F_{5\text{B}23}, F_{5\text{B}24}, F_{5\text{B}14}, F_{5\text{B}15}, F_{5\text{B}33}, F_{5\text{B}25}, F_{5\text{B}26}, F_{5\text{B}34}, F_{5\text{B}35}, F_{5\text{B}30}, F_{5\text{B}31}, F_{5\text{B}36}\}\,. \tag{5.12}$$

Note that there is a significant overlap between these sets, as we need to make sure that our bases are closed under the reduction.

We further want to explain the 'prime'-notation for the last two groups: In these, we were not able to fully cast them into the ϵ-form, because the choice of basis integrals leads to non-integer powers in the matrices $\hat{A}(z,\epsilon)$. The algorithm we use to change our basis cannot easily account for this and fails to find a transformation. Trying to solve this problem, we changed our basis by changing powers of denominators in the integrals with the most lines, but we were not able to find a basis that eludes the problematic entries in the differential equation matrix. Because of this, we eliminated the last three from each group and computed them separately via the differential equation in original form. This will be discussed in Sect. 5.7. The new integral-vectors without these read:

$$\vec{F}_{105} = \{F_{5\text{B}1}, F_{5\text{B}2}, F_{5\text{B}13}, F_{5\text{B}22}, F_{5\text{B}23}, F_{5\text{B}15}, F_{5\text{B}16}, F_{5\text{B}24}, F_{5\text{B}25}, F_{5\text{B}26}, F_{5\text{B}27}, F_{5\text{B}28}, F_{5\text{B}29}\}\,, \tag{5.13}$$

and

$$\vec{F}_{106} = \{F_{5\text{B}1}, F_{5\text{B}2}, F_{5\text{B}13}, F_{5\text{B}22}, F_{5\text{B}23}, F_{5\text{B}24}, F_{5\text{B}14}, F_{5\text{B}15}, F_{5\text{B}33}, F_{5\text{B}25}, F_{5\text{B}26}, F_{5\text{B}34}, F_{5\text{B}35}\}\,. \tag{5.14}$$

With these adjustments, we can define the respective systems as:

$$\partial_z \vec{F}_i = \hat{A}_i(z,\epsilon)\vec{F}_i \qquad i \in \{101, 102, 103, 104, 105, 106\}\,. \tag{5.15}$$

These systems are now transformed into a basis, where they take the form

$$\partial_z \vec{G}_i = \epsilon \hat{A}_{i,\epsilon}(z)\vec{G}_i\,, \tag{5.16}$$

via the transformation matrices $\hat{T}$ that fulfil:

$$\vec{G} = \hat{T}^{-1}\vec{F}\,. \tag{5.17}$$

The matrices $\hat{T}_1$–$\hat{T}_6$ can be found in Appendix G.

The new differential equation systems that the transformations yielded were found to have poles in $z = 0, 1, 2$. For this reason, we rewrote them in terms of $\bar{z}$ to get an alphabet of $\bar{z} = -1, 0, 1$, enabling us to solve the equations in terms of HPLs. After the change of variables, we can write the systems as (notice the minus sign from the transformation):

$$\partial_{\bar{z}}\vec{G}_i = -\epsilon\hat{A}_{i,\epsilon}(\bar{z})\vec{G}_i\,, \tag{5.18}$$

with the matrices $\hat{A}_{i,\epsilon}$ splitting into:

$$\hat{A}_{i,\epsilon} = \frac{1}{\bar{z}}\hat{A}_{i,\bar{z}} + \frac{1}{1-\bar{z}}\hat{A}_{i,1-\bar{z}} + \frac{1}{1+\bar{z}}\hat{A}_{i,1+\bar{z}}\,. \tag{5.19}$$

The complete differential equation matrices can be found in Appendix D.

Contrary to the four-body integrals, we only use the ϵ-basis to solve the integrals iteratively in terms of HPLs, not fixing the boundary conditions there. This has its reason in the method we chose for determining the latter, which we now want to discuss.

5.6 Calculation of the Boundary Conditions

For the determination of the boundary conditions for our five-body integrals, we chose a method that differs from the ones we used in the four-body case.
When comparing the parametrizations of the phase space, we see that because of the extra particle it is not an easy task to avoid introducing roots into the integrals, which quickly makes analytic calculation, or even an expanded result via Mellin-Barnes representations, cumbersome, if not impossible.
For this reason, we used a method that avoids having to calculate the integrals with the cut on the energy of the photon, i.e. dropping the $\delta(z - s_{1235})$ in the phase space

integrals. Let us now explain in detail how this works.

5.6.1 Determination of Constants by Integration Over z

The basic idea of this method is that if we integrate a kernel over the whole phase space without the cut on the photon energy, the result has to be equal to the one with the δ-function that is introduced to account for the cut, when the latter is integrated over z from zero to one.
This integration, which normally only ranges from zero to the cut value δ, completes the full phase space and we can read off the correct constant part of the solutions we got from the differential equations by comparison. This method was introduced in Ref. [114] and has been successfully applied, for example, in Ref. [115].
Introducing the notation $\widetilde{F}$ for the boundary conditions, we can write:

$$\int_0^1 dz\, F_i(z,\epsilon) = \int_0^1 dz\, (\hat{T}(z,\epsilon)\vec{G}(z,\epsilon))_i = \widetilde{F}_i(\epsilon)\,. \tag{5.20}$$

The matrix $\hat{T}$ is the transformation matrix between the original and the ϵ-basis and the vector $\vec{G}$ contains the solutions of the differential equations with undetermined constant parts.
Note that $F_i(z,\epsilon)$ and $\widetilde{F}_i(\epsilon)$ are related by the removal of the propagator from reversed unitarity that originates in the $\delta(p_{1235}^2 - z\, m_b^2)$ (in case that there are no poles in the transformation when integrating over z, more on that in the next section).
For the right-hand side of the equation, the cut-less integrals $\widetilde{F}_i$, we can furthermore apply a second integration-by-parts reduction:

$$\widetilde{F}_i(\epsilon) \overset{\text{IBP}}{=} \sum_j c_{ij}(\epsilon)\, H_j(\epsilon)\,. \tag{5.21}$$

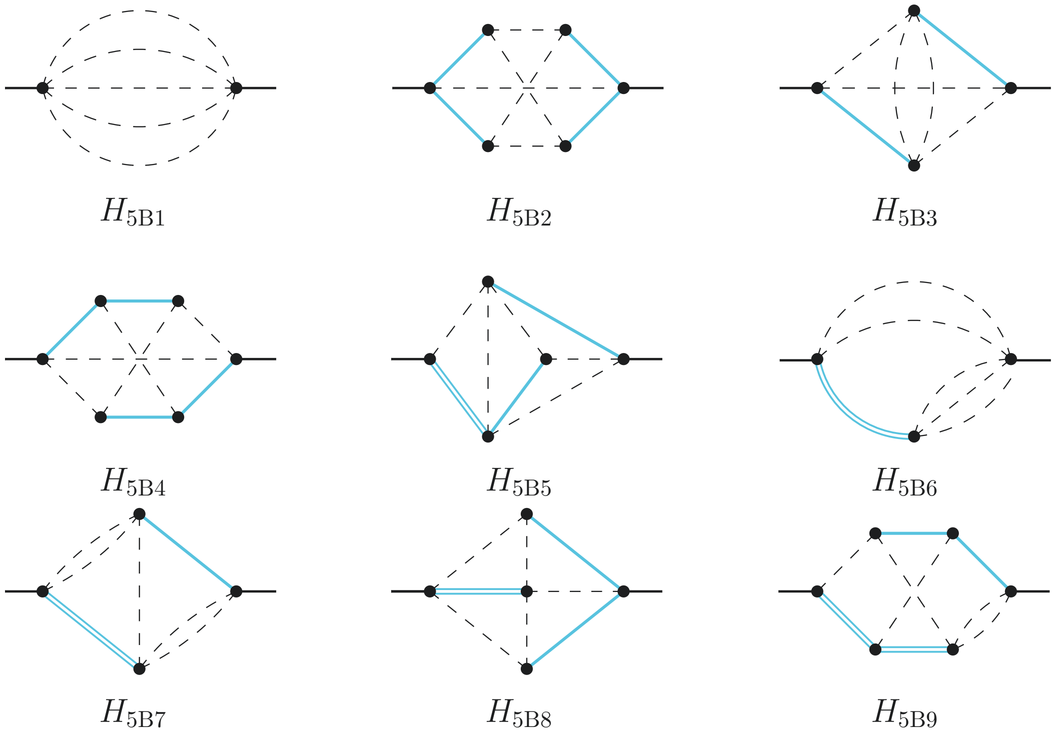
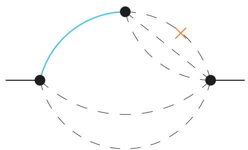


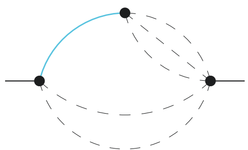
Figure 5.3: The full set of master integrals we encounter in the reduction without the cut on the photon energy. The dashed lines indicated the cut propagators from reversed unitarity relations, the solid light blue (double-)lines indicate (massive) propagators.

Diagrammatically, our approach looks as follows:

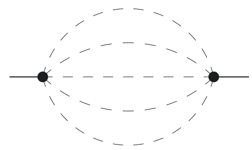
$$\int_0^1 dz$$



$$=$$



$$\stackrel{\text{IBP}}{=} \frac{2(4\epsilon-3)(5\epsilon-4)(5\epsilon-3)}{\epsilon(2\epsilon-1)(3\epsilon-1)} \tag{5.22}$$



For the reduction, we are using the families defined in Eq. (5.5) with the propagator $[p_{1235}^2 - z\, m_b^2]$ replaced by $[p_{1235}^2]$.
Eliminating the dependence on z significantly lowers the number of master integrals we encounter for these reductions. We are left with a total of nine integrals, which

are shown in Fig. 5.3. In the propagator notation, they read:

$$
\begin{aligned}
H_{5\mathrm{B}1} &= F_{101}[1,1,1,1,1,0;0,0,0,0,0,0,0,0]\,,\\
H_{5\mathrm{B}2} &= F_{101}[1,1,1,1,1,0;0,1,1,1,1,0,0,0]\,,\\[1em]
H_{5\mathrm{B}3} &= F_{102}[1,1,1,1,1,0;1,1,0,0,0,0,0,0]\,,\\
H_{5\mathrm{B}4} &= F_{102}[1,1,1,1,1,0;1,1,1,1,0,0,0,0]\,,\\[1em]
H_{5\mathrm{B}5} &= F_{104}[1,1,1,1,1,0;1,1,1,0,0,0,0,0]\,,\\[1em]
H_{5\mathrm{B}6} &= F_{105}[1,1,1,1,1,0;0,1,0,0,0,0,0,0]\,,\\
H_{5\mathrm{B}7} &= F_{105}[1,1,1,1,1,0;0,1,0,1,0,0,0,0]\,,\\
H_{5\mathrm{B}8} &= F_{105}[1,1,1,1,1,0;1,0,0,1,0,1,0,0]\,,\\
H_{5\mathrm{B}9} &= F_{105}[1,1,1,1,1,0;1,1,1,0,0,1,0,0]\,.
\end{aligned}
$$

The massless integrals in Fig. 5.3 were already calculated in Ref. [116] and we could use these results to calculate the boundary conditions for the first three families. For the other three families, which contain massive lines, we calculated them ourselves via Mellin-Barnes representations as a series in ϵ.

Collecting the results, we arrive at the following solutions for the base of integrals we use to construct the boundary conditions:

$$
\begin{aligned}
H_{5\mathrm{B}1} = n_0(\epsilon)\Bigg[&\frac{1}{144}+\frac{71}{576}\epsilon+\left(\frac{26815}{20736}-\frac{\pi^2}{54}\right)\epsilon^2+\left(-\frac{5}{12}\zeta(3)+\frac{872675}{82944}-\frac{71}{216}\pi^2\right)\epsilon^3\\
&+\left(-\frac{355}{48}\zeta(3)+\frac{220099831}{2985984}-\frac{26815}{7776}\pi^2+\frac{17}{2160}\pi^4\right)\epsilon^4+\mathcal{O}(\epsilon^5)\Bigg]\,,\\
H_{5\mathrm{B}2} = \frac{n_0(\epsilon)}{\epsilon^5}\Bigg[&\frac{1}{4}-\frac{1}{4}\epsilon+\left(\frac{7}{4}-\frac{29\pi^2}{36}\right)\epsilon^2+\frac{1}{36}\left(-834\zeta(3)-117+29\pi^2\right)\epsilon^3
\end{aligned}
$$

$$+ \left(\frac{139}{6}\zeta(3) + \frac{55}{4} - \frac{203}{36}\pi^2 + \frac{137}{1080}\pi^4\right)\epsilon^4 + \mathcal{O}(\epsilon^5)\Bigg],$$

$$H_{5B3} = n_0(\epsilon)\Bigg[- \frac{1}{2} + \frac{\pi^2}{12} + \left(\frac{9}{2}\zeta(3) - \frac{43}{4} + \frac{29}{24}\pi^2\right)\epsilon + \Big(\frac{261}{4}\zeta(3) - \frac{1071}{8} + \frac{581}{48}\pi^2 + \frac{7}{120}\pi^4\Big)\epsilon^2 + \Big(\frac{4893}{8}\zeta(3) - \frac{31}{2}\pi^2\zeta(3) + \frac{207}{2}\zeta(5) - \frac{20283}{16} + \frac{10105}{96}\pi^2 + \frac{203}{240}\pi^4\Big)\epsilon^3 + \mathcal{O}(\epsilon^4)\Bigg],$$

$$H_{5B4} = \frac{n_0(\epsilon)}{\epsilon^5}\Bigg[\frac{5}{3} - \frac{5}{3}\epsilon + \left(\frac{35}{3} - \frac{40}{9}\pi^2\right)\epsilon^2 + \left(-\frac{296}{3}\zeta(3) - \frac{65}{3} + \frac{40}{9}\pi^2\right)\epsilon^3 + \left(\frac{296}{3}\zeta(3) + \frac{275}{3} - \frac{280}{9}\pi^2 + 2\pi^4\right)\epsilon^4 + \mathcal{O}(\epsilon^5)\Bigg],$$

$$H_{5B5} = \frac{n_0(\epsilon)}{\epsilon}\Bigg[2\zeta(3) + \left(22\zeta(3) + \frac{7\pi^4}{30}\right)\epsilon + \mathcal{O}(\epsilon^2)\Bigg],$$

$$H_{5B6} = n_0(\epsilon)\Bigg[- \frac{5}{6} + \frac{\pi^2}{12} + \left(\frac{13}{2}\zeta(3) - \frac{2219}{144} + \frac{3}{4}\pi^2\right)\epsilon + \Big(\frac{117}{2}\zeta(3) - \frac{144383}{864} + \frac{893}{144}\pi^2 + \frac{25}{72}\pi^4\Big)\epsilon^2 + \Big(\frac{2883}{8}\zeta(3) - \frac{35}{2}\pi^2\zeta(3) + \frac{563}{2}\zeta(5) - \frac{3616801}{2592} + \frac{6157}{108}\pi^2 + \frac{25}{8}\pi^4\Big)\epsilon^3 + \Big(\frac{25993}{12}\zeta(3) - \frac{315}{2}\pi^2\zeta(3) - \frac{403}{2}\zeta(3)^2 + \frac{5067}{2}\zeta(5) - \frac{155039803}{15552} + \frac{2577347}{5184}\pi^2 + \frac{1501}{96}\pi^4 + \frac{9841}{22680}\pi^6\Big)\epsilon^4 + \Big(\frac{4045609}{288}\zeta(3) - \frac{23255}{24}\pi^2\zeta(3) + \frac{1201}{180}\pi^4\zeta(3) - \frac{3627}{2}\zeta(3)^2 + \frac{113053}{8}\zeta(5) - \frac{4459}{6}\pi^2\zeta(5) + \frac{13867}{2}\zeta(7) - \frac{12007861673}{186624} + \frac{59947519}{15552}\pi^2 + \frac{105527}{2160}\pi^4 + \frac{9841}{2520}\pi^6\Big)\epsilon^5 + \mathcal{O}(\epsilon^6)\Bigg],$$

$$H_{5B7} = n_0(\epsilon)\Bigg[- 2\zeta(3) + \frac{\pi^2}{12} + \frac{3}{2} + \left(-\frac{17}{2}\zeta(3) + \frac{121}{4} + \frac{11}{8}\pi^2 - \frac{13}{36}\pi^4\right)\epsilon + \left(\frac{175}{4}\zeta(3) + \frac{5}{3}\pi^2\zeta(3) - 294\zeta(5) + \frac{2837}{8} + \frac{449}{48}\pi^2 - \frac{85}{36}\pi^4\right)\epsilon^2 + \mathcal{O}(\epsilon^3)\Bigg],$$

$$H_{5B8} = \frac{n_0(\epsilon)}{\epsilon^2}\Bigg[- \zeta(3) + \left(-7\zeta(3) - \frac{\pi^4}{8}\right)\epsilon$$

$$+\Big(-39\zeta(3)+\frac{13}{6}\pi^2\zeta(3)-75\zeta(5)-\frac{7}{8}\pi^4\Big)\epsilon^2+\mathcal{O}(\epsilon^3)\Bigg],$$

$$H_{5B9}=\frac{n_0(\epsilon)}{\epsilon^5}\Bigg[-\frac{1}{36}-\frac{1}{9}\epsilon+\left(\frac{53}{432}\pi^2-\frac{1}{3}\right)\epsilon^2+\left(\frac{43}{8}\zeta(3)-\frac{8}{9}+\frac{53}{108}\pi^2\right)\epsilon^3$$
$$+\left(\frac{43}{2}\zeta(3)-\frac{20}{9}+\frac{53}{36}\pi^2+\frac{143}{1080}\pi^4\right)\epsilon^4+\mathcal{O}(\epsilon^5)\Bigg].$$

Combining Eqs. (5.20) and (5.21) leads to:

$$\int_0^1 dz\, F_i(z,\epsilon)=\sum_j c_{ij}(\epsilon)\,H_j(\epsilon)\,. \tag{5.23}$$

On the left-hand side, we plug in our general solutions from the differential equations and integrate over z. On the other side we can use the integrals we just determined. By comparing the general solution with the expressions on the right, we can extract the analytic values for the constants order-by-order in ϵ.
Note that this method has one subtlety tied to the transformation from the ϵ- to the original basis. The entries of $\hat{T}(z,\epsilon)$ in Eq. (5.20) are functions of z and thus can contain poles in this parameter, e.g. being proportional to z^{-a} and $(1-z)^{-b}$, leading to the integrations being non-convergent. The origin of these poles is the fact that the ϵ-expansion and the integration over z do not commute for every integral.
The next section thus will be about the solution to this problem and the treatment of these poles.

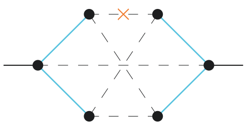
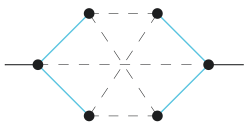
5.6.2 Avoiding Poles in the Integration

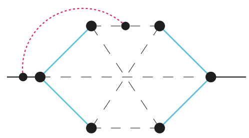
For the fixing of the boundary conditions, we need the integrations over z to be finite in order to get meaningful results. To ensure convergence, we identify the problematic denominators z^{-a} and $(1-z)^{-b}$ and multiply by their inverse to avoid poles in the integration region. This multiplication makes the terms finite, but since we changed the left-hand side of Eq. (5.20), we also have to find a proper adjustment for the right-hand side.

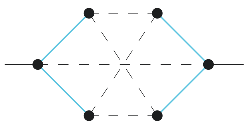
To do this, we are using a property of the δ-function that allows us to replace the z under the integration:

$$\int_0^1 dz \int dPS_5\, \delta(z - s_{1235})\, f(z) = \int_0^1 dz \int dPS_5\, \delta(z - s_{1235})\, f(s_{1235}) = \int dPS_5\, f(s_{1235})\,, \tag{5.24}$$

where $f(z)$ is a polynomial in z.
For our case, this introduces diagrams with addidtional numerators s_{1235}^n, depending on the power of the poles in z.
This is best illustrated for the boundary condition of $F_{5\mathrm{B}5}$, where we encounter a pole of $(1-z)^{-1}$:

$$\int_0^1 dz\,(1-z)$$

$$= (1 - s_{1235})$$

(5.25)

$$= -$$


$$\overset{\mathrm{IBP}}{=} \frac{2\epsilon}{5\epsilon - 1}$$

. (5.26)

As for the other diagrams in this thesis, the dotted red line here indicates a numerator N, given by the difference of the two connected momenta l_1 and l_2: $N = (l_1 - l_2)^2 - m_1^2 - m_2^2$.
This means that for the integrals where we encounter poles $z^{-a}(1-z)^{-b}$ in the transformation, we multiply the expression by $z^a(1-z)^b$, the integrated version of which we then call $\widetilde{F}_{5\mathrm{B}i}$. To avoid ambiguities, the polynomial we multiply to the expression is always the one with the minimal power of a and b that eliminates the problematic poles.

5.6.3 Results for the Boundary Conditions

Circumventing the divergences stemming from poles introduced by the transformation with the method from the last section, we can fix all boundary conditions for the integrals $F_{5\mathrm{B}1}$–$F_{5\mathrm{B}36}$. We give them here up to the respective order that is needed in our diagrams, the normalization factors are defined in Eq. (4.65):

$$\widetilde{F}_{5\mathrm{B}1} = n_0(\epsilon)\Bigg[\frac{1}{144} + \epsilon\frac{71}{576} + \epsilon^2\left(\frac{26815}{20736} - \frac{\pi^2}{54}\right) + \epsilon^3\left(\frac{872675}{82944} - \frac{71}{216}\pi^2 - \frac{5}{12}\zeta(3)\right) + \epsilon^4\left(\frac{220099831}{2985984} - \frac{26815}{7776}\pi^2 + \frac{17}{2160}\pi^4 - \frac{355}{48}\zeta(3)\right) + \mathcal{O}(\epsilon^5)\Bigg],$$

$$\widetilde{F}_{5\mathrm{B}2} = n_0(\epsilon)\left[\frac{1}{24} + \epsilon\frac{7}{9} + \epsilon^2\left(\frac{7357}{864} - \frac{\pi^2}{9}\right) + \epsilon^3\left(\frac{46517}{648} - \frac{56\pi^2}{27} - \frac{5\zeta(3)}{2}\right) + \mathcal{O}(\epsilon^4)\right],$$

$$\widetilde{F}_{5\mathrm{B}3} = \frac{n_0(\epsilon)}{\epsilon}\left[-\frac{1}{2} - \frac{37\epsilon}{4} + \epsilon^2\left(-\frac{813}{8} + \frac{4\pi^2}{3}\right) + \mathcal{O}(\epsilon^3)\right],$$

$$\widetilde{F}_{5\mathrm{B}4} = \frac{n_0(\epsilon)}{\epsilon^3}\Bigg[\frac{1}{6} + \epsilon\frac{4}{3} + \epsilon^2\left(\frac{49}{6} - \frac{4\pi^2}{9}\right) - \frac{2}{9}\epsilon^3\left(-201 + 16\pi^2 + 45\zeta(3)\right) + \epsilon^4\left(\frac{1397}{6} - \frac{196}{9}\pi^2 + \frac{17}{90}\pi^4 - 80\zeta(3)\right) + \mathcal{O}(\epsilon^5)\Bigg],$$

$$\widetilde{F}_{5\mathrm{B}5} = \frac{n_0(\epsilon)}{\epsilon^4}\Bigg[-\frac{10}{3} - \epsilon\frac{40}{3} + \epsilon^2\left(-90 + \frac{80\pi^2}{9}\right) + \epsilon^3\left(-\frac{1220}{3} + \frac{320}{9}\pi^2 + \frac{592}{3}\zeta(3)\right) + \mathcal{O}(\epsilon^4)\Bigg],$$

$$\widetilde{F}_{5\mathrm{B}6} = n_0(\epsilon)\left[\frac{1}{12}\left(-6 + \pi^2\right) + \epsilon\left(-\frac{43}{4} + \frac{29\pi^2}{24} + \frac{9\zeta(3)}{2}\right) + \mathcal{O}(\epsilon^2)\right],$$

$$\widetilde{F}_{5B7} = \frac{n_0(\epsilon)}{\epsilon^2}\left[\frac{1}{6} + \left(2 + \frac{\pi^2}{18}\right)\epsilon + \frac{1}{18}\epsilon^2\left(279 + \pi^2 + 54\zeta(3)\right) + \mathcal{O}(\epsilon^3)\right],$$

$$\widetilde{F}_{5B8} = n_0(\epsilon)\left[\frac{1}{288}\left(-15 + 2\pi^2\right) + \epsilon\left(-\frac{581}{576} + \frac{19}{216}\pi^2 + \frac{3}{8}\zeta(3)\right) + \mathcal{O}(\epsilon^2)\right],$$

$$\widetilde{F}_{5B9} = n_0(\epsilon)\bar{z}\left[\frac{1}{24} + \epsilon\frac{31}{36} + \epsilon^2\left(\frac{8917}{864} - \frac{\pi^2}{9}\right) + \mathcal{O}(\epsilon^3)\right],$$

$$\widetilde{F}_{5B10} = \frac{n_0(\epsilon)}{\epsilon^2}\left[-\frac{4\pi^2}{9} + \epsilon\left(-\frac{16}{9}\pi^2 - \frac{76}{3}\zeta(3)\right) + \mathcal{O}(\epsilon^2)\right],$$

$$\widetilde{F}_{5B11} = \frac{n_0(\epsilon)}{\epsilon^3}\left[-\frac{\pi^2}{36} + \epsilon\left(-\frac{7}{36}\pi^2 - \frac{3}{2}\zeta(3)\right) + \mathcal{O}(\epsilon^2)\right],$$

$$\widetilde{F}_{5B12} = \frac{n_0(\epsilon)}{\epsilon^4}\left[-\frac{2}{3} - \frac{8\epsilon}{3} + \epsilon^2\left(-18 + \frac{20}{9}\pi^2\right)\right.$$
$$\left. + \epsilon^3\left(-\frac{244}{3} + \frac{80}{9}\pi^2 + \frac{196}{3}\zeta(3)\right) + \mathcal{O}(\epsilon^4)\right],$$

$$\widetilde{F}_{5B13} = n_0(\epsilon)\left[-\frac{1}{36} - \epsilon\frac{37}{72} - \epsilon^2\left(\frac{7237}{1296} - \frac{2}{27}\pi^2\right)\right.$$
$$\left. - \epsilon^3\left(\frac{121273}{2592} - \frac{37}{27}\pi^2 - \frac{5}{3}\zeta(3)\right) + \mathcal{O}(\epsilon^4)\right],$$

$$\widetilde{F}_{5B14} = \frac{n_0(\epsilon)}{\epsilon}\left[\frac{1}{4} + \epsilon\frac{35}{8} + \epsilon^2\left(\frac{731}{16} - \frac{2}{3}\pi^2\right)\right.$$

$$+\,\epsilon^3\left(\frac{11903}{32}-\frac{35}{3}\pi^2-15\zeta(3)\right)+\mathcal{O}(\epsilon^4)\Bigg]\,,$$

$$\widetilde{F}_{5\mathrm{B}15}=n_0(\epsilon)\left[-\frac{1}{4}-\epsilon\frac{41}{8}+\mathcal{O}(\epsilon^2)\right],$$

$$\widetilde{F}_{5\mathrm{B}16}=\frac{n_0(\epsilon)}{\epsilon^2}\Bigg[-\frac{1}{8}-\epsilon\frac{35}{16}-\epsilon^2\left(\frac{739}{32}-\frac{\pi^2}{3}\right)$$
$$-\,\epsilon^3\left(\frac{12231}{64}-\frac{35}{6}\pi^2-\frac{15}{2}\zeta(3)\right)+\mathcal{O}(\epsilon^5)\Bigg]\,,$$

$$\widetilde{F}_{5\mathrm{B}17}=n_0(\epsilon)\left[-\frac{1}{4}-\epsilon\frac{11}{4}-\epsilon^2\left(\frac{83}{4}-\frac{2\pi^2}{3}\right)-\epsilon^3\left(\frac{535}{4}-\frac{22\pi^2}{3}-15\zeta(3)\right)+\mathcal{O}(\epsilon^4)\right],$$

$$\widetilde{F}_{5\mathrm{B}18}=\frac{n_0(\epsilon)}{\epsilon}\left[2\zeta(3)+\epsilon\left(\frac{7}{30}\pi^4+22\zeta(3)\right)+\mathcal{O}(\epsilon^2)\right],$$

$$\widetilde{F}_{5\mathrm{B}19}=\frac{n_0(\epsilon)}{\epsilon}\left[-\frac{19}{36}+\frac{4}{9}-\zeta(3)\epsilon\left(\frac{39}{4}-\frac{7\pi^4}{135}-4\zeta(3)\right)+\mathcal{O}(\epsilon^2)\right],$$

$$\widetilde{F}_{5\mathrm{B}20}=\frac{n_0(\epsilon)}{\epsilon^4}\left[-\frac{1}{6}-\frac{11\epsilon}{6}+\epsilon^2\left(-\frac{89}{6}+\frac{4}{9}\pi^2\right)+\mathcal{O}(\epsilon^3)\right],$$

$$\widetilde{F}_{5\mathrm{B}21}=\frac{n_0(\epsilon)}{\epsilon^5}\left[\frac{1}{4}+\epsilon\frac{9}{4}+\epsilon^2\left(\frac{67}{4}-\frac{2}{3}\pi^2\right)+\epsilon^3\left(\frac{451}{4}-6\pi^2-15\zeta(3)\right)+\mathcal{O}(\epsilon^4)\right],$$

$$\widetilde{F}_{5\mathrm{B}22}=n_0(\epsilon)\left[-\frac{1}{144}+\epsilon\left(-\frac{689}{1728}+\frac{\pi^2}{36}\right)-\epsilon^2\left(\frac{138187}{20736}-\frac{8\pi^2}{27}-\frac{13\zeta(3)}{6}\right)+\mathcal{O}(\epsilon^3)\right],$$

$$\widetilde{F}_{5B23} = n_0(\epsilon)\left[\frac{7}{8} - \frac{\pi^2}{12} + \epsilon\left(\frac{133}{8} - \frac{19}{24}\pi^2 - \frac{13}{2}\zeta(3)\right) + \epsilon^2\left(\frac{5907}{32} - \frac{325}{48}\pi^2 - \frac{25}{72}\pi^4 - \frac{247}{4}\zeta(3)\right) + + \epsilon^3\left(\frac{50333}{32} - \frac{2021}{32}\pi^2 - \frac{475}{144}\pi^4 - \frac{3189}{8}\zeta(3) + \frac{35}{2}\pi^2\zeta(3) - \frac{563}{2}\zeta(5)\right) + \mathcal{O}(\epsilon^4)\right],$$

$$\widetilde{F}_{5B24} = \frac{n_0(\epsilon)}{\epsilon}\left[-\frac{3}{2} + \frac{\pi^2}{6} - \epsilon\left(\frac{105}{4} - \frac{4}{3}\pi^2 - 13\zeta(3)\right) - \epsilon^2\left(\frac{2189}{8} - \frac{21}{2}\pi^2 - \frac{25}{36}\pi^4 - 104\zeta(3)\right) - \epsilon^3\left(\frac{35521}{16} - \frac{283\pi^2}{3} - \frac{50\pi^4}{9} - 597\zeta(3) + 35\pi^2\zeta(3) - 563\zeta(5)\right) + \mathcal{O}(\epsilon^4)\right],$$

$$\widetilde{F}_{5B25} = n_0(\epsilon)\left[-\frac{1}{3} + \frac{\pi^2}{36} - \epsilon\left(\frac{35}{9} - \frac{73}{216}\pi^2 + \frac{1}{2}\zeta(3)\right) + \mathcal{O}(\epsilon^2)\right],$$

$$\widetilde{F}_{5B26} = \frac{n_0(\epsilon)}{\epsilon}\left[\frac{1}{2} + \epsilon\left(7 + \frac{1}{6}\pi^2 + 2\zeta(3)\right) + \mathcal{O}(\epsilon^2)\right],$$

$$\widetilde{F}_{5B27} = \frac{n_0(\epsilon)}{\epsilon^2}\left[-\zeta(3) + \epsilon\left(-\frac{1}{8}\pi^4 - 7\zeta(3)\right) + \epsilon^2\left(-\frac{7}{8}\pi^4 - 39\zeta(3) + \frac{13}{6}\pi^2\zeta(3) - 75\zeta(5)\right) + \mathcal{O}(\epsilon^3)\right],$$

$$\widetilde{F}_{5B28} = \frac{n_0(\epsilon)}{\epsilon^3}\left[-\frac{1}{2} - \epsilon\frac{15}{2} - \epsilon^2\left(\frac{139}{2} - \frac{4}{3}\pi^2 + 2\zeta(3)\right) - \epsilon^3\left(\frac{1031}{2} - 20\pi^2 + \frac{1}{4}\pi^4 - 12\zeta(3)\right) + \mathcal{O}(\epsilon^4)\right],$$

$$\widetilde{F}_{5B29} = n_0(\epsilon)\left[-\frac{157}{2592} + \frac{\pi^2}{135} + \mathcal{O}(\epsilon)\right],$$

$$\widetilde{F}_{5B30} = n_0(\epsilon)\left[2\zeta(3) - \frac{\pi^2}{6} + \mathcal{O}(\epsilon)\right],$$

$$\widetilde{F}_{5B31} = \frac{n_0(\epsilon)}{\epsilon^2\bar{z}}\left[\frac{1}{6} + \epsilon\frac{19}{6} + \epsilon^2\Big(\frac{193}{6} - \frac{\pi^2}{6}\Big) + \mathcal{O}(\epsilon^3)\right],$$

$$\widetilde{F}_{5B32} = \frac{n_0(\epsilon)}{\epsilon^4\bar{z}}\left[\frac{1}{12} + \epsilon\frac{11}{12} + \epsilon^2\left(\frac{83}{12} - \frac{11}{36}\pi^2\right) + \epsilon^3\left(\frac{535}{12} - \frac{121}{36}\pi^2 - \frac{23}{2}\zeta(3)\right) + \mathcal{O}(\epsilon^4)\right],$$

$$\widetilde{F}_{5B33} = n_0(\epsilon)\left[-\frac{1}{48} - \epsilon\frac{271}{576} + \mathcal{O}(\epsilon^2)\right],$$

$$\widetilde{F}_{5B34} = n_0(\epsilon)\left[0 + \mathcal{O}(\epsilon)\right],$$

$$\widetilde{F}_{5B35} = \frac{n_0(\epsilon)}{\epsilon^2}\left[-\frac{\pi^2}{12} + \epsilon\left(-\frac{3}{4}\pi^2 - \frac{15}{2}\zeta(3)\right) + \mathcal{O}(\epsilon^2)\right],$$

$$\widetilde{F}_{5B36} = \frac{n_0(\epsilon)}{\epsilon^3}\left[-\frac{1}{6} + \epsilon\left(-\frac{13}{6} - \frac{\pi^2}{12}\right) + \epsilon^2\left(-\frac{115}{6} - \frac{17}{36}\pi^2 - \frac{11}{2}\zeta(3)\right) + \mathcal{O}(\epsilon^3)\right].$$

With the boundary conditions as the last missing piece, we can then calculate all integrals from the families $\vec{F}_{101}$–$\vec{F}_{106}$. These can be found in Appendix B.
Note that the six families do not include all master integrals, as we had to take out some of them to make the transformation to the ϵ-basis possible. Four of them have to be calculated separately, which we will discuss in the next section.

5.7 The Remaining Integrals

The integrals F_{5B30}, F_{5B31}, F_{5B32} and F_{5B36} are solved via the differential equations in the original basis. These equations are written in full length in Appendix E, their general form is:

$$\partial_z F_{5B30} = R_{30}(\epsilon, z) - \frac{2(2z\epsilon + 2\epsilon + 1)}{(z+3)\bar{z}} F_{5B30} - \frac{2}{z+3} F_{5B31}\,, \tag{5.27}$$

$$\partial_z F_{5B31} = R_{31}(\epsilon, z) + \frac{(3\epsilon - 1)(4\epsilon - 1)}{(z+3)\bar{z}} F_{5B30} - \frac{2(z + 6\epsilon + 1)}{(z+3)\bar{z}} F_{5B31}\,, \tag{5.28}$$

$$\partial_z F_{5B32} = R_{32}(\epsilon, z) + \frac{(3\epsilon - 1)}{z\bar{z}} F_{5B30} + \frac{1}{z\bar{z}} F_{5B31} - \frac{(5z\epsilon + z - 3\epsilon)}{z\bar{z}} F_{5B32}\,, \tag{5.29}$$

$$\partial_z F_{5B36} = R_{36}(\epsilon, z) + \frac{1}{\bar{z}} F_{5B31} - \frac{(2\epsilon + 1)}{\bar{z}} F_{5B36}\,. \tag{5.30}$$

The functions $R_i(\epsilon, z)$ contain all the parts proportional to the other integrals, which are known from the last section. This means that the task at hand is the solution of inhomogeneous differential equations of first order.

There is one complication though, as we see that we cannot work our way iteratively from the top to the bottom of the equations in Eqs. (5.27) – (5.30). The reason for this is the non-decoupling of the first two equations, which have to be solved at the same time.

For this, we first use a variable transformation of

$$x = \sqrt{\frac{1-z}{z-3}}\,, \tag{5.31}$$

which simplifies the alphabet.

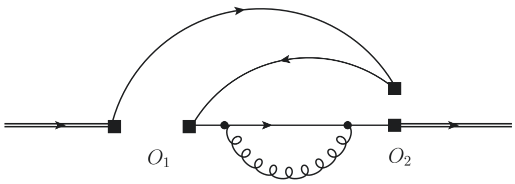
After this we can solve these order-by-order in ϵ via Goncharov polylogarithms. We see that the solutions have letters including $\pm\frac{i}{\sqrt{3}} = \pm r_3$, which also explains why it was not possible for us to find a transformation to the ϵ-base. (There may be a way to find a transformation when applying the change of variables from Eq. (5.31) to

the whole systems $\vec{F}'_{105}$ and $\vec{F}'_{106}$ and finding a ϵ-basis for the price of complicating the calculation of the lower-line integrals.)
After the first two of the remaining integrals are computed in this way, the other two are following straightforwardly since they are not coupled anymore. The results can be found together with the other master integrals in Appendix B.

With the master integrals done, we now can insert them into the diagrams and combine the four- and five-body squared amplitudes. After this combination, the infrared divergences due to gluons cancel out, but we still encounter poles in ϵ. These have two different origins: On the one hand, we still suffer from ultraviolet divergences, and on the other hand, there are still infrared collinear divergences from photons contained in the expression. These will be cancelled by renormalization and the use of splitting functions, respectively, which will be discussed in chapters 6 and 7.

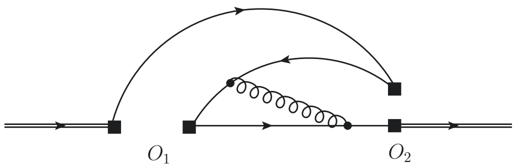
5.8 Color Factors

Let us discuss here the different color structures that are partaking in the squared amplitude. As we discussed in the beginning of the calculation, we calculate the expressions without explicit color factors and in the end multiply by the corresponding factors to get, for example the insertions $(P_3 \times P_4)$, $(P_4 \times P_3)$ and $(P_4 \times P_4)$ from the calculation of $(P_3 \times P_3)$.
When taking the color traces, we have to differentiate between the cases of one and two traces, both of which can be found in Fig. 5.4 and Fig. 5.5, respectively.
N_c is the number of colors, C_F and C_A are the Casimir operators of the fundamental and the adjoint representation of the $SU(3)$ group, respectively. In our case, the numeric values are $N_c = 3$, $C_F = \frac{N_c^2-1}{2N_c} = \frac{4}{3}$ and $C_A = N_c = 3$.



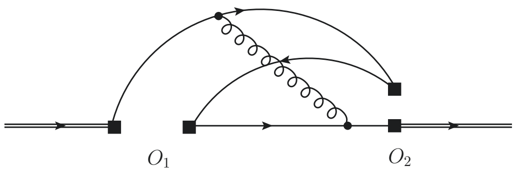
$\mathrm{Tr}(\mathrm{O_2 T_c T_c O_1 O_2 O_1})$

O_2 \ O_1	$T_a \otimes T_a$	$\mathbb{1} \otimes \mathbb{1}$
$T_b \otimes T_b$	$-\frac{1}{2}C_F^2$	$C_F^2 C_A$
$\mathbb{1} \otimes \mathbb{1}$	$C_F^2 C_A$	$C_F C_A$



$\mathrm{Tr}(\mathrm{O_2 T_c O_1 T_c O_2 O_1})$

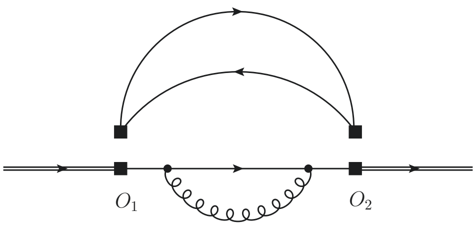
O_2 \ O_1	$T_a \otimes T_a$	$\mathbb{1} \otimes \mathbb{1}$
$T_b \otimes T_b$	$\frac{1}{4}C_F\,(C_A - 2C_F)$	$C_F^2 C_A$
$\mathbb{1} \otimes \mathbb{1}$	$-\frac{1}{2}C_F$	$C_F C_A$



$\mathrm{Tr}(\mathrm{O_2 T_c O_1 O_2 T_c O_1})$

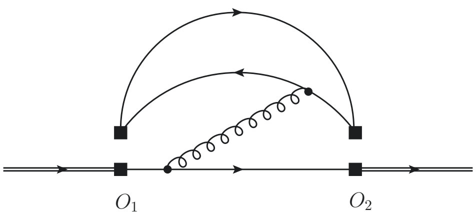
O_2 \ O_1	$T_a \otimes T_a$	$\mathbb{1} \otimes \mathbb{1}$
$T_b \otimes T_b$	$\frac{1}{2}C_F\,(C_A - C_F)$	$-\frac{1}{2}C_F$
$\mathbb{1} \otimes \mathbb{1}$	$-\frac{1}{2}C_F$	$C_F C_A$

Figure 5.4: All occurring color traces in the case of one fermion line. The color of the gluon is denoted by T_c, while O_1 and O_2 are the color structures of the operators P_1–P_6.



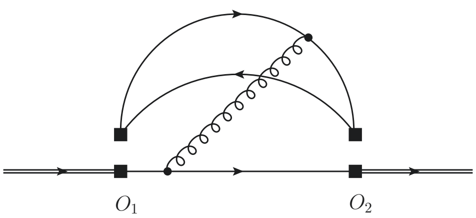
$\mathrm{Tr}(\mathrm{O_1 O_2})\mathrm{Tr}(\mathrm{O_2 T_c T_c O_1})$

O_2 \ O_1	$T_a \otimes T_a$	$\mathbb{1} \otimes \mathbb{1}$
$T_b \otimes T_b$	$\frac{1}{2} C_F^2 C_A$	0
$\mathbb{1} \otimes \mathbb{1}$	0	$C_F^2 C_A$



$\mathrm{Tr}(\mathrm{O_2 O_1 T_c})\mathrm{Tr}(\mathrm{O_2 T_c O_1})$

O_2 \ O_1	$T_a \otimes T_a$	$\mathbb{1} \otimes \mathbb{1}$
$T_b \otimes T_b$	$\frac{1}{4}\left(C_F C_A^2 - 2C_F\right)$	$\frac{1}{2} C_F C_A$
$\mathbb{1} \otimes \mathbb{1}$	$\frac{1}{2} C_F C_A$	0



$\mathrm{Tr}(\mathrm{O_2 T_c O_1})\mathrm{Tr}(\mathrm{O_2 T_c O_1})$

O_2 \ O_1	$T_a \otimes T_a$	$\mathbb{1} \otimes \mathbb{1}$
$T_b \otimes T_b$	$-\frac{1}{2} C_F$	$\frac{1}{2} C_F C_A$
$\mathbb{1} \otimes \mathbb{1}$	$\frac{1}{2} C_F C_A$	0

Figure 5.5: All occurring color traces in the case of two fermion lines. The color of the gluon is denoted by T_c, while O_1 and O_2 are the color structures of the operators P_1–P_6.

Chapter 6

Renormalization

After we calculated the bare contributions, our result still suffers from poles up to $1/\epsilon^2$.
These have two different sources: The first one are the collinear infrared divergences from the photons that do not cancel in the process of summing the virtual and real part of the amplitude and the other source are ultraviolet divergences. The latter are going to be adressed in this section.

6.1 Standard Model Renormalization

In a first step, we renormalize the parameters of the Standard Model Lagrangian. For this, we define the occurring parameters X as **bare** quantities, denoting them as X_B. If we analyze the tree-level four-body diagrams, we see that the relevant quantities are the wave functions of the light (massless) and heavy (massive) quarks $\Psi_{l,B}$ and $\Psi_{h,B}$, the photon vertex V_B, the mass of the b quark $m_{b,B}$ and the wave-function of the photon $A_{\mu,B}$. The corresponding counterterm diagrams are shown in Fig. 6.1 **a)** – **c)**. The respective contributions will be discussed in detail in the next sections.
These quantities are related to the physical parameters by Z-factors, which we define

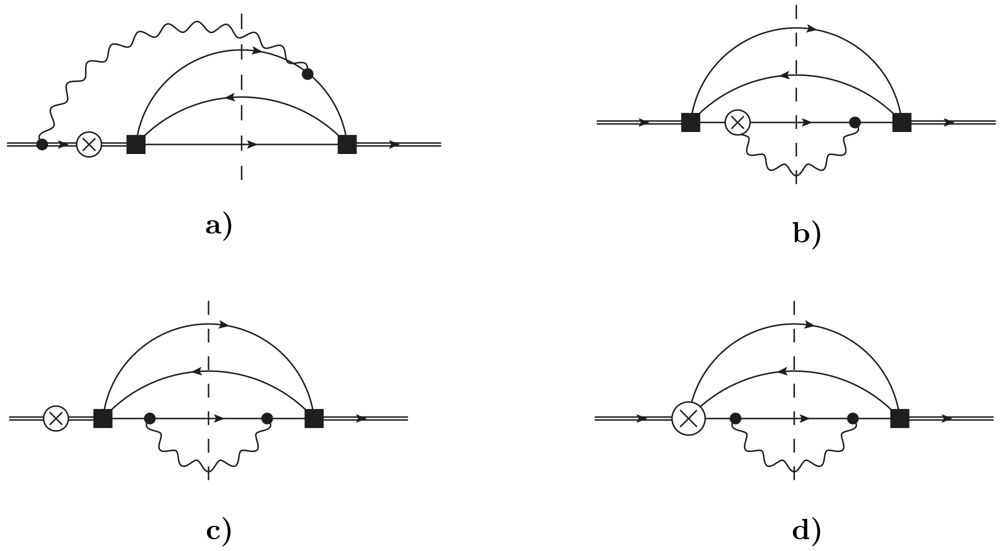


Figure 6.1: All types of contributions that are needed to cancel the UV divergences of the four- and five-body diagrams calculated up until now:
Diagrams of type **a)** show the insertion of a renormalized quark propagator, splitting into parts proportional to the mass- and wave function renormalization constants. Additionally, we encounter **b)** renormalization of the photon vertex, **c)** wave function renormalization of the external legs and, finally, **d)** operator renormalization.

as follows:

$$\begin{aligned} \Psi_{h,B} &= Z_h^{1/2}\,\Psi_h\,, & \Psi_{l,B} &= Z_l^{1/2}\,\Psi_l\,, \\ m_{b,B} &= Z_m\,m_b\,, & A_{\mu,B} &= Z_3^{1/2}\,A_\mu\,, \\ V_B &= Z_V^{1/2}\,V\,. \end{aligned} \tag{6.1}$$

These factors can be expressed in a series in α_s:

$$Z_X = 1 + \sum_{k=1}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^k Z_X^{(k)}\,. \tag{6.2}$$

As we are calculating the next-to-leading order in α_s, we are interested in the $Z_X^{(1)}$ expressions, which can can be found in the literature [42, 43, 117]. For the $\mathcal{O}(\alpha_s)$ correction to the photon vertex, a simple comparison of the diagrams shows that it is related to the $\mathcal{O}(\alpha_e)$ correction by a multiplication of the color factor C_F.

Our choice of renormalization scheme is the so called on-shell scheme (or OS scheme). At $\mathcal{O}(\alpha_s)$, the factors (in Feynman gauge, $\xi = 1$) can be given as an all-order expression in ϵ:

$$
\begin{aligned}
Z_h^{(1)} &= -C_F \frac{(3-2\epsilon)\Gamma(\epsilon)}{1-2\epsilon} e^{\epsilon(L_\mu+\gamma_E)}\,, \\
Z_V^{(1)} &= Z_h^{(1)}\,, \\
Z_m^{(1)} &= Z_h^{(1)}\,, \\
Z_l^{(1)} &= 0\,, && (6.3) \\
Z_3^{(1)} &= 0\,, && (6.4)
\end{aligned}
$$

where γ_E is the Euler-Mascheroni constant. We also introduced the shorthand $L_\mu = \log(\frac{\mu^2}{m_b^2})$ for the logarithm incorporating the renormalization scale μ.
Note that we do not need to renormalize the photon wave function at this order, as the corrections in the strong coupling only start at $\mathcal{O}(\alpha_e\alpha_s)$. This is because gluons can not directly couple to the photon and need e.g. a quark loop to couple to.

In the following discussion we will only renormalize the left side of the cut. This is done similar to the calculation of the four-body diagrams, where we only calculated the loop on the left side and then took the complex conjugate to get the expression for the right side. Here, we also only insert the counterterms to the left and build the right side insertions by conjugation.

6.1.1 Quark Propagator Renormalization

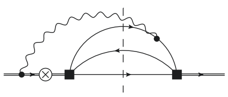
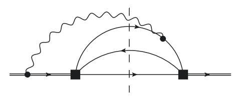
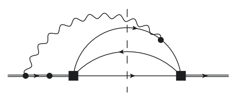
Occurring divergences from quark propagators are renormalized by counterterm insertions of the type in Fig. 6.1 **a)**. Inserting a counterterm changes the propagator expression in the following way:

$$
\left(i\frac{\not{p}+m_q}{p^2-m_q^2}\right) \Rightarrow \left(i\frac{\not{p}+m_q}{p^2-m_q^2}\right)\left(i[(Z_q-1)\not{p}-(Z_qZ_m-1)m_q]\right)\left(i\frac{\not{p}+m_q}{p^2-m_q^2}\right)\,,
$$

where q is either l for the light quarks or h for the b quark propagator.
Inserting the relation from Eq. (6.2) this becomes:

$$\left(i\frac{\not{p}+m_q}{p^2-m_q^2}\right)\left(\frac{\alpha_s}{4\pi}\right)\left(i\left[Z_h^{(1)}(\not{p}-m_q)\right]-i\left[Z_m^{(1)}m_q\right]\right)\left(i\frac{\not{p}+m_q}{p^2-m_q^2}\right)\,. \tag{6.5}$$

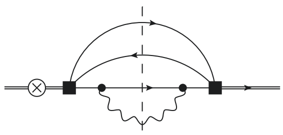
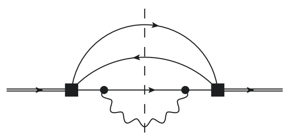
Here we observe that the first part is directly proportional to the original propagator again, whereas the second part leads to a new type of diagrams with a dot on the propagator. Taking one of the occurring diagrams as an example, we can write this as:

 $= \left(\frac{\alpha_s}{4\pi}\right)\Bigg[-Z_h^{(1)}$  $- i\, Z_h^{(1)}\, m_q$  $\Bigg]\,.$

As the second term is proportional to the quark mass, we see that only the diagrams with the b quark propagator play a role for the mass renormalization. Furthermore, we see that the first term is simply a multiplication of the tree-level squared amplitude by the corresponding Z-factor.

6.1.2 Quark Wave Function Renormalization

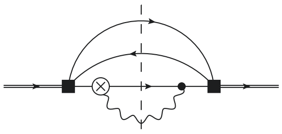
The external quark spinors also need to be renormalized. For this, we replace the external legs via the relation in Eq. (6.1). Plugging in the relation from Eq. (6.2) leads to:

 $= \left(\frac{\alpha_s}{4\pi}\right)\Bigg[\frac{1}{2}Z_h^{(1)}$  $\Bigg]\,.$

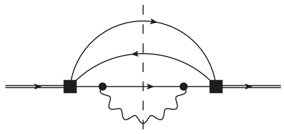
Since only the heavy quark contribution is non-zero, we get a prefactor of $\frac{1}{2}$ from the expansion of the square root in Eq. (6.1).

6.1.3 Photon Vertex Renormalization

As for the wave function, the photon vertex renormalization also is multiplicative. We can write the insertion as:



$$= \left(\frac{\alpha_s}{4\pi}\right) \left[Z_h^{(1)} \right.$$



$$\left. \vphantom{Z_h^{(1)}} \right] ,$$

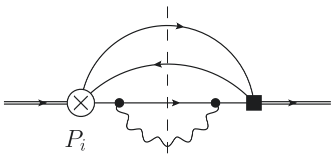
which cancels with the first part of the mass renormalization, cf. Eq. (6.5).

6.2 Operator Renormalization

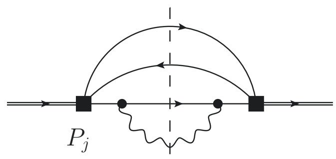
After renormalizing the Standard Model Parameters, we have eliminated almost all sources of UV divergences. What is left now are the ones from the insertion of the operators. To calculate the relevant contributions to our process, we again have to determine the $\mathcal{O}(\alpha_s)$ corrections to the tree-level amplitudes, when inserting the bare operators $P_{i,B}$ to the left of the cut (as before, we get the corresponding right insertion by complex conjugation) [62]:

$$\begin{aligned}
\sum_{i=1u,2u,3,4,5,6} \mathcal{C}_i P_{i,B} &= \sum_{i,j=1u,2u,3,4,5,6} \mathcal{C}_i (Z_{PP})_{ij} P_j + \sum_{\substack{i=1u,2u,3,4,5,6 \\ j=1,2,3,4}} \mathcal{C}_i (Z_{PE})_{ij} E_j \\
&= \sum_{i,j=1u,2u,3,4,5,6} \mathcal{C}_i \left(\delta_{ij} + \frac{1}{\epsilon}\frac{\alpha_s}{4\pi} (\delta Z_{PP})_{ij} \right) P_j \\
&+ \sum_{\substack{i=1u,2u,3,4,5,6 \\ j=1,2,3,4}} \mathcal{C}_i \left(\delta_{ij} + \frac{1}{\epsilon}\frac{\alpha_s}{4\pi} (\delta Z_{PE})_{ij} \right) E_j \\
&= \sum_{i=1u,2u,3,4,5,6} \mathcal{C}_i \, P_i + \sum_{i=1,2,3,4} \mathcal{C}_i \, E_i \\
&+ \frac{1}{\epsilon}\frac{\alpha_s}{4\pi} \Bigg[\sum_{i,j=1u,2u,3,4,5,6} \mathcal{C}_i \, (\delta Z_{PP})_{ij} \, P_j + \sum_{\substack{i=1u,2u,3,4,5,6 \\ j=1,2,3,4}} \mathcal{C}_i \, (\delta Z_{PE})_{ij} \, E_j \Bigg] ,
\end{aligned} \tag{6.6}$$

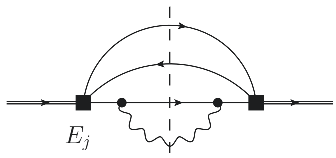
where the δZ_{ij} are entries of the operator mixing matrices that will be discussed in the next section. The operators denoted by E_j are the evanescent operators defined in Eq. (2.17). These operators vanish in four dimensions, i.e. their contributions start at $\mathcal{O}(\epsilon)$. But, when calculating expressions that are not UV-finite in and of themselves, they multiply UV divergences, which leads to finite terms. Since we have not included them in the calculation of the bare amplitudes, we now account for this effect here in the renormalization step.
The first term in the second to last row of the Eq. (6.6) was already calculated in Ref. [10] and the second term vanishes in the limit $\epsilon \to 0$. For our calculation, we only need the two terms in the last row, as they are of $\mathcal{O}(\alpha_s)$. As before, we can express the procedure diagrammatically, for the insertions of the operator P_i to the left of the cut, the diagrams we have to calculate take the following form:



$$= \left(\frac{\alpha_s}{4\pi\epsilon}\right)\Bigg[\sum_{j=1u,2u,3,4,5,6} (\delta Z_{PP})_{ij}$$



$$+ \sum_{j=1,2,3,4} (\delta Z_{PE})_{ij}$$



$$\Bigg] .$$

6.2.1 The Z Matrices for Operator Mixing

The mixing matrices that are relevant for our calculation can be determined without calculation of additional diagrams. For the sum of all possible insertions, the matrix has already been calculated in Ref. [117].
As this calculation is formally a completion of the one in Ref. [62], we have to be careful as to not double-count certain elements. To circumvent this, we take the entries of the δZ_{PP} matrix that were already used there and subtract them from the ones in Ref. [117], the results of which lead to the matrix we need. As Ref. [62] did not need evanescent contributions, we take our δZ_{PE} as is from Ref. [117].

Both matrices take the following form:

$$(\delta Z_{PP}) = \begin{pmatrix} -2 & \frac{4}{3} & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -10 & 0 & 1 \\ 0 & 0 & -\frac{20}{9} & -\frac{26}{3} & \frac{2}{9} & \frac{5}{12} \\ 0 & 0 & 0 & -64 & 0 & 10 \\ 0 & 0 & -\frac{128}{9} & -\frac{80}{3} & \frac{20}{9} & -\frac{1}{3} \end{pmatrix}, \tag{6.7}$$

$$(\delta Z_{PE}) = \begin{pmatrix} \frac{5}{12} & \frac{2}{9} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{2}{9} & \frac{5}{12} \end{pmatrix}. \tag{6.8}$$

6.3 Cancellation of the UV Poles

Adding all previous diagrams up and dressing them with the corresponding Z factors leads to four-body tree-level kernels that we need to integrate. The diagram expressions are generated with the same toolkit as before (`QGRAF` and `Form`) and can then be integrated in `Mathematica`. For this, we use the parametrization we introduced in Eq. (4.16). Because we only encounter tree-level diagrams here, the factorizing parametrization makes this a straightforward task.

After calculating all the above counterterm contributions to our expression, we end up with a UV finite correction for each entry of G_{ij} (cf. Eq. (2.20)) that includes all four- and five-body corrections at $\mathcal{O}(\alpha_s)$. As we discussed before, the expressions that remain after the renormalization still contain collinear divergent pieces, which are discussed in the next section.

Chapter 7

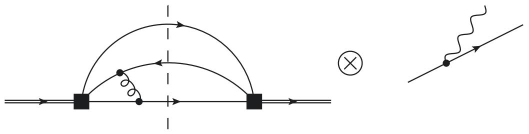
Regularization of the Collinear Divergences via Splitting Functions

7.1 Switching the Regularization Scheme

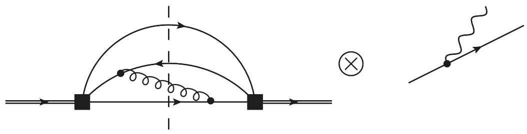
At this last step of the calculation, our intermediate results still contain poles in the dimensional regularization parameter ϵ. These poles arise from the regions of phase space where the photon is collinear to the light quarks and are an artifact of the latter being treated massless. Treating them as massive would avoid these collinear divergences for the price of making the integrals more involved. At the current state of the art, the calculation of the analytical results with only the b quark being massive was a considerable challenge already, so we want to avoid additional masses. Given this, we now want to trade this artificial regulator for a more natural one, i.e. a physical cut-off that is related to the light meson masses.

For this, we make use of the fact that the amplitudes **factorize** in the quasi-collinear limit [10]. This means that the amplitudes with a photon being emitted from the quark leg q_n ($b \to q_1\bar{q}_2q_3\gamma(g)$) can be expressed as the amplitude without the photon multiplied by a splitting function f_n that describes the emission of a quasi-collinear photon from the quark q_n ($b \to q_1\bar{q}_2q_3(g) \otimes f_n$).

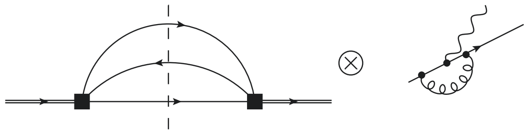
These **splitting functions** then contain the collinear divergent parts, which can



a)



b)



c)

Figure 7.1: The three types of diagrams that are calculated to regularize the collinear logarithms:
Diagrams **a)** show the three-body one-loop contributions and diagrams **b)** the four-body contributions with a real gluon emission. In these two, only the leading order contribution of the splitting function needs to be considered. These correspond to the four-body one-loop and the five-body real emission diagrams, respectively, the photon emission substituted by the convolution with a splitting function.
In diagram **c)**, we see that we also encounter the case of three-body tree-level diagrams that are convoluted with the NLO contributions to the splitting function.

either be regulated in dimensional regularization or via quark masses. Comparing these leads to a relation between the schemes that makes it possible to switch from one to the other [10]:

$$\frac{d\Gamma_m}{dz} = \frac{d\Gamma_\epsilon}{dz} + \frac{d\Gamma_{\text{shift}}}{dz}\,. \tag{7.1}$$

7.2 The Leading Order Splitting Functions

The first elements in the switching of the regularization schemes are the diagrams in the first row of Fig. 7.1, where we substitute the photon in NLO diagrams by a leading order splitting function. This splitting function is given to $\mathcal{O}(\epsilon^0)$ in e.g. Ref. [10]. This does not suffice for our case, as we encounter a more complicated pole structure at our order of calculation. To illustrate the process, the following section is an overview over the calculation of an all orders expression for the leading order splitting function, which we will call $\Delta f^{(0)}$.

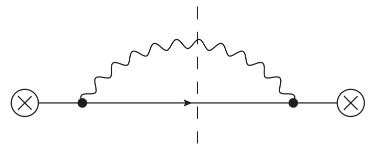


Figure 7.2: In light-cone gauge, we only encounter one diagram in the calculation of the leading order splitting function, which is the tree level emission of a photon from the quark line.

In the collinear limit, the process of the emission of a photon from the quark leg factorizes, which can be written as [118]:

$$|\mathcal{M}(p, k, ...)|^2 \simeq \sigma_2^c(x)\,|\mathcal{M}(\tilde{p}, ...)|^2\,. \tag{7.2}$$

Diagrammatically, this is depicted in Fig. 7.1.
Here, x is the momentum fraction, while p and k are the momenta of the quark and the photon, respectively. We define $\tilde{p}$ via its relation to the quark momentum in the collinear limit, i.e. $\tilde{p} \to p/\bar{x}$.

In the following, as we are calculating splitting amplitudes in the collinear limit [119, 120], we employ methods of the soft-collinear effective theory (SCET). As a first step, we introduce the two light-like vectors n and $\bar{n}$:

$$n = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \bar{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}. \tag{7.3}$$

Further, we can relate our matrix element to the imaginary part of the collinear propagator [121, 122]:

$$\sigma_2^c(x) = \frac{1}{\pi}\mathrm{Im}\left[i\mathrm{Tr}\left[\frac{\bar{\not{n}}}{4}\mathcal{M}_2\right]\right],$$

where $\mathcal{M}_2$ is constructed from the propagator diagram in Fig. 7.2.

Working in light-cone gauge $\bar{n} \cdot A = 0$, which sets the Wilson lines to unity in the calculation, the diagram from Fig. 7.2 translates into the following intermediate expression:

$$\mathrm{Tr}\left[\frac{\not{\bar{n}}}{4}\frac{\not{n}\not{\bar{n}}}{4}\left(\frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2}(ie\tilde{\mu}^{\epsilon}\gamma^{\mu})\frac{i(\not{p}+m)}{p^2-m^2}(ie\tilde{\mu}^{\epsilon}\gamma^{\nu})\frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2}\right)\frac{\not{\bar{n}}\not{n}}{4}\right]\times$$
$$\frac{i}{k^2}\left(-g_{\mu\nu}+\frac{\bar{n}^{\mu}k^{\nu}+\bar{n}^{\nu}k^{\mu}}{\bar{n}\cdot k}\right), \tag{7.4}$$

where the so-called collinear projection operator $\frac{\not{n}\not{\bar{n}}}{4}$ is introduced [121].
We now use Cutkosky rules to apply the optical theorem, which amounts to the replacements $\frac{1}{k^2} \to 2\pi i\delta(k^2)\theta(k^0)$ and $\frac{1}{p^2-m^2} \to 2\pi i\delta(p^2-m^2)\theta(p^0)$. The factor (2π) is absorbed in the phase space prefactor:

$$\sigma_2^c(x) = \frac{2e^2\tilde{\mu}^{2\epsilon}}{((k+p)^2-m^2)^2}\frac{1}{64}\left(-g_{\mu\nu}+\frac{\bar{n}^{\mu}k^{\nu}+\bar{n}^{\nu}k^{\mu}}{\bar{n}\cdot k}\right)\times$$
$$\mathrm{Tr}\left[\not{\bar{n}}\not{n}\not{\bar{n}}(\not{k}+\not{p}+m)\gamma^{\mu}(\not{p}+m)\gamma^{\nu}(\not{k}+\not{p}+m)\not{\bar{n}}\not{n}\right]. \tag{7.5}$$

We additionally, along the lines of the classical Sudakov parametrization, introduce $k_\perp$, the transverse momentum of the photon. The vectors of the two particles then take the following form:

$$k^{\mu} = x\frac{n^{\mu}}{2} + k_{\perp} - \frac{k_{\perp}^2}{x}\frac{\bar{n}^{\mu}}{2}, \tag{7.6}$$
$$p^{\mu} = \bar{x}\frac{n^{\mu}}{2} - k_{\perp} - \frac{k_{\perp}^2-m^2}{\bar{x}}\frac{\bar{n}^{\mu}}{2}. \tag{7.7}$$

Overall, our vectors satisfy the following relations:

$$\begin{aligned} n^2 &= 0\,, & \bar{n}^2 &= 0\,, & n\cdot\bar{n} &= 2\,, \\ p^2 &= m^2\,, & k^2 &= 0\,, & (p+k)^2 &= s \\ \bar{n}\cdot p &= \bar{x}\,, & \bar{n}\cdot k &= x\,. \end{aligned} \tag{7.8}$$

With these relations, Eq. (7.5) simplifies to

$$\sigma_2^c(x) = \frac{2e^2\tilde{\mu}^{2\epsilon}}{(s-m^2)^2}\left[\left(s-m^2\right)\left(\frac{1+\bar{x}^2}{x} - \epsilon x\right) - 2m^2\right]. \tag{7.9}$$

To arrive at the final splitting functions we now have to integrate $\sigma_2^c(x)$ over s in the cases of the quark being massive on the one hand and it being massless on the other. The two-particle quasi-collinear phase space $\widetilde{\mathrm{PS}}_2$, generalized to the massive case, can be written as [123]:

$$\widetilde{\mathrm{PS}}_2 = S_\Gamma \int_{\frac{m^2}{\bar{x}}}^{\infty} ds \left(s - \frac{m^2}{\bar{x}}\right)^{-\epsilon} dx\,(x\bar{x})^{-\epsilon} \tag{7.10}$$

7.2.1 The Massless Case

For the massless case, the calculation turns out to be very simple. As we take the mass of the quark to zero, we only encounter scale-less integrals:

$$f^{(0)} = S_\Gamma \int_0^\infty ds\, s^{-\epsilon}\, dx\,(x\bar{x})^{-\epsilon}\, \sigma_2^c(x)\Big|_{m^2=0} = 0 \tag{7.11}$$

7.2.2 The Massive Case

For a massive quark, the integration over s looks as follows:

$$\begin{aligned} f^{(m)} &= S_\Gamma \int_{\frac{m^2}{\bar{x}}}^{\infty} ds \left(s - \frac{m^2}{\bar{x}}\right)^{-\epsilon} dx\,(x\bar{x})^{-\epsilon}\,\sigma_2^c(x) \\ &= 2e^2\tilde{\mu}^{2\epsilon}\, S_\Gamma\, dx\,(x\bar{x})^{-\epsilon}\left[\left(\frac{1+\bar{x}^2}{x} - \epsilon x\right)\pi\csc(\pi\epsilon)\left(\frac{m^2 x}{\bar{x}}\right)^{-\epsilon}\right. \\ &\qquad \left. - 2m^2\pi\epsilon\csc(\pi\epsilon)\left(\frac{m^2 x}{\bar{x}}\right)^{-1-\epsilon}\right]. \end{aligned} \tag{7.12}$$

7.2.3 Result for the Leading Order Splitting Function

Adding and simplifying the two contributions leads to the final result:

$$\begin{aligned}\Delta f^{(0)} &= f^{(m)} - f^{(0)} \\ &= \frac{e^2}{(4\pi)^2} \frac{2\pi e^{\gamma_E \epsilon}(1-\epsilon)(1+\bar{x}^2)\csc(\pi\epsilon)}{x\Gamma(1-\epsilon)} \left(\frac{mx}{\mu}\right)^{-2\epsilon} . \end{aligned} \tag{7.13}$$

We then use the relation between the momentum fraction x and the cut on the photon energy z:

$$x = \frac{\bar{z}}{1-s_{ij}} , \tag{7.14}$$

where i and j are the momentum labels of the two final-state quarks that are not involved in the collinear splitting. Transforming this variable and denoting the photon momentum by p_4 and the momentum of quark q_i by p_i brings us to the final shift-relation:

$$\begin{aligned}\frac{d\Gamma_{\text{shift}}}{dz} &= \frac{1}{2m_b}\frac{1}{2N_c} \int dPS_3\, \mathcal{K}_3(s_{ij}) \frac{\alpha_e}{2\pi\bar{z}} \left\{ Q_1^2 \left[1 + \frac{(z-s_{23})^2}{(1-s_{23})^2} \right] \right. \\ &\times \left. \left[\pi(1-\epsilon)\csc(\pi\epsilon) \left(\frac{m_{q_1}(1-z)}{\mu(1-s_{23})} \right)^{-2\epsilon} \right] \Theta(z-s_{23}) + (\text{cyclic}) \right\} . \end{aligned} \tag{7.15}$$

Expanding this result up to $\mathcal{O}(\epsilon^0)$ yields the result from Ref. [10].
The **cyclic** in Eq. (7.15) means that we have to sum over the splitting functions being attached to every of the three quark lines with the respective charges, masses and momenta adjusted.
$\mathcal{K}_3(s_{ij})$ in Eq. (7.15) denotes the spin-summed squared matrix element of the b decaying into the three light quarks at one-loop order. To simplify the occurring expressions, we first carried out a Passarino-Veltman reduction, after which we only had to solve about half a dozen of one-loop integrals. After this, the integration over the three-body phase space is done. The corresponding parametrization we used, which made an automatic handling of this contribution possible, was already given in Sect. 3.3. An example for a corresponding diagram can be found in Fig. 7.1 a).

For the regularization of the five-body real radiation diagrams, we find a similar relation:

$$\frac{d\Gamma_{\text{shift}}}{dz} = \frac{1}{2m_b}\frac{1}{2N_c}\int dPS_4\,\mathcal{K}_4(s_{ij})\frac{\alpha_e}{2\pi\bar{z}}\left\{Q_1^2\left[1+\frac{(z-s_{235})^2}{(1-s_{235})^2}\right]\right.$$
$$\left.\times\left[\pi(1-\epsilon)\csc(\pi\epsilon)\left(\frac{m_{q_1}(1-z)}{\mu(1-s_{235})}\right)^{-2\epsilon}\right]\Theta(z-s_{235})+(\text{cyclic})\right\}. \quad (7.16)$$

The $\mathcal{K}_4(s_{ij})$ are tree-level diagrams with an additional gluon (with momentum p_5) in the final state and we sum over the emission from the three quarks again. These diagrams are shown in Fig. 7.1 b).

In contrast to the three-body case, the case of the four-body phase space in Eq. (7.16) is not as straightforward. There are multiple arrangements of momenta to consider, but in the end we also found a factorizing phase space for each of them. The important point to make this achievable was to realize that in the secluded part of this calculation, the restrictions on the renaming of the photon momentum are not as prevalent as in the case of the bare integrals. The parametrization we use is a modification of the one given in Ref. [71], where the phase space takes the following form:

$$dPS_4 = \frac{2^{-3D}\pi^{1-3/2D}(m_b^2)^{3/2D-4}}{\Gamma(\frac{D-1}{2})\Gamma(\frac{D-2}{2})\Gamma(\frac{D-3}{2})}\,d\chi\,dz_1\,dt\,dv\,dy_{134}\,\big(\chi\bar{\chi}\big)^{-\frac{1}{2}-\epsilon}\big(z_1 t\bar{t}v\bar{v}\big)^{-\epsilon}\big(y_{134}\bar{y}_{134}\bar{z}_1\big)^{1-2\epsilon} \quad (7.17)$$

and the momentum invariants transform as:

$$\begin{aligned}
s_{24} &= s_{124} - s_{14} - s_{12}\,, & s_{13} &= s_{134} - s_{14} - s_{34}\,,\\
s_{12} &= y_{134}\,z_1\,, & s_{23} &= v\,y_{134}\,\bar{z}_1\,,\\
s_{14} &= \chi\big(s_{14,b} - s_{14,a}\big) + s_{14,a}\,, & s_{34} &= t\,\bar{z}_1\,\bar{y}_{134}\,,\\
s_{14,a} &= \bar{y}_{134}\big(\bar{t}\,\bar{v} + t\,v\,z_1 - 2\sqrt{t\,\bar{t}\,v\,\bar{v}\,z_1}\big)\,, & s_{134} &= y_{134}\,,\\
s_{14,b} &= \bar{y}_{134}\big(\bar{t}\,\bar{v} + t\,v\,z_1 + 2\sqrt{t\,\bar{t}\,v\,\bar{v}\,z_1}\big)\,, & s_{124} &= 1 - \bar{z}_1\big(t\,\bar{y}_{134} - y_{134}\big)\,.
\end{aligned}$$

With this parametrization it is possible for us to rename the invariants in each momentum configuration in such a way that having the square root expressions in the denominator can be avoided, while at the same time the argument of $\Theta(z - s_{ijk})$ in

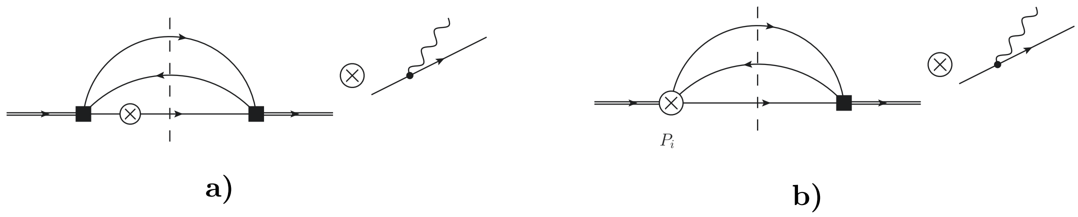


Figure 7.3: Types of diagrams needed to renormalize the UV divergences of the diagrams involving the splitting functions. Note that the crossed circles in the diagrams denote the insertion of counterterms while the ones between the diagrams denote the convolution of the diagrams with the leading order splitting functions.
a): Renormalization of the wave functions of the participating quarks. This is the only Standard Model parameter that needs to be renormalized for these diagrams.
b): Renormalization of the operator insertions.

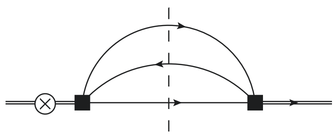
Eq. (7.16) is kept simple.
The argument of this Θ function leads to the triple invariant being the last integration that is done. Here, we expand our results in ϵ first before the last integration over y_{134} from 0 to z, because we are not able to get a full analytic expression to all orders. We checked that this last integration does not introduce new divergences, otherwise this interchange of expansion and integration would be forbidden.

7.3 Renormalization of the Regularization Diagrams

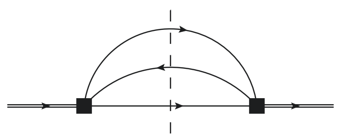
The renormalization procedure for the splitting function kernels is carried out very similar to Chapter 6, but needs substantially fewer pieces than the one for the bare diagrams.
As the diagrams do not contain photons, we do not have to concern ourselves with vertex, propagator or mass renormalization, as neither of these occur in the amplitudes. The only parts that are relevant are shown in Fig. 7.3, indicating that we only need to renormalize the quark wave-functions and the operator insertions.

Note that in the following, the convolution with the leading order splitting function is taken as implicit to avoid cluttering the equations.

For the renormalization of the wave function, we compute the following pieces:

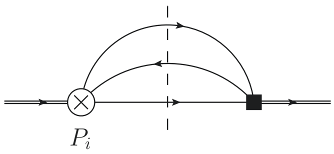


$$= \left(\frac{\alpha_s}{4\pi}\right)\left[\frac{1}{2}Z_h^{(1)}\right.$$

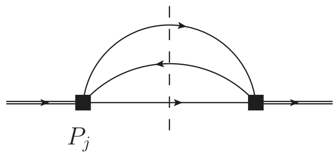


$$\left.\vphantom{\frac{1}{2}}\right],$$

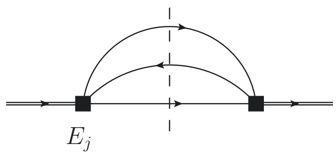
and for the operator insertions, the diagrammatic representation of the counterterms takes the following form:



$$= \left(\frac{\alpha_s}{4\pi\epsilon}\right)\left[\sum_{j=1u,2u,3,4,5,6}(\delta Z_{PP})_{ij}\right.$$



$$+\sum_{j=1,2,3,4}(\delta Z_{PE})_{ij}$$



$$\left.\vphantom{\sum}\right],$$

where the coefficients (δZ_{PP}) and (δZ_{PE}) are the entries of the matrix defined in Sect. 6.2.1.
As we mentioned before, these insertions are constructed similarly to the ones done in the first renormalization step, only now we encounter the additional splitting functions in the integrals.
The integration kernels that we obtain from these diagrams are then plugged into Eq. (7.15), only with the $\mathcal{K}_3(s_{ij})$ not being one-loop but tree-level expressions.

7.4 The Splitting Functions at Next-To-Leading Order

To get a fully consistent picture at $\mathcal{O}(\alpha_s)$ in the end, we not only need the leading order splitting functions combined with the NLO diagrams, but also the NLO splitting functions combined with the leading order diagrams. The latter case is shown in Fig. 7.1 c).
To calculate the NLO part of $\frac{d\Gamma_{\text{shift}}}{dz}$, one has to consider the $\mathcal{O}(\alpha_s)$ corrections to the

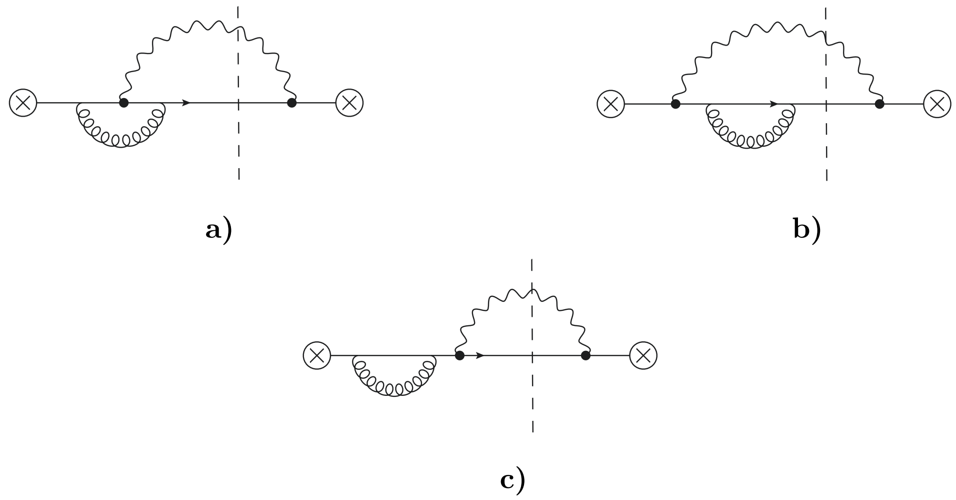


Figure 7.4: The three diagrams that contribute to the real-virtual part of the splitting function at next-to-leading order in light-cone gauge.

collinear emission. As in the calculation of the bare diagrams, we have to add the real-virtual and the real-real parts up for the cancellation of unregulated infrared divergences. These contributions are shown in Fig. 7.4 for the real-virtual and in Fig. 7.5 for the real-real case. Parallel to the LO case, these need to be computed with a mass and a dimensional regulator, respectively, and then subtracted. We have seen, in the course of calculating tree-level contributions, that the non-massive part of the calculation vanishes due to scaleless integrals, which also holds true for the next-to-leading order part. Thus only the massive contribution has to be calculated for the final result. There is an active effort going on in the calculation of these missing pieces [124] and, for completeness, we will outline the methods that are used despite not giving a final result in this work.

The Real-Virtual Contributions

For the real-virtual contributions, we again utilize the expression from Eq. (7.10). As a first step, we carry out the loop integrals and then integrate the result over the quasi-collinear phase space. For the most part, especially for the contributions from more complicated loop-integrals, this is done as an expansion in ϵ via Mellin-Barnes representations.

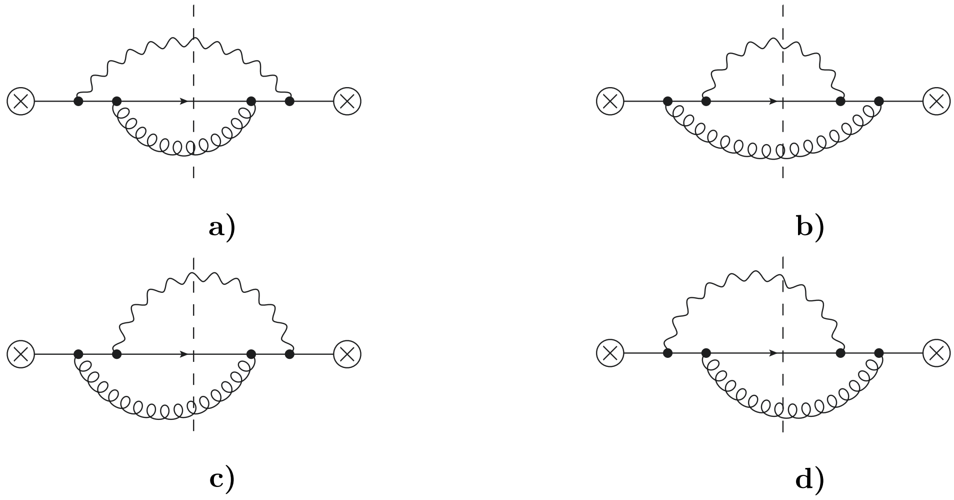


Figure 7.5: At next-to-leading order, in light-cone gauge, we encounter four different diagrams that contribute to the real-real part of the splitting function.

The Real-Real Contributions

For the real-real contributions, we trade the loop integration for a more complicated phase space. Calling the photon momentum p_1, the gluon momentum p_2, and the quark momentum p_3, we can define the momentum invariants as $s_{ij} = 2\,p_i \cdot p_j$ and $s_{123} = s_{12} + s_{13} + s_{23}$. Their respective momentum fractions are denoted by z_i. With these, we can express the three-particle quasi-collinear phase space as [124]:

$$d\widetilde{\mathrm{PS}}_3 = \int dz_1\, dz_2\, dz_3\, ds_{12}\, d\tilde{s}_{13}\, d\tilde{s}_{23}\, d\tilde{s}_{123}\, \delta(1 - z_1 - z_2 - z_3)\delta(\tilde{s}_{123} - \tilde{s}_{13} - \tilde{s}_{23} - s_{12}) \\ \times \frac{4}{(4\pi)^{5-2\epsilon}\Gamma(1-2\epsilon)}\, \Theta(-\widetilde{\Delta}_m)\, (-\widetilde{\Delta}_m)^{-\frac{1}{2}-\epsilon}, \tag{7.18}$$

where the tilde-notation $\tilde{s}_x = s_x - m^2$ is introduced for some of the invariants and the Gram determinant $\widetilde{\Delta}_m$ is defined by:

$$\widetilde{\Delta}_m = \Big[(z_3 s_{12} - z_1 \tilde{s}_{23} - z_2 \tilde{s}_{13})^2 - 4 z_1 z_2 \tilde{s}_{13} \tilde{s}_{23}\Big] + 4m^2 s_{12} z_1 z_2\,. \tag{7.19}$$

The momentum invariants in Eq. (7.18) have the following lower bounds:

$$s_{12} \geq 0\,, \qquad \tilde{s}_{13} \geq \frac{z_1}{z_3} m^2\,,$$
$$\tilde{s}_{23} \geq \frac{z_2}{z_3} m^2\,, \qquad \tilde{s}_{123} \geq \frac{\bar{z}_3}{z_3} m^2\,,$$

while the momentum fractions z_i are decoupled and integrate from 0 to 1.
As in the two-body phase space, we want to stay differential in the momentum fraction of the photon, here z_1. After the other integrations are carried out, we use the relation from Eq. (7.14):

$$z_1 = \frac{\bar{z}}{1 - s_{ij}}\,, \tag{7.20}$$

where i and j are the indices of the two quark legs that the photon is not emitted from.

Cross-Checks of the Intermediate Results

As a cross-check, giving us a robust confirmation that the remaining divergences only stem from the missing next-to-leading order splitting function, we can check the operator insertions that mix a color singlet and a color octet operator, e.g. $(P_3 \times P_4)$. Looking at diagrams such as Fig. 7.1 c), we can see that the tree-level diagrams $b \to q_1 q_2 q_3$ vanish after performing the color algebra (as they are $\propto \mathrm{Tr}(T^a)$), meaning that the expressions containing the NLO splitting functions become zero. Explicitly, this can be checked in $G_{ij}^{(I)}$ in Chapter 8, where these color-mixing entries vanish and the results are thus finite.
Additionally, the remaining poles are only proportional to Q_d^2 or Q_u^2 and not $Q_u Q_d$. As we have seen in Eq. (7.16), splitting functions could not account for the latter, as they are always $\propto Q_i^2$.
Furthermore, we can draw conclusions from the preliminary results for the real-virtual part of the splitting function at NLO. What we noted is that, although the diagrams in themselves generate poles at ϵ^{-3}, these highest poles cancel. At ϵ^{-2}, we encounter functions up to weight one, going up to weight three for the finite piece. As we will see in Chapter 8, this is exactly the form of the remaining pieces.

Observing all of the aforementioned behaviour, we can be sufficiently sure that the rest of the calculation is self-consistent, and despite still seeing some poles, our treatment gets rid of all divergent pieces from other sources.

With this, we conclude the discussion of all the different pieces of the calculation. Adding these all up, we are able to construct our contribution to the differential rate. The only step left is now the integration of the differential rate over z to express everything in terms of δ, which is related to the actual energy cut-off in the experiment.
The next section will give a brief overview over these integrations and finally show the results for the matrix G_{ij}.

Chapter 8

Results

The combination of all the aforementioned parts of the calculation is used to construct G_{ij}, our correction to the multi-parton contribution matrix $\hat{G}_{ij}^{(1)}$ defined in Eq. (2.21). Except for the missing NLO splitting function pieces, which were discussed in Sect. 7.4, this formally completes the partonic multi-body corrections at $\mathcal{O}(\alpha_s)$.
We split our resulting matrix in two parts, namely $G_{ij}^{(I)}$, where the operator insertions that contain two dirac traces are collected and $G_{ij}^{(II)}$ for insertions that contain a single dirac trace.
All results are also published in digital form on `GitHub`, the link to which can be found in Ref. [113]. The matrix can be constructed after downloading all the files and executing the notebook `ConstructGij.m`.

8.1 Building Blocks

We now want to present the 30 building blocks T_i of the final matrix. We only give the first 25 blocks in explicit written form for the sake of brevity. The last five building blocks contain up to over a thousand integrated GPL functions and are too lengthy to write down here. However, to make this work self-consistent, we put forward to a conceptual description in Sects. 8.1.2 and 8.1.3. With this, the building blocks can be constructed from pieces that are already given in this thesis.
All the building blocks are additionally given in electronic form in the notebook

`ConstructGij.m`, downloadable at Ref. [113]. There they are given in differential form as `DB[i]` and in their integrated form `IB[i]`.

8.1.1 Building Blocks 1-25

The integration of the first 25 building blocks is pretty straightforward. In Chapter 5, we saw that the variable $\bar{z}$ is a more natural choice for our problem than z. Switching the variable also calls for the change of integration limits. Denoting the differential building blocks by $D_i(\bar{z})$, the final integration formula is given by:

$$T_i = \int_{\bar{\delta}}^{1} d\bar{z}\, D_i(\bar{z})\,. \tag{8.1}$$

The results of these integrations are shown here:

$$\begin{aligned} T_1 = &-\frac{1}{108}\big(\bar{\delta}-1\big)\big(8\bar{\delta}^3-4\bar{\delta}^2+29\bar{\delta}-45\big)-\frac{1}{9}\bar{\delta}\big(\bar{\delta}^3-4\bar{\delta}^2+3\bar{\delta}-5\big)H_0\big(\bar{\delta}\big)\\ &-\frac{1}{9}\big(\bar{\delta}-1\big)\big(\bar{\delta}^3-3\bar{\delta}^2+3\bar{\delta}-7\big)H_1\big(\bar{\delta}\big)-\frac{2}{3}H_2\big(\bar{\delta}\big)+\frac{\pi^2}{9}\,,\end{aligned}$$

$$\begin{aligned} T_2 = &-\frac{1}{54}\big(\bar{\delta}-1\big)\big(16\bar{\delta}^3-8\bar{\delta}^2+34\bar{\delta}-45\big)-\frac{1}{18}\bar{\delta}\big(8\bar{\delta}^3-24\bar{\delta}^2+15\bar{\delta}-22\big)H_0\big(\bar{\delta}\big)\\ &-\frac{1}{9}\big(\bar{\delta}-1\big)\big(4\bar{\delta}^3-8\bar{\delta}^2+7\bar{\delta}-15\big)H_1\big(\bar{\delta}\big)-\frac{4}{3}H_2\big(\bar{\delta}\big)+\frac{2\pi^2}{9}\,,\end{aligned}$$

$$\begin{aligned} T_3 = &\frac{1}{324}\big(-68\bar{\delta}^3+12\bar{\delta}^2+825\bar{\delta}-769\big)+\frac{1}{18}\bar{\delta}\big(32\bar{\delta}^3-96\bar{\delta}^2+45\bar{\delta}-66\big)H_{0,0}\big(\bar{\delta}\big)\\ &+\frac{1}{9}\big(4\bar{\delta}^4-12\bar{\delta}^3+15\bar{\delta}^2-22\bar{\delta}+15\big)H_{1,0}\big(\bar{\delta}\big)+\frac{1}{18}\bar{\delta}\big(16\bar{\delta}^3-48\bar{\delta}^2+45\bar{\delta}-66\big)H_2\big(\bar{\delta}\big)\\ &+\frac{1}{9}\big(-4\bar{\delta}^4+12\bar{\delta}^3-15\bar{\delta}^2+22\bar{\delta}-15\big)H_{1,1}\big(\bar{\delta}\big)+\frac{4}{3}H_{2,0}\big(\bar{\delta}\big)-\frac{4}{3}H_{2,1}\big(\bar{\delta}\big)\\ &+\frac{8}{3}H_3\big(\bar{\delta}\big)+\frac{4}{3}\zeta(3)+\frac{53}{108}\pi^2+\frac{1}{108}\bar{\delta}\big(-36\bar{\delta}^3+160\bar{\delta}^2-93\bar{\delta}+216\big)H_0\big(\bar{\delta}\big)\\ &+\frac{1}{108}\big(-36\bar{\delta}^4+128\bar{\delta}^3-177\bar{\delta}^2+576\bar{\delta}-491\big)H_1\big(\bar{\delta}\big)\,,\end{aligned}$$

$$\begin{aligned} T_4 = &\frac{1}{9}\bar{\delta}\big(4\bar{\delta}^3-16\bar{\delta}^2+9\bar{\delta}-15\big)H_{0,0}\big(\bar{\delta}\big)+\frac{1}{9}\big(\bar{\delta}^4-4\bar{\delta}^3+6\bar{\delta}^2-10\bar{\delta}+7\big)H_{1,0}\big(\bar{\delta}\big)\\ &+\frac{1}{9}\big(-\bar{\delta}^4+4\bar{\delta}^3-6\bar{\delta}^2+10\bar{\delta}-7\big)H_{1,1}\big(\bar{\delta}\big)+\frac{2}{3}H_{2,0}\big(\bar{\delta}\big)-\frac{2}{3}H_{2,1}\big(\bar{\delta}\big)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{648}\big(-34\bar{\delta}^3-3\bar{\delta}^2+786\bar{\delta}-749\big)+\frac{1}{108}\bar{\delta}\big(-9\bar{\delta}^3+52\bar{\delta}^2-36\bar{\delta}+96\big)H_0\big(\bar{\delta}\big)\\
&+\frac{1}{108}\big(-9\bar{\delta}^4+44\bar{\delta}^3-72\bar{\delta}^2+264\bar{\delta}-227\big)H_1\big(\bar{\delta}\big)+\frac{1}{9}\bar{\delta}\big(2\bar{\delta}^3-8\bar{\delta}^2+9\bar{\delta}-15\big)H_2\big(\bar{\delta}\big)\\
&+\frac{4}{3}H_3\big(\bar{\delta}\big)+\frac{2\zeta(3)}{3}+\frac{2\pi^2}{9}\,,
\end{aligned}
$$

$$
\begin{aligned}
T_5=&\frac{1}{648}\big(-812\bar{\delta}^4+1444\bar{\delta}^3-645\bar{\delta}^2-1066\bar{\delta}+1079\big)-\frac{11}{3}H_2\big(\bar{\delta}\big)\\
&+\frac{1}{108}\pi^2\big(-32\bar{\delta}^4+96\bar{\delta}^3-105\bar{\delta}^2+154\bar{\delta}-47\big)\\
&+\big(\frac{1}{27}\big(-14\bar{\delta}^4+105\bar{\delta}^3-117\bar{\delta}^2+88\bar{\delta}+90\big)-\frac{8\pi^2}{9}\big)H_0\big(\bar{\delta}\big)\\
&+\frac{1}{108}\big(-152\bar{\delta}^4+548\bar{\delta}^3-837\bar{\delta}^2+1034\bar{\delta}-593\big)H_1\big(\bar{\delta}\big)\,,
\end{aligned}
$$

$$
\begin{aligned}
T_6=&\frac{1}{1296}\Big(-298\bar{\delta}^4+542\bar{\delta}^3-213\bar{\delta}^2-890\bar{\delta}+859\Big)-\frac{35}{18}H_2\big(\bar{\delta}\big)\\
&+\frac{1}{108}\pi^2\big(-8\bar{\delta}^4+32\bar{\delta}^3-42\bar{\delta}^2+70\bar{\delta}-17\big)\\
&+\left(\frac{1}{216}\big(-\bar{\delta}^4+248\bar{\delta}^3-360\bar{\delta}^2+320\bar{\delta}+360\big)-\frac{4\pi^2}{9}\right)H_0\big(\bar{\delta}\big)\\
&+\frac{1}{216}\big(-49\bar{\delta}^4+312\bar{\delta}^3-630\bar{\delta}^2+940\bar{\delta}-573\big)H_1\big(\bar{\delta}\big)\,,
\end{aligned}
$$

$$
\begin{aligned}
T_7=&\frac{1}{288}\big(61\bar{\delta}^4-232\bar{\delta}^3+330\bar{\delta}^2-760\bar{\delta}+601\big)+\frac{17}{12}H_0\big(\bar{\delta}\big)\\
&+\frac{1}{12}\big(\bar{\delta}^4-4\bar{\delta}^3+6\bar{\delta}^2-10\bar{\delta}+7\big)H_1\big(\bar{\delta}\big)+\frac{1}{2}H_2\big(\bar{\delta}\big)-\frac{\pi^2}{12}\,,
\end{aligned}
$$

$$
\begin{aligned}
T_8=&\frac{1}{36}\big(70\bar{\delta}^4-186\bar{\delta}^3+189\bar{\delta}^2-358\bar{\delta}+285\big)+\frac{31}{6}H_0\big(\bar{\delta}\big)\\
&+\frac{1}{6}\big(4\bar{\delta}^4-12\bar{\delta}^3+15\bar{\delta}^2-22\bar{\delta}+15\big)H_1\big(\bar{\delta}\big)+2H_2\big(\bar{\delta}\big)-\frac{\pi^2}{3}\,,
\end{aligned}
$$

$$
\begin{aligned}
T_9=&\frac{2}{9}\big(\bar{\delta}-3\big)\bar{\delta}^3H_{0,0}\big(\bar{\delta}\big)+\frac{1}{18}\big(4\bar{\delta}^4-12\bar{\delta}^3+15\bar{\delta}^2-22\bar{\delta}+15\big)H_{1,0}\big(\bar{\delta}\big)\\
&+\frac{1}{9}\big(-4\bar{\delta}^4+12\bar{\delta}^3-15\bar{\delta}^2+22\bar{\delta}-15\big)H_{1,1}\big(\bar{\delta}\big)+\frac{2}{3}H_{2,0}\big(\bar{\delta}\big)-\frac{4}{3}H_{2,1}\big(\bar{\delta}\big)\\
&+\frac{1}{648}\big(-2772\bar{\delta}^4+7432\bar{\delta}^3-12111\bar{\delta}^2+18288\bar{\delta}-10837\big)\\
&+\frac{1}{108}\pi^2\big(8\bar{\delta}^4-24\bar{\delta}^3-15\bar{\delta}^2-64\bar{\delta}-26\big)+\frac{8\zeta(3)}{3}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{108}\big(74\bar{\delta}^4-140\bar{\delta}^3+153\bar{\delta}^2-162\bar{\delta}-1386\big)H_0\big(\bar{\delta}\big)\\
&+\frac{1}{54}\big(\bar{\delta}^4-54\bar{\delta}^3-84\bar{\delta}^2+35\bar{\delta}+102\big)H_1\big(\bar{\delta}\big)\\
&+\frac{1}{18}\big(-8\bar{\delta}^4+24\bar{\delta}^3+15\bar{\delta}^2+64\bar{\delta}+26\big)H_2\big(\bar{\delta}\big)\,,
\end{aligned}
$$

$$
\begin{aligned}
T_{10}=&-\frac{1}{18}\big(\bar{\delta}-4\big)\bar{\delta}^3H_{0,0}\big(\bar{\delta}\big)+\frac{1}{18}\big(-\bar{\delta}^4+4\bar{\delta}^3-6\bar{\delta}^2+10\bar{\delta}-7\big)H_{1,0}\big(\bar{\delta}\big)\\
&+\frac{1}{9}\big(\bar{\delta}^4-4\bar{\delta}^3+6\bar{\delta}^2-10\bar{\delta}+7\big)H_{1,1}\big(\bar{\delta}\big)-\frac{1}{3}H_{2,0}\big(\bar{\delta}\big)\\
&+\frac{2}{3}H_{2,1}\big(\bar{\delta}\big)+\frac{1}{2592}\Big(-1485\bar{\delta}^4+7976\bar{\delta}^3-3666\bar{\delta}^2+9504\bar{\delta}-12329\Big)\\
&+\frac{1}{216}\pi^2\big(-4\bar{\delta}^4+16\bar{\delta}^3+39\bar{\delta}^2+110\bar{\delta}+52\big)-\frac{4}{3}\zeta(3)\\
&+\frac{1}{216}\big(-4\bar{\delta}^4+16\bar{\delta}^3-21\bar{\delta}^2+156\bar{\delta}-693\big)H_0\big(\bar{\delta}\big)\\
&+\frac{1}{216}\big(-22\bar{\delta}^4+180\bar{\delta}^3+315\bar{\delta}^2+52\bar{\delta}-525\big)H_1\big(\bar{\delta}\big)\\
&+\frac{1}{36}\big(4\bar{\delta}^4-16\bar{\delta}^3-39\bar{\delta}^2-110\bar{\delta}-52\big)H_2\big(\bar{\delta}\big)\,,
\end{aligned}
$$

$$
\begin{aligned}
T_{11}=&\,\frac{4}{9}\bar{\delta}\big(10\bar{\delta}^3-30\bar{\delta}^2+15\bar{\delta}-22\big)H_{0,0}\big(\bar{\delta}\big)+\frac{2}{9}\big(4\bar{\delta}^4-12\bar{\delta}^3+15\bar{\delta}^2-22\bar{\delta}+15\big)H_{1,0}\big(\bar{\delta}\big)\\
&-\frac{4}{9}\big(4\bar{\delta}^4-12\bar{\delta}^3+15\bar{\delta}^2-22\bar{\delta}+15\big)H_{1,1}\big(\bar{\delta}\big)+\frac{8}{3}H_{2,0}\big(\bar{\delta}\big)+\frac{16}{3}H_3\big(\bar{\delta}\big)\\
&-\frac{16}{3}H_{2,1}\big(\bar{\delta}\big)+\frac{1}{324}\big(-524\bar{\delta}^4+416\bar{\delta}^3+4017\bar{\delta}^2+4856\bar{\delta}-8765\big)\\
&+\frac{1}{54}\pi^2\big(8\bar{\delta}^4-24\bar{\delta}^3+69\bar{\delta}^2+54\bar{\delta}+354\big)+\frac{16}{3}\zeta(3)\\
&+\big(\frac{1}{54}\big(-292\bar{\delta}^4+1096\bar{\delta}^3-1041\bar{\delta}^2+798\bar{\delta}+486\big)+\frac{4\pi^2}{9}\big)H_0\big(\bar{\delta}\big)\\
&+\frac{1}{27}\big(-226\bar{\delta}^4+712\bar{\delta}^3-762\bar{\delta}^2+1819\bar{\delta}-1543\big)H_1\big(\bar{\delta}\big)\\
&+\frac{1}{9}\big(16\bar{\delta}^4-48\bar{\delta}^3+21\bar{\delta}^2-186\bar{\delta}-264\big)H_2\big(\bar{\delta}\big)\,,
\end{aligned}
$$

$$
\begin{aligned}
T_{12}=&\,\frac{2}{9}\bar{\delta}\big(3\bar{\delta}^3-12\bar{\delta}^2+6\bar{\delta}-10\big)H_{0,0}\big(\bar{\delta}\big)+\frac{2}{9}\big(\bar{\delta}^4-4\bar{\delta}^3+6\bar{\delta}^2-10\bar{\delta}+7\big)H_{1,0}\big(\bar{\delta}\big)\\
&-\frac{4}{9}\big(\bar{\delta}^4-4\bar{\delta}^3+6\bar{\delta}^2-10\bar{\delta}+7\big)H_{1,1}\big(\bar{\delta}\big)+\frac{4}{3}H_{2,0}\big(\bar{\delta}\big)-\frac{8}{3}H_{2,1}\big(\bar{\delta}\big)\\
&+\frac{1}{648}\big(-41\bar{\delta}^4-794\bar{\delta}^3+216\bar{\delta}^2+4472\bar{\delta}-3853\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{54}\pi^2\big(3\bar{\delta}^4-12\bar{\delta}^3+6\bar{\delta}^2-25\bar{\delta}+66\big)+4\zeta(3)\\
&+\Big(\frac{1}{12}\big(-5\bar{\delta}^4+32\bar{\delta}^3-37\bar{\delta}^2+22\bar{\delta}+27\big)+\frac{\pi^2}{9}\Big)H_0\big(\bar{\delta}\big)\\
&+\frac{1}{108}\big(-85\bar{\delta}^4+320\bar{\delta}^3-729\bar{\delta}^2+1582\bar{\delta}-1088\big)H_1\big(\bar{\delta}\big)\\
&+\Big(\frac{4\bar{\delta}^2}{3}-\frac{5\bar{\delta}}{9}-5\Big)H_2\big(\bar{\delta}\big)+\frac{4}{3}H_3\big(\bar{\delta}\big)\,,
\end{aligned}
$$

$$
\begin{aligned}
T_{13}=&\,5\pi^2\big(3\bar{\delta}^2+6\bar{\delta}+2\big)-\frac{5}{4}\big(53\bar{\delta}^4-288\bar{\delta}^3+394\bar{\delta}^2-792\bar{\delta}+633\big)\\
&-30\big(3\bar{\delta}^2+6\bar{\delta}+2\big)H_2\big(\bar{\delta}\big)+5\big(6\bar{\delta}^4-20\bar{\delta}^3+27\bar{\delta}^2-111\big)H_0\big(\bar{\delta}\big)\\
&+5\big(4\bar{\delta}^3+27\bar{\delta}^2-31\big)H_1\big(\bar{\delta}\big)\,,
\end{aligned}
$$

$$
T_{14}=\frac{8\bar{\delta}^4}{3}-8\bar{\delta}^3+10\bar{\delta}^2-\frac{44\bar{\delta}}{3}+8H_0\big(\bar{\delta}\big)+10\,,
$$

$$
T_{15}=\frac{2}{3}\big(\bar{\delta}^4-4\bar{\delta}^3+6\bar{\delta}^2-10\bar{\delta}+7\big)+4H_0\big(\bar{\delta}\big)\,,
$$

$$
T_{16}=\frac{1}{3}\big(1-\bar{\delta}^3\big)+H_0\big(\bar{\delta}\big)\,,
$$

$$
\begin{aligned}
T_{17}=&-\frac{20}{3}\big(6\bar{\delta}^2-20\bar{\delta}+27\big)\bar{\delta}^2H_{0,0}\big(\bar{\delta}\big)-\frac{20}{3}\big(2\bar{\delta}^3-3\bar{\delta}^2+26\bar{\delta}-28\big)H_{1,0}\big(\bar{\delta}\big)\\
&+\frac{40}{3}\big(4\bar{\delta}^3+27\bar{\delta}^2-31\big)H_{1,1}\big(\bar{\delta}\big)+480H_{2,0}\big(\bar{\delta}\big)-80\big(3\bar{\delta}^2+6\bar{\delta}+2\big)H_{2,1}\big(\bar{\delta}\big)\\
&+\frac{1}{36}\big(-12097\bar{\delta}^4+65204\bar{\delta}^3-79020\bar{\delta}^2+244300\bar{\delta}-218867\big)\\
&+\frac{4}{9}\pi^2\big(-30\bar{\delta}^4+110\bar{\delta}^3-90\bar{\delta}^2+270\bar{\delta}+363\big)+40\zeta(3)\big(3\bar{\delta}^2+12\bar{\delta}+28\big)\\
&+120\bar{\delta}^2H_3\big(\bar{\delta}\big)+\Big(\frac{1315\bar{\delta}^5-5495\bar{\delta}^4+9070\bar{\delta}^3-10770\bar{\delta}^2-25704\bar{\delta}+31704}{9\big(\bar{\delta}-1\big)}\\
&+\frac{80}{3}\pi^2\Big)H_0\big(\bar{\delta}\big)+\frac{1}{9}\big(-947\bar{\delta}^4+6808\bar{\delta}^3-4062\bar{\delta}^2+14180\bar{\delta}-15979\big)H_1\big(\bar{\delta}\big)\\
&+\frac{4}{3}\big(30\bar{\delta}^4-140\bar{\delta}^3-45\bar{\delta}^2-190\bar{\delta}-886\big)H_2\big(\bar{\delta}\big)\,,
\end{aligned}
$$

$$
T_{18}=\frac{1}{18}\big(-69\bar{\delta}^4+92\bar{\delta}^3-201\bar{\delta}^2-24\bar{\delta}+202\big)+\frac{71}{3}H_0\big(\bar{\delta}\big)\,,
$$

$$
\begin{aligned}
T_{19} = {} & \frac{1}{15552}\big(- 29844\bar{\delta}^4 - 12250\bar{\delta}^3 - 236331\bar{\delta}^2 - 7743354\bar{\delta} + 1052779\big) \\
& + \frac{1}{1296}\pi^2\big(- 2380\bar{\delta}^4 + 8224\bar{\delta}^3 - 9819\bar{\delta}^2 + 13488\bar{\delta} - 14567\big) - \frac{2}{9}\pi^2 H_{-2}\big(\bar{\delta}\big) \\
& + \frac{1}{27}\pi^2\big(- \bar{\delta}^4 - 6\bar{\delta}^3 - 15\bar{\delta}^2 - 22\bar{\delta} - 12\big) H_{-1}\big(\bar{\delta}\big) \\
& + \Big(\frac{1}{216}\pi^2\big(152\bar{\delta}^4 - 408\bar{\delta}^3 + 255\bar{\delta}^2 - 374\bar{\delta} - 1200\big) \\
& + \frac{1}{2592}\big(9133\bar{\delta}^4 - 3312\bar{\delta}^3 - 11976\bar{\delta}^2 + 42180\bar{\delta} + 36360\big) - \frac{8}{3}\zeta(3)\Big) H_0\big(\bar{\delta}\big) \\
& + \Big(\frac{1}{36}\pi^2\big(- 8\bar{\delta}^4 + 20\bar{\delta}^3 - 15\bar{\delta}^2 + 22\bar{\delta} - 19\big) \\
& + \frac{1}{2592}\big(6709\bar{\delta}^4 - 11528\bar{\delta}^3 - 56394\bar{\delta}^2 - 40936\bar{\delta} + 102149\big)\Big) H_1\big(\bar{\delta}\big) \\
& + \big(\frac{1}{216}\big(496\bar{\delta}^4 - 1688\bar{\delta}^3 + 3087\bar{\delta}^2 + 260\bar{\delta} + 4419\big) - \frac{5\pi^2}{9}\big) H_2\big(\bar{\delta}\big) \\
& + \frac{1}{36}\big(- 40\bar{\delta}^4 + 120\bar{\delta}^3 - 69\bar{\delta}^2 + 154\bar{\delta} + 132\big) H_3\big(\bar{\delta}\big) - \frac{8}{3} H_4\big(\bar{\delta}\big) \\
& + \frac{4}{9}\bar{\delta}\big(\bar{\delta}^3 + 6\bar{\delta}^2 + 22\big) H_{-2,0}\big(\bar{\delta}\big) + \frac{1}{54}\big(41\bar{\delta}^4 + 24\bar{\delta}^3 + 384\bar{\delta}^2 + 356\bar{\delta} - 45\big) H_{-1,0}\big(\bar{\delta}\big) \\
& + \big(\frac{1}{216}\big(- 8\bar{\delta}^4 - 1480\bar{\delta}^3 - 507\bar{\delta}^2 - 1884\bar{\delta} - 720\big) + \frac{8\pi^2}{9}\big) H_{0,0}\big(\bar{\delta}\big) \\
& + \frac{1}{54}\big(70\bar{\delta}^4 - 193\bar{\delta}^3 + 264\bar{\delta}^2 - 457\bar{\delta} + 316\big) H_{1,0}\big(\bar{\delta}\big) \\
& + \frac{1}{108}\big(262\bar{\delta}^4 - 738\bar{\delta}^3 + 51\bar{\delta}^2 - 2377\bar{\delta} + 2802\big) H_{1,1}\big(\bar{\delta}\big) \\
& + \frac{1}{18}\big(4\bar{\delta}^4 - 12\bar{\delta}^3 + 15\bar{\delta}^2 - 22\bar{\delta} + 15\big) H_{1,2}\big(\bar{\delta}\big) + \frac{1}{18}\big(- 15\bar{\delta}^2 + 22\bar{\delta} + 33\big) H_{2,0}\big(\bar{\delta}\big) \\
& - \frac{2}{3} H_{2,1,1}\big(\bar{\delta}\big) + 2 H_{2,1,0}\big(\bar{\delta}\big) + \frac{1}{36}\big(111\bar{\delta}^2 + 458\bar{\delta} + 602\big) H_{2,1}\big(\bar{\delta}\big) \\
& + \frac{4}{3} H_{3,1}\big(\bar{\delta}\big) - \frac{4}{9}\big(\bar{\delta}^4 + 6\bar{\delta}^3 + 15\bar{\delta}^2 + 22\bar{\delta} + 12\big) H_{-1,-1,0}\big(\bar{\delta}\big) \\
& + \frac{2}{3} H_{2,2}\big(\bar{\delta}\big) + \frac{2}{9}\big(\bar{\delta}^4 + 6\bar{\delta}^3 + 15\bar{\delta}^2 + 22\bar{\delta} + 12\big) H_{-1,0,0}\big(\bar{\delta}\big) \\
& + \frac{4}{3} H_{-2,0,0}\big(\bar{\delta}\big) + \frac{1}{36}\bar{\delta}\big(- 96\bar{\delta}^3 + 240\bar{\delta}^2 - 75\bar{\delta} + 110\big) H_{0,0,0}\big(\bar{\delta}\big) \\
& - \frac{8}{3} H_{-2,-1,0}\big(\bar{\delta}\big) - \frac{2}{9}\big(\bar{\delta}^4 - 6\bar{\delta}^3 + 15\bar{\delta}^2 - 22\bar{\delta} + 12\big) H_{1,0,0}\big(\bar{\delta}\big) \\
& - \frac{4}{3} H_{3,0}\big(\bar{\delta}\big) + \frac{1}{6}\big(4\bar{\delta}^4 - 12\bar{\delta}^3 + 15\bar{\delta}^2 - 22\bar{\delta} + 15\big) H_{1,1,0}\big(\bar{\delta}\big) \\
& - \frac{4}{3} H_{2,0,0}\big(\bar{\delta}\big) + \frac{1}{18}\big(- 4\bar{\delta}^4 + 12\bar{\delta}^3 - 15\bar{\delta}^2 + 22\bar{\delta} - 15\big) H_{1,1,1}\big(\bar{\delta}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{18}\big(16\bar{\delta}^4 + 12\bar{\delta}^3 - 21\bar{\delta}^2 - 130\bar{\delta} - 295\big)\zeta(3) + \frac{16}{135}\pi^4\,, \\
T_{20} =\ & \frac{1}{720}\pi^4\big(13\bar{\delta}^3 - 85\bar{\delta}^2 + 12\big) + \frac{1}{18}\pi^2\big(-13\bar{\delta}^3 + 15\bar{\delta}^2 + 2\big)H_{-2}\big(\bar{\delta}\big) \\
&+ \frac{1}{5184}\pi^2\big(-4415\bar{\delta}^4 + 10820\bar{\delta}^3 - 23508\bar{\delta}^2 + 28580\bar{\delta} - 29974\big) \\
&+ \frac{1}{31104}\big(186021\bar{\delta}^4 - 1081876\bar{\delta}^3 + 1176762\bar{\delta}^2 - 2216430\bar{\delta} + 1935523\big) \\
&+ \frac{1}{72}\pi^2\big(-5\bar{\delta}^4 + 100\bar{\delta}^3 + 114\bar{\delta}^2 + 28\bar{\delta} + 19\big)H_{-1}\big(\bar{\delta}\big) \\
&+ \Big(\frac{1}{432}\pi^2\big(190\bar{\delta}^4 - 480\bar{\delta}^3 - 393\bar{\delta}^2 - 374\bar{\delta} - 1072\big) \\
&+ \frac{1}{5184}\big(3560\bar{\delta}^4 + 13778\bar{\delta}^3 + 62649\bar{\delta}^2 + 61986\bar{\delta} + 125856\big) - \frac{14}{3}\zeta(3)\Big)H_0\big(\bar{\delta}\big) \\
&+ \Big(\frac{1}{432}\pi^2\big(-41\bar{\delta}^4 + 468\bar{\delta}^3 - 552\bar{\delta}^2 + 392\bar{\delta} - 267\big) \\
&+ \frac{1}{5184}\big(17264\bar{\delta}^4 - 55230\bar{\delta}^3 + 53931\bar{\delta}^2 - 137402\bar{\delta} + 121437\big)\Big)H_1\big(\bar{\delta}\big) \\
&+ \frac{1}{6}\big(-13\bar{\delta}^3 + 15\bar{\delta}^2 - 8\big)H_4\big(\bar{\delta}\big) + \Big(\frac{1}{36}\pi^2\big(-13\bar{\delta}^3 - 15\bar{\delta}^2 - 16\big) \\
&+ \frac{1}{864}\big(631\bar{\delta}^4 + 4464\bar{\delta}^3 + 7206\bar{\delta}^2 - 6028\bar{\delta} + 12212\big)\Big)H_2\big(\bar{\delta}\big) \\
&+ \frac{1}{72}\big(-79\bar{\delta}^4 + 492\bar{\delta}^3 + 183\bar{\delta}^2 + 178\bar{\delta} + 132\big)H_3\big(\bar{\delta}\big) \\
&+ \big(\frac{13\bar{\delta}^3}{3} - 5\bar{\delta}^2 - \frac{2}{3}\big)H_{-2,2}\big(\bar{\delta}\big) + \frac{1}{12}\big(5\bar{\delta}^4 - 100\bar{\delta}^3 - 114\bar{\delta}^2 - 28\bar{\delta} - 19\big)H_{-1,2}\big(\bar{\delta}\big) \\
&+ \Big(\pi^2\big(\frac{4}{9} - \frac{5}{6}\bar{\delta}^2\big) + \frac{1}{864}\big(1219\bar{\delta}^4 - 8580\bar{\delta}^3 + 9060\bar{\delta}^2 - 5388\bar{\delta} - 1440\big)\Big)H_{0,0}\big(\bar{\delta}\big) \\
&+ \frac{1}{864}\big(1171\bar{\delta}^4 - 6964\bar{\delta}^3 + 7062\bar{\delta}^2 - 3328\bar{\delta} + 2047\big)H_{1,0}\big(\bar{\delta}\big) \\
&+ \frac{1}{864}\big(967\bar{\delta}^4 + 5264\bar{\delta}^3 - 4872\bar{\delta}^2 - 12460\bar{\delta} + 11101\big)H_{1,1}\big(\bar{\delta}\big) \\
&+ \frac{1}{36}\big(4\bar{\delta}^4 - 12\bar{\delta}^3 + 15\bar{\delta}^2 - 22\bar{\delta} + 15\big)H_{1,2}\big(\bar{\delta}\big) \\
&+ \frac{1}{36}\big(8\bar{\delta}^4 + 276\bar{\delta}^3 - 15\bar{\delta}^2 + 106\bar{\delta} - 8\big)H_{2,0}\big(\bar{\delta}\big) \\
&+ \frac{1}{72}\big(55\bar{\delta}^4 - 420\bar{\delta}^3 - 249\bar{\delta}^2 + 90\bar{\delta} + 568\big)H_{2,1}\big(\bar{\delta}\big) + \frac{1}{3}H_{2,2}\big(\bar{\delta}\big) \\
&+ \frac{1}{3}\big(-13\bar{\delta}^3 - 2\big)H_{3,0}\big(\bar{\delta}\big) + \frac{1}{6}\big(13\bar{\delta}^3 - 15\bar{\delta}^2 + 4\big)H_{3,1}\big(\bar{\delta}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{7}{3}H_{2,1,1}\big(\bar\delta\big)+\frac{1}{24}\big(-5\bar\delta^4+100\bar\delta^3+114\bar\delta^2+28\bar\delta+19\big)H_{-1,0,0}\big(\bar\delta\big)\\
&+\frac{1}{72}\bar\delta\big(-56\bar\delta^3+168\bar\delta^2-75\bar\delta+110\big)H_{0,0,0}\big(\bar\delta\big)+\frac{1}{6}\big(-13\bar\delta^3+15\bar\delta^2+2\big)H_{-2,0,0}\big(\bar\delta\big)\\
&+\frac{1}{36}\big(4\bar\delta^4-12\bar\delta^3+15\bar\delta^2-22\bar\delta+15\big)H_{1,0,0}\big(\bar\delta\big)+\frac{1}{6}\big(-13\bar\delta^3-15\bar\delta^2+4\big)H_{2,1,0}\big(\bar\delta\big)\\
&+\frac{1}{24}\big(13\bar\delta^4+76\bar\delta^3-84\bar\delta^2-16\bar\delta+11\big)H_{1,1,0}\big(\bar\delta\big)+\frac{1}{3}H_{2,0,0}\big(\bar\delta\big)\\
&+\frac{7}{36}\big(4\bar\delta^4-12\bar\delta^3+15\bar\delta^2-22\bar\delta+15\big)H_{1,1,1}\big(\bar\delta\big)\\
&+\frac{1}{36}\big(-20\bar\delta^4+60\bar\delta^3-177\bar\delta^2+174\bar\delta-546\big)\zeta(3)\,,
\end{aligned}
$$

$$
\begin{aligned}
T_{21}=&\,\frac{400}{3}H_{3,0}\big(\bar\delta\big)\bar\delta^3+\frac{4}{3}\big(50\bar\delta+57\big)H_{2,1,0}\big(\bar\delta\big)\bar\delta^2\\
&+\frac{1}{81}\big(837\bar\delta^3-8144\bar\delta^2-8154\bar\delta-2808\big)H_3\big(\bar\delta\big)\bar\delta\\
&-\frac{8}{81}\big(18\bar\delta^3+133\bar\delta^2-855\big)H_{0,0,0}\big(\bar\delta\big)\bar\delta+\pi^4\Big(-\frac{5\bar\delta^3}{9}+\frac{323\bar\delta^2}{90}+\frac{61}{405}\Big)\\
&+\frac{1}{174960}\big(-1774629\bar\delta^4+1026567888\bar\delta^3-994290120\bar\delta^2-153106920\bar\delta+13831625\big)\\
&+\frac{1}{29160}\pi^2\big(-3348\bar\delta^5-153045\bar\delta^4+924400\bar\delta^3+464940\bar\delta^2-276480\bar\delta+122725\big)\\
&+\frac{1}{27}\pi^2\big(89\bar\delta^4-984\bar\delta^3-978\bar\delta^2+20\bar\delta-75\big)H_{-1}\big(\bar\delta\big)+\frac{4}{9}\pi^2\big(50\bar\delta^3-57\bar\delta^2-4\big)H_{-2}\big(\bar\delta\big)\\
&+\Big(\frac{11169\bar\delta^5-786367\bar\delta^4-58989140\bar\delta^3+75376080\bar\delta^2-14103360\bar\delta+549000}{14580\big(\bar\delta-1\big)}\\
&+\frac{1}{243}\pi^2\big(-765\bar\delta^4-196\bar\delta^3+8424\bar\delta^2+1260\bar\delta-72\big)+\frac{16}{3}\zeta(3)\Big)H_0\big(\bar\delta\big)\\
&+\Big(\frac{1}{4860}\big(-2973\bar\delta^4+691816\bar\delta^3-1074586\bar\delta^2+128308\bar\delta+257435\big)\\
&+\frac{1}{27}\pi^2\big(-46\bar\delta^4-498\bar\delta^3+519\bar\delta^2-26\bar\delta+51\big)\Big)H_1\big(\bar\delta\big)\\
&+\Big(\frac{1}{2430}\big(1674\bar\delta^5+83565\bar\delta^4-708220\bar\delta^3-91800\bar\delta^2+219420\bar\delta-14265\big)\\
&+\frac{2}{9}\pi^2\big(50\bar\delta^3+57\bar\delta^2+4\big)\Big)H_2\big(\bar\delta\big)+\frac{4}{9}\big(150\bar\delta^3-171\bar\delta^2-82\big)H_4\big(\bar\delta\big)-\frac{16}{3}H_{-3,0}\big(\bar\delta\big)\\
&+\frac{2}{9}\big(8\bar\delta^4+40\bar\delta^3-6\bar\delta^2+76\bar\delta-11\big)H_{-2,0}\big(\bar\delta\big)-\frac{8}{3}\big(50\bar\delta^3-57\bar\delta^2-2\big)H_{-2,2}\big(\bar\delta\big)\\
&+\frac{1}{54}\big(103\bar\delta^4-80\bar\delta^3+540\bar\delta^2+808\bar\delta+85\big)H_{-1,0}\big(\bar\delta\big)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2}{9}\big(93\bar{\delta}^4 - 964\bar{\delta}^3 - 936\bar{\delta}^2 + 84\bar{\delta} - 37\big) H_{-1,2}\big(\bar{\delta}\big) + \Big(\frac{4}{27}\pi^2\big(171\bar{\delta}^2 + 28\big) \\
&+ \frac{\bar{\delta}\big(- 837\bar{\delta}^5 - 41598\bar{\delta}^4 + 277795\bar{\delta}^3 - 581275\bar{\delta}^2 + 237735\bar{\delta} + 109620\big)}{1215\big(\bar{\delta} - 1\big)}\Big) H_{0,0}\big(\bar{\delta}\big) \\
&+ \frac{1}{810}\big(- 558\bar{\delta}^5 - 26745\bar{\delta}^4 + 164780\bar{\delta}^3 - 221850\bar{\delta}^2 + 74610\bar{\delta} + 8833\big) H_{1,0}\big(\bar{\delta}\big) \\
&+ \big(\frac{31\bar{\delta}^5}{45} + \frac{619\bar{\delta}^4}{18} - \frac{74362\bar{\delta}^3}{243} + \frac{2939\bar{\delta}^2}{9} - 77\bar{\delta} + \frac{51961}{2430}\big) H_{1,1}\big(\bar{\delta}\big) \\
&- \frac{2}{9}\big(964\bar{\delta}^3 - 84\bar{\delta} - 95\big) H_{2,0}\big(\bar{\delta}\big) + \big(- \frac{200\bar{\delta}^3}{3} + 76\bar{\delta}^2 + \frac{200}{9}\big) H_{3,1}\big(\bar{\delta}\big) \\
&+ \frac{1}{81}\big(- 837\bar{\delta}^4 + 8410\bar{\delta}^3 + 8424\bar{\delta}^2 + 630\bar{\delta} + 9\big) H_{2,1}\big(\bar{\delta}\big) - \frac{32}{3} H_{-2,-1,0}\big(\bar{\delta}\big) \\
&+ \frac{4}{3}\big(50\bar{\delta}^3 - 57\bar{\delta}^2 + 2\big) H_{-2,0,0}\big(\bar{\delta}\big) - \frac{8}{9}\big(2\bar{\delta}^4 + 10\bar{\delta}^3 + 21\bar{\delta}^2 + 32\bar{\delta} + 19\big) H_{-1,-1,0}\big(\bar{\delta}\big) \\
&- \frac{8}{3} H_{2,0,0}\big(\bar{\delta}\big) + \frac{1}{9}\big(101\bar{\delta}^4 - 924\bar{\delta}^3 - 852\bar{\delta}^2 + 212\bar{\delta} + 39\big) H_{-1,0,0}\big(\bar{\delta}\big) \\
&+ \frac{608}{9} H_{0,0,0,0}\big(\bar{\delta}\big) + \frac{1}{27}\big(- 45\bar{\delta}^4 + 88\bar{\delta}^3 - 234\bar{\delta}^2 + 408\bar{\delta} + 39\big) H_{1,0,0}\big(\bar{\delta}\big) \\
&- \frac{2}{9}\big(50\bar{\delta}^4 + 478\bar{\delta}^3 - 477\bar{\delta}^2 - 38\bar{\delta} - 13\big) H_{1,1,0}\big(\bar{\delta}\big) + \frac{2}{81}\big(180\bar{\delta}^4 + 594\bar{\delta}^2 + 2057\big)\zeta(3)\,,
\end{aligned}
$$

$$
\begin{aligned}
T_{22} = &\frac{1}{1399680}\big(- 651969\bar{\delta}^4 + 4669168\bar{\delta}^3 + 9770598\bar{\delta}^2 - 1150044412\bar{\delta} + 101618375\big) \\
&+ \frac{1}{116640}\pi^2\big(- 864\bar{\delta}^5 - 49860\bar{\delta}^4 + 226880\bar{\delta}^3 - 368280\bar{\delta}^2 + 461880\bar{\delta} - 404785\big) \\
&+ \Big(\frac{1}{972}\pi^2\big(207\bar{\delta}^4 - 884\bar{\delta}^3 + 711\bar{\delta}^2 + 243\bar{\delta} - 3258\big) - 8\zeta(3) \\
&- \frac{-79614\bar{\delta}^5 + 28492\bar{\delta}^4 + 407099\bar{\delta}^3 - 4398183\bar{\delta}^2 + 2771226\bar{\delta} + 1304460}{116640\big(\bar{\delta} - 1\big)}\Big) H_0\big(\bar{\delta}\big) \\
&+ \Big(\frac{1}{38880}\big(15612\bar{\delta}^4 + 89566\bar{\delta}^3 + 25919\bar{\delta}^2 - 998882\bar{\delta} + 867785\big) \\
&- \frac{1}{54}\pi^2\big(\bar{\delta} - 1\big)\big(4\bar{\delta}^3 - 12\bar{\delta}^2 + 9\bar{\delta} - 28\big)\Big) H_1\big(\bar{\delta}\big) \\
&+ \big(\frac{2\bar{\delta}^5}{45} + \frac{31\bar{\delta}^4}{36} - \frac{2083\bar{\delta}^3}{486} + \frac{389\bar{\delta}^2}{72} - \frac{17\bar{\delta}}{27} - \frac{\pi^2}{2} + \frac{1723}{432}\big) H_2\big(\bar{\delta}\big) \\
&+ \frac{1}{162}\big(- 99\bar{\delta}^4 + 472\bar{\delta}^3 - 585\bar{\delta}^2 - 1197\bar{\delta} + 324\big) H_3\big(\bar{\delta}\big) - \frac{94}{9} H_4\big(\bar{\delta}\big) \\
&+ \frac{1}{108}\big(- 13\bar{\delta}^4 - 88\bar{\delta}^3 - 144\bar{\delta}^2 - 40\bar{\delta} + 29\big) H_{-1,0}\big(\bar{\delta}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(\frac{-216\bar{\delta}^6+1251\bar{\delta}^5-2665\bar{\delta}^4+1135\bar{\delta}^3-111105\bar{\delta}^2+104220\bar{\delta}+8100}{4860\big(\bar{\delta}-1\big)}\\
&+\frac{40}{27}\pi^2\Big)H_{0,0}\big(\bar{\delta}\big)-\frac{4}{3}H_{-3,0}\big(\bar{\delta}\big)+\frac{1}{9}\big(-6\bar{\delta}^2-16\bar{\delta}-7\big)H_{-2,0}\big(\bar{\delta}\big)\\
&+\frac{1}{6480}\big(-288\bar{\delta}^5+1875\bar{\delta}^4-1240\bar{\delta}^3-7650\bar{\delta}^2+16200\bar{\delta}-11357\big)H_{1,0}\big(\bar{\delta}\big)\\
&+\frac{1}{19440}\big(864\bar{\delta}^5+18405\bar{\delta}^4-50320\bar{\delta}^3+129870\bar{\delta}^2-390060\bar{\delta}+291241\big)H_{1,1}\big(\bar{\delta}\big)\\
&+\frac{1}{18}\big(\bar{\delta}^4-4\bar{\delta}^3+6\bar{\delta}^2-10\bar{\delta}+7\big)H_{1,2}\big(\bar{\delta}\big)+\frac{1}{36}\big(8\bar{\delta}^4-32\bar{\delta}^3-12\bar{\delta}^2+20\bar{\delta}-49\big)H_{2,0}\big(\bar{\delta}\big)\\
&+\frac{1}{324}\big(180\bar{\delta}^4-796\bar{\delta}^3+612\bar{\delta}^2-162\bar{\delta}+1791\big)H_{2,1}\big(\bar{\delta}\big)+\frac{1}{3}H_{2,2}\big(\bar{\delta}\big)\\
&-\frac{2}{3}H_{3,0}\big(\bar{\delta}\big)+\frac{56}{9}H_{3,1}\big(\bar{\delta}\big)+\frac{1}{162}\bar{\delta}\big(-36\bar{\delta}^3-8\bar{\delta}^2+549\bar{\delta}+2961\big)H_{0,0,0}\big(\bar{\delta}\big)\\
&+\frac{1}{54}\big(3\bar{\delta}^4-20\bar{\delta}^3+36\bar{\delta}^2+45\big)H_{1,0,0}\big(\bar{\delta}\big)+\frac{1}{9}\big(\bar{\delta}^4-4\bar{\delta}^3+9\bar{\delta}^2-13\bar{\delta}+7\big)H_{1,1,0}\big(\bar{\delta}\big)\\
&+\frac{5}{6}\big(\bar{\delta}^4-4\bar{\delta}^3+6\bar{\delta}^2-10\bar{\delta}+7\big)H_{1,1,1}\big(\bar{\delta}\big)+\frac{4}{3}H_{2,0,0}\big(\bar{\delta}\big)+\frac{1}{3}H_{2,1,0}\big(\bar{\delta}\big)+5H_{2,1,1}\big(\bar{\delta}\big)\\
&+\frac{152}{9}H_{0,0,0,0}\big(\bar{\delta}\big)+\big(-\frac{\bar{\delta}^4}{2}+2\bar{\delta}^3-\frac{19\bar{\delta}^2}{3}+\frac{125\bar{\delta}}{9}-\frac{5509}{324}\big)\zeta(3)+\frac{37}{1620}\pi^4\,,
\end{aligned}
$$

$$
\begin{aligned}
T_{23}=&\frac{72160}{3}H_{3,0}\big(\bar{\delta}\big)\bar{\delta}^3+\frac{16}{3}\big(2255\bar{\delta}+2568\big)H_{2,1,0}\big(\bar{\delta}\big)\bar{\delta}^2\\
&-\frac{1088}{81}\big(18\bar{\delta}^3+133\bar{\delta}^2-855\big)H_{0,0,0}\big(\bar{\delta}\big)\bar{\delta}+\pi^4\big(-\frac{902\bar{\delta}^3}{9}+\frac{29104\bar{\delta}^2}{45}+\frac{8296}{405}\big)\\
&-\frac{1}{10935}\big(210250089\bar{\delta}^4-1149167458\bar{\delta}^3+1116315567\bar{\delta}^2+1376771122\bar{\delta}-125338400\big)\\
&-\frac{1}{3645}\pi^2\big(78786\bar{\delta}^5+3642615\bar{\delta}^4-20501900\bar{\delta}^3-9677880\bar{\delta}^2+4781160\bar{\delta}-1719125\big)\\
&+\frac{16}{9}\pi^2\big(2255\bar{\delta}^3-2568\bar{\delta}^2-136\big)H_{-2}\big(\bar{\delta}\big)\\
&+\frac{8}{27}\pi^2\big(2098\bar{\delta}^4-21708\bar{\delta}^3-21081\bar{\delta}^2+1600\bar{\delta}-1125\big)H_{-1}\big(\bar{\delta}\big)\\
&+\Big(\frac{2\big(245358\bar{\delta}^5-176872494\bar{\delta}^4-131358673\bar{\delta}^3+1636574312\bar{\delta}^2-244190522\bar{\delta}+9647550\big)}{3645\big(\bar{\delta}-1\big)}\\
&-\frac{16}{243}\pi^2\big(9135\bar{\delta}^4+1666\bar{\delta}^3-92259\bar{\delta}^2-10710\bar{\delta}+1152\big)+\frac{2176}{3}\zeta(3)\Big)H_0\big(\bar{\delta}\big)\\
&+\Big(-\frac{4}{27}\pi^2\big(2149\bar{\delta}^4+21912\bar{\delta}^3-22101\bar{\delta}^2-376\bar{\delta}-1584\big)\\
&-\frac{2}{1215}\big(75786\bar{\delta}^4-153965122\bar{\delta}^3+242847772\bar{\delta}^2-4409006\bar{\delta}-4555045\big)\Big)H_1\big(\bar{\delta}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big(\frac{4}{1215}\big(39393\bar{\delta}^5 + 1941030\bar{\delta}^4 - 157441440\bar{\delta}^3 - 1801575\bar{\delta}^2 \\
&+ 4530015\bar{\delta} - 152055\big) + \frac{8}{9}\pi^2\big(2255\bar{\delta}^3 + 2568\bar{\delta}^2 + 136\big)\Big)H_2\big(\bar{\delta}\big) \\
&+ \frac{16}{81}\big(9747\bar{\delta}^4 - 91634\bar{\delta}^3 - 89964\bar{\delta}^2 - 19008\bar{\delta} + 135\big)H_3\big(\bar{\delta}\big) - \frac{2176}{3}H_{-3,0}\big(\bar{\delta}\big) \\
&+ \frac{16}{9}\big(6765\bar{\delta}^3 - 7704\bar{\delta}^2 - 2788\big)H_4\big(\bar{\delta}\big) - \frac{32}{3}\big(2255\bar{\delta}^3 - 2568\bar{\delta}^2 - 68\big)H_{-2,2}\big(\bar{\delta}\big) \\
&+ \frac{272}{9}\big(8\bar{\delta}^4 + 40\bar{\delta}^3 - 6\bar{\delta}^2 + 76\bar{\delta} - 11\big)H_{-2,0}\big(\bar{\delta}\big) \\
&+ \frac{68}{27}\big(103\bar{\delta}^4 - 80\bar{\delta}^3 + 540\bar{\delta}^2 + 808\bar{\delta} + 85\big)H_{-1,0}\big(\bar{\delta}\big) \\
&- \frac{16}{9}\big(2166\bar{\delta}^4 - 21368\bar{\delta}^3 - 20367\bar{\delta}^2 + 2688\bar{\delta} - 479\big)H_{-1,2}\big(\bar{\delta}\big) \\
&+ \Big(\frac{128}{27}\pi^2\big(963\bar{\delta}^2 + 119\big) - \frac{4\bar{\delta}}{1215\big(\bar{\delta} - 1\big)}\big(39393\bar{\delta}^5 + 1919772\bar{\delta}^4 - 123414455\bar{\delta}^3 \\
&+ 258167750\bar{\delta}^2 - 117158840\bar{\delta} - 3767580\big)\Big)H_{0,0}\big(\bar{\delta}\big) - \frac{4}{405}\Big(13131\bar{\delta}^5 + 626790\bar{\delta}^4 \\
&- 3700360\bar{\delta}^3 + 4997925\bar{\delta}^2 - 1766520\bar{\delta} - 156506\Big)H_{1,0}\big(\bar{\delta}\big) + \frac{4}{1215}\Big(39393\bar{\delta}^5 \\
&+ 1941030\bar{\delta}^4 - 166648340\bar{\delta}^3 + 178093335\bar{\delta}^2 - 39564445\bar{\delta} + 815027\Big)H_{1,1}\big(\bar{\delta}\big) \\
&- \frac{16}{9}\big(21368\bar{\delta}^3 - 2688\bar{\delta} - 1615\big)H_{2,0}\big(\bar{\delta}\big) + \frac{16}{3}\big(2255\bar{\delta}^3 - 2568\bar{\delta}^2 + 68\big)H_{-2,0,0}\big(\bar{\delta}\big) \\
&- \frac{8}{81}\big(19494\bar{\delta}^4 - 187790\bar{\delta}^3 - 182088\bar{\delta}^2 - 990\bar{\delta} + 1197\big)H_{2,1}\big(\bar{\delta}\big) \\
&+ \frac{16}{9}\big(-6765\bar{\delta}^3 + 7704\bar{\delta}^2 + 1700\big)H_{3,1}\big(\bar{\delta}\big) - \frac{4352}{3}H_{-2,-1,0}\big(\bar{\delta}\big) \\
&- \frac{1088}{3}H_{2,0,0}\big(\bar{\delta}\big) - \frac{1088}{9}\big(2\bar{\delta}^4 + 10\bar{\delta}^3 + 21\bar{\delta}^2 + 32\bar{\delta} + 19\big)H_{-1,-1,0}\big(\bar{\delta}\big) \\
&+ \frac{8}{9}\big(2302\bar{\delta}^4 - 20688\bar{\delta}^3 - 18939\bar{\delta}^2 + 4864\bar{\delta} + 813\big)H_{-1,0,0}\big(\bar{\delta}\big) \\
&- \frac{136}{27}\big(45\bar{\delta}^4 - 88\bar{\delta}^3 + 234\bar{\delta}^2 - 408\bar{\delta} - 39\big)H_{1,0,0}\big(\bar{\delta}\big) \\
&- \frac{8}{9}\big(2285\bar{\delta}^4 + 21232\bar{\delta}^3 - 20673\bar{\delta}^2 - 2552\bar{\delta} - 292\big)H_{1,1,0}\big(\bar{\delta}\big) \\
&+ \frac{82688}{9}H_{0,0,0,0}\big(\bar{\delta}\big) + \frac{16}{81}\big(3060\bar{\delta}^4 + 11313\bar{\delta}^2 + 35509\big)\zeta(3)\,,
\end{aligned}
$$

$$
T_{24} = -\frac{1}{6}\big(97\bar{\delta} + 111\big)H_{2,1,0}\big(\bar{\delta}\big)\bar{\delta}^2 + \frac{1}{162}\big(-405\bar{\delta}^3 + 3947\bar{\delta}^2 + 3897\bar{\delta} + 972\big)H_3\big(\bar{\delta}\big)\bar{\delta}
$$

$$
\begin{aligned}
&-\frac{97}{3}H_{3,0}(\bar\delta)\bar\delta^3+\frac{1}{162}\big(54\bar\delta^3+494\bar\delta^2+171\bar\delta-2736\big)H_{0,0,0}(\bar\delta)\bar\delta\\
&+\frac{1}{1399680}\big(3762369\bar\delta^4-201351568\bar\delta^3+194361012\bar\delta^2+215333382\bar\delta-18201515\big)\\
&+\frac{1}{233280}\pi^2\big(6588\bar\delta^5+304020\bar\delta^4-1797460\bar\delta^3-845370\bar\delta^2+463860\bar\delta-223585\big)\\
&+\frac{1}{2160}\pi^4\big(291\bar\delta^3-1887\bar\delta^2-61\big)+\frac{1}{18}\pi^2\big(-97\bar\delta^3+111\bar\delta^2+6\big)H_{-2}(\bar\delta)\\
&+\frac{1}{108}\pi^2\big(-87\bar\delta^4+946\bar\delta^3+918\bar\delta^2-66\bar\delta+49\big)H_{-1}(\bar\delta)\\
&+\Big(\frac{-56628\bar\delta^5+3141374\bar\delta^4+227511113\bar\delta^3-28385001\bar\delta^2+4694742\bar\delta-2162880}{233280(\bar\delta-1)}\\
&+\frac{1}{972}\pi^2\big(756\bar\delta^4+182\bar\delta^3-8037\bar\delta^2-1008\bar\delta-81\big)-\zeta(3)\Big)H_0(\bar\delta)\\
&+\Big(\frac{1}{432}\pi^2\big(181\bar\delta^4+1908\bar\delta^3-1926\bar\delta^2-28\bar\delta-135\big)\\
&+\frac{7476\bar\delta^4-2692442\bar\delta^3+4124837\bar\delta^2-668666\bar\delta-771205}{77760}\Big)H_1(\bar\delta)\\
&+\Big(\frac{1}{38880}\big(-6588\bar\delta^5-324675\bar\delta^4+2741720\bar\delta^3+346140\bar\delta^2-778500\bar\delta+47430\big)\\
&+\frac{1}{36}\pi^2\big(-97\bar\delta^3-111\bar\delta^2-6\big)\Big)H_2(\bar\delta)+\frac{1}{6}\big(-97\bar\delta^3+111\bar\delta^2+41\big)H_4(\bar\delta)+H_{-3,0}(\bar\delta)\\
&+\frac{1}{36}\big(-12\bar\delta^4-56\bar\delta^3+6\bar\delta^2-80\bar\delta+15\big)H_{-2,0}(\bar\delta)+\big(\frac{97\bar\delta^3}{3}-37\bar\delta^2-1\big)H_{-2,2}(\bar\delta)\\
&+\frac{1}{432}\big(-137\bar\delta^4+120\bar\delta^3-456\bar\delta^2-944\bar\delta-231\big)H_{-1,0}(\bar\delta)\\
&+\Big(\pi^2\big(-\frac{37}{6}\bar\delta^2-\frac{7}{9}\big)+\frac{\bar\delta}{38880(\bar\delta-1)}\big(6588\bar\delta^5+318357\bar\delta^4-21408885\bar\delta^3+4444030\bar\delta^2\\
&-1910790\bar\delta-727380\big)\Big)H_{0,0}(\bar\delta)+\big(5\bar\delta^4-\frac{466\bar\delta^3}{9}-\frac{99\bar\delta^2}{2}+6\bar\delta-\frac{23}{18}\big)H_{-1,2}(\bar\delta)\\
&+\frac{1}{12960}\big(2196\bar\delta^5+104205\bar\delta^4-638500\bar\delta^3+869130\bar\delta^2-293580\bar\delta-40871\big)H_{1,0}(\bar\delta)\\
&-\frac{1}{38880}\big(6588\bar\delta^5+324675\bar\delta^4-2876600\bar\delta^3+3102030\bar\delta^2-698220\bar\delta+141527\big)H_{1,1}(\bar\delta)\\
&+\frac{1}{648}\big(1620\bar\delta^4-16282\bar\delta^3-15705\bar\delta^2-360\bar\delta+63\big)H_{2,1}(\bar\delta)\\
&+\big(\frac{466\bar\delta^3}{9}-6\bar\delta-\frac{55}{12}\big)H_{2,0}(\bar\delta)+\frac{1}{6}\big(97\bar\delta^3-111\bar\delta^2-25\big)H_{3,1}(\bar\delta)+2H_{-2,-1,0}(\bar\delta)\\
&-\frac{1}{6}\big(97\bar\delta^3-111\bar\delta^2+3\big)H_{-2,0,0}(\bar\delta)+\frac{1}{9}\big(3\bar\delta^4+14\bar\delta^3+27\bar\delta^2+42\bar\delta+26\big)H_{-1,-1,0}(\bar\delta)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}H_{2,0,0}(\bar\delta)+\frac{1}{36}\big(-96\bar\delta^4+904\bar\delta^3+837\bar\delta^2-192\bar\delta-29\big)H_{-1,0,0}(\bar\delta)\\
&-\frac{38}{3}H_{0,0,0,0}(\bar\delta)+\frac{1}{216}\big(75\bar\delta^4-100\bar\delta^3+288\bar\delta^2-492\bar\delta-219\big)H_{1,0,0}(\bar\delta)\\
&+\frac{1}{72}\big(193\bar\delta^4+1852\bar\delta^3-1818\bar\delta^2-196\bar\delta-31\big)H_{1,1,0}(\bar\delta)-\big(\frac{5\bar\delta^4}{6}+3\bar\delta^2+\frac{6505}{648}\big)\zeta(3)\,,
\end{aligned}
$$

$$
\begin{aligned}
T_{25}=&\frac{1}{1399680}\Big(-585011116\bar\delta^4+1636912120\bar\delta^3-2064217230\bar\delta^2+8547341220\bar\delta\\
&-753450655\Big)+\frac{1}{233280}\pi^2\Big(7668\bar\delta^5+2109555\bar\delta^4-6523580\bar\delta^3+8385120\bar\delta^2\\
&-11063340\bar\delta+11586760\Big)-\frac{1}{216}\pi^2\big(53\bar\delta^4+312\bar\delta^3+678\bar\delta^2+728\bar\delta+309\big)H_{-1}(\bar\delta)\\
&+\Big(\frac{-944559\bar\delta^5+219647\bar\delta^4+37311164\bar\delta^3-167666580\bar\delta^2-81207540\bar\delta+21876840}{116640(\bar\delta-1)}\\
&+\frac{1}{3888}\pi^2\big(-11916\bar\delta^4+39016\bar\delta^3-9639\bar\delta^2+25254\bar\delta+87264\big)+39\zeta(3)\Big)H_0(\bar\delta)\\
&+\Big(\frac{1}{77760}\big(-2086194\bar\delta^4+3638078\bar\delta^3-1488503\bar\delta^2+163255594\bar\delta-16388975\big)\\
&+\frac{1}{144}\pi^2\big(187\bar\delta^4-596\bar\delta^3+806\bar\delta^2-1104\bar\delta+707\big)\Big)H_1(\bar\delta)+\Big(\frac{1}{38880}\big(-7668\bar\delta^5\\
&-602325\bar\delta^4+1498360\bar\delta^3-2455110\bar\delta^2+1446840\bar\delta-4565160\big)+\frac{133\pi^2}{36}\Big)H_2(\bar\delta)\\
&+\frac{1}{648}\big(4347\bar\delta^4-19244\bar\delta^3+4563\bar\delta^2-9018\bar\delta-10692\big)H_3(\bar\delta)+\frac{190}{9}H_4(\bar\delta)\\
&+\frac{4}{3}H_{-3,0}(\bar\delta)+\frac{1}{18}\big(-10\bar\delta^4-52\bar\delta^3+6\bar\delta^2-120\bar\delta+11\big)H_{-2,0}(\bar\delta)+\frac{13}{3}H_{-2,2}(\bar\delta)\\
&+\frac{1}{54}\big(-36\bar\delta^4+14\bar\delta^3-231\bar\delta^2-291\bar\delta-10\big)H_{-1,0}(\bar\delta)-\frac{4}{9}\pi^2H_{-2}(\bar\delta)\\
&+\Big(\frac{3834\bar\delta^6-52299\bar\delta^5+671495\bar\delta^4-805370\bar\delta^3+1686690\bar\delta^2-1218510\bar\delta-291600}{19440(\bar\delta-1)}\\
&-\frac{136}{27}\pi^2\Big)H_{0,0}(\bar\delta)+\big(\frac{7\bar\delta^4}{4}+\frac{91\bar\delta^3}{9}+22\bar\delta^2+25\bar\delta+\frac{409}{36}\big)H_{-1,2}(\bar\delta)\\
&+\frac{1}{6480}\big(1278\bar\delta^5-20685\bar\delta^4+85670\bar\delta^3+30195\bar\delta^2+74580\bar\delta-168278\big)H_{1,0}(\bar\delta)\\
&+\frac{1}{38880}\Big(-7668\bar\delta^5-707265\bar\delta^4+1269880\bar\delta^3-1683720\bar\delta^2+5699160\bar\delta\\
&-4570387\Big)H_{1,1}(\bar\delta)+\frac{1}{4}\big(-4\bar\delta^4+12\bar\delta^3-15\bar\delta^2+22\bar\delta-15\big)H_{1,2}(\bar\delta)\\
&-3H_{2,2}(\bar\delta)+\frac{1}{72}\big(-124\bar\delta^4-344\bar\delta^3+270\bar\delta^2-2196\bar\delta-337\big)H_{2,0}(\bar\delta)
\end{aligned}
$$

$$
\begin{aligned}
&+6H_{3,0}(\bar{\delta})+\frac{1}{648}\big(-2223\bar{\delta}^4+12448\bar{\delta}^3+729\bar{\delta}^2-8478\bar{\delta}-45018\big)H_{2,1}(\bar{\delta})\\
&-\frac{104}{9}H_{3,1}(\bar{\delta})-\frac{23}{6}H_{-2,0,0}(\bar{\delta})+\frac{1}{9}\big(5\bar{\delta}^4+26\bar{\delta}^3+57\bar{\delta}^2+86\bar{\delta}+50\big)H_{-1,-1,0}(\bar{\delta})\\
&+\frac{10}{3}H_{-2,-1,0}(\bar{\delta})+\frac{1}{72}\big(-83\bar{\delta}^4-468\bar{\delta}^3-1020\bar{\delta}^2-1244\bar{\delta}-609\big)H_{-1,0,0}(\bar{\delta})\\
&-\frac{152}{9}H_{0,0,0,0}(\bar{\delta})+\frac{1}{648}\bar{\delta}\big(5166\bar{\delta}^3-12128\bar{\delta}^2+6075\bar{\delta}-22590\big)H_{0,0,0}(\bar{\delta})\\
&-\frac{11}{6}H_{2,0,0}(\bar{\delta})+\frac{1}{108}\big(-42\bar{\delta}^4+164\bar{\delta}^3-45\bar{\delta}^2+12\bar{\delta}-345\big)H_{1,0,0}(\bar{\delta})\\
&-\frac{37}{6}H_{2,1,0}(\bar{\delta})+\frac{1}{72}\big(-139\bar{\delta}^4+268\bar{\delta}^3-54\bar{\delta}^2+304\bar{\delta}-379\big)H_{1,1,0}(\bar{\delta})\\
&-\frac{59}{3}H_{2,1,1}(\bar{\delta})+\frac{1}{36}\big(-212\bar{\delta}^4+660\bar{\delta}^3-849\bar{\delta}^2+1274\bar{\delta}-873\big)H_{1,1,1}(\bar{\delta})\\
&+\frac{1}{324}\big(990\bar{\delta}^4-4428\bar{\delta}^3+11718\bar{\delta}^2-13266\bar{\delta}+37781\big)\zeta(3)-\frac{677}{3240}\pi^4\,.
\end{aligned}
$$

8.1.2 Building Block 26

Building block 26 is a special case. It occurs in the differential G_{ij} matrix as

$$
\begin{aligned}
D_{26}(\bar{z})=\frac{40}{81(\bar{z}-4)}\Big(&54H_{1,0}(\bar{z})-27H_{1,1}(\bar{z})+57H_{2,1}(\bar{z})+228H_{0,0,0}(\bar{z})\\
&+36H_{1,0,0}(\bar{z})+14\pi^2H_0(\bar{z})-9H_2(\bar{z})-114H_3(\bar{z})-4\pi^2\Big)\,. \qquad (8.2)
\end{aligned}
$$

We see that here the HPLs with the alphabet of $\{0,1,-1\}$ are mixed with a denominator of $\bar{z}-4$. In principle we could just integrate it, resulting in GPLs of argument $\bar{z}$ with alphabet $\{0,1,-1,4\}$. We introduced the variable transformation

$$
z=\frac{4x^2}{x^2-1}
$$

in the five-body integrals before and thus we want to express all occuring GPLs in terms of this variable x for uniformity.
The price we pay for this uniform alphabet is a quadratic occurence of x in the arguments of the HPLs in Eq. (8.2). We will show here on an example function how this can be simplified to only include linear arguments of x again:

For simplicity, we take a function of weight one, $H_1\left(\frac{4x^2}{x^2-1}\right)$. We can now derivate the term with respect to x, simplify the occuring terms and then integrate it over x again while fixing the boundary conditions with the help of the initial expression. Defining $f(x)$ as

$$f(x) = H_1\left(\frac{4x^2}{x^2-1}\right),$$

we can carry out the differentiation by x. Simplification of the result leads to:

$$\frac{d}{dx}f(x) = \frac{8x}{(x-1)(x+1)\left(3x^2+1\right)}.$$

We can now integrate this again, which yields:

$$f(x) = \int dx\, \frac{8x}{(x-1)(x+1)\left(3x^2+1\right)} = c + \log(3x^2+1) - \log(1-x^2),$$

which can be linearized by using the properties of logarithms:

$$f(x) = c + \log(1+i\sqrt{3}x) + \log(1-i\sqrt{3}x) - \log(1+x) - \log(1-x).$$

Here $c = 0$ can be determined by comparing the original function at e.g. $x = 1/7$. We then use

$$\log(1-x) = G_1(x)$$

and the global scaling for GPLs

$$G_1(ax) = G_{1/a}(x)$$

to arrive at

$$f(x) = G_1(x) + G_{-1}(x) - G_{r_3}(x) - G_{-r_3}(x),$$

where we again see the letters

$$r_3 = \frac{i}{\sqrt{3}}, \tag{8.3}$$

that we encountered in the five-body integrals F_{5B30} and F_{5B31}.

For functions of higher weight, this procedure can be applied multiple times iteratively, which allows us to express all the occurring HPLs in terms of GPLs with the alphabet $\{0, 1, -1, r_3, -r_3\}$.

We can plug this now back in the original expression in Eq. (8.2) for integration. Afterwards, we can carry out the integration algorithmically with the help of `PolyLogTools` [92].
Note that the integration variable and limits have to be changed, since we traded the old variable $\bar{z}$ for x. We have to introduce new limits to the integration and multiply by the jacobian determinant $\frac{d\bar{z}}{dx}$:

$$\int_{\bar{\delta}}^{1} d\bar{z}\, D_i(\bar{z}) = \int_{x_\delta}^{r_3} dx \left(\frac{d\bar{z}}{dx}\right) D_i(x) = \int_{x_\delta}^{r_3} dx \left(-\frac{8x}{\left(x^2-1\right)^2}\right) D_i(x)\,.$$

Here, we defined the lower bound of the integration

$$x_\delta = \sqrt{\frac{\bar{\delta}}{\bar{\delta}-4}}\,. \tag{8.4}$$

The final result for this becomes very lengthy as the linearization of the argument takes more and more terms for higher weights. For this reason we only give the rule for construction here, the result can be found in electronic form as `IB[26]` in the notebook on the aforementioned `GitHub` page.

8.1.3 Building Blocks 27-30

Building blocks 27 to 30 are constructed from the finite parts of the integrals $F_{5\mathrm{B}30}$ and $F_{5\mathrm{B}31}$ given in Appendix B, which we denote in the following by $F^{(0)}_{5\mathrm{B}30}$ and $F^{(0)}_{5\mathrm{B}31}$.

In differential form they look as follows:

$$D_{27}(x) = -\frac{2x^8 + x^6 - 51x^4 + 19x^2 - 3}{162x^2\left(x^2-1\right)^2} F_{5B31}^{(0)} ,$$
$$D_{28}(x) = -\frac{5x^8 + 79x^6 + 54x^4 - 29x^2 + 3}{81x^2\left(x^2-1\right)^2} F_{5B31}^{(0)} ,$$
$$D_{29}(x) = \frac{5x^8 + 228x^6 + 318x^4 - 108x^2 + 5}{9\left(x^2-1\right)^3} F_{5B30}^{(0)} ,$$
$$D_{30}(x) = -\frac{x^8 + 15x^6 - 99x^4 + 21x^2 - 2}{9\left(x^2-1\right)^3} F_{5B30}^{(0)} .$$

We integrate these expressions using the same transformation of the integration parameters as in the last section, again using `PolyLogTools` [92].
As these expressions contain complex GPLs with high weights and are already very lengthy in differential form, we made them available in electronical form as `IB[27]`-`IB[30]` on `GitHub`.

Having all the building blocks for the final result at hand, we can now construct G_{ij}. In the next sections, we will give the different entries of the matrix and the relations between them.

8.2 $G_{ij}^{(I)}$

First, let us write down the entries of $G_{ij}^{(I)}$, which is populated by the contributions from all the diagrams with two dirac traces. Note that all entries of the 6×6 matrix, which are not given here, vanish.
In the following, logarithms containing the renormalization scale μ are abbreviated by

$$L_\mu = \log\left(\frac{\mu^2}{m_b^2}\right) , \tag{8.5}$$

and the collinear logarithms by

$$L_q = \log\left(\frac{m_b^2}{m_q^2}\right) . \tag{8.6}$$

Furthermore, n_0 denotes the number of light flavors , i.e. $n_0 = 3$ for up, down and strange in our case.

8.2.1 Current-Current Insertions

$$\begin{aligned}
G_{11}^I =\; & \frac{1}{\epsilon^2} C_F^2 \Big[\frac{1}{4}\, Q_d^2\, T_1 + \frac{1}{4}\, Q_u^2\, T_2\Big] \\
& + \frac{1}{\epsilon} C_F^2 \Big[Q_d^2 \Big(\frac{1}{2}\, T_1 \big(2\, L_\mu + L_q - 2\big) + \frac{1}{2}\, T_2 + \frac{1}{4}\, T_4 + \frac{1}{4}\, T_6\Big) \\
& + Q_u^2 \Big(\frac{1}{2}\, T_2 \big(2\, L_\mu + L_q - 1\big) + T_1 + \frac{1}{4}\, T_3 + \frac{1}{4}\, T_5\Big)\Big] \\
& + C_F^2 \Big[Q_d^2 \Big(\frac{1}{4}\, T_1 \big(8\, L_\mu^2 - 16\, L_\mu + 8\, L_\mu\, L_q + L_q^2 - 8\, L_q + 6\big) - T_{10}\, L_q + \frac{1}{2}\, T_{12}\, L_q \\
& + \frac{1}{4}\, T_2 \big(8\, L_\mu + 4\, L_q - 3\big) + \frac{1}{864}\, T_{15} \big(432\, L_\mu\, L_q - 504\, L_q - 95\big) - \frac{1}{120}\, T_{13}\, L_q \\
& + T_4 \big(L_\mu - 1\big) + \frac{1}{864}\, T_{14} \big(72\, L_q - 19\big) + T_6 \big(L_\mu - 1\big) - \frac{4}{45}\, T_7 \big(45\, L_\mu - 89\big) + \frac{1}{2}\, T_3 \\
& + \frac{1}{2}\, T_5 - \frac{19}{9}\, T_8 - \frac{397}{108}\, T_{16} + \frac{1}{120}\, T_{17} + \frac{1}{8}\, T_{18} - \frac{2}{3}\, T_{19} - \frac{14}{3}\, T_{20} + \frac{1}{3}\, T_{21} \\
& + \frac{1}{2}\, T_{22} + 2\, T_{24} - \frac{2}{3}\, T_{25} - \frac{1}{10}\, T_{26} + T_{27} + 2\, T_{30}\Big) \\
& + Q_d Q_u \Big(\frac{1}{108}\, T_{16} \big(65 - 144\, L_\mu\big) - \frac{1}{864}\, T_{14} - \frac{5}{864}\, T_{15} + \frac{1}{24}\, T_{18} - \frac{17}{30}\, T_{21} \\
& + \frac{1}{240}\, T_{23} - 2\, T_{24} + \frac{11}{20}\, T_{26} - \frac{1}{2}\, T_{27} + T_{28} + 2\, T_{29} - T_{30}\Big) \\
& + Q_u^2 \Big(\frac{1}{2}\, T_1 \big(8\, L_\mu + 4\, L_q - 3\big) + \frac{1}{4}\, T_2 \big(8\, L_\mu^2 - 8\, L_\mu + 8\, L_\mu\, L_q + L_q^2 - 4\, L_q + 3\big) \\
& + \frac{1}{4}\, T_{14} \big(2\, L_\mu - 7\big)\, L_q - 2\, T_9\, L_q - 3\, T_{10}\, L_q + \frac{1}{4}\, T_{11}\, L_q + \frac{1}{120}\, T_{13}\, L_q - \frac{13}{12}\, T_{15}\, L_q \\
& + \frac{1}{2}\, T_3 \big(2\, L_\mu - 1\big) + \frac{1}{2}\, T_5 \big(2\, L_\mu - 1\big) + \frac{1}{3}\, T_8 \big(5 - 6\, L_\mu\big) + T_4 + T_6 + \frac{1}{2}\, T_{19} \\
& - 18\, T_{20} - \frac{1}{2}\, T_{21} - 2\, T_{25}\Big)\Big] \\
& + C_F C_A \Big[Q_u^2 \Big(- \frac{1}{12}\, T_{14} \big(3\, L_\mu - 8\big)\, L_q + T_9\, L_q + T_{10} \frac{3\, L_q}{2} - \frac{1}{240}\, T_{13}\, L_q
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{13}{24}\,T_{15}\,L_q + T_8\,L_\mu + 9\,T_{20} + \frac{1}{4}\,T_{21} + T_{25}\Big) \\
&+ Q_dQ_u\Big(T_{16}\frac{2\,L_\mu}{3} + T_{24} - T_{26}\frac{11}{40} + \frac{1}{4}\,T_{27} - \frac{1}{2}\,T_{28} - T_{29} + \frac{1}{2}\,T_{30}\Big) \\
&+ Q_d^2\Big(\frac{1}{1728}\,T_{15}\big(-432\,L_\mu\,L_q + 144\,L_q + 95\big) + \frac{1}{2}\,T_{10}\,L_q + \frac{1}{240}\,T_{13}\,L_q \\
&+ \frac{1}{1728}\,T_{14}\big(19 - 72\,L_q\big) + \frac{1}{45}\,T_7\big(90\,L_\mu - 103\big) + \frac{19}{18}\,T_8 + \frac{397}{216}\,T_{16} - \frac{1}{240}\,T_{17} \\
&- \frac{1}{16}\,T_{18} + \frac{1}{3}\,T_{19} + \frac{7}{3}\,T_{20} - \frac{1}{6}\,T_{21} - T_{24} + \frac{1}{3}\,T_{25} + \frac{1}{40}\,T_{26} - \frac{1}{4}\,T_{27} - \frac{1}{2}\,T_{30}\Big)\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
G_{12}^I = C_F\Big[&Q_u^2\Big(\frac{1}{216}\,T_{14}\big(54\,L_\mu\,L_q - 126\,L_q - 1\big) + \frac{4}{9}\,T_7 - T_9\,L_q - T_{10}\,L_q \\
&+ \frac{1}{360}\,T_{13}\,L_q - \frac{5}{216}\,T_{15}\big(18\,L_q + 1\big) + \frac{1}{9}\,T_8\big(5 - 9\,L_\mu\big) - \frac{1}{27}\,T_{16} + \frac{1}{3}\,T_{19} \\
&- \frac{20}{3}\,T_{20} - \frac{1}{6}\,T_{21} - \frac{2}{3}\,T_{25}\Big) \\
&+ Q_dQ_u\Big(\frac{1}{108}\,T_{16}\big(127 - 72\,L_\mu\big) + \frac{1}{864}\,T_{14} + \frac{5}{864}\,T_{15} - \frac{1}{24}\,T_{18} - \frac{1}{4}\,T_{21} \\
&- 2\,T_{24} + \frac{3}{10}\,T_{26} - \frac{1}{2}\,T_{27} + \frac{1}{2}\,T_{28} + T_{29} - T_{30}\Big) \\
&+ Q_d^2\Big(\frac{1}{864}\,T_{15}\big(216\,L_\mu\,L_q - 144\,L_q - 35\big) - T_{10}\,L_q - \frac{1}{360}\,T_{13}\,L_q \\
&- \frac{2}{45}\,T_7\big(45\,L_\mu - 34\big) - \frac{16}{9}\,T_8 - \frac{7}{864}\,T_{14} - \frac{409}{108}\,T_{16} + \frac{1}{240}\,T_{17} + \frac{1}{8}\,T_{18} \\
&- \frac{2}{3}\,T_{19} - \frac{14}{3}\,T_{20} + \frac{T_{21}}{3} + 2\,T_{24} - \frac{2}{3}\,T_{25} - \frac{1}{20}\,T_{26} + \frac{1}{2}\,T_{27} + T_{30}\Big)\Big]\,,
\end{aligned}
$$

$$
G_{21}^I = G_{12}^I\,,
$$

$$
\begin{aligned}
G_{22}^I = &\frac{1}{\epsilon^2}C_FC_A\Big[\frac{1}{2}\,Q_d^2\,T_1 + \frac{1}{2}\,Q_u^2\,T_2\Big] \\
&+ \frac{1}{\epsilon}C_FC_A\Big[Q_d^2\Big(T_1\,(2\,L_\mu + L_q - 2) + T_2 + \frac{1}{2}\,T_4 + \frac{1}{2}\,T_6\Big) \\
&+ Q_u^2\Big(T_2\,(2\,L_\mu + L_q - 1) + 2\,T_1 + \frac{1}{2}\,T_3 + \frac{1}{2}\,T_5\Big)\Big] \\
&+ C_FC_A\Big[Q_u^2\Big(\frac{1}{2}\,T_2\,\big(8\,L_\mu^2 - 8\,L_\mu + 8\,L_\mu\,L_q + L_q^2 - 4\,L_q + 3\big) - 2\,T_{10}\,L_q \\
&+ T_1\,(8\,L_\mu + 4\,L_q - 3) + \frac{1}{2}\,T_{11}\,L_q + \frac{1}{180}\,T_{13}\,L_q + \frac{1}{54}\,T_{14}\,(1 - 63\,L_q) \\
&+ \frac{1}{54}\,T_{15}\,(5 - 27\,L_q) + T_3\,(2\,L_\mu - 1) + T_5\,(2\,L_\mu - 1) + 2\,T_4 + 2\,T_6
\end{aligned}
$$

$$- \frac{16}{9} T_7 + \frac{10}{9} T_8 + \frac{4}{27} T_{16} - \frac{1}{3} T_{19} - \frac{28}{3} T_{20} - \frac{1}{3} T_{21} - \frac{4}{3} T_{25}\Big)$$
$$+ Q_d Q_u \Big(- \frac{1}{144} T_{14} - \frac{5}{144} T_{15} - \frac{7}{2} T_{16} + \frac{T_{18}}{4} - \frac{2}{15} T_{21} + \frac{1}{120} T_{23} + 4\, T_{24}$$
$$- \frac{1}{10} T_{26} + T_{27} + 2\, T_{30}\Big)$$
$$+ Q_d^2 \Big(\frac{1}{2} T_1 \left(8\, L_\mu^2 - 16\, L_\mu + 8\, L_\mu\, L_q + L_q^2 - 8\, L_q + 6\right) + \frac{1}{2} T_2 \left(8\, L_\mu + 4\, L_q - 3\right)$$
$$+ 2\, T_{10}\, L_q + T_{12}\, L_q - \frac{1}{180} T_{13}\, L_q + \frac{1}{432} T_{14} \left(72\, L_q - 5\right) - \frac{1}{432} T_{15} \left(216\, L_q + 25\right)$$
$$+ 2\, T_4 \left(L_\mu - 1\right) + 2\, T_6 \left(L_\mu - 1\right) + T_3 + T_5 + \frac{88}{9} T_7 + \frac{26}{9} T_8 + \frac{421}{54} T_{16} - \frac{T_{18}}{4}$$
$$+ \frac{4}{3} T_{19} + \frac{28}{3} T_{20} - \frac{2}{3} T_{21} + T_{22} - 4\, T_{24} + \frac{4}{3} T_{25}\Big)\Big] \,.$$

8.2.2 Penguin-Penguin Insertions

$$G_{44}^I = \frac{1}{\epsilon^2} C_F^2 \Big[\frac{1}{2} Q_d^2\, n_0\, T_1 + Q_q^2\, \frac{1}{2} T_2 \Big]$$
$$+ \frac{1}{\epsilon} C_F^2 \Big[Q_d^2\, n_0 \Big(T_1 \left(2\, L_\mu + L_q\right) + \frac{1}{2} T_4 + \frac{1}{2} T_6 \Big)$$
$$+ Q_q^2 \Big(T_2 \left(2\, L_\mu + L_q\right) + \frac{1}{2} T_3 + \frac{1}{2} T_5 \Big)\Big]$$
$$+ C_F^2 \Big[Q_d^2\, n_0 \Big(\frac{1}{2} T_1 \left(8\, L_\mu^2 + 8\, L_\mu\, L_q + L_q^2\right) + T_{12}\, L_q + 2\, T_4\, L_\mu + 2\, T_6\, L_\mu$$
$$+ T_{22} - \frac{T_{26}}{10} + T_{27} + 2\, T_{30}\Big)$$
$$+ Q_d Q_q \Big(- \frac{8}{3} T_{16}\, L_\mu + T_{21} + T_{26} + 2\, T_{28} + 4\, T_{29}\Big)$$
$$+ Q_q^2 \Big(\frac{1}{2} T_2 \left(8\, L_\mu^2 + 8\, L_\mu\, L_q + L_q^2\right) + \frac{1}{2} T_{11}\, L_q + 2\, T_3\, L_\mu + 2\, T_5\, L_\mu + T_{19}\Big)\Big]$$
$$+ C_F C_A \Big[Q_d^2\, n_0 \Big(\frac{1}{864} T_{15} \left(35 - 108\, L_\mu\, L_q\right) + T_{10}\, L_q + \frac{1}{45} T_7 \left(45\, L_\mu + 61\right)$$
$$+ \frac{16}{9} T_8 + \frac{7}{864} T_{14} + \frac{409}{108} T_{16} - \frac{1}{480} T_{17} - \frac{1}{8} T_{18} + \frac{2}{3} T_{19} + \frac{14}{3} T_{20}$$
$$- \frac{1}{3} T_{21} - 2\, T_{24} + \frac{2}{3} T_{25} + \frac{T_{26}}{20} - \frac{T_{27}}{2} - T_{30}\Big)$$
$$+ Q_d Q_q \Big(T_{16}\, L_\mu - \frac{3}{8} T_{21} - \frac{3}{8} T_{26} - \frac{3}{4} T_{28} - \frac{3}{2} T_{29}\Big)$$

$$+ Q_q^2\Big(- \frac{1}{8}\,T_{14}\,L_\mu\,L_q + \frac{1}{2}\,T_9\,L_q + \frac{1}{2}\,T_8\,L_\mu - \frac{1}{2}\,T_{19} + T_{20}\Big)\Big]\,,$$

$$G_{43}^I = C_F\Big[Q_d^2\,n_0\Big(\frac{T_{15}\,L_q}{2} - 4\,T_7\Big) + Q_q^2\Big(\frac{1}{2}T_{14}\;L_q - 2\,T_8\Big)$$
$$+ Q_dQ_q\Big(- \frac{4}{3}T_{16}\,L_\mu + \frac{1}{2}T_{21} + \frac{1}{2}T_{26} + T_{28} + 2\,T_{29}\Big)\Big]\,,$$

$$G_{34}^I = G_{43}^I\,,$$

$$G_{33}^I = \frac{1}{\epsilon^2}C_FC_A\Big[Q_d^2\,n_0\,T_1 + Q_q^2\,T_2\Big]$$
$$+ \frac{1}{\epsilon}C_FC_A\Big[Q_d^2\,n_0\Big(2\,T_1\,(2\,L_\mu + L_q) + T_4 + T_6\Big)$$
$$+ Q_q^2\Big(2\,T_2\,(2\,L_\mu + L_q) + T_3 + T_5\Big)\Big]$$
$$+ C_FC_A\Big[Q_d^2\,n_0\Big(T_1\,\big(8\,L_\mu^2 + 8\,L_\mu\,L_q + L_q^2\big) + 2\,T_{12}\,L_q - 2\,T_{15}\,L_q$$
$$+ 4\,T_4\,L_\mu + 4\,T_6\,L_\mu + 16\,T_7 + 2\,T_{22} - \frac{1}{5}\,T_{26} + 2\,T_{27} + 4\,T_{30}\Big)$$
$$+ Q_q^2\Big(T_2\,\big(8\,L_\mu^2 + 8\,L_\mu\,L_q + L_q^2\big) + T_{11}\,L_q - 2\,T_{14}\,L_q + 4\,T_3\,L_\mu$$
$$+ 4\,T_5\,L_\mu + 8\,T_8 + 2\,T_{19}\Big)\Big]\,,$$

$$G_{46}^I = \frac{1}{\epsilon^2}C_F^2\Big[5\,Q_d^2\,n_0\,T_1 + 5\,Q_q^2\,T_2\Big]$$
$$+ \frac{1}{\epsilon}C_F^2\Big[Q_d^2\,n_0\Big(T_1\,(20\,L_\mu + 10\,L_q - 3) + 5\,T_4 + 5\,T_6\Big)$$
$$+ Q_q^2\Big(T_2\,(20\,L_\mu + 10\,L_q - 3) + 5\,T_3 + 5\,T_5\Big)\Big]$$
$$+ C_F^2\Big[Q_d^2\,n_0\Big(T_1\,\big(40\,L_\mu^2 - 12\,L_\mu + 40\,L_\mu\,L_q + 5\,L_q^2 - 6\,L_q\big) + 6\,T_{15}\,L_\mu\,L_q$$
$$+ 10\,T_{12}\,L_q - \frac{2}{15}\,T_{13}\,L_q + T_4\,(20\,L_\mu - 3) + T_6\,(20\,L_\mu - 3)$$
$$- \frac{16}{5}\,T_7\,(15\,L_\mu - 23) + \frac{1}{10}\,T_{17} + 10\,T_{22} - T_{26} + 10\,T_{27} + 20\,T_{30}\Big)$$
$$+ Q_dQ_q\Big(- \frac{80}{3}\,T_{16}\,L_\mu + \frac{16}{5}\,T_{21} + \frac{1}{20}\,T_{23} + 10\,T_{26} + 20\,T_{28} + 40\,T_{29}\Big)$$

$$
\begin{aligned}
&+ Q_q^2\Big(T_2\left(40\,L_\mu^2 - 12\,L_\mu + 40\,L_\mu\,L_q + 5\,L_q^2 - 6\,L_q\right) - 24\,T_9\,L_q - 48\,T_{10}\,L_q \\
&+ \frac{2}{9}\,T_{14}\,(27\,L_\mu\,L_q - 90\,L_q - 1) + 5\,T_{11}\,L_q + \frac{2}{15}\,T_{13}\,L_q - \frac{10}{9}\,T_{15}\,(18\,L_q + 1) \\
&+ T_3\,(20\,L_\mu - 3) + T_5\,(20\,L_\mu - 3) - \frac{8}{3}\,T_8\,(9\,L_\mu + 2) + \frac{64}{3}T_7 - \frac{16}{9}T_{16} + 2\,T_{19} \\
&- 272\,T_{20} - 8\,T_{21} - 32\,T_{25}\Big)\Big] \\
&+ C_F C_A\Big[Q_d^2\,n_0\Big(-\frac{7}{432}\,T_{15}\,(216\,L_\mu\,L_q - 25) + 10\,T_{10}\,L_q + \frac{1}{20}\,T_{13}\,L_q \\
&+ \frac{4}{45}\,T_7\,(315\,L_\mu - 158) + \frac{160}{9}T_8 + \frac{35}{432}T_{14} + \frac{2045}{54}T_{16} - \frac{7}{120}T_{17} - \frac{5}{4}T_{18} \\
&+ \frac{20}{3}T_{19} + \frac{140}{3}T_{20} - \frac{10}{3}T_{21} - 20\,T_{24} + \frac{20}{3}T_{25} + \frac{T_{26}}{2} - 5\,T_{27} - 10\,T_{30}\Big) \\
&+ Q_d Q_q\Big(\frac{1}{6}\,T_{16}\,(72\,L_\mu - 95) - \frac{1}{48}\,T_{14} - \frac{5}{48}\,T_{15} + \frac{3}{4}\,T_{18} - \frac{3}{2}\,T_{21} + 12\,T_{24} \\
&- \frac{24}{5}\,T_{26} + 3\,T_{27} - 9\,T_{28} - 18\,T_{29} + 6\,T_{30}\Big) \\
&+ Q_q^2\Big(\frac{1}{12}\,T_{14}\,(-42\,L_\mu\,L_q + 90\,L_q + 1) - 8\,T_7 + 14\,T_9\,L_q + 18\,T_{10}\,L_q - \frac{1}{20}\,T_{13}\,L_q \\
&+ \frac{5}{12}\,T_{15}\,(18\,L_q + 1) + 2\,T_8\,(7\,L_\mu + 1) + \frac{2\,T_{16}}{3} - 2\,T_{19} + 112\,T_{20} + 3\,T_{21} + 12\,T_{25}\Big)\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
G_{45}^I = {}& C_F\Big[Q_d^2\,n_0\Big(T_{15}\,(3\,L_\mu + 5)\;L_q - \frac{1}{15}\,T_{13}\,L_q - \frac{8}{5}\,T_7\,(15\,L_\mu + 2) + \frac{1}{20}\,T_{17}\Big) \\
&+ Q_d Q_q\Big(-\frac{40}{3}\,T_{16}\,(L_\mu + 1) + 5\,T_{21} + 5\,T_{26} + 10\,T_{28} + 20\,T_{29}\Big) \\
&+ Q_q^2\Big(\frac{1}{9}\,T_{14}\,(27\,L_\mu\,L_q - 45\,L_q - 1) - 12\,T_9\,L_q - 24\,T_{10}\,L_q + \frac{1}{15}\,T_{13}\,L_q + \frac{32}{3}\,T_7 \\
&- \frac{4}{3}\,T_8\,(9\,L_\mu + 17) - \frac{5}{9}\,T_{15}\,(18\,L_q + 1) - \frac{8}{9}T_{16} - 4\,T_{19} - 136\,T_{20} - 4\,T_{21} - 16\,T_{25}\Big)\Big]\,,
\end{aligned}
$$

$$
G_{36}^I = G_{45}^I\,,
$$

$$
\begin{aligned}
G_{35}^I = {}& \frac{1}{\epsilon^2}C_F C_A\Big[10\,Q_d^2\,n_0\,T_1 + 10\,Q_q^2\,T_2\Big] \\
&+ \frac{1}{\epsilon}C_F C_A\Big[Q_d^2\,n_0\Big(2\,T_1\,(20\,L_\mu + 10\,L_q - 3) + 10\,T_4 + 10\,T_6\Big) \\
&+ Q_q^2\Big(2\,T_2\,(20\,L_\mu + 10\,L_q - 3) + 10\,T_3 + 10\,T_5\Big)\Big] \\
&+ C_F C_A\Big[Q_d^2\,n_0\Big(2\,T_1\left(40\,L_\mu^2 - 12\,L_\mu + 40\,L_\mu\,L_q + 5\,L_q^2 - 6\,L_q\right) + 20\,T_{12}\,L_q + 20\,T_{22}
\end{aligned}
$$

$$
\begin{aligned}
&- 20\, T_{15}\, L_q + 2\, T_4\, (20\, L_\mu - 3) + 2\, T_6\, (20\, L_\mu - 3) + 160\, T_7 - 2\, T_{26} + 20\, T_{27} + 40\, T_{30}\Big) \\
&+ Q_d Q_q \Big(\frac{160}{3}\, T_{16} - \frac{68}{5}\, T_{21} + \frac{1}{10}\, T_{23} \Big) \\
&+ Q_q^2 \Big(2\, T_2\, \big(40\, L_\mu^2 - 12\, L_\mu + 40\, L_\mu\, L_q + 5\, L_q^2 - 6\, L_q\big) + 10\, T_{11}\, L_q - 20\, T_{14}\, L_q \\
&+ 2\, T_3\, (20\, L_\mu - 3) + 2\, T_5\, (20\, L_\mu - 3) + 80\, T_8 + 20\, T_{19} \Big) \Big] \,,
\end{aligned}
$$

$$G_{46}^I = G_{64}^I \,,$$

$$G_{54}^I = G_{45}^I \,,$$

$$G_{63}^I = G_{36}^I \,,$$

$$G_{53}^I = G_{35}^I \,,$$

$$
\begin{aligned}
G_{66}^I = &\frac{1}{\epsilon^2} C_F^2 \Big[68\, Q_d^2\, n_0\, T_1 + 68\, Q_q^2\, T_2 \Big] \\
&+ \frac{1}{\epsilon} C_F^2 \Big[Q_d^2\, n_0 \Big(2\, T_1\, (136\, L_\mu + 68\, L_q - 69) + 24\, T_2 + 68\, T_4 + 68\, T_6 \Big) \\
&+ Q_q^2 \Big(2\, T_2\, (136\, L_\mu + 68\, L_q - 57) + 48\, T_1 + 68\, T_3 + 68\, T_5 \Big) \Big] \\
&+ C_F^2 \Big[Q_d^2\, n_0 \Big(2\, T_1\, \big(272\, L_\mu^2 - 276\, L_\mu + 272\, L_\mu\, L_q + 34\, L_q^2 - 138\, L_q + 63\big) \\
&+ 48\, T_2\, (2\, L_\mu + L_q - 1) + 24\, T_{15}\, (5\, L_\mu + 1)\, L_q + 136\, T_{12}\, L_q - \frac{8}{3}\, T_{13}\, L_q \\
&+ 2\, T_4\, (136\, L_\mu - 69) + 2\, T_6\, (136\, L_\mu - 69) - 320\, T_7\, (3\, L_\mu - 4) + 24\, T_3 + 24\, T_5 \\
&+ 2\, T_{17} + 136\, T_{22} - \frac{68}{5}\, T_{26} + 136\, T_{27} + 272\, T_{30} \Big) \\
&+ Q_d Q_q \Big(- \frac{1088}{3}\, T_{16}\, L_\mu + T_{23} + 136\, T_{26} + 272\, T_{28} + 544\, T_{29} \Big) \\
&+ Q_q^2 \Big(2\, T_2\, \big(272\, L_\mu^2 - 228\, L_\mu + 272\, L_\mu\, L_q + 34\, L_q^2 - 114\, L_q + 39\big) \\
&+ 96\, T_1\, (2\, L_\mu + L_q - 1) + \frac{8}{9}\, T_{14}\, (135\, L_\mu\, L_q - 423\, L_q - 5) - 480\, T_9\, L_q
\end{aligned}
$$

$$
\begin{aligned}
&- 960\, T_{10}\, L_q + 68\, T_{11}\, L_q + \frac{8}{3}\, T_{13}\, L_q - \frac{200}{9}\, T_{15}\, (18\, L_q + 1) + 2\, T_3\, (136\, L_\mu - 57) \\
&+ 2\, T_5\, (136\, L_\mu - 57) - \frac{32}{3}\, T_8\, (45\, L_\mu + 19) + 48\, T_4 + 48\, T_6 + \frac{1280}{3}\, T_7 \\
&- \frac{320\, T_{16}}{9} - 24\, T_{19} - 5440\, T_{20} - 160\, T_{21} - 640\, T_{25}\Big)\Big] \\
&+ C_F C_A \Big[Q_d^2\, n_0 \Big(\frac{1}{108}\, T_{15}\, (595 - 6696\, L_\mu\, L_q) + 136\, T_{10}\, L_q + \, T_{13}\, L_q \\
&+ \frac{16}{45}\, T_7\, (1395\, L_\mu - 1034) + \frac{2176}{9}\, T_8 + \frac{119}{108}\, T_{14} + \frac{13906}{27}\, T_{16} \\
&- \frac{31}{30}\, T_{17} - 17\, T_{18} + \frac{272}{3}\, T_{19} + \frac{1904}{3}\, T_{20} - \frac{136}{3}\, T_{21} - 272\, T_{24} \\
&+ \frac{272}{3}\, T_{25} + \frac{34}{5}\, T_{26} - 68\, T_{27} - 136\, T_{30}\Big) \\
&+ Q_d Q_q \Big(\frac{2}{3}\, T_{16}\, (264\, L_\mu - 475) - \frac{5}{12}\, T_{14} - \frac{25}{12}\, T_{15} + 15\, T_{18} - 6\, T_{21} + 240\, T_{24} \\
&- 72\, T_{26} + 60\, T_{27} - 132\, T_{28} - 264\, T_{29} + 120\, T_{30}\Big) \\
&+ Q_q^2 \Big(\frac{1}{3}\, T_{14}\, (-186\, L_\mu\, L_q + 450\, L_q + 5) + 248\, T_9\, L_q + 360\, T_{10}\, L_q - \, T_{13}\, L_q \\
&+ \frac{25}{3}\, T_{15}\, (18\, L_q + 1) + 8\, T_8\, (31\, L_\mu + 5) - 160\, T_7 + \frac{40}{3}\, T_{16} - 8\, T_{19} \\
&+ 2176\, T_{20} + 60\, T_{21} + 240\, T_{25}\Big)\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
G^I_{65} =\ & C_F \Big[Q_d^2\, n_0 \Big(4\, T_{15}\, (15\, L_\mu + 23)\ L_q - \frac{4}{3}\, T_{13}\, L_q - 480\, T_7\, L_\mu + \, T_{17}\Big) \\
&+ Q_d Q_q \Big(- \frac{32}{3}\, T_{16}\, (17\, L_\mu + 25) + 68\, T_{21} + 68\, T_{26} + 136\, T_{28} + 272\, T_{29}\Big) \\
&+ Q_q^2 \Big(\frac{4}{9}\, T_{14}\, (135\, L_\mu\, L_q - 243\, L_q - 5) - 240\, T_9\, L_q - 480\, T_{10}\, L_q + \frac{4}{3}\, T_{13}\, L_q \\
&- \frac{100}{9}\, T_{15}\, (18\, L_q + 1) - \frac{16}{3}\, T_8\, (45\, L_\mu + 79) + \frac{640}{3}\, T_7 - \frac{160}{9}\, T_{16} \\
&- 80\, T_{19} - 2720\, T_{20} - 80\, T_{21} - 320\, T_{25}\Big)\Big]\,,
\end{aligned}
$$

$$
G^I_{56} = G^I_{65}\,,
$$

$$
\begin{aligned}
G^I_{55} =\ & \frac{1}{\epsilon^2} C_F C_A \Big[136\, Q_d^2\, n_0\, T_1 + 136\, Q_q^2\, T_2\Big] \\
&+ \frac{1}{\epsilon} C_F C_A \Big[Q_d^2\, n_0 \Big(4\, T_1\, (136\, L_\mu + 68\, L_q - 69) + 48\, T_2 + 136\, T_4 + 136\, T_6\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ Q_q^2\Big(4\,T_2\,(136\,L_\mu + 68\,L_q - 57) + 96\,T_1 + 136\,T_3 + 136\,T_5\Big)\Big]\\
&+ C_F C_A\Big[Q_d^2\,n_0\Big(4\,T_1\,\big(272\,L_\mu^2 - 276\,L_\mu + 272\,L_\mu\,L_q + 34\,L_q^2 - 138\,L_q + 63\big)\\
&+ 96\,T_2\,(2\,L_\mu + \,L_q - 1) + 272\,T_{12}\,L_q - 320\,T_{15}\,L_q + 48\,T_3 + 4\,T_4\,(136\,L_\mu - 69)\\
&+ 4\,T_6\,(136\,L_\mu - 69) + 48\,T_5 + 2560\,T_7 + 272\,T_{22} - \frac{136}{5}\,T_{26} + 272\,T_{27} + 544\,T_{30}\Big)\\
&+ Q_d Q_q\Big(\frac{3200}{3}\,T_{16} - 272\,T_{21} + 2\,T_{23}\Big)\\
&+ Q_q^2\Big(4\,T_2\,\big(272\,L_\mu^2 - 228\,L_\mu + 272\,L_\mu\,L_q + 34\,L_q^2 - 114\,L_q + 39\big)\\
&+ 192\,T_1\,(2\,L_\mu + \,L_q - 1) + 136\,T_{11}\,L_q - 320\,T_{14}\,L_q + 4\,T_3\,(136\,L_\mu - 57)\\
&+ 4\,T_5\,(136\,L_\mu - 57) + 96\,T_4 + 96\,T_6 + 1280\,T_8 + 272\,T_{19}\Big)\Big]\,.
\end{aligned}
$$

8.3 $G_{ij}^{(II)}$

Writing down the second type of insertions, we noted that we can relate almost all the insertions back to the first block of current-penguin insertions. This stems from the fact that we only encounter a single dirac trace in these, which means that the degrees of freedom for the algebra are much more narrow. Defining three additional relational functions and giving the first block explicitly, we are able to express the rest of the elements without introducing new expressions.

8.3.1 Auxiliary Functions

$$
\begin{aligned}
f_1 &= \frac{1}{8}Q_d^2\left(\big(5\,T_1 - 2\,T_2\big)\left(\frac{1}{\epsilon} + 4\,L_\mu + 2\,L_q - 1\right) + T_1 - 2\,T_3 + 5\,T_4 - 2\,T_5 + 5\,T_6\right)\\
&\quad + \frac{1}{8}Q_u^2\left(\big(3\,T_2 - 4\,T_1\big)\left(\frac{1}{\epsilon} + 4\,L_\mu + 2\,L_q - 1\right) + T_2 + 3\,T_3 - 4\,T_4 + 3\,T_5 - 4\,T_6\right),\\
f_2 &= \frac{1}{16}\left(Q_d^2\big(8\,T_7 - T_{21}L_q\big) + Q_d Q_u\,\frac{8}{3}\,T_{22} + Q_u^2\big(4\,T_8 - T_{20}L_q\big)\right),\\
f_3 &= Q_d^2\Big(T_1 + T_2\Big)\,.
\end{aligned}
$$

8.3.2 Current-Penguin Insertions

$$
\begin{aligned}
G_{13}^{(II)} &= \frac{1}{\epsilon^2} C_F^2 \Big[\frac{1}{2}\, Q_d^2\, T_1 + \frac{1}{2}\, Q_u^2\, T_2\Big] \\
&+ \frac{1}{\epsilon} C_F^2 \Big[Q_d^2 \Big(\frac{1}{2}\, T_1\, (4\, L_\mu + 2\, L_q - 1) + \frac{1}{2}\, T_4 + \frac{1}{2}\, T_6\Big) \\
&+ Q_u^2 \Big(\frac{1}{2}\, T_2\, (4\, L_\mu + 2\, L_q - 1) + \frac{1}{2}\, T_3 + \frac{1}{2}\, T_5\Big)\Big] \\
&+ C_F^2 \Big[Q_d^2 \Big(\frac{1}{2}\, T_1\, \big(8\, L_\mu^2 - 4\, L_\mu + 8\, L_\mu\, L_q + L_q^2 - 2\, L_q\big) + \frac{1}{2}\, T_{15}\, (L_\mu - 1)\, L_q \\
&+ T_{12}\, L_q - \frac{1}{90}\, T_{13}\, L_q + \frac{1}{2}\, T_4\, (4\, L_\mu - 1) + \frac{1}{2}\, T_6\, (4\, L_\mu - 1) \\
&- \frac{4}{15}\, T_7\, (15\, L_\mu - 38) + \frac{1}{120}\, T_{17} + T_{22} - \frac{1}{10}\, T_{26} + T_{27} + 2\, T_{30}\Big) \\
&+ Q_d Q_u \Big(- \frac{4}{9}\, T_{16}\, (3\, L_\mu - 5) - \frac{19}{30}\, T_{21} + \frac{1}{120}\, T_{23} + \frac{1}{2}\, T_{26} + T_{28} + 2\, T_{29}\Big) \\
&+ Q_u^2 \Big(\frac{1}{2}\, T_2\, \big(8\, L_\mu^2 - 4\, L_\mu + 8\, L_\mu\, L_q + L_q^2 - 2\, L_q\big) - 2\, T_9\, L_q - 4\, T_{10}\, L_q \\
&+ \frac{1}{54}\, T_{14}\, (27\, L_\mu\, L_q - 117\, L_q - 1) + \frac{1}{2}\, T_{11}\, L_q + \frac{1}{90}\, T_{13}\, L_q \\
&- \frac{5}{54}\, T_{15}\, (18\, L_q + 1) + \frac{1}{2}\, T_3\, (4\, L_\mu - 1) + \frac{1}{2}\, T_5\, (4\, L_\mu - 1) - \frac{2}{9}\, T_8\, (9\, L_\mu - 7) \\
&+ \frac{16}{9}\, T_7 - \frac{4}{27}\, T_{16} + \frac{1}{3}\, T_{19} - \frac{68}{3}\, T_{20} - \frac{2}{3}\, T_{21} - \frac{8}{3}\, T_{25}\Big)\Big] \\
&+ C_F C_A \Big[Q_u^2 \Big(\frac{1}{216}\, T_{14}\, (-54\, L_\mu\, L_q + 135\, L_q + 2) + T_9\, L_q + 2\, T_{10}\, L_q \\
&- \frac{1}{180}\, T_{13}\, L_q + \frac{5}{108}\, T_{15}\, (18\, L_q + 1) + \frac{1}{18}\, T_8\, (18\, L_\mu + 19) - \frac{8}{9}\, T_7 + \frac{2}{27}\, T_{16} \\
&+ \frac{1}{3}\, T_{19} + \frac{34}{3}\, T_{20} + \frac{1}{3}\, T_{21} + \frac{4}{3}\, T_{25}\Big) \\
&+ Q_d Q_u \Big(\frac{1}{9}\, T_{16}\, (6\, L_\mu + 5) - \frac{1}{4}\, T_{21} - \frac{1}{4}\, T_{26} - \frac{1}{2}\, T_{28} - T_{29}\Big) \\
&+ Q_d^2 \Big(- \frac{1}{24}\, T_{15}\, (6\, L_\mu + 5)\, L_q + \frac{1}{180}\, T_{13}\, L_q + \frac{1}{5}\, T_7\, (10\, L_\mu - 7) - \frac{1}{240}\, T_{17}\Big)\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
G_{14}^{(II)} &= \frac{1}{\epsilon^2} \Big(2\, C_F^3 - C_A C_F^2\Big) \Big[\frac{1}{4}\, Q_d^2\, T_1 + \frac{1}{4}\, Q_u^2\, T_2\Big] \\
&+ \frac{1}{\epsilon} \Big(2\, C_F^3 - C_A C_F^2\Big) \Big[Q_d^2 \Big(\frac{1}{4}\, T_1 \Big(4\, L_\mu + 2\, L_q - 1\Big) + \frac{1}{4}\, T_4 + \frac{1}{4}\, T_6\Big) \\
&+ Q_u^2 \Big(\frac{1}{4}\, T_2 \Big(4\, L_\mu + 2\, L_q - 1\Big) + \frac{1}{4}\, T_3 + \frac{1}{4}\, T_5\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big(2\,C_F^3 - C_A C_F^2\Big)\Big[Q_d^2\Big(\frac{1}{4}\,T_1\Big(8\,L_\mu^2 - 4\,L_\mu + 8\,L_\mu\,L_q + \,L_q^2 - 2\,L_q\Big)\\
&+ \frac{1}{4}\,T_{15}\Big(\,L_\mu - 1\Big)\,L_q + \frac{1}{2}\,T_{12}\,L_q - \frac{1}{180}\,T_{13}\,L_q + \frac{1}{4}\,T_4\Big(4\,L_\mu - 1\Big)\\
&+ \frac{1}{4}\,T_6\Big(4\,L_\mu - 1\Big) - \frac{2}{15}\,T_7\Big(15\,L_\mu - 38\Big) + \frac{1}{240}\,T_{17} + \frac{1}{2}\,T_{22} - \frac{1}{20}\,T_{26}\\
&+ \frac{1}{2}\,T_{27} + \,T_{30}\Big)\\
&+ Q_d Q_u\Big(- \frac{2}{9}\,T_{16}\Big(3\,L_\mu - 5\Big) - \frac{19}{60}\,T_{21} + \frac{1}{240}\,T_{23} + \frac{1}{4}\,T_{26} + \frac{1}{2}\,T_{28} + \,T_{29}\Big)\\
&+ Q_u^2\Big(\frac{1}{4}\,T_2\Big(8\,L_\mu^2 - 4\,L_\mu + 8\,L_\mu\,L_q + \,L_q^2 - 2\,L_q\Big) - \,T_9\,L_q - 2\,T_{10}\,L_q\\
&+ \frac{1}{108}\,T_{14}\Big(27\,L_\mu\,L_q - 117\,L_q - 1\Big) + \frac{1}{4}\,T_{11}\,L_q + \frac{1}{180}\,T_{13}\,L_q\\
&- \frac{5}{108}\,T_{15}\Big(18\,L_q + 1\Big) + \frac{1}{4}\,T_3\Big(4\,L_\mu - 1\Big) + \frac{1}{4}\,T_5\Big(4\,L_\mu - 1\Big)\\
&+ \frac{1}{9}\,T_8\Big(7 - 9\,L_\mu\Big) + \frac{8}{9}\,T_7 - \frac{2}{27}\,T_{16} + \frac{1}{6}\,T_{19} - \frac{34}{3}\,T_{20} - \frac{1}{3}\,T_{21} - \frac{4}{3}\,T_{25}\Big)\Big]\\
&+ C_F\Big[Q_u^2\Big(\frac{1}{144}\,T_{14}\Big(36\,L_\mu\,L_q - 90\,L_q - 1\Big) - \,T_9\,L_q - \frac{3}{2}\,T_{10}\,L_q\\
&+ \frac{1}{240}\,T_{13}\,L_q - \frac{5}{144}\,T_{15}\Big(18\,L_q + 1\Big) + \frac{1}{6}\,T_8\Big(- 6\,L_\mu - 1\Big) + \frac{2}{3}\,T_7 - \frac{1}{18}\,T_{16}\\
&- 9\,T_{20} - \frac{1}{4}\,T_{21} - \,T_{25}\Big)\\
&+ Q_d Q_u\Big(\frac{1}{72}\,T_{16}\Big(95 - 48\,L_\mu\Big) + \frac{1}{576}\,T_{14} + \frac{5}{576}\,T_{15} - \frac{1}{16}\,T_{18} - \,T_{24}\\
&+ \frac{11}{40}\,T_{26} - \frac{1}{4}\,T_{27} + \frac{1}{2}\,T_{28} + \,T_{29} - \frac{1}{2}\,T_{30}\Big)\\
&+ Q_d^2\Big(\frac{1}{1728}\,T_{15}\,(432\,L_\mu\,L_q - 35) - \frac{1}{2}\,T_{10}\,L_q - \frac{1}{240}\,T_{13}\,L_q\\
&+ \frac{1}{45}\,T_7\Big(73 - 90\,L_\mu\Big) - \frac{8}{9}\,T_8 - \frac{7}{1728}\,T_{14} - \frac{409}{216}\,T_{16} + \frac{1}{240}\,T_{17} + \frac{1}{16}\,T_{18}\\
&- \frac{1}{3}\,T_{19} - \frac{7}{3}\,T_{20} + \frac{1}{6}\,T_{21} + \,T_{24} - \frac{1}{3}\,T_{25} - \frac{1}{40}\,T_{26} + \frac{1}{4}\,T_{27} + \frac{1}{2}\,T_{30}\Big)\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
G_{23}^{(II)} &= \frac{1}{\epsilon^2}C_F\Big[\frac{1}{2}\,Q_d^2\,T_1 + \frac{1}{2}\,Q_u^2\,T_2\Big]\\
&+ \frac{1}{\epsilon}C_F\Big[Q_d^2\Big(\frac{1}{2}\,T_1\Big(4\,L_\mu + 2\,L_q - 1\Big) + \frac{1}{2}\,T_4 + \frac{1}{2}\,T_6\Big)\\
&+ Q_u^2\Big(\frac{1}{2}\,T_2\Big(4\,L_\mu + 2\,L_q - 1\Big) + \frac{1}{2}\,T_3 + \frac{1}{2}\,T_5\Big)\Big]\\
&+ C_F\Big[Q_d^2\Big(\frac{1}{2}\,T_1\Big(8\,L_\mu^2 - 4\,L_\mu + 8\,L_\mu\,L_q + \,L_q^2 - 2\,L_q\Big) + \frac{1}{2}\,T_{15}\Big(\,L_\mu - 1\Big)\,L_q
\end{aligned}
$$

$$
\begin{aligned}
&+ T_{12}\,L_q - \frac{1}{90}\,T_{13}\,L_q + \frac{1}{2}\,T_4\Big(4\,L_\mu - 1\Big) + \frac{1}{2}\,T_6\Big(4\,L_\mu - 1\Big) - \frac{4}{15}\,T_7\Big(15\,L_\mu - 38\Big)\\
&+ \frac{1}{120}\,T_{17} + T_{22} - \frac{1}{10}\,T_{26} + T_{27} + 2\,T_{30}\Big)\\
&+ Q_dQ_u\Big(- \frac{4}{9}\,T_{16}\Big(3\,L_\mu - 5\Big) - \frac{19}{30}\,T_{21} + \frac{1}{120}\,T_{23} + \frac{1}{2}\,T_{26} + T_{28} + 2\,T_{29}\Big)\\
&+ Q_u^2\Big(\frac{1}{2}\,T_2\Big(8\,L_\mu^2 - 4\,L_\mu + 8\,L_\mu\,L_q + L_q^2 - 2\,L_q\Big) - 2\,T_9\,L_q - 4\,T_{10}\,L_q\\
&+ \frac{1}{54}\,T_{14}\Big(27\,L_\mu\,L_q - 117\,L_q - 1\Big) + \frac{1}{2}\,T_{11}\,L_q + \frac{1}{90}\,T_{13}\,L_q\\
&- \frac{5}{54}\,T_{15}\Big(18\,L_q + 1\Big) + \frac{1}{2}\,T_3\Big(4\,L_\mu - 1\Big) + \frac{1}{2}\,T_5\Big(4\,L_\mu - 1\Big) - \frac{2}{9}\,T_8\Big(9\,L_\mu - 7\Big)\\
&+ \frac{16}{9}\,T_7 - \frac{4}{27}\,T_{16} + \frac{1}{3}\,T_{19} - \frac{68}{3}\,T_{20} - \frac{2}{3}\,T_{21} - \frac{8}{3}\,T_{25}\Big)\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
G_{24}^{(II)} &= \frac{1}{\epsilon^2}C_F^2\Big[\frac{1}{2}\,Q_d^2\,T_1 + \frac{1}{2}\,Q_u^2\,T_2\Big]\\
&+ \frac{1}{\epsilon}C_F^2\Big[Q_d^2\Big(\frac{1}{2}\,T_1\Big(4\,L_\mu + 2\,L_q - 1\Big) + \frac{1}{2}\,T_4 + \frac{1}{2}\,T_6\Big)\\
&+ Q_u^2\Big(\frac{1}{2}\,T_2\Big(4\,L_\mu + 2\,L_q - 1\Big) + \frac{1}{2}\,T_3 + \frac{1}{2}\,T_5\Big)\Big]\\
&+ C_F^2\Big[Q_d^2\Big(\frac{1}{2}\,T_1\Big(8\,L_\mu^2 - 4\,L_\mu + 8\,L_\mu\,L_q + L_q^2 - 2\,L_q\Big) + \frac{1}{2}\,T_{15}\Big(L_\mu - 1\Big)\,L_q\\
&+ T_{12}\,L_q - \frac{1}{90}\,T_{13}\,L_q + \frac{1}{2}\,T_4\Big(4\,L_\mu - 1\Big) + \frac{1}{2}\,T_6\Big(4\,L_\mu - 1\Big)\\
&- \frac{4}{15}\,T_7\Big(15\,L_\mu - 38\Big) + \frac{T_{17}}{120} + T_{22} - \frac{T_{26}}{10} + T_{27} + 2\,T_{30}\Big)\\
&+ Q_dQ_u\Big(- \frac{4}{9}\,T_{16}\Big(3\,L_\mu - 5\Big) - \frac{19}{30}\,T_{21} + \frac{1}{120}\,T_{23} + \frac{1}{2}\,T_{26} + T_{28} + 2\,T_{29}\Big)\\
&+ Q_u^2\Big(\frac{1}{2}\,T_2\Big(8\,L_\mu^2 - 4\,L_\mu + 8\,L_\mu\,L_q + L_q^2 - 2\,L_q\Big) - 2\,T_9\,L_q - 4\,T_{10}\,L_q\\
&+ \frac{1}{54}\,T_{14}\Big(27\,L_\mu\,L_q - 117\,L_q - 1\Big) + \frac{1}{2}\,T_{11}\,L_q + \frac{1}{90}\,T_{13}\,L_q\\
&- \frac{5}{54}\,T_{15}\Big(18\,L_q + 1\Big) + \frac{1}{2}\,T_3\Big(4\,L_\mu - 1\Big) + \frac{1}{2}\,T_5\Big(4\,L_\mu - 1\Big)\\
&- \frac{2}{9}\,T_8\Big(9\,L_\mu - 7\Big) + \frac{16}{9}\,T_7 - \frac{4}{27}\,T_{16} + \frac{1}{3}\,T_{19} - \frac{68}{3}\,T_{20} - \frac{2}{3}\,T_{21} - \frac{8}{3}\,T_{25}\Big)\Big]\\
&+ C_FC_A\Big[Q_u^2\Big(\frac{1}{216}\,T_{14}\Big(- 54\,L_\mu\,L_q + 135\,L_q + 1\Big) + T_9\,L_q + T_{10}\,L_q\\
&- \frac{1}{360}\,T_{13}\,L_q + \frac{5}{216}\,T_{15}\Big(18\,L_q + 1\Big) + \frac{1}{18}\,T_8\Big(18\,L_\mu - 13\Big) - \frac{4}{9}\,T_7 + \frac{1}{27}\,T_{16}\\
&- \frac{1}{3}\,T_{19} + \frac{20}{3}\,T_{20} + \frac{1}{6}\,T_{21} + \frac{2}{3}\,T_{25}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ Q_d Q_u \Big(\frac{1}{36}\, T_{16} \Big(24\, L_\mu - 115 \Big) - \frac{1}{288}\, T_{14} - \frac{5}{288}\, T_{15} + \frac{1}{8}\, T_{18} + \frac{1}{4}\, T_{21} \\
&+ 2\, T_{24} - \frac{3}{10}\, T_{26} + \frac{1}{2}\, T_{27} - \frac{1}{2}\, T_{28} - T_{29} + T_{30} \Big) \\
&+ Q_d^2 \Big(\frac{1}{864}\, T_{15} \Big(-216\, L_\mu\, L_q + 180\, L_q + 35 \Big) + T_{10}\, L_q + \frac{1}{360}\, T_{13}\, L_q \\
&+ \frac{1}{45}\, T_7 \Big(90\, L_\mu - 83 \Big) + \frac{16}{9}\, T_8 + \frac{7}{864}\, T_{14} + \frac{409}{108}\, T_{16} - \frac{1}{240}\, T_{17} - \frac{1}{8}\, T_{18} \\
&+ \frac{2}{3}\, T_{19} + \frac{14}{3}\, T_{20} - \frac{1}{3}\, T_{21} - 2\, T_{24} + \frac{2}{3}\, T_{25} + \frac{1}{20}\, T_{26} - \frac{1}{2}\, T_{27} - T_{30} \Big) \Big] \,,
\end{aligned}
$$

$$
\begin{aligned}
G_{31}^{(II)} &= G_{13}^{(II)} \,, \\
G_{32}^{(II)} &= G_{23}^{(II)} \,, \\
G_{41}^{(II)} &= G_{14}^{(II)} \,, \\
G_{42}^{(II)} &= G_{24}^{(II)} \,,
\end{aligned}
$$

$$
\begin{aligned}
G_{15}^{(II)} &= 16\, G_{13}^{(II)} - 16\, C_F^2\, f_1 + 16\, C_F C_A\, f_2 \,, \\
G_{16}^{(II)} &= 16\, G_{14}^{(II)} + 8 \left(C_F^2 C_A - C_F^3 \right) f_1 \,, \\
G_{25}^{(II)} &= 16\, G_{23}^{(II)} - 16\, C_F\, f_1 \,, \\
G_{26}^{(II)} &= 16\, G_{24}^{(II)} - 16\, C_F^2\, f_1 - 16\, C_F C_A\, f_2 \,,
\end{aligned}
$$

$$
\begin{aligned}
G_{51}^{(II)} &= G_{15}^{(II)} \,, \\
G_{61}^{(II)} &= G_{16}^{(II)} \,, \\
G_{52}^{(II)} &= G_{25}^{(II)} \,,
\end{aligned}
$$

$$G_{62}^{(II)} = G_{26}^{(II)} \,.$$

8.3.3 Penguin-Penguin Insertions

In the penguin-penguin operator part of insertion (II), we only encounter Q_d^2. The notation $G_{ij}^{(II)}(Q_u \to Q_d)$ denotes that we take the expression and simply change all occurring Q_u to Q_d.

$$G_{33}^{(II)} = G_{23}^{(II)}(Q_u \to Q_d) \,,$$

$$G_{34}^{(II)} = G_{13}^{(II)}(Q_u \to Q_d) \,,$$

$$G_{43}^{(II)} = G_{24}^{(II)}(Q_u \to Q_d) \,,$$

$$G_{44}^{(II)} = G_{14}^{(II)}(Q_u \to Q_d) \,,$$

$$G_{35}^{(II)} = G_{25}^{(II)}(Q_u \to Q_d) \,,$$

$$G_{36}^{(II)} = G_{26}^{(II)}(Q_u \to Q_d) \,,$$

$$G_{45}^{(II)} = G_{15}^{(II)}(Q_u \to Q_d) \,,$$

$$G_{46}^{(II)} = G_{16}^{(II)}(Q_u \to Q_d) \,,$$

$$G_{53}^{(II)} = G_{35}^{(II)} \,,$$

$$G_{63}^{(II)} = G_{36}^{(II)} \,,$$

$$G_{54}^{(II)} = G_{45}^{(II)} \,,$$

$$G_{64}^{(II)} = G_{46}^{(II)} \,,$$

$$G_{55}^{(II)} = \Big[256\, G_{23}^{(II)} - C_F\Big(896\, f_1 - 40\, f_3\Big)\Big](Q_u \to Q_d)\,,$$

$$G_{56}^{(II)} = \Big[256\, G_{13}^{(II)} - C_F^2\Big(896\, f_1 - 40\, f_3\Big)\Big](Q_u \to Q_d)\,,$$

$$G_{65}^{(II)} = G_{56}^{(II)}\,,$$

$$G_{66}^{(II)} = \Big[256\, G_{14}^{(II)} + \Big(C_F^2 C_A - 2\, C_F^3\Big)\Big(448\, f_1 - 20\, f_3\Big)\Big](Q_u \to Q_d)\,.$$

8.4 Next Steps

In this last section, we want to discuss the steps that immediately follow from our result. The goal of this effort is to finalize the calculation of the perturbative contributions to the inclusive decay rate of the process $\bar{B} \to X_s\gamma$ at $\mathcal{O}(\alpha_s)$.

Completion of the Regularization

As we have mentioned before in Sect. 7.4, there is one piece left to calculate to regularize the last divergent pieces that are still present in G_{ij}. This is an ongoing effort and we hope to be able to supplement these terms in the near future. This will render our result finite and we can go to the next step.

Combination of the Results

In this step, we construct the complete matrix $\widetilde{G}_{ij}(\mu_b, \delta)$ by adding up all known results and with this the decay rate $\Gamma(b \to X_s^{\text{parton}}\gamma)_{E_\gamma > E_0}$ from Eq. (2.18).
To get the full contributions at $\mathcal{O}(\alpha_s)$, we also need the Wilson coefficients to this order. They can be found in the literature [61] and after writing them as

$$\mathcal{C}_j(\mu) = \mathcal{C}_j(\mu)^{(0)} + \frac{\alpha_s(\mu)}{4\pi}\mathcal{C}_j(\mu)^{(1)} + \mathcal{O}(\alpha_s^2)\,,$$

we can construct the expression for the decay rate up to $\mathcal{O}(\alpha_s)$ in the following way:

$$\Gamma(b \to X_s^{\text{parton}}\gamma)_{E_\gamma > E_0} = \Gamma_0 \sum_{i,j} \Bigg[\mathcal{C}_i^*(\mu)^{(0)}\, \mathcal{C}_j(\mu)^{(0)}\, \widetilde{G}_{ij}(\mu, \delta)^{(0)}$$

$$+\frac{\alpha_s(\mu)}{4\pi}\Bigg(\Big(\mathcal{C}_i^*(\mu)^{(1)}\,\mathcal{C}_j(\mu)^{(0)}+\mathcal{C}_i^*(\mu)^{(0)}\,\mathcal{C}_j(\mu)^{(1)}\Big)\widetilde{G}_{ij}(\mu,\delta)^{(0)}$$
$$+\,\mathcal{C}_i^*(\mu)^{(0)}\,\mathcal{C}_j(\mu)^{(0)}\,\widetilde{G}_{ij}(\mu,\delta)^{(1)}\Bigg)\Bigg]+\mathcal{O}(\alpha_s^2)\,,$$

where the last line, together with all the contributions that were calculated previously, contains the results from this thesis.

We claim that this then completes the perturbative contributions to this order and it can be checked that the result is renormalization group invariant. After combining all these terms, the dependece of the physical quantity $\Gamma(b\to X_s^{\text{parton}}\gamma)_{E_\gamma>E_0}$ on the renormalization scale μ should vanish up to $\mathcal{O}(\alpha_s)$. All in all, this should lower the dependece of the complete result on the choice of μ:

$$\frac{d}{d\mu}\Gamma(b\to X_s^{\text{parton}}\gamma)_{E_\gamma>E_0}=\mathcal{O}(\alpha_s^2)\,. \tag{8.7}$$

Chapter 9

Conclusion

The inclusive rare decay $\bar{B} \to X_s\gamma$ is one of the most precise observables in flavor physics and is commonly used as a standard candle to compare experimental results to theory. The efforts of computing the branching fraction of this process have been going on for multiple decades and there is a large amount of contributions up to NNLO in QCD [10–38]. Despite these efforts, the perturbative part of the decay width, $\Gamma(b \to X_s^{\text{parton}}\gamma)_{E_\gamma > E_0}$, is not yet know completely at $\mathcal{O}(\alpha_s)$. Missing up until now were some of the contributions from four-particle decays, where due to the computational complexity of occurring integrals, only estimates were available previously.

One of the tools that have proven to be the most useful for theoretical predictions in the flavor sector is the effective weak theory. Thus, we used this framework to compute a large part of the contributions that were missing for the completion of the perturbative part at $\mathcal{O}(\alpha_s)$, where we calculated the four-body one-loop contributions of $b \to s\gamma q\bar{q}$ using integration-by-parts (IBP) reduction and differential equation methods.

We supplemented these with their real-emission counterparts, $b \to s\gamma q\bar{q}g$, for the cancellation of infrared divergences caused by the gluon. We also used IBP reductions on these five-body integrals, but the computation of the resulting master integrals via differential equations took a different form. Especially the calculation of the boundary conditions via integration over (partially) divergent kernels, which was used successfully for the bremsstrahlung contributions in this work, is a very

interesting (and also comparatively young) technique in this field.
For the complete calculation, we adopted the (slightly modified) KKS scheme for the treatment of γ_5. After implementing this, it was possible to carry out the calculation of the affected current-current block of the matrix G_{ij} parallel to the other combinations of operators.
Following the calculation of the bare expressions, we went over the processes of renormalization and regularization. As part of the regularization chapter, we also discussed the last missing piece in this calculation, which is related to the contribution of the next-to-leading order splitting function convoluted with tree-level three-body diagrams. We also supplemented cross-checks to substantiate the claim that this is the last missing piece.
In the final chapter, we gave the matrix G_{ij} that is needed in the construction of the decay width $\Gamma(b \to X_s^{\text{parton}}\gamma)_{E_\gamma > E_0}$. Besides writing them down here, we also made the matrix and all our master integrals public online in electronic form.
Lastly, we discussed the next steps in the ongoing calculational effort that are needed for a final, numerical update on the decay width. These include the finalization of the computation of the NLO splitting function and the combination of our novel results with the already available theoretical predictions, where the renormalization-scale-dependence will be shifted from $\mathcal{O}(\alpha_s)$ to $\mathcal{O}(\alpha_s^2)$.
Summing up the aforementioned parts, this thesis represents a large step in the direction of completing the $\mathcal{O}(\alpha_s)$ contributions to $\Gamma(b \to X_s^{\text{parton}}\gamma)_{E_\gamma > E_0}$, making a final result possible in the near future.

Appendix A

Master Integrals in the Four-Body Case

$$
\begin{aligned}
G_{4\mathrm{B}1} = \frac{n_0(\epsilon) i}{\epsilon}\Bigg[& -1 + \epsilon\Big(3H_0(z) - 2H_1(z) - i\pi - 12\Big) \\
& + \epsilon^2\Big(3i\pi H_0(z) + 36H_0(z) - 2i\pi H_1(z) - 24H_1(z) + 6H_2(z) - 9H_{0,0}(z) \\
& + 6H_{1,0}(z) - 4H_{1,1}(z) + \frac{5\pi^2}{3} - 12i\pi - 88\Big) \\
& + \epsilon^3\Big(-5\pi^2 H_0(z) + 36i\pi H_0(z) + 264H_0(z) + \frac{10}{3}\pi^2 H_1(z) - 24i\pi H_1(z) \\
& - 176H_1(z) + 6i\pi H_2(z) + 72H_2(z) - 18H_3(z) - 9i\pi H_{0,0}(z) - 108H_{0,0}(z) \\
& + 6i\pi H_{1,0}(z) + 72H_{1,0}(z) - 4i\pi H_{1,1}(z) - 48H_{1,1}(z) + 12H_{1,2}(z) - 18H_{2,0}(z) \\
& + 12H_{2,1}(z) + 27H_{0,0,0}(z) - 18H_{1,0,0}(z) + 12H_{1,1,0}(z) - 8H_{1,1,1}(z) + 24\zeta(3) \\
& + \frac{4}{3}i\pi^3 + 20\pi^2 - 88i\pi - 512\Big) + \mathcal{O}(\epsilon^4)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4\mathrm{B}2} = n_1(\epsilon)\Bigg[& -\frac{4}{3} + \epsilon\Big(\frac{8}{3}H_0(z) - \frac{8}{3}H_1(z) - \frac{4}{3}i\pi\Big) + \epsilon^2\Big(\frac{8}{3}i\pi H_0(z) - \frac{8}{3}i\pi H_1(z) \\
& + \frac{20}{3}H_2(z) - \frac{16}{3}H_{0,0}(z) + \frac{16}{3}H_{1,0}(z) - \frac{16}{3}H_{1,1}(z) + 2\pi^2\Big) \\
& + \epsilon^3\Big(-4\pi^2 H_0(z) + 4\pi^2 H_1(z) + \frac{20}{3}i\pi H_2(z) - \frac{40}{3}H_3(z) - \frac{16}{3}i\pi H_{0,0}(z)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{16}{3}i\pi H_{1,0}(z)-\frac{16}{3}i\pi H_{1,1}(z)+\frac{40}{3}H_{1,2}(z)-\frac{40}{3}H_{2,0}(z)+\frac{56}{3}H_{2,1}(z)\\
&+\frac{32}{3}H_{0,0,0}(z)-\frac{32}{3}H_{1,0,0}(z)+\frac{32}{3}H_{1,1,0}(z)-\frac{32}{3}H_{1,1,1}(z)+\frac{80}{3}\zeta(3)+\frac{14}{9}i\pi^3\Big)\\
&+\epsilon^4\Big(-\frac{8}{3}S_{2,2}(z)-\frac{160}{3}\zeta(3)H_0(z)-\frac{28}{9}i\pi^3H_0(z)+\frac{160}{3}\zeta(3)H_1(z)+\frac{28}{9}i\pi^3H_1(z)\\
&-10\pi^2H_2(z)-\frac{40}{3}i\pi H_3(z)+\frac{80}{3}H_4(z)+8\pi^2H_{0,0}(z)-8\pi^2H_{1,0}(z)+8\pi^2H_{1,1}(z)\\
&+\frac{40}{3}i\pi H_{1,2}(z)-\frac{80}{3}H_{1,3}(z)-\frac{40}{3}i\pi H_{2,0}(z)+\frac{56}{3}i\pi H_{2,1}(z)-\frac{100}{3}H_{2,2}(z)\\
&+\frac{80}{3}H_{3,0}(z)-\frac{104}{3}H_{3,1}(z)+\frac{32}{3}i\pi H_{0,0,0}(z)-\frac{32}{3}i\pi H_{1,0,0}(z)+\frac{32}{3}i\pi H_{1,1,0}(z)\\
&-\frac{32}{3}i\pi H_{1,1,1}(z)+\frac{80}{3}H_{1,1,2}(z)-\frac{80}{3}H_{1,2,0}(z)+\frac{112}{3}H_{1,2,1}(z)+\frac{80}{3}H_{2,0,0}(z)\\
&-\frac{112}{3}H_{2,1,0}(z)+\frac{176}{3}H_{2,1,1}(z)-\frac{64}{3}H_{0,0,0,0}(z)+\frac{64}{3}H_{1,0,0,0}(z)-\frac{64}{3}H_{1,1,0,0}(z)\\
&+\frac{64}{3}H_{1,1,1,0}(z)-\frac{64}{3}H_{1,1,1,1}(z)+\frac{80}{3}i\pi\zeta(3)-\frac{49}{90}\pi^4\Big)+\mathcal{O}(\epsilon^5)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4\text{B}3}=n_1(\epsilon)\Bigg[&\frac{8}{3}\epsilon H_1(z)+\epsilon^2\Big(\frac{8}{3}i\pi H_1(z)-\frac{16}{3}H_{1,0}(z)+16H_{1,1}(z)\Big)\\
&+\epsilon^3\Big(-4\pi^2H_1(z)-\frac{16}{3}i\pi H_{1,0}(z)+16i\pi H_{1,1}(z)-\frac{40}{3}H_{1,2}(z)+\frac{32}{3}H_{1,0,0}(z)\\
&-32H_{1,1,0}(z)+\frac{224}{3}H_{1,1,1}(z)\Big)\\
&+\epsilon^4\Big(-\frac{32}{3}S_{2,2}(z)-\frac{160}{3}\zeta(3)H_1(z)-\frac{28}{9}i\pi^3H_1(z)+8\pi^2H_{1,0}(z)-24\pi^2H_{1,1}(z)\\
&-\frac{40}{3}i\pi H_{1,2}(z)+\frac{80}{3}H_{1,3}(z)+\frac{32}{3}H_{3,1}(z)+\frac{32}{3}i\pi H_{1,0,0}(z)-32i\pi H_{1,1,0}(z)\\
&+\frac{224}{3}i\pi H_{1,1,1}(z)-80H_{1,1,2}(z)+\frac{80}{3}H_{1,2,0}(z)-\frac{112}{3}H_{1,2,1}(z)-\frac{64}{3}H_{1,0,0,0}(z)\\
&+64H_{1,1,0,0}(z)-\frac{448}{3}H_{1,1,1,0}(z)+320H_{1,1,1,1}(z)\Big)+\mathcal{O}(\epsilon^5)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4\text{B}4}=n_1(\epsilon)\Bigg[&-\frac{4}{3}+\epsilon\Big(\frac{8}{3}H_0(z)-4H_1(z)-\frac{4}{3}i\pi\Big)+\epsilon^2\Big(\frac{8}{3}i\pi H_0(z)-4i\pi H_1(z)\\
&+8H_2(z)-\frac{16}{3}H_{0,0}(z)+8H_{1,0}(z)-12H_{1,1}(z)+\frac{20}{9}\pi^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\epsilon^3\Big(-\frac{40}{9}\pi^2H_0(z)+\frac{20}{3}\pi^2H_1(z)+8i\pi H_2(z)-16H_3(z)-\frac{16}{3}i\pi H_{0,0}(z)\\
&+8i\pi H_{1,0}(z)-12i\pi H_{1,1}(z)+24H_{1,2}(z)-16H_{2,0}(z)+24H_{2,1}(z)\\
&+\frac{32}{3}H_{0,0,0}(z)-16H_{1,0,0}(z)+24H_{1,1,0}(z)-36H_{1,1,1}(z)+32\zeta(3)+\frac{16}{9}i\pi^3\Big)\\
&+\epsilon^4\Big(-64\zeta(3)H_0(z)-\frac{32}{9}i\pi^3H_0(z)+96\zeta(3)H_1(z)+\frac{16}{3}i\pi^3H_1(z)\\
&-\frac{40}{3}\pi^2H_2(z)-16i\pi H_3(z)+32H_4(z)+\frac{80}{9}\pi^2H_{0,0}(z)-\frac{40}{3}\pi^2H_{1,0}(z)\\
&+20\pi^2H_{1,1}(z)+24i\pi H_{1,2}(z)-48H_{1,3}(z)-16i\pi H_{2,0}(z)+24i\pi H_{2,1}(z)\\
&-48H_{2,2}(z)+32H_{3,0}(z)-48H_{3,1}(z)+\frac{32}{3}i\pi H_{0,0,0}(z)-16i\pi H_{1,0,0}(z)\\
&+24i\pi H_{1,1,0}(z)-36i\pi H_{1,1,1}(z)+72H_{1,1,2}(z)-48H_{1,2,0}(z)+72H_{1,2,1}(z)\\
&+32H_{2,0,0}(z)-48H_{2,1,0}(z)+72H_{2,1,1}(z)-\frac{64}{3}H_{0,0,0,0}(z)+32H_{1,0,0,0}(z)\\
&-48H_{1,1,0,0}(z)+72H_{1,1,1,0}(z)-108H_{1,1,1,1}(z)+32i\pi\zeta(3)-\frac{32}{45}\pi^4\Big)+\mathcal{O}(\epsilon^5)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4\mathrm{B}5}=n_1(\epsilon)\Bigg[&\epsilon^2\Big(8H_2(z)+4H_{1,1}(z)-\frac{2}{3}\pi^2\Big)\\
&+\epsilon^3\Big(\frac{4}{3}\pi^2H_0(z)+8i\pi H_2(z)-28H_3(z)+8H_{1,2}(z)+4i\pi H_{1,1}(z)-16H_{2,0}(z)\\
&+32H_{2,1}(z)-8H_{1,1,0}(z)+36H_{1,1,1}(z)-16\zeta(3)-\frac{2}{3}i\pi^3\Big)\\
&+\epsilon^4\Big(-8S_{2,2}(z)+32\zeta(3)H_0(z)+\frac{4}{3}i\pi^3H_0(z)-14\pi^2H_2(z)-28i\pi H_3(z)\\
&+92H_4(z)-\frac{8}{3}\pi^2H_{0,0}(z)-8\pi^2H_{1,1}(z)+8i\pi H_{1,2}(z)-24H_{1,3}(z)-16i\pi H_{2,0}(z)\\
&+32i\pi H_{2,1}(z)-76H_{2,2}(z)+56H_{3,0}(z)-104H_{3,1}(z)-8i\pi H_{1,1,0}(z)\\
&+36i\pi H_{1,1,1}(z)-16H_{1,2,0}(z)+32H_{1,2,1}(z)+32H_{2,0,0}(z)-64H_{2,1,0}(z)\\
&+80H_{2,1,1}(z)+16H_{1,1,0,0}(z)-72H_{1,1,1,0}(z)+220H_{1,1,1,1}(z)-16i\pi\zeta(3)+\frac{\pi^4}{2}\Big)\\
&+\mathcal{O}(\epsilon^5)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4\text{B}6} = n_1(\epsilon)\Bigg[&\epsilon^2 \frac{8}{3} H_2(z) \\
&+ \epsilon^3 \Big(\frac{4}{3}\pi^2 H_1(z) + \frac{8}{3} i\pi H_2(z) - 8H_3(z) + \frac{8}{3} H_{1,2}(z) - \frac{16}{3} H_{2,0}(z) + \frac{32}{3} H_{2,1}(z) \Big) \\
&+ \epsilon^4 \Big(\frac{32}{3} S_{2,2}(z) + 32\zeta(3) H_1(z) + \frac{4}{3} i\pi^3 H_1(z) - \frac{16}{3}\pi^2 H_2(z) - 8i\pi H_3(z) \\
&+ 24H_4(z) - \frac{8}{3}\pi^2 H_{1,0}(z) + \frac{16}{3}\pi^2 H_{1,1}(z) + \frac{8}{3} i\pi H_{1,2}(z) - \frac{16}{3} i\pi H_{2,0}(z) \\
&+ \frac{32}{3} i\pi H_{2,1}(z) - \frac{80}{3} H_{2,2}(z) + 16H_{3,0}(z) - \frac{128}{3} H_{3,1}(z) + \frac{32}{3} H_{1,1,2}(z) \\
&- \frac{16}{3} H_{1,2,0}(z) + \frac{32}{3} H_{1,2,1}(z) + \frac{32}{3} H_{2,0,0}(z) - \frac{64}{3} H_{2,1,0}(z) + \frac{80}{3} H_{2,1,1}(z) \Big) \\
&+ \mathcal{O}(\epsilon^5) \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4\text{B}7} = n_1(\epsilon)\Bigg[&4 + \epsilon\Big(- 12H_0(z) + 8H_1(z) + 4i\pi \Big) + \epsilon^2 \Big(- 12i\pi H_0(z) + 8i\pi H_1(z) \\
&- 32H_2(z) + 36H_{0,0}(z) - 24H_{1,0}(z) + 8H_{1,1}(z) - \frac{64\pi^2}{9} \Big) + \\
&+ \epsilon^3 \Big(\frac{64}{3}\pi^2 H_0(z) - \frac{128}{9}\pi^2 H_1(z) - 32i\pi H_2(z) + \frac{272H_3(z)}{3} + 36i\pi H_{0,0}(z) \\
&- 24i\pi H_{1,0}(z) + 8i\pi H_{1,1}(z) - \frac{208}{3} H_{1,2}(z) + 88H_{2,0}(z) - 80H_{2,1}(z) \\
&- 108H_{0,0,0}(z) + 72H_{1,0,0}(z) - 32H_{1,1,0}(z) - 40H_{1,1,1}(z) - \frac{328\zeta(3)}{3} - \frac{52i\pi^3}{9} \Big) \\
&+ \mathcal{O}(\epsilon^4) \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4\text{B}8} = n_2(\epsilon)\Bigg[&- \frac{1}{120} + \epsilon\Big(\frac{1}{60} H_0(z) - \frac{1}{60} H_1(z) \Big) \\
&+ \epsilon^2 \Big(\frac{1}{30} H_2(z) - \frac{1}{30} H_{0,0}(z) + \frac{1}{30} H_{1,0}(z) - \frac{1}{30} H_{1,1}(z) + \frac{\pi^2}{180} \Big) \\
&+ \epsilon^3 \Big(- \frac{1}{90}\pi^2 H_0(z) + \frac{1}{90}\pi^2 H_1(z) - \frac{1}{15} H_3(z) + \frac{1}{15} H_{1,2}(z) - \frac{1}{15} H_{2,0}(z) \\
&+ \frac{1}{15} H_{2,1}(z) + \frac{1}{15} H_{0,0,0}(z) - \frac{1}{15} H_{1,0,0}(z) + \frac{1}{15} H_{1,1,0}(z) - \frac{1}{15} H_{1,1,1}(z)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{10}\zeta(3)\Big) + \epsilon^4\Big(- \frac{1}{5}\zeta(3)H_0(z) + \frac{1}{5}\zeta(3)H_1(z) - \frac{1}{45}\pi^2 H_2(z) \\
&+ \frac{2}{15}H_4(z) + \frac{1}{45}\pi^2 H_{0,0}(z) - \frac{1}{45}\pi^2 H_{1,0}(z) + \frac{1}{45}\pi^2 H_{1,1}(z) - \frac{2}{15}H_{1,3}(z) \\
&- \frac{2}{15}H_{2,2}(z) + \frac{2}{15}H_{3,0}(z) - \frac{2}{15}H_{3,1}(z) + \frac{2}{15}H_{1,1,2}(z) - \frac{2}{15}H_{1,2,0}(z) \\
&+ \frac{2}{15}H_{1,2,1}(z) + \frac{2}{15}H_{2,0,0}(z) - \frac{2}{15}H_{2,1,0}(z) + \frac{2}{15}H_{2,1,1}(z) - \frac{2}{15}H_{0,0,0,0}(z) \\
&+ \frac{2}{15}H_{1,0,0,0}(z) - \frac{2}{15}H_{1,1,0,0}(z) + \frac{2}{15}H_{1,1,1,0}(z) - \frac{2}{15}H_{1,1,1,1}(z) + \frac{\pi^4}{1800}\Big) \\
&+ \epsilon^5\Big(- \frac{1}{900}\pi^4 H_0(z) + \frac{1}{900}\pi^4 H_1(z) - \frac{2}{5}\zeta(3)H_2(z) + \frac{2}{45}\pi^2 H_3(z) \\
&- \frac{4}{15}H_5(z) + \frac{2}{5}\zeta(3)H_{0,0}(z) - \frac{2}{5}\zeta(3)H_{1,0}(z) + \frac{2}{5}\zeta(3)H_{1,1}(z) - \frac{2}{45}\pi^2 H_{1,2}(z) \\
&+ \frac{4}{15}H_{1,4}(z) + \frac{2}{45}\pi^2 H_{2,0}(z) - \frac{2}{45}\pi^2 H_{2,1}(z) + \frac{4}{15}H_{2,3}(z) + \frac{4}{15}H_{3,2}(z) \\
&- \frac{4}{15}H_{4,0}(z) + \frac{4}{15}H_{4,1}(z) - \frac{2}{45}\pi^2 H_{0,0,0}(z) + \frac{2}{45}\pi^2 H_{1,0,0}(z) - \frac{2}{45}\pi^2 H_{1,1,0}(z) \\
&+ \frac{2}{45}\pi^2 H_{1,1,1}(z) - \frac{4}{15}H_{1,1,3}(z) - \frac{4}{15}H_{1,2,2}(z) + \frac{4}{15}H_{1,3,0}(z) - \frac{4}{15}H_{1,3,1}(z) \\
&- \frac{4}{15}H_{2,1,2}(z) + \frac{4}{15}H_{2,2,0}(z) - \frac{4}{15}H_{2,2,1}(z) - \frac{4}{15}H_{3,0,0}(z) + \frac{4}{15}H_{3,1,0}(z) \\
&- \frac{4}{15}H_{3,1,1}(z) + \frac{4}{15}H_{1,1,1,2}(z) - \frac{4}{15}H_{1,1,2,0}(z) + \frac{4}{15}H_{1,1,2,1}(z) + \frac{4}{15}H_{1,2,0,0}(z) \\
&- \frac{4}{15}H_{1,2,1,0}(z) + \frac{4}{15}H_{1,2,1,1}(z) - \frac{4}{15}H_{2,0,0,0}(z) + \frac{4}{15}H_{2,1,0,0}(z) - \frac{4}{15}H_{2,1,1,0}(z) \\
&+ \frac{4}{15}H_{2,1,1,1}(z) + \frac{4}{15}H_{0,0,0,0,0}(z) - \frac{4}{15}H_{1,0,0,0,0}(z) + \frac{4}{15}H_{1,1,0,0,0}(z) \\
&- \frac{4}{15}H_{1,1,1,0,0}(z) + \frac{4}{15}H_{1,1,1,1,0}(z) - \frac{4}{15}H_{1,1,1,1,1}(z) + \frac{1}{2}\zeta(5) - \frac{1}{15}\pi^2\zeta(3)\Big) \\
&+ \mathcal{O}(\epsilon^6)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4B9} = n_2(\epsilon)\Bigg[&\frac{11}{360} + \epsilon\Big(\frac{11}{180}H_1(z) - \frac{11}{360}H_0(z)\Big) \\
&+ \epsilon^2\Big(- \frac{11}{180}H_2(z) - \frac{11}{120}H_{0,0}(z) - \frac{11}{36}H_{1,0}(z) + \frac{11}{90}H_{1,1}(z) - \frac{77}{2160}\pi^2\Big) \\
&+ \epsilon^3\Big(\frac{77}{1080}\pi^2 H_0(z) - \frac{11}{720}\pi^2 H_1(z) - \frac{11}{60}H_3(z) - \frac{11}{18}H_{1,2}(z)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{11}{72}H_{2,0}(z)-\frac{11}{90}H_{2,1}(z)+\frac{341}{360}H_{0,0,0}(z)+\frac{55}{36}H_{1,0,0}(z)-\frac{55}{72}H_{1,1,0}(z)\\
&+\frac{11}{45}H_{1,1,1}(z)-\frac{187}{360}\zeta(3)\Big)\\
&+\epsilon^4\Big(-\frac{803}{360}S_{2,2}(z)+\frac{11}{180}\zeta(3)H_0(z)-\frac{187}{72}\zeta(3)H_1(z)+\frac{77}{2160}\pi^2H_2(z)\\
&+\frac{341}{180}H_4(z)-\frac{77}{540}\pi^2H_{0,0}(z)+\frac{11}{360}\pi^2H_{1,0}(z)-\frac{11}{360}\pi^2H_{1,1}(z)+\frac{55}{18}H_{1,3}(z)\\
&+\frac{11}{36}H_{2,2}(z)+\frac{11}{30}H_{3,0}(z)+\frac{671}{360}H_{3,1}(z)-\frac{55}{36}H_{1,1,2}(z)+\frac{517}{360}H_{1,2,0}(z)\\
&-\frac{11}{9}H_{1,2,1}(z)-\frac{11}{24}H_{2,0,0}(z)+\frac{121}{360}H_{2,1,0}(z)-\frac{11}{45}H_{2,1,1}(z)-\frac{2057}{360}H_{0,0,0,0}(z)\\
&-\frac{275}{36}H_{1,0,0,0}(z)+\frac{33}{8}H_{1,1,0,0}(z)-\frac{11}{6}H_{1,1,1,0}(z)+\frac{22}{45}H_{1,1,1,1}(z)-\frac{77}{14400}\pi^4\Big)\\
&+\mathcal{O}(\epsilon^5)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4\mathrm{B}10}=n_2(\epsilon)\Bigg[&\epsilon\frac{11}{144}H_0(z)+\epsilon^2\Big(\frac{11}{72}H_2(z)-\frac{77}{144}H_{0,0}(z)-\frac{11}{180}H_{1,0}(z)-\frac{11}{432}\pi^2\Big)\\
&+\epsilon^3\Big(\frac{11}{216}\pi^2H_0(z)+\frac{11}{1080}\pi^2H_1(z)-\frac{77}{72}H_3(z)-\frac{11}{90}H_{1,2}(z)-\frac{11}{36}H_{2,0}(z)\\
&+\frac{11}{36}H_{2,1}(z)+\frac{143}{48}H_{0,0,0}(z)+\frac{77}{180}H_{1,0,0}(z)-\frac{11}{60}H_{1,1,0}(z)+\frac{11}{72}\zeta(3)\Big)\\
&+\epsilon^4\Big(-\frac{143}{90}S_{2,2}(z)-\frac{11}{4}\zeta(3)H_0(z)-\frac{11}{20}\zeta(3)H_1(z)-\frac{55}{432}\pi^2H_2(z)\\
&+\frac{143}{24}H_4(z)-\frac{11}{108}\pi^2H_{0,0}(z)-\frac{11}{540}\pi^2H_{1,0}(z)+\frac{11}{540}\pi^2H_{1,1}(z)+\frac{77}{90}H_{1,3}(z)\\
&-\frac{11}{18}H_{2,2}(z)+\frac{55}{24}H_{3,0}(z)-\frac{11}{20}H_{3,1}(z)-\frac{11}{30}H_{1,1,2}(z)+\frac{11}{36}H_{1,2,0}(z)\\
&-\frac{11}{45}H_{1,2,1}(z)+\frac{11}{18}H_{2,0,0}(z)-\frac{11}{24}H_{2,1,0}(z)+\frac{11}{18}H_{2,1,1}(z)-\frac{2233}{144}H_{0,0,0,0}(z)\\
&-\frac{143}{60}H_{1,0,0,0}(z)+\frac{77}{60}H_{1,1,0,0}(z)-\frac{22}{45}H_{1,1,1,0}(z)+\frac{77}{8640}\pi^4\Big)\\
&+\epsilon^5\Big(-\frac{11}{30}S_{2,3}(z)-\frac{99}{40}S_{3,2}(z)+\frac{187}{30}S_{2,2}(z)H_0(z)-\frac{121}{1440}\pi^4H_0(z)\\
&-\frac{44}{45}S_{2,2}(z)H_1(z)-\frac{451}{21600}\pi^4H_1(z)-\frac{209}{72}\zeta(3)H_2(z)+\frac{187}{216}\pi^2H_3(z)\\
&-\frac{2233}{72}H_5(z)+\frac{319}{18}\zeta(3)H_{0,0}(z)+\frac{55}{18}\zeta(3)H_{1,0}(z)-\frac{187}{90}\zeta(3)H_{1,1}(z)
\end{aligned}
$$

$$+ \frac{121}{1080}\pi^2 H_{1,2}(z) - \frac{143}{30}H_{1,4}(z) + \frac{55}{216}\pi^2 H_{2,0}(z) - \frac{55}{216}\pi^2 H_{2,1}(z) + \frac{11}{9}H_{2,3}(z)$$
$$- \frac{33}{20}H_{3,2}(z) - \frac{935}{72}H_{4,0}(z) - \frac{517}{120}H_{4,1}(z) + \frac{11}{54}\pi^2 H_{0,0,0}(z) + \frac{11}{270}\pi^2 H_{1,0,0}(z)$$
$$- \frac{11}{270}\pi^2 H_{1,1,0}(z) + \frac{11}{270}\pi^2 H_{1,1,1}(z) + \frac{77}{30}H_{1,1,3}(z) + \frac{11}{18}H_{1,2,2}(z) - \frac{88}{45}H_{1,3,0}(z)$$
$$+ \frac{121}{45}H_{1,3,1}(z) - \frac{11}{12}H_{2,1,2}(z) + \frac{11}{8}H_{2,2,0}(z) - \frac{11}{45}H_{2,2,1}(z) - \frac{385}{72}H_{3,0,0}(z)$$
$$- \frac{77}{30}H_{3,1,0}(z) - \frac{44}{45}H_{3,1,1}(z) - \frac{44}{45}H_{1,1,1,2}(z) + \frac{44}{45}H_{1,1,2,0}(z) - \frac{11}{15}H_{1,1,2,1}(z)$$
$$- \frac{11}{12}H_{1,2,0,0}(z) + \frac{77}{180}H_{1,2,1,0}(z) - \frac{22}{45}H_{1,2,1,1}(z) - \frac{11}{9}H_{2,0,0,0}(z) + \frac{11}{72}H_{2,1,0,0}(z)$$
$$- \frac{11}{18}H_{2,1,1,0}(z) + \frac{11}{9}H_{2,1,1,1}(z) + \frac{11341}{144}H_{0,0,0,0,0}(z) + \frac{2233}{180}H_{1,0,0,0,0}(z)$$
$$- \frac{143}{20}H_{1,1,0,0,0}(z) + \frac{154}{45}H_{1,1,1,0,0}(z) - \frac{11}{9}H_{1,1,1,1,0}(z) + \frac{605}{144}\zeta(5) + \frac{11}{18}\pi^2\zeta(3)\Big)$$
$$+ \mathcal{O}(\epsilon^6)\Bigg],$$

$$G_{4B11} = n_2(\epsilon)\Bigg[- \epsilon\frac{11}{180}H_0(z) + \epsilon^2\Big(- \frac{11}{90}H_2(z) + \frac{77}{180}H_{0,0}(z) + \frac{11}{90}H_{1,0}(z) + \frac{11}{540}\pi^2\Big)$$
$$+ \epsilon^3\Big(- \frac{11}{270}\pi^2 H_0(z) - \frac{11}{540}\pi^2 H_1(z) + \frac{77}{90}H_3(z) + \frac{11}{45}H_{1,2}(z) + \frac{11}{45}H_{2,0}(z)$$
$$- \frac{11}{45}H_{2,1}(z) - \frac{143}{60}H_{0,0,0}(z) - \frac{77}{90}H_{1,0,0}(z) + \frac{11}{30}H_{1,1,0}(z) - \frac{11}{90}\zeta(3)\Big)$$
$$+ \epsilon^4\Big(\frac{187}{90}S_{2,2}(z) + \frac{11}{5}\zeta(3)H_0(z) + \frac{11}{10}\zeta(3)H_1(z) + \frac{11}{108}\pi^2 H_2(z) - \frac{143}{30}H_4(z)$$
$$+ \frac{11}{135}\pi^2 H_{0,0}(z) + \frac{11}{270}\pi^2 H_{1,0}(z) - \frac{11}{270}\pi^2 H_{1,1}(z) - \frac{77}{45}H_{1,3}(z) + \frac{22}{45}H_{2,2}(z)$$
$$- \frac{11}{6}H_{3,0}(z) - \frac{11}{30}H_{3,1}(z) + \frac{11}{15}H_{1,1,2}(z) - \frac{11}{18}H_{1,2,0}(z) + \frac{22}{45}H_{1,2,1}(z)$$
$$- \frac{22}{45}H_{2,0,0}(z) + \frac{11}{30}H_{2,1,0}(z) - \frac{22}{45}H_{2,1,1}(z) + \frac{2233}{180}H_{0,0,0,0}(z)$$
$$+ \frac{143}{30}H_{1,0,0,0}(z) - \frac{77}{30}H_{1,1,0,0}(z) + \frac{44}{45}H_{1,1,1,0}(z) - \frac{77}{10800}\pi^4\Big)$$
$$+ \epsilon^5\Big(- \frac{44}{15}S_{2,3}(z) + \frac{33}{5}S_{3,2}(z) - \frac{341}{30}S_{2,2}(z)H_0(z) + \frac{121}{1800}\pi^4 H_0(z)$$
$$+ \frac{88}{45}S_{2,2}(z)H_1(z) + \frac{451}{10800}\pi^4 H_1(z) + \frac{209}{90}\zeta(3)H_2(z) - \frac{187}{270}\pi^2 H_3(z)$$

$$
\begin{aligned}
&+ \frac{2233}{90}H_5(z) - \frac{638}{45}\zeta(3)H_{0,0}(z) - \frac{55}{9}\zeta(3)H_{1,0}(z) + \frac{187}{45}\zeta(3)H_{1,1}(z) \\
&- \frac{121}{540}\pi^2 H_{1,2}(z) + \frac{143}{15}H_{1,4}(z) - \frac{11}{54}\pi^2 H_{2,0}(z) + \frac{11}{54}\pi^2 H_{2,1}(z) - \frac{44}{45}H_{2,3}(z) \\
&+ \frac{77}{10}H_{3,2}(z) + \frac{187}{18}H_{4,0}(z) + \frac{539}{30}H_{4,1}(z) - \frac{22}{135}\pi^2 H_{0,0,0}(z) - \frac{11}{135}\pi^2 H_{1,0,0}(z) \\
&+ \frac{11}{135}\pi^2 H_{1,1,0}(z) - \frac{11}{135}\pi^2 H_{1,1,1}(z) - \frac{77}{15}H_{1,1,3}(z) - \frac{11}{9}H_{1,2,2}(z) + \frac{176}{45}H_{1,3,0}(z) \\
&- \frac{242}{45}H_{1,3,1}(z) + \frac{11}{15}H_{2,1,2}(z) - \frac{11}{10}H_{2,2,0}(z) - \frac{44}{45}H_{2,2,1}(z) + \frac{77}{18}H_{3,0,0}(z) \\
&+ \frac{253}{30}H_{3,1,0}(z) + \frac{22}{45}H_{3,1,1}(z) + \frac{88}{45}H_{1,1,1,2}(z) - \frac{88}{45}H_{1,1,2,0}(z) + \frac{22}{15}H_{1,1,2,1}(z) \\
&+ \frac{11}{6}H_{1,2,0,0}(z) - \frac{77}{90}H_{1,2,1,0}(z) + \frac{44}{45}H_{1,2,1,1}(z) + \frac{44}{45}H_{2,0,0,0}(z) - \frac{11}{90}H_{2,1,0,0}(z) \\
&+ \frac{22}{45}H_{2,1,1,0}(z) - \frac{44}{45}H_{2,1,1,1}(z) - \frac{11341}{180}H_{0,0,0,0,0}(z) - \frac{2233}{90}H_{1,0,0,0,0}(z) \\
&+ \frac{143}{10}H_{1,1,0,0,0}(z) - \frac{308}{45}H_{1,1,1,0,0}(z) + \frac{22}{9}H_{1,1,1,1,0}(z) - \frac{121}{36}\zeta(5) - \frac{22}{45}\pi^2\zeta(3)\Big) \\
&+ \mathcal{O}(\epsilon^6)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4B12} = n_2(\epsilon)\Bigg[&\frac{1}{20} + \epsilon\Big(\frac{1}{5}H_1(z) - \frac{1}{10}H_0(z)\Big) \\
&+ \epsilon^2\Big(\frac{1}{10}H_2(z) + \frac{1}{5}H_{0,0}(z) - \frac{2}{5}H_{1,0}(z) + H_{1,1}(z) - \frac{\pi^2}{30}\Big) \\
&+ \epsilon^3\Big(\frac{1}{15}\pi^2 H_0(z) - \frac{2}{15}\pi^2 H_1(z) + \frac{1}{2}H_3(z) + \frac{1}{5}H_{1,2}(z) - \frac{1}{5}H_{2,0}(z) + 2H_{2,1}(z) \\
&- \frac{2}{5}H_{0,0,0}(z) + \frac{4}{5}H_{1,0,0}(z) - 2H_{1,1,0}(z) + \frac{28}{5}H_{1,1,1}(z) - \frac{3}{5}\zeta(3)\Big) \\
&+ \epsilon^4\Big(- \frac{26}{5}S_{2,2}(z) + \frac{6}{5}\zeta(3)H_0(z) - \frac{12}{5}\zeta(3)H_1(z) - \frac{1}{15}\pi^2 H_2(z) - \frac{1}{2}H_4(z) \\
&- \frac{2}{15}\pi^2 H_{0,0}(z) + \frac{4}{15}\pi^2 H_{1,0}(z) - \frac{2}{3}\pi^2 H_{1,1}(z) + \frac{11}{5}H_{1,3}(z) + H_{2,2}(z) \\
&- H_{3,0}(z) + \frac{34}{5}H_{3,1}(z) + \frac{2}{5}H_{1,1,2}(z) - \frac{2}{5}H_{1,2,0}(z) + \frac{32}{5}H_{1,2,1}(z) \\
&+ \frac{2}{5}H_{2,0,0}(z) - 4H_{2,1,0}(z) + \frac{74}{5}H_{2,1,1}(z) + \frac{4}{5}H_{0,0,0,0}(z) - \frac{8}{5}H_{1,0,0,0}(z) \\
&+ 4H_{1,1,0,0}(z) - \frac{56}{5}H_{1,1,1,0}(z) + \frac{164}{5}H_{1,1,1,1}(z) - \frac{\pi^4}{300}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\epsilon^5\Big(\frac{136}{5}S_{2,3}(z)-\frac{2}{5}S_{3,2}(z)+\frac{52}{5}S_{2,2}(z)H_0(z)+\frac{1}{150}\pi^4H_0(z)-\frac{124}{5}S_{2,2}(z)H_1(z)\\
&-\frac{1}{75}\pi^4H_1(z)-\frac{6}{5}\zeta(3)H_2(z)-\frac{1}{3}\pi^2H_3(z)+\frac{17}{10}H_5(z)-\frac{12}{5}\zeta(3)H_{0,0}(z)\\
&+\frac{24}{5}\zeta(3)H_{1,0}(z)-12\zeta(3)H_{1,1}(z)-\frac{2}{15}\pi^2H_{1,2}(z)-\frac{11}{5}H_{1,4}(z)+\frac{2}{15}\pi^2H_{2,0}(z)\\
&-\frac{4}{3}\pi^2H_{2,1}(z)+\frac{13}{5}H_{2,3}(z)-\frac{57}{5}H_{3,2}(z)+H_{4,0}(z)-30H_{4,1}(z)+\frac{4}{15}\pi^2H_{0,0,0}(z)\\
&-\frac{8}{15}\pi^2H_{1,0,0}(z)+\frac{4}{3}\pi^2H_{1,1,0}(z)-\frac{56}{15}\pi^2H_{1,1,1}(z)+\frac{58}{5}H_{1,1,3}(z)+\frac{22}{5}H_{1,2,2}(z)\\
&-\frac{22}{5}H_{1,3,0}(z)+\frac{164}{5}H_{1,3,1}(z)+2H_{2,1,2}(z)-2H_{2,2,0}(z)+\frac{176}{5}H_{2,2,1}(z)+2H_{3,0,0}(z)\\
&-\frac{68}{5}H_{3,1,0}(z)+54H_{3,1,1}(z)+\frac{4}{5}H_{1,1,1,2}(z)-\frac{4}{5}H_{1,1,2,0}(z)+\frac{136}{5}H_{1,1,2,1}(z)\\
&+\frac{4}{5}H_{1,2,0,0}(z)-\frac{64}{5}H_{1,2,1,0}(z)+\frac{244}{5}H_{1,2,1,1}(z)-\frac{4}{5}H_{2,0,0,0}(z)+8H_{2,1,0,0}(z)\\
&-\frac{148}{5}H_{2,1,1,0}(z)+\frac{472}{5}H_{2,1,1,1}(z)-\frac{8}{5}H_{0,0,0,0,0}(z)+\frac{16}{5}H_{1,0,0,0,0}(z)-8H_{1,1,0,0,0}(z)\\
&+\frac{112}{5}H_{1,1,1,0,0}(z)-\frac{328}{5}H_{1,1,1,1,0}(z)+\frac{976}{5}H_{1,1,1,1,1}(z)-3\zeta(5)+\frac{2\pi^2\zeta(3)}{5}\Big)\\
&+\mathcal{O}(\epsilon^6)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4\mathrm{B}13}=n_2(\epsilon)\Bigg[&-\frac{1}{60}+\epsilon\Big(\frac{1}{30}H_0(z)-\frac{1}{10}H_1(z)\Big)\\
&+\epsilon^2\Big(-\frac{1}{5}H_2(z)-\frac{1}{15}H_{0,0}(z)+\frac{1}{5}H_{1,0}(z)-\frac{3}{5}H_{1,1}(z)+\frac{\pi^2}{90}\Big)\\
&+\epsilon^3\Big(-\frac{1}{45}\pi^2H_0(z)+\frac{1}{15}\pi^2H_1(z)-\frac{H_3(z)}{5}-\frac{2}{5}H_{1,2}(z)+\frac{2}{5}H_{2,0}(z)\\
&-2H_{2,1}(z)+\frac{2}{15}H_{0,0,0}(z)-\frac{2}{5}H_{1,0,0}(z)+\frac{6}{5}H_{1,1,0}(z)-\frac{18}{5}H_{1,1,1}(z)+\frac{1}{5}\zeta(3)\Big)\\
&+\epsilon^4\Big(4S_{2,2}(z)-\frac{2}{5}\zeta(3)H_0(z)+\frac{6}{5}\zeta(3)H_1(z)+\frac{2}{15}\pi^2H_2(z)+\frac{2}{45}\pi^2H_{0,0}(z)\\
&-\frac{2}{15}\pi^2H_{1,0}(z)+\frac{2}{5}\pi^2H_{1,1}(z)-\frac{6}{5}H_{1,3}(z)-\frac{2}{5}H_{2,2}(z)+\frac{2}{5}H_{3,0}(z)\\
&-\frac{24}{5}H_{3,1}(z)-\frac{4}{5}H_{1,1,2}(z)+\frac{4}{5}H_{1,2,0}(z)-\frac{28}{5}H_{1,2,1}(z)-\frac{4}{5}H_{2,0,0}(z)\\
&+4H_{2,1,0}(z)-\frac{68}{5}H_{2,1,1}(z)-\frac{4}{15}H_{0,0,0,0}(z)+\frac{4}{5}H_{1,0,0,0}(z)-\frac{12}{5}H_{1,1,0,0}(z)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{36}{5}H_{1,1,1,0}(z)-\frac{108}{5}H_{1,1,1,1}(z)+\frac{\pi^4}{900}\Big)\\
&+\epsilon^5\Big(-\frac{112}{5}S_{2,3}(z)-\frac{16}{5}S_{3,2}(z)-8S_{2,2}(z)H_0(z)-\frac{1}{450}\pi^4H_0(z)\\
&+\frac{88}{5}S_{2,2}(z)H_1(z)+\frac{1}{150}\pi^4H_1(z)+\frac{12}{5}\zeta(3)H_2(z)+\frac{2}{15}\pi^2H_3(z)-\frac{3}{5}H_5(z)\\
&+\frac{4}{5}\zeta(3)H_{0,0}(z)-\frac{12}{5}\zeta(3)H_{1,0}(z)+\frac{36}{5}\zeta(3)H_{1,1}(z)+\frac{4}{15}\pi^2H_{1,2}(z)\\
&+\frac{4}{5}H_{1,4}(z)-\frac{4}{15}\pi^2H_{2,0}(z)+\frac{4}{3}\pi^2H_{2,1}(z)-\frac{16}{5}H_{2,3}(z)+8H_{3,2}(z)+\frac{128}{5}H_{4,1}(z)\\
&-\frac{4}{45}\pi^2H_{0,0,0}(z)+\frac{4}{15}\pi^2H_{1,0,0}(z)-\frac{4}{5}\pi^2H_{1,1,0}(z)+\frac{12}{5}\pi^2H_{1,1,1}(z)-\frac{36}{5}H_{1,1,3}(z)\\
&-\frac{12}{5}H_{1,2,2}(z)+\frac{12}{5}H_{1,3,0}(z)-\frac{112}{5}H_{1,3,1}(z)-\frac{4}{5}H_{2,1,2}(z)+\frac{4}{5}H_{2,2,0}(z)\\
&-\frac{128}{5}H_{2,2,1}(z)-\frac{4}{5}H_{3,0,0}(z)+\frac{48}{5}H_{3,1,0}(z)-\frac{172}{5}H_{3,1,1}(z)-\frac{8}{5}H_{1,1,1,2}(z)\\
&+\frac{8}{5}H_{1,1,2,0}(z)-\frac{104}{5}H_{1,1,2,1}(z)-\frac{8}{5}H_{1,2,0,0}(z)+\frac{56}{5}H_{1,2,1,0}(z)\\
&-40H_{1,2,1,1}(z)+\frac{8}{5}H_{2,0,0,0}(z)-8H_{2,1,0,0}(z)+\frac{136}{5}H_{2,1,1,0}(z)-\frac{424}{5}H_{2,1,1,1}(z)\\
&+\frac{8}{15}H_{0,0,0,0,0}(z)-\frac{8}{5}H_{1,0,0,0,0}(z)+\frac{24}{5}H_{1,1,0,0,0}(z)-\frac{72}{5}H_{1,1,1,0,0}(z)\\
&+\frac{216}{5}H_{1,1,1,1,0}(z)-\frac{648}{5}H_{1,1,1,1,1}(z)+\zeta(5)-\frac{2}{15}\pi^2\zeta(3)\Big)+\mathcal{O}(\epsilon^6)\Big],
\end{aligned}
$$

$$
\begin{aligned}
G_{4\mathrm{B}14}=n_2(\epsilon)\Big[&-\frac{1}{270}+\epsilon\Big(\frac{1}{135}H_0(z)-\frac{1}{90}H_1(z)-\frac{1}{270}i\pi\Big)+\epsilon^2\Big(\frac{1}{135}i\pi H_0(z)\\
&-\frac{1}{90}i\pi H_1(z)+\frac{H_2(z)}{45}-\frac{2}{135}H_{0,0}(z)+\frac{1}{45}H_{1,0}(z)-\frac{1}{30}H_{1,1}(z)+\frac{\pi^2}{162}\Big)\\
&+\epsilon^3\Big(-\frac{1}{81}\pi^2H_0(z)+\frac{1}{54}\pi^2H_1(z)+\frac{1}{45}i\pi H_2(z)-\frac{2H_3(z)}{45}-\frac{2}{135}i\pi H_{0,0}(z)\\
&+\frac{1}{45}i\pi H_{1,0}(z)-\frac{1}{30}i\pi H_{1,1}(z)+\frac{1}{15}H_{1,2}(z)-\frac{2}{45}H_{2,0}(z)+\frac{1}{15}H_{2,1}(z)\\
&+\frac{4}{135}H_{0,0,0}(z)-\frac{2}{45}H_{1,0,0}(z)+\frac{1}{15}H_{1,1,0}(z)-\frac{1}{10}H_{1,1,1}(z)+\frac{4}{45}\zeta(3)+\frac{2}{405}i\pi^3\Big)\\
&+\epsilon^4\Big(-\frac{8}{45}\zeta(3)H_0(z)-\frac{4}{405}i\pi^3H_0(z)+\frac{4}{15}\zeta(3)H_1(z)+\frac{2}{135}i\pi^3H_1(z)\\
&-\frac{1}{27}\pi^2H_2(z)-\frac{2}{45}i\pi H_3(z)+\frac{4H_4(z)}{45}+\frac{2}{81}\pi^2H_{0,0}(z)-\frac{1}{27}\pi^2H_{1,0}(z)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{18}\pi^2 H_{1,1}(z) + \frac{1}{15} i\pi H_{1,2}(z) - \frac{2}{15} H_{1,3}(z) - \frac{2}{45} i\pi H_{2,0}(z) + \frac{1}{15} i\pi H_{2,1}(z) \\
&- \frac{2}{15} H_{2,2}(z) + \frac{4}{45} H_{3,0}(z) - \frac{2}{15} H_{3,1}(z) + \frac{4}{135} i\pi H_{0,0,0}(z) - \frac{2}{45} i\pi H_{1,0,0}(z) \\
&+ \frac{1}{15} i\pi H_{1,1,0}(z) - \frac{1}{10} i\pi H_{1,1,1}(z) + \frac{1}{5} H_{1,1,2}(z) - \frac{2}{15} H_{1,2,0}(z) + \frac{1}{5} H_{1,2,1}(z) \\
&+ \frac{4}{45} H_{2,0,0}(z) - \frac{2}{15} H_{2,1,0}(z) + \frac{1}{5} H_{2,1,1}(z) - \frac{8}{135} H_{0,0,0,0}(z) + \frac{4}{45} H_{1,0,0,0}(z) \\
&- \frac{2}{15} H_{1,1,0,0}(z) + \frac{1}{5} H_{1,1,1,0}(z) - \frac{3}{10} H_{1,1,1,1}(z) + \frac{4}{45} i\pi\zeta(3) - \frac{4}{2025}\pi^4 \Big) \\
&+ \epsilon^5 \Big(- \frac{8}{45} i\pi\zeta(3) H_0(z) + \frac{8}{2025}\pi^4 H_0(z) + \frac{4}{15} i\pi\zeta(3) H_1(z) - \frac{4}{675}\pi^4 H_1(z) \\
&- \frac{8}{15}\zeta(3) H_2(z) - \frac{4}{135} i\pi^3 H_2(z) + \frac{2}{27}\pi^2 H_3(z) + \frac{4}{45} i\pi H_4(z) - \frac{8}{45} H_5(z) \\
&+ \frac{16}{45}\zeta(3) H_{0,0}(z) + \frac{8}{405} i\pi^3 H_{0,0}(z) - \frac{8}{15}\zeta(3) H_{1,0}(z) - \frac{4}{135} i\pi^3 H_{1,0}(z) \\
&+ \frac{4}{5}\zeta(3) H_{1,1}(z) + \frac{2}{45} i\pi^3 H_{1,1}(z) - \frac{1}{9}\pi^2 H_{1,2}(z) - \frac{2}{15} i\pi H_{1,3}(z) + \frac{4}{15} H_{1,4}(z) \\
&+ \frac{2}{27}\pi^2 H_{2,0}(z) - \frac{1}{9}\pi^2 H_{2,1}(z) - \frac{2}{15} i\pi H_{2,2}(z) + \frac{4}{15} H_{2,3}(z) + \frac{4}{45} i\pi H_{3,0}(z) \\
&- \frac{2}{15} i\pi H_{3,1}(z) + \frac{4}{15} H_{3,2}(z) - \frac{8}{45} H_{4,0}(z) + \frac{4}{15} H_{4,1}(z) - \frac{4}{81}\pi^2 H_{0,0,0}(z) \\
&+ \frac{2}{27}\pi^2 H_{1,0,0}(z) - \frac{1}{9}\pi^2 H_{1,1,0}(z) + \frac{1}{6}\pi^2 H_{1,1,1}(z) + \frac{1}{5} i\pi H_{1,1,2}(z) - \frac{2}{5} H_{1,1,3}(z) \\
&- \frac{2}{15} i\pi H_{1,2,0}(z) + \frac{1}{5} i\pi H_{1,2,1}(z) - \frac{2}{5} H_{1,2,2}(z) + \frac{4}{15} H_{1,3,0}(z) - \frac{2}{5} H_{1,3,1}(z) \\
&+ \frac{4}{45} i\pi H_{2,0,0}(z) - \frac{2}{15} i\pi H_{2,1,0}(z) + \frac{1}{5} i\pi H_{2,1,1}(z) - \frac{2}{5} H_{2,1,2}(z) + \frac{4}{15} H_{2,2,0}(z) \\
&- \frac{2}{5} H_{2,2,1}(z) - \frac{8}{45} H_{3,0,0}(z) + \frac{4}{15} H_{3,1,0}(z) - \frac{2}{5} H_{3,1,1}(z) - \frac{8}{135} i\pi H_{0,0,0,0}(z) \\
&+ \frac{4}{45} i\pi H_{1,0,0,0}(z) - \frac{2}{15} i\pi H_{1,1,0,0}(z) + \frac{1}{5} i\pi H_{1,1,1,0}(z) - \frac{3}{10} i\pi H_{1,1,1,1}(z) \\
&+ \frac{3}{5} H_{1,1,1,2}(z) - \frac{2}{5} H_{1,1,2,0}(z) + \frac{3}{5} H_{1,1,2,1}(z) + \frac{4}{15} H_{1,2,0,0}(z) - \frac{2}{5} H_{1,2,1,0}(z) \\
&+ \frac{3}{5} H_{1,2,1,1}(z) - \frac{8}{45} H_{2,0,0,0}(z) + \frac{4}{15} H_{2,1,0,0}(z) - \frac{2}{5} H_{2,1,1,0}(z) + \frac{3}{5} H_{2,1,1,1}(z) \\
&+ \frac{16}{135} H_{0,0,0,0,0}(z) - \frac{8}{45} H_{1,0,0,0,0}(z) + \frac{4}{15} H_{1,1,0,0,0}(z) - \frac{2}{5} H_{1,1,1,0,0}(z) \\
&+ \frac{3}{5} H_{1,1,1,1,0}(z) - \frac{9}{10} H_{1,1,1,1,1}(z) + \frac{4}{5}\zeta(5) - \frac{4}{27}\pi^2\zeta(3) - \frac{i\pi^5}{2430} \Big) + \mathcal{O}(\epsilon^6) \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4\mathrm{B}15} = n_2(\epsilon)\bigg[& -\frac{1}{60}\epsilon H_1(z) + \epsilon^2\Big(\frac{H_2(z)}{15} + \frac{2}{15}H_{1,0}(z) - \frac{2}{15}H_{1,1}(z)\Big) \\
& + \epsilon^3\Big(-\frac{7}{360}\pi^2 H_1(z) + \frac{H_3(z)}{15} + \frac{1}{3}H_{1,2}(z) - \frac{2}{15}H_{2,0}(z) + \frac{8}{15}H_{2,1}(z) \\
& - \frac{23}{30}H_{1,0,0}(z) + \frac{29}{60}H_{1,1,0}(z) - \frac{13}{15}H_{1,1,1}(z)\Big) \\
& + \epsilon^4\Big(\frac{19}{20}S_{2,2}(z) + \frac{31}{60}\zeta(3)H_1(z) - \frac{2}{45}\pi^2 H_2(z) + \frac{1}{15}H_4(z) + \frac{7}{180}\pi^2 H_{1,0}(z) \\
& + \frac{1}{36}\pi^2 H_{1,1}(z) - \frac{22}{15}H_{1,3}(z) + \frac{2}{15}H_{2,2}(z) - \frac{2}{15}H_{3,0}(z) - \frac{5}{12}H_{3,1}(z) + \frac{7}{10}H_{1,1,2}(z) \\
& - \frac{41}{60}H_{1,2,0}(z) + \frac{16}{15}H_{1,2,1}(z) + \frac{4}{15}H_{2,0,0}(z) - \frac{16}{15}H_{2,1,0}(z) + \frac{52}{15}H_{2,1,1}(z) \\
& + \frac{121}{30}H_{1,0,0,0}(z) - \frac{41}{20}H_{1,1,0,0}(z) + \frac{11}{5}H_{1,1,1,0}(z) - \frac{16}{3}H_{1,1,1,1}(z)\Big) \\
& + \epsilon^5\Big(\frac{11}{5}S_{2,3}(z) + \frac{821}{60}S_{3,2}(z) - \frac{143}{20}S_{2,2}(z)H_0(z) + \frac{16}{5}S_{2,2}(z)H_1(z) \\
& + \frac{119}{7200}\pi^4 H_1(z) - \frac{4}{5}\zeta(3)H_2(z) - \frac{2}{45}\pi^2 H_3(z) + \frac{1}{15}H_5(z) - \frac{127}{30}\zeta(3)H_{1,0}(z) \\
& + \frac{5}{2}\zeta(3)H_{1,1}(z) - \frac{31}{120}\pi^2 H_{1,2}(z) + \frac{122}{15}H_{1,4}(z) + \frac{4}{45}\pi^2 H_{2,0}(z) - \frac{16}{45}\pi^2 H_{2,1}(z) \\
& + \frac{14}{15}H_{2,3}(z) + \frac{437}{60}H_{3,2}(z) - \frac{2}{15}H_{4,0}(z) + \frac{83}{10}H_{4,1}(z) - \frac{7}{90}\pi^2 H_{1,0,0}(z) \\
& - \frac{1}{18}\pi^2 H_{1,1,0}(z) + \frac{41}{90}\pi^2 H_{1,1,1}(z) - \frac{131}{30}H_{1,1,3}(z) - \frac{29}{30}H_{1,2,2}(z) + \frac{19}{6}H_{1,3,0}(z) \\
& - \frac{86}{15}H_{1,3,1}(z) + \frac{4}{15}H_{2,1,2}(z) - \frac{4}{15}H_{2,2,0}(z) - \frac{8}{15}H_{2,2,1}(z) + \frac{4}{15}H_{3,0,0}(z) \\
& + \frac{73}{12}H_{3,1,0}(z) - \frac{25}{3}H_{3,1,1}(z) + \frac{22}{15}H_{1,1,1,2}(z) - \frac{22}{15}H_{1,1,2,0}(z) - \frac{1}{5}H_{1,1,2,1}(z) \\
& + \frac{29}{20}H_{1,2,0,0}(z) - \frac{39}{20}H_{1,2,1,0}(z) + \frac{68}{15}H_{1,2,1,1}(z) - \frac{8}{15}H_{2,0,0,0}(z) \\
& + \frac{32}{15}H_{2,1,0,0}(z) - \frac{104}{15}H_{2,1,1,0}(z) + \frac{64}{3}H_{2,1,1,1}(z) - \frac{617}{30}H_{1,0,0,0,0}(z) \\
& + \frac{571}{60}H_{1,1,0,0,0}(z) - \frac{101}{15}H_{1,1,1,0,0}(z) + \frac{35}{3}H_{1,1,1,1,0}(z) - \frac{484}{15}H_{1,1,1,1,1}(z)\Big) \\
& + \mathcal{O}(\epsilon^6)\bigg],
\end{aligned}
$$

$$
G_{4\mathrm{B}16} = n_2(\epsilon)\bigg[-\epsilon\frac{1}{60}H_1(z) + \epsilon^2\Big(\frac{1}{15}H_2(z) + \frac{1}{30}H_{1,0}(z) - \frac{2}{15}H_{1,1}(z)\Big)
$$

$$
\begin{aligned}
&+\epsilon^3\Big(\frac{1}{90}\pi^2 H_1(z)-\frac{4H_3(z)}{15}+\frac{2}{15}H_{1,2}(z)-\frac{2}{15}H_{2,0}(z)+\frac{8}{15}H_{2,1}(z)\\
&-\frac{1}{15}H_{1,0,0}(z)+\frac{4}{15}H_{1,1,0}(z)-\frac{13}{15}H_{1,1,1}(z)\Big)\\
&+\epsilon^4\Big(-\frac{16}{15}S_{2,2}(z)+\frac{1}{5}\zeta(3)H_1(z)-\frac{2}{45}\pi^2 H_2(z)+\frac{1}{15}H_4(z)-\frac{1}{45}\pi^2 H_{1,0}(z)\\
&+\frac{4}{45}\pi^2 H_{1,1}(z)-\frac{11}{15}H_{1,3}(z)-\frac{8}{15}H_{2,2}(z)+\frac{8}{15}H_{3,0}(z)-\frac{16}{15}H_{3,1}(z)\\
&+\frac{4}{15}H_{1,1,2}(z)-\frac{4}{15}H_{1,2,0}(z)+\frac{2}{3}H_{1,2,1}(z)+\frac{4}{15}H_{2,0,0}(z)-\frac{16}{15}H_{2,1,0}(z)\\
&+\frac{52}{15}H_{2,1,1}(z)+\frac{2}{15}H_{1,0,0,0}(z)-\frac{8}{15}H_{1,1,0,0}(z)+\frac{26}{15}H_{1,1,1,0}(z)-\frac{16}{3}H_{1,1,1,1}(z)\Big)\\
&+\epsilon^5\Big(-\frac{16}{15}S_{2,3}(z)+\frac{2}{15}S_{3,2}(z)+\frac{32}{15}S_{2,2}(z)H_0(z)+\frac{4}{15}S_{2,2}(z)H_1(z)\\
&+\frac{1}{900}\pi^4 H_1(z)-\frac{4}{5}\zeta(3)H_2(z)+\frac{8}{45}\pi^2 H_3(z)-\frac{4H_5(z)}{15}-\frac{2}{5}\zeta(3)H_{1,0}(z)\\
&+\frac{8}{5}\zeta(3)H_{1,1}(z)-\frac{4}{45}\pi^2 H_{1,2}(z)+\frac{1}{3}H_{1,4}(z)+\frac{4}{45}\pi^2 H_{2,0}(z)-\frac{16}{45}\pi^2 H_{2,1}(z)\\
&+\frac{14}{15}H_{2,3}(z)-2H_{3,2}(z)-\frac{2}{15}H_{4,0}(z)-6H_{4,1}(z)+\frac{2}{45}\pi^2 H_{1,0,0}(z)\\
&-\frac{8}{45}\pi^2 H_{1,1,0}(z)+\frac{26}{45}\pi^2 H_{1,1,1}(z)-\frac{8}{3}H_{1,1,3}(z)-\frac{22}{15}H_{1,2,2}(z)+\frac{22}{15}H_{1,3,0}(z)\\
&-\frac{16}{3}H_{1,3,1}(z)-\frac{16}{15}H_{2,1,2}(z)+\frac{16}{15}H_{2,2,0}(z)-\frac{44}{15}H_{2,2,1}(z)-\frac{16}{15}H_{3,0,0}(z)\\
&+\frac{32}{15}H_{3,1,0}(z)-\frac{68}{5}H_{3,1,1}(z)+\frac{8}{15}H_{1,1,1,2}(z)-\frac{8}{15}H_{1,1,2,0}(z)-\frac{16}{15}H_{1,1,2,1}(z)\\
&+\frac{8}{15}H_{1,2,0,0}(z)-\frac{4}{3}H_{1,2,1,0}(z)+\frac{56}{15}H_{1,2,1,1}(z)-\frac{8}{15}H_{2,0,0,0}(z)+\frac{32}{15}H_{2,1,0,0}(z)\\
&-\frac{104}{15}H_{2,1,1,0}(z)+\frac{64}{3}H_{2,1,1,1}(z)-\frac{4}{15}H_{1,0,0,0,0}(z)+\frac{16}{15}H_{1,1,0,0,0}(z)\\
&-\frac{52}{15}H_{1,1,1,0,0}(z)+\frac{32}{3}H_{1,1,1,1,0}(z)-\frac{484}{15}H_{1,1,1,1,1}(z)\Big)+\mathcal{O}(\epsilon^6)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4\mathrm{B}17}=n_2(\epsilon)\Bigg[&-\frac{1}{10}\epsilon^2 H_{1,0}(z)+\epsilon^3\Big(-\frac{1}{40}\pi^2 H_1(z)-\frac{1}{5}H_{1,2}(z)+\frac{7}{10}H_{1,0,0}(z)-\frac{11}{20}H_{1,1,0}(z)\Big)\\
&+\epsilon^4\Big(-\frac{47}{20}S_{2,2}(z)-\frac{53}{20}\zeta(3)H_1(z)+\frac{1}{20}\pi^2 H_{1,0}(z)-\frac{1}{20}\pi^2 H_{1,1}(z)+\frac{7}{5}H_{1,3}(z)\\
&+\frac{47}{20}H_{3,1}(z)-\frac{11}{10}H_{1,1,2}(z)+\frac{3}{4}H_{1,2,0}(z)-\frac{2}{5}H_{1,2,1}(z)-\frac{39}{10}H_{1,0,0,0}(z)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{77}{20}H_{1,1,0,0}(z)-\frac{9}{5}H_{1,1,1,0}(z)\Big)\\
&+\epsilon^5\Big(\frac{57}{5}S_{2,3}(z)-\frac{171}{20}S_{3,2}(z)+\frac{299}{20}S_{2,2}(z)H_0(z)-\frac{28}{5}S_{2,2}(z)H_1(z)\\
&-\frac{217}{2400}\pi^4H_1(z)+\frac{17}{2}\zeta(3)H_{1,0}(z)-\frac{109}{10}\zeta(3)H_{1,1}(z)+\frac{9}{40}\pi^2H_{1,2}(z)\\
&-\frac{39}{5}H_{1,4}(z)-\frac{299}{20}H_{3,2}(z)-\frac{363}{10}H_{4,1}(z)-\frac{1}{10}\pi^2H_{1,0,0}(z)+\frac{1}{10}\pi^2H_{1,1,0}(z)\\
&-\frac{1}{10}\pi^2H_{1,1,1}(z)+\frac{77}{10}H_{1,1,3}(z)+\frac{3}{2}H_{1,2,2}(z)-\frac{37}{10}H_{1,3,0}(z)+\frac{42}{5}H_{1,3,1}(z)\\
&+\frac{28}{5}H_{2,2,1}(z)-\frac{299}{20}H_{3,1,0}(z)+\frac{27}{5}H_{3,1,1}(z)-\frac{18}{5}H_{1,1,1,2}(z)+\frac{18}{5}H_{1,1,2,0}(z)\\
&-\frac{11}{5}H_{1,1,2,1}(z)-\frac{13}{4}H_{1,2,0,0}(z)+\frac{19}{20}H_{1,2,1,0}(z)-\frac{4}{5}H_{1,2,1,1}(z)\\
&+\frac{203}{10}H_{1,0,0,0,0}(z)-\frac{429}{20}H_{1,1,0,0,0}(z)+\frac{63}{5}H_{1,1,1,0,0}(z)-5H_{1,1,1,1,0}(z)\Big)\\
&+\mathcal{O}(\epsilon^6)\Bigg]\,,
\end{aligned}
$$

$$
\begin{aligned}
G_{4\mathrm{B}18}=n_2(\epsilon)\Bigg[&\epsilon^2\Big(-\frac{1}{60}i\pi H_1(z)-\frac{1}{60}H_2(z)-\frac{1}{60}H_{1,1}(z)\Big)\\
&+\epsilon^3\Big(\frac{1}{60}\pi^2H_1(z)-\frac{1}{60}H_3(z)+\frac{1}{30}i\pi H_{1,0}(z)-\frac{7}{60}i\pi H_{1,1}(z)-\frac{1}{30}H_{1,2}(z)\\
&+\frac{1}{30}H_{2,0}(z)-\frac{2}{15}H_{2,1}(z)+\frac{1}{30}H_{1,1,0}(z)-\frac{3}{20}H_{1,1,1}(z)\Big)\\
&+\epsilon^4\Big(\frac{2}{15}S_{2,2}(z)+\frac{1}{5}\zeta(3)H_1(z)+\frac{1}{45}i\pi^3H_1(z)+\frac{1}{90}\pi^2H_2(z)-\frac{1}{60}H_4(z)\\
&-\frac{1}{30}\pi^2H_{1,0}(z)+\frac{23}{180}\pi^2H_{1,1}(z)+\frac{1}{10}i\pi H_{1,2}(z)-\frac{2}{15}H_{1,3}(z)-\frac{1}{30}H_{2,2}(z)\\
&+\frac{1}{30}H_{3,0}(z)-\frac{4}{15}H_{3,1}(z)-\frac{1}{15}i\pi H_{1,0,0}(z)+\frac{7}{30}i\pi H_{1,1,0}(z)-\frac{37}{60}i\pi H_{1,1,1}(z)\\
&+\frac{1}{30}H_{1,1,2}(z)+\frac{1}{15}H_{1,2,0}(z)-\frac{11}{30}H_{1,2,1}(z)-\frac{1}{15}H_{2,0,0}(z)+\frac{4}{15}H_{2,1,0}(z)\\
&-\frac{13}{15}H_{2,1,1}(z)-\frac{1}{15}H_{1,1,0,0}(z)+\frac{3}{10}H_{1,1,1,0}(z)-\frac{11}{12}H_{1,1,1,1}(z)\Big)+\mathcal{O}(\epsilon^5)\Bigg]\,,
\end{aligned}
$$

$$
G_{4\mathrm{B}19}=n_2(\epsilon)\Bigg[\epsilon^4\Big(\zeta(3)H_1(z)+\frac{1}{6}i\pi^3H_1(z)+H_{1,3}(z)+H_{1,1,2}(z)\Big)
$$

$$+\,\epsilon^5\Big(12S_{2,3}(z)-10S_{3,2}(z)+4S_{2,2}(z)H_1(z)+9i\pi\zeta(3)H_1(z)-\frac{1}{30}\pi^4H_1(z)$$
$$-\,2\zeta(3)H_{1,0}(z)-\frac{1}{3}i\pi^3H_{1,0}(z)+7\zeta(3)H_{1,1}(z)+\frac{7}{6}i\pi^3H_{1,1}(z)-H_{1,4}(z)$$
$$+\,10H_{4,1}(z)+8H_{1,1,3}(z)-2H_{1,3,0}(z)+4H_{1,3,1}(z)-4H_{2,2,1}(z)$$
$$-\,24H_{3,1,1}(z)+9H_{1,1,1,2}(z)-2H_{1,1,2,0}(z)+8H_{1,1,2,1}(z)\Big)+\mathcal{O}(\epsilon^6)\Bigg],$$

$$G_{4\mathrm{B}20}=n_2(\epsilon)\Bigg[\frac{5}{72}+\epsilon\Big(-\frac{25}{72}H_0(z)+\frac{5}{36}H_1(z)+\frac{i\pi}{18}\Big)$$
$$+\,\epsilon^2\Big(-\frac{5}{18}i\pi H_0(z)+\frac{1}{9}i\pi H_1(z)-\frac{11H_2(z)}{12}+\frac{125}{72}H_{0,0}(z)$$
$$-\,\frac{7}{36}H_{1,0}(z)+\frac{1}{18}H_{1,1}(z)-\frac{\pi^2}{54}\Big)$$
$$+\,\epsilon^3\Big(\frac{5}{54}\pi^2H_0(z)-\frac{7}{27}\pi^2H_1(z)-\frac{2}{3}i\pi H_2(z)+\frac{55}{12}H_3(z)+\frac{25}{18}i\pi H_{0,0}(z)$$
$$-\,\frac{2}{9}i\pi H_{1,0}(z)+\frac{1}{9}i\pi H_{1,1}(z)-\frac{5}{6}H_{1,2}(z)+\frac{7}{4}H_{2,0}(z)-\frac{13}{6}H_{2,1}(z)$$
$$-\,\frac{625}{72}H_{0,0,0}(z)-\frac{1}{36}H_{1,0,0}(z)+\frac{1}{18}H_{1,1,0}(z)-\frac{16}{9}H_{1,1,1}(z)-\frac{7}{3}\zeta(3)-\frac{i\pi^3}{54}\Big)$$
$$+\,\mathcal{O}(\epsilon^4)\Bigg],$$

$$G_{4\mathrm{B}21}=n_2(\epsilon)\Bigg[-\frac{1}{6}+\epsilon\Big(\frac{1}{3}H_0(z)-\frac{1}{3}H_1(z)-\frac{i\pi}{3}\Big)$$
$$+\,\epsilon^2\Big(\frac{2}{3}i\pi H_0(z)-\frac{2}{3}i\pi H_1(z)+2H_2(z)-\frac{2}{3}H_{0,0}(z)+\frac{2}{3}H_{1,0}(z)+\frac{2}{3}H_{1,1}(z)+\frac{4}{9}\pi^2\Big)$$
$$+\,\epsilon^3\Big(-\frac{8}{9}\pi^2H_0(z)+\frac{8}{9}\pi^2H_1(z)+2i\pi H_2(z)-\frac{4}{3}i\pi H_{0,0}(z)+\frac{4}{3}i\pi H_{1,0}(z)$$
$$-\,\frac{2}{3}i\pi H_{1,1}(z)+8H_{1,2}(z)-4H_{2,0}(z)+10H_{2,1}(z)+\frac{4}{3}H_{0,0,0}(z)-\frac{4}{3}H_{1,0,0}(z)$$
$$-\,\frac{4}{3}H_{1,1,0}(z)+\frac{38}{3}H_{1,1,1}(z)+8\zeta(3)+\frac{7i\pi^3}{9}\Big)$$
$$+\,\epsilon^4\Big(-\frac{20}{3}S_{2,2}(z)-16\zeta(3)H_0(z)-\frac{14}{9}i\pi^3H_0(z)+16\zeta(3)H_1(z)+\frac{14}{9}i\pi^3H_1(z)$$
$$-\,\frac{10}{3}\pi^2H_2(z)-4i\pi H_3(z)+4H_4(z)+\frac{16}{9}\pi^2H_{0,0}(z)-\frac{16}{9}\pi^2H_{1,0}(z)+\frac{2}{9}\pi^2H_{1,1}(z)$$

$$
\begin{aligned}
&+ 4i\pi H_{1,2}(z) + 4H_{1,3}(z) - 4i\pi H_{2,0}(z) + 6i\pi H_{2,1}(z) - 8H_{2,2}(z) + \frac{56}{3}H_{3,1}(z) \\
&+ \frac{8}{3}i\pi H_{0,0,0}(z) - \frac{8}{3}i\pi H_{1,0,0}(z) + \frac{4}{3}i\pi H_{1,1,0}(z) + \frac{10}{3}i\pi H_{1,1,1}(z) + 24H_{1,1,2}(z) \\
&- 16H_{1,2,0}(z) + 52H_{1,2,1}(z) + 8H_{2,0,0}(z) - 20H_{2,1,0}(z) + 50H_{2,1,1}(z) \\
&- \frac{8}{3}H_{0,0,0,0}(z) + \frac{8}{3}H_{1,0,0,0}(z) + \frac{8}{3}H_{1,1,0,0}(z) - \frac{76}{3}H_{1,1,1,0}(z) + \frac{302}{3}H_{1,1,1,1}(z) \\
&+ 26i\pi\zeta(3) - \frac{23\pi^4}{90}\Big) + \mathcal{O}(\epsilon^5)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4\mathrm{B}22} = n_2(\epsilon)\Bigg[&-\frac{5}{24} + \epsilon\Big(\frac{25}{24}H_0(z) - \frac{5}{12}H_1(z) - \frac{i\pi}{6}\Big) + \epsilon^2\Big(\frac{5}{6}i\pi H_0(z) - \frac{1}{3}i\pi H_1(z) \\
&+ \frac{11}{4}H_2(z) - \frac{125}{24}H_{0,0}(z) + \frac{25}{12}H_{1,0}(z) - \frac{1}{6}H_{1,1}(z) + \frac{\pi^2}{18}\Big) \\
&+ \epsilon^3\Big(- \frac{5}{18}\pi^2 H_0(z) + \frac{1}{9}\pi^2 H_1(z) + 2i\pi H_2(z) - \frac{55}{4}H_3(z) - \frac{25}{6}i\pi H_{0,0}(z) \\
&+ \frac{5}{3}i\pi H_{1,0}(z) - \frac{1}{3}i\pi H_{1,1}(z) + \frac{11}{2}H_{1,2}(z) - \frac{21}{4}H_{2,0}(z) + \frac{13}{2}H_{2,1}(z) \\
&+ \frac{625}{24}H_{0,0,0}(z) - \frac{125}{12}H_{1,0,0}(z) + \frac{17}{6}H_{1,1,0}(z) + \frac{16}{3}H_{1,1,1}(z) + 7\zeta(3) + \frac{i\pi^3}{18}\Big) \\
&+ \mathcal{O}(\epsilon^4)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4\mathrm{B}23} = n_3(\epsilon)\Bigg[&\frac{1}{40} + \epsilon\Big(\frac{1}{20}H_1(z) - \frac{1}{20}H_0(z)\Big) \\
&+ \epsilon^2\Big(- \frac{1}{10}H_2(z) + \frac{1}{10}H_{0,0}(z) - \frac{1}{10}H_{1,0}(z) + \frac{1}{10}H_{1,1}(z) - \frac{\pi^2}{60}\Big) \\
&+ \epsilon^3\Big(\frac{1}{30}\pi^2 H_0(z) - \frac{1}{30}\pi^2 H_1(z) + \frac{1}{5}H_3(z) - \frac{1}{5}H_{1,2}(z) + \frac{1}{5}H_{2,0}(z) \\
&- \frac{1}{5}H_{2,1}(z) - \frac{1}{5}H_{0,0,0}(z) + \frac{1}{5}H_{1,0,0}(z) - \frac{1}{5}H_{1,1,0}(z) + \frac{1}{5}H_{1,1,1}(z) - \frac{3}{10}\zeta(3)\Big) \\
&+ \epsilon^4\Big(\frac{3}{5}\zeta(3)H_0(z) - \frac{3}{5}\zeta(3)H_1(z) + \frac{1}{15}\pi^2 H_2(z) - \frac{2}{5}H_4(z) - \frac{1}{15}\pi^2 H_{0,0}(z) \\
&+ \frac{1}{15}\pi^2 H_{1,0}(z) - \frac{1}{15}\pi^2 H_{1,1}(z) + \frac{2}{5}H_{1,3}(z) + \frac{2}{5}H_{2,2}(z) - \frac{2}{5}H_{3,0}(z) \\
&+ \frac{2}{5}H_{3,1}(z) - \frac{2}{5}H_{1,1,2}(z) + \frac{2}{5}H_{1,2,0}(z) - \frac{2}{5}H_{1,2,1}(z) - \frac{2}{5}H_{2,0,0}(z)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{5}H_{2,1,0}(z)-\frac{2}{5}H_{2,1,1}(z)+\frac{2}{5}H_{0,0,0,0}(z)-\frac{2}{5}H_{1,0,0,0}(z)+\frac{2}{5}H_{1,1,0,0}(z)\\
&-\frac{2}{5}H_{1,1,1,0}(z)+\frac{2}{5}H_{1,1,1,1}(z)-\frac{\pi^4}{600}\Big)\\
&+\epsilon^5\Big(\frac{1}{300}\pi^4H_0(z)-\frac{1}{300}\pi^4H_1(z)+\frac{6}{5}\zeta(3)H_2(z)-\frac{2}{15}\pi^2H_3(z)+\frac{4}{5}H_5(z)\\
&-\frac{6}{5}\zeta(3)H_{0,0}(z)+\frac{6}{5}\zeta(3)H_{1,0}(z)-\frac{6}{5}\zeta(3)H_{1,1}(z)+\frac{2}{15}\pi^2H_{1,2}(z)-\frac{4}{5}H_{1,4}(z)\\
&-\frac{2}{15}\pi^2H_{2,0}(z)+\frac{2}{15}\pi^2H_{2,1}(z)-\frac{4}{5}H_{2,3}(z)-\frac{4}{5}H_{3,2}(z)+\frac{4}{5}H_{4,0}(z)\\
&-\frac{4}{5}H_{4,1}(z)+\frac{2}{15}\pi^2H_{0,0,0}(z)-\frac{2}{15}\pi^2H_{1,0,0}(z)+\frac{2}{15}\pi^2H_{1,1,0}(z)\\
&-\frac{2}{15}\pi^2H_{1,1,1}(z)+\frac{4}{5}H_{1,1,3}(z)+\frac{4}{5}H_{1,2,2}(z)-\frac{4}{5}H_{1,3,0}(z)+\frac{4}{5}H_{1,3,1}(z)\\
&+\frac{4}{5}H_{2,1,2}(z)-\frac{4}{5}H_{2,2,0}(z)+\frac{4}{5}H_{2,2,1}(z)+\frac{4}{5}H_{3,0,0}(z)-\frac{4}{5}H_{3,1,0}(z)\\
&+\frac{4}{5}H_{3,1,1}(z)-\frac{4}{5}H_{1,1,1,2}(z)+\frac{4}{5}H_{1,1,2,0}(z)-\frac{4}{5}H_{1,1,2,1}(z)-\frac{4}{5}H_{1,2,0,0}(z)\\
&+\frac{4}{5}H_{1,2,1,0}(z)-\frac{4}{5}H_{1,2,1,1}(z)+\frac{4}{5}H_{2,0,0,0}(z)-\frac{4}{5}H_{2,1,0,0}(z)+\frac{4}{5}H_{2,1,1,0}(z)\\
&-\frac{4}{5}H_{2,1,1,1}(z)-\frac{4}{5}H_{0,0,0,0,0}(z)+\frac{4}{5}H_{1,0,0,0,0}(z)-\frac{4}{5}H_{1,1,0,0,0}(z)\\
&+\frac{4}{5}H_{1,1,1,0,0}(z)-\frac{4}{5}H_{1,1,1,1,0}(z)+\frac{4}{5}H_{1,1,1,1,1}(z)-\frac{3}{2}\zeta(5)+\frac{1}{5}\pi^2\zeta(3)\Big)+\mathcal{O}(\epsilon^6)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4\mathrm{B}24}=n_3(\epsilon)\Bigg[&-\frac{11}{120}+\epsilon\Big(\frac{11}{120}H_0(z)-\frac{11}{60}H_1(z)\Big)\\
&+\epsilon^2\Big(\frac{11H_2(z)}{60}+\frac{11}{40}H_{0,0}(z)+\frac{11}{12}H_{1,0}(z)-\frac{11}{30}H_{1,1}(z)+\frac{77}{720}\pi^2\Big)\\
&+\epsilon^3\Big(-\frac{77}{360}\pi^2H_0(z)+\frac{11}{240}\pi^2H_1(z)+\frac{11}{20}H_3(z)+\frac{11}{6}H_{1,2}(z)\\
&-\frac{11}{24}H_{2,0}(z)+\frac{11}{30}H_{2,1}(z)-\frac{341}{120}H_{0,0,0}(z)-\frac{55}{12}H_{1,0,0}(z)+\frac{55}{24}H_{1,1,0}(z)\\
&-\frac{11}{15}H_{1,1,1}(z)+\frac{187}{120}\zeta(3)\Big)+\mathcal{O}(\epsilon^4)\Bigg],
\end{aligned}
$$

$$
G_{4\mathrm{B}25}=n_3(\epsilon)\Bigg[-\epsilon\frac{11}{48}H_0(z)+\epsilon^2\Big(-\frac{11}{24}H_2(z)+\frac{77}{48}H_{0,0}(z)+\frac{11}{60}H_{1,0}(z)+\frac{11}{144}\pi^2\Big)
$$

$$
\begin{aligned}
&+\epsilon^3\Big(-\frac{11}{72}\pi^2H_0(z)-\frac{11}{360}\pi^2H_1(z)+\frac{77}{24}H_3(z)+\frac{11}{30}H_{1,2}(z)+\frac{11}{12}H_{2,0}(z)\\
&-\frac{11}{12}H_{2,1}(z)-\frac{143}{16}H_{0,0,0}(z)-\frac{77}{60}H_{1,0,0}(z)+\frac{11}{20}H_{1,1,0}(z)-\frac{11}{24}\zeta(3)\Big)\\
&+\epsilon^4\Big(\frac{143}{30}S_{2,2}(z)+\frac{33}{4}\zeta(3)H_0(z)+\frac{33}{20}\zeta(3)H_1(z)+\frac{55}{144}\pi^2H_2(z)\\
&-\frac{143}{8}H_4(z)+\frac{11}{36}\pi^2H_{0,0}(z)+\frac{11}{180}\pi^2H_{1,0}(z)-\frac{11}{180}\pi^2H_{1,1}(z)-\frac{77}{30}H_{1,3}(z)\\
&+\frac{11}{6}H_{2,2}(z)-\frac{55}{8}H_{3,0}(z)+\frac{33}{20}H_{3,1}(z)+\frac{11}{10}H_{1,1,2}(z)-\frac{11}{12}H_{1,2,0}(z)\\
&+\frac{11}{15}H_{1,2,1}(z)-\frac{11}{6}H_{2,0,0}(z)+\frac{11}{8}H_{2,1,0}(z)-\frac{11}{6}H_{2,1,1}(z)+\frac{2233}{48}H_{0,0,0,0}(z)\\
&+\frac{143}{20}H_{1,0,0,0}(z)-\frac{77}{20}H_{1,1,0,0}(z)+\frac{22}{15}H_{1,1,1,0}(z)-\frac{77}{2880}\pi^4\Big)+\mathcal{O}(\epsilon^5)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4\mathrm{B}26}=n_3(\epsilon)\Bigg[&\epsilon\frac{11}{60}H_0(z)+\epsilon^2\Big(\frac{11}{30}H_2(z)-\frac{77}{60}H_{0,0}(z)-\frac{11}{30}H_{1,0}(z)-\frac{11}{180}\pi^2\Big)\\
&+\epsilon^3\Big(\frac{11}{90}\pi^2H_0(z)+\frac{11}{180}\pi^2H_1(z)-\frac{77H_3(z)}{30}-\frac{11}{15}H_{1,2}(z)-\frac{11}{15}H_{2,0}(z)\\
&+\frac{11}{15}H_{2,1}(z)+\frac{143}{20}H_{0,0,0}(z)+\frac{77}{30}H_{1,0,0}(z)-\frac{11}{10}H_{1,1,0}(z)+\frac{11}{30}\zeta(3)\Big)\\
&+\epsilon^4\Big(-\frac{187}{30}S_{2,2}(z)-\frac{33}{5}\zeta(3)H_0(z)-\frac{33}{10}\zeta(3)H_1(z)-\frac{11}{36}\pi^2H_2(z)+\frac{143}{10}H_4(z)\\
&-\frac{11}{45}\pi^2H_{0,0}(z)-\frac{11}{90}\pi^2H_{1,0}(z)+\frac{11}{90}\pi^2H_{1,1}(z)+\frac{77}{15}H_{1,3}(z)-\frac{22}{15}H_{2,2}(z)\\
&+\frac{11}{2}H_{3,0}(z)+\frac{11}{10}H_{3,1}(z)-\frac{11}{5}H_{1,1,2}(z)+\frac{11}{6}H_{1,2,0}(z)-\frac{22}{15}H_{1,2,1}(z)\\
&+\frac{22}{15}H_{2,0,0}(z)-\frac{11}{10}H_{2,1,0}(z)+\frac{22}{15}H_{2,1,1}(z)-\frac{2233}{60}H_{0,0,0,0}(z)-\frac{143}{10}H_{1,0,0,0}(z)\\
&+\frac{77}{10}H_{1,1,0,0}(z)-\frac{44}{15}H_{1,1,1,0}(z)+\frac{77}{3600}\pi^4\Big)+\mathcal{O}(\epsilon^5)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4\mathrm{B}27}=n_3(\epsilon)\Bigg[&\frac{1}{350}+\epsilon\Big(\frac{1}{1050}H_1(z)-\frac{1}{175}H_0(z)\Big)\\
&+\epsilon^2\Big(-\frac{37}{1050}H_2(z)+\frac{2}{175}H_{0,0}(z)-\frac{1}{525}H_{1,0}(z)-\frac{2}{75}H_{1,1}(z)-\frac{\pi^2}{525}\Big)\\
&+\epsilon^3\Big(\frac{2}{525}\pi^2H_0(z)-\frac{1}{1575}\pi^2H_1(z)+\frac{19}{1050}H_3(z)-\frac{37}{525}H_{1,2}(z)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{37}{525}H_{2,0}(z)-\frac{16}{75}H_{2,1}(z)-\frac{4}{175}H_{0,0,0}(z)+\frac{2}{525}H_{1,0,0}(z)+\frac{4}{75}H_{1,1,0}(z)\\
&-\frac{118}{525}H_{1,1,1}(z)-\frac{6}{175}\zeta(3)\Big)\\
&+\epsilon^4\Big(\frac{34}{105}S_{2,2}(z)+\frac{12}{175}\zeta(3)H_0(z)-\frac{2}{175}\zeta(3)H_1(z)+\frac{37}{1575}\pi^2H_2(z)\\
&-\frac{73}{1050}H_4(z)-\frac{4}{525}\pi^2H_{0,0}(z)+\frac{2}{1575}\pi^2H_{1,0}(z)+\frac{4}{225}\pi^2H_{1,1}(z)\\
&-\frac{11}{525}H_{1,3}(z)+\frac{19}{525}H_{2,2}(z)-\frac{19}{525}H_{3,0}(z)-\frac{166}{525}H_{3,1}(z)-\frac{74}{525}H_{1,1,2}(z)\\
&+\frac{74}{525}H_{1,2,0}(z)-\frac{284}{525}H_{1,2,1}(z)-\frac{74}{525}H_{2,0,0}(z)+\frac{32}{75}H_{2,1,0}(z)\\
&-\frac{674}{525}H_{2,1,1}(z)+\frac{8}{175}H_{0,0,0,0}(z)-\frac{4}{525}H_{1,0,0,0}(z)-\frac{8}{75}H_{1,1,0,0}(z)\\
&+\frac{236}{525}H_{1,1,1,0}(z)-\frac{776}{525}H_{1,1,1,1}(z)-\frac{\pi^4}{5250}\Big)\\
&+\epsilon^5\Big(-\frac{40}{21}S_{2,3}(z)-\frac{10}{21}S_{3,2}(z)-\frac{68}{105}S_{2,2}(z)H_0(z)+\frac{1}{2625}\pi^4H_0(z)\\
&+\frac{4}{3}S_{2,2}(z)H_1(z)-\frac{1}{15750}\pi^4H_1(z)+\frac{74}{175}\zeta(3)H_2(z)-\frac{19}{1575}\pi^2H_3(z)\\
&+\frac{13}{150}H_5(z)-\frac{24}{175}\zeta(3)H_{0,0}(z)+\frac{4}{175}\zeta(3)H_{1,0}(z)+\frac{8}{25}\zeta(3)H_{1,1}(z)\\
&+\frac{74}{1575}\pi^2H_{1,2}(z)-\frac{43}{525}H_{1,4}(z)-\frac{74}{1575}\pi^2H_{2,0}(z)+\frac{32}{225}\pi^2H_{2,1}(z)\\
&-\frac{223}{525}H_{2,3}(z)+\frac{89}{175}H_{3,2}(z)+\frac{73}{525}H_{4,0}(z)+\frac{374}{175}H_{4,1}(z)+\frac{8}{525}\pi^2H_{0,0,0}(z)\\
&-\frac{4}{1575}\pi^2H_{1,0,0}(z)-\frac{8}{225}\pi^2H_{1,1,0}(z)+\frac{236}{1575}\pi^2H_{1,1,1}(z)-\frac{202}{525}H_{1,1,3}(z)\\
&-\frac{22}{525}H_{1,2,2}(z)+\frac{22}{525}H_{1,3,0}(z)-\frac{116}{75}H_{1,3,1}(z)+\frac{38}{525}H_{2,1,2}(z)-\frac{38}{525}H_{2,2,0}(z)\\
&-\frac{992}{525}H_{2,2,1}(z)+\frac{38}{525}H_{3,0,0}(z)+\frac{332}{525}H_{3,1,0}(z)-\frac{394}{175}H_{3,1,1}(z)-\frac{148}{525}H_{1,1,1,2}(z)\\
&+\frac{148}{525}H_{1,1,2,0}(z)-\frac{928}{525}H_{1,1,2,1}(z)-\frac{148}{525}H_{1,2,0,0}(z)+\frac{568}{525}H_{1,2,1,0}(z)\\
&-\frac{1828}{525}H_{1,2,1,1}(z)+\frac{148}{525}H_{2,0,0,0}(z)-\frac{64}{75}H_{2,1,0,0}(z)+\frac{1348}{525}H_{2,1,1,0}(z)\\
&-\frac{4048}{525}H_{2,1,1,1}(z)-\frac{16}{175}H_{0,0,0,0,0}(z)+\frac{8}{525}H_{1,0,0,0,0}(z)+\frac{16}{75}H_{1,1,0,0,0}(z)\\
&-\frac{472}{525}H_{1,1,1,0,0}(z)+\frac{1552}{525}H_{1,1,1,1,0}(z)-\frac{4792}{525}H_{1,1,1,1,1}(z)-\frac{6}{35}\zeta(5)+\frac{4}{175}\pi^2\zeta(3)\Big)
\end{aligned}
$$

$$
+\mathcal{O}(\epsilon^6)\Bigg],
$$

$$
\begin{aligned}
G_{4B28} = n_3(\epsilon)\Bigg[&\frac{1}{60}+\epsilon\Big(\frac{1}{20}H_1(z)-\frac{1}{30}H_0(z)\Big)+\\
&+\epsilon^2\Big(-\frac{1}{12}H_2(z)+\frac{1}{15}H_{0,0}(z)-\frac{1}{10}H_{1,0}(z)+\frac{1}{6}H_{1,1}(z)-\frac{\pi^2}{90}\Big)\\
&+\epsilon^3\Big(\frac{1}{45}\pi^2H_0(z)-\frac{1}{30}\pi^2H_1(z)+\frac{3}{20}H_3(z)-\frac{4}{15}H_{1,2}(z)+\frac{1}{6}H_{2,0}(z)\\
&-\frac{7}{30}H_{2,1}(z)-\frac{2}{15}H_{0,0,0}(z)+\frac{1}{5}H_{1,0,0}(z)-\frac{1}{3}H_{1,1,0}(z)+\frac{3}{5}H_{1,1,1}(z)-\frac{1}{5}\zeta(3)\Big)\\
&+\epsilon^4\Big(-\frac{1}{30}S_{2,2}(z)+\frac{2}{5}\zeta(3)H_0(z)-\frac{3}{5}\zeta(3)H_1(z)+\frac{1}{18}\pi^2H_2(z)-\frac{17}{60}H_4(z)\\
&-\frac{2}{45}\pi^2H_{0,0}(z)+\frac{1}{15}\pi^2H_{1,0}(z)-\frac{1}{9}\pi^2H_{1,1}(z)+\frac{7}{15}H_{1,3}(z)+\frac{2}{5}H_{2,2}(z)\\
&-\frac{3}{10}H_{3,0}(z)+\frac{2}{5}H_{3,1}(z)-\frac{14}{15}H_{1,1,2}(z)+\frac{8}{15}H_{1,2,0}(z)-\frac{4}{5}H_{1,2,1}(z)-\frac{1}{3}H_{2,0,0}(z)\\
&+\frac{7}{15}H_{2,1,0}(z)-\frac{11}{15}H_{2,1,1}(z)+\frac{4}{15}H_{0,0,0,0}(z)-\frac{2}{5}H_{1,0,0,0}(z)+\frac{2}{3}H_{1,1,0,0}(z)\\
&-\frac{6}{5}H_{1,1,1,0}(z)+\frac{34}{15}H_{1,1,1,1}(z)-\frac{\pi^4}{900}\Big)+\mathcal{O}(\epsilon^5)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4B29} = n_3(\epsilon)\Bigg[&\epsilon^2\frac{3}{10}H_{1,0}(z)+\epsilon^3\Big(\frac{3}{40}\pi^2H_1(z)+\frac{3}{5}H_{1,2}(z)-\frac{21}{10}H_{1,0,0}(z)+\frac{33}{20}H_{1,1,0}(z)\Big)\\
&+\epsilon^4\Big(\frac{141}{20}S_{2,2}(z)+\frac{159}{20}\zeta(3)H_1(z)-\frac{3}{20}\pi^2H_{1,0}(z)+\frac{3}{20}\pi^2H_{1,1}(z)\\
&-\frac{21}{5}H_{1,3}(z)-\frac{141}{20}H_{3,1}(z)+\frac{33}{10}H_{1,1,2}(z)-\frac{9}{4}H_{1,2,0}(z)+\frac{6}{5}H_{1,2,1}(z)\\
&+\frac{117}{10}H_{1,0,0,0}(z)-\frac{231}{20}H_{1,1,0,0}(z)+\frac{27}{5}H_{1,1,1,0}(z)\Big)+\mathcal{O}(\epsilon^5)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{4B30} = n_3(\epsilon)\Bigg[&\frac{1}{3150}+\epsilon\Big(-\frac{1}{1575}H_0(z)-\frac{1}{1050}H_1(z)\Big)\\
&+\epsilon^2\Big(-\frac{4}{525}H_2(z)+\frac{2}{1575}H_{0,0}(z)+\frac{1}{525}H_{1,0}(z)-\frac{2}{175}H_{1,1}(z)-\frac{\pi^2}{4725}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\epsilon^3\Big(\frac{2}{4725}\pi^2H_0(z)+\frac{1}{1575}\pi^2H_1(z)+\frac{1}{1050}H_3(z)-\frac{8}{525}H_{1,2}(z)\\
&+\frac{8}{525}H_{2,0}(z)-\frac{4}{75}H_{2,1}(z)-\frac{4}{1575}H_{0,0,0}(z)-\frac{2}{525}H_{1,0,0}(z)\\
&+\frac{4}{175}H_{1,1,0}(z)-\frac{2}{25}H_{1,1,1}(z)-\frac{2}{525}\zeta(3)\Big)\\
&+\epsilon^4\Big(\frac{2}{21}S_{2,2}(z)+\frac{4}{525}\zeta(3)H_0(z)+\frac{2}{175}\zeta(3)H_1(z)+\frac{8}{1575}\pi^2H_2(z)\\
&-\frac{2}{175}H_4(z)-\frac{4}{4725}\pi^2H_{0,0}(z)-\frac{2}{1575}\pi^2H_{1,0}(z)+\frac{4}{525}\pi^2H_{1,1}(z)\\
&-\frac{3}{175}H_{1,3}(z)+\frac{1}{525}H_{2,2}(z)-\frac{1}{525}H_{3,0}(z)-\frac{18}{175}H_{3,1}(z)-\frac{16}{525}H_{1,1,2}(z)\\
&+\frac{16}{525}H_{1,2,0}(z)-\frac{76}{525}H_{1,2,1}(z)-\frac{16}{525}H_{2,0,0}(z)+\frac{8}{75}H_{2,1,0}(z)\\
&-\frac{176}{525}H_{2,1,1}(z)+\frac{8}{1575}H_{0,0,0,0}(z)+\frac{4}{525}H_{1,0,0,0}(z)-\frac{8}{175}H_{1,1,0,0}(z)\\
&+\frac{4}{25}H_{1,1,1,0}(z)-\frac{88}{175}H_{1,1,1,1}(z)-\frac{\pi^4}{47250}\Big)\\
&+\epsilon^5\Big(-\frac{8}{15}S_{2,3}(z)-\frac{8}{105}S_{3,2}(z)-\frac{4}{21}S_{2,2}(z)H_0(z)+\frac{1}{23625}\pi^4H_0(z)\\
&+\frac{44}{105}S_{2,2}(z)H_1(z)+\frac{1}{15750}\pi^4H_1(z)+\frac{16}{175}\zeta(3)H_2(z)-\frac{1}{1575}\pi^2H_3(z)\\
&+\frac{3}{350}H_5(z)-\frac{8}{525}\zeta(3)H_{0,0}(z)-\frac{4}{175}\zeta(3)H_{1,0}(z)+\frac{24}{175}\zeta(3)H_{1,1}(z)\\
&+\frac{16}{1575}\pi^2H_{1,2}(z)-\frac{2}{525}H_{1,4}(z)-\frac{16}{1575}\pi^2H_{2,0}(z)+\frac{8}{225}\pi^2H_{2,1}(z)\\
&-\frac{52}{525}H_{2,3}(z)+\frac{88}{525}H_{3,2}(z)+\frac{4}{175}H_{4,0}(z)+\frac{44}{75}H_{4,1}(z)+\frac{8}{4725}\pi^2H_{0,0,0}(z)\\
&+\frac{4}{1575}\pi^2H_{1,0,0}(z)-\frac{8}{525}\pi^2H_{1,1,0}(z)+\frac{4}{75}\pi^2H_{1,1,1}(z)-\frac{26}{175}H_{1,1,3}(z)\\
&-\frac{6}{175}H_{1,2,2}(z)+\frac{6}{175}H_{1,3,0}(z)-\frac{268}{525}H_{1,3,1}(z)+\frac{2}{525}H_{2,1,2}(z)-\frac{2}{525}H_{2,2,0}(z)\\
&-\frac{44}{75}H_{2,2,1}(z)+\frac{2}{525}H_{3,0,0}(z)+\frac{36}{175}H_{3,1,0}(z)-\frac{418}{525}H_{3,1,1}(z)-\frac{32}{525}H_{1,1,1,2}(z)\\
&+\frac{32}{525}H_{1,1,2,0}(z)-\frac{272}{525}H_{1,1,2,1}(z)-\frac{32}{525}H_{1,2,0,0}(z)+\frac{152}{525}H_{1,2,1,0}(z)\\
&-\frac{512}{525}H_{1,2,1,1}(z)+\frac{32}{525}H_{2,0,0,0}(z)-\frac{16}{75}H_{2,1,0,0}(z)+\frac{352}{525}H_{2,1,1,0}(z)\\
&-\frac{1072}{525}H_{2,1,1,1}(z)-\frac{16}{1575}H_{0,0,0,0,0}(z)-\frac{8}{525}H_{1,0,0,0,0}(z)+\frac{16}{175}H_{1,1,0,0,0}(z)\\
&-\frac{8}{25}H_{1,1,1,0,0}(z)+\frac{176}{175}H_{1,1,1,1,0}(z)-\frac{536}{175}H_{1,1,1,1,1}(z)-\frac{2}{105}\zeta(5)+\frac{4}{1575}\pi^2\zeta(3)\Big)
\end{aligned}
$$

$$+\mathcal{O}(\epsilon^6)\Bigg],$$

$$\begin{aligned}
G_{4\mathrm{B}31} = n_3(\epsilon)\Bigg[&\epsilon\frac{1}{20}H_1(z)+\epsilon^2\Big(-\frac{1}{5}H_2(z)-\frac{1}{10}H_{1,0}(z)+\frac{2}{5}H_{1,1}(z)\Big)\\
&+\epsilon^3\Big(-\frac{1}{30}\pi^2H_1(z)+\frac{4}{5}H_3(z)-\frac{2}{5}H_{1,2}(z)+\frac{2}{5}H_{2,0}(z)-\frac{8}{5}H_{2,1}(z)\\
&+\frac{1}{5}H_{1,0,0}(z)-\frac{4}{5}H_{1,1,0}(z)+\frac{13}{5}H_{1,1,1}(z)\Big)\\
&+\epsilon^4\Big(\frac{16}{5}S_{2,2}(z)-\frac{3}{5}\zeta(3)H_1(z)+\frac{2}{15}\pi^2H_2(z)-\frac{1}{5}H_4(z)+\frac{1}{15}\pi^2H_{1,0}(z)\\
&-\frac{4}{15}\pi^2H_{1,1}(z)+\frac{11}{5}H_{1,3}(z)+\frac{8}{5}H_{2,2}(z)-\frac{8}{5}H_{3,0}(z)+\frac{16}{5}H_{3,1}(z)\\
&-\frac{4}{5}H_{1,1,2}(z)+\frac{4}{5}H_{1,2,0}(z)-2H_{1,2,1}(z)-\frac{4}{5}H_{2,0,0}(z)+\frac{16}{5}H_{2,1,0}(z)\\
&-\frac{52}{5}H_{2,1,1}(z)-\frac{2}{5}H_{1,0,0,0}(z)+\frac{8}{5}H_{1,1,0,0}(z)-\frac{26}{5}H_{1,1,1,0}(z)+16H_{1,1,1,1}(z)\Big)\\
&+\mathcal{O}(\epsilon^5)\Bigg],
\end{aligned}$$

$$\begin{aligned}
G_{4\mathrm{B}32} = n_3(\epsilon)\Bigg[&\epsilon\Big(-\frac{1}{5}H_0(z)-\frac{1}{5}H_1(z)\Big)\\
&+\epsilon^2\Big(-\frac{8}{5}H_2(z)+\frac{7}{5}H_{0,0}(z)+\frac{2}{5}H_{1,0}(z)-\frac{8}{5}H_{1,1}(z)+\frac{7}{60}\pi^2\Big)\\
&+\epsilon^3\Big(-\frac{7}{30}\pi^2H_0(z)+\frac{7}{20}\pi^2H_1(z)+\frac{13}{5}H_3(z)-\frac{12}{5}H_{1,2}(z)+\frac{29}{10}H_{2,0}(z)\\
&-\frac{52}{5}H_{2,1}(z)-\frac{39}{5}H_{0,0,0}(z)-\frac{4}{5}H_{1,0,0}(z)+\frac{39}{10}H_{1,1,0}(z)-\frac{52}{5}H_{1,1,1}(z)+\frac{7}{10}\zeta(3)\Big)\\
&+\epsilon^4\Big(\frac{211}{10}S_{2,2}(z)+5\zeta(3)H_0(z)+\frac{19}{2}\zeta(3)H_1(z)+\frac{11}{12}\pi^2H_2(z)-\frac{89}{5}H_4(z)\\
&+\frac{7}{15}\pi^2H_{0,0}(z)-\frac{7}{10}\pi^2H_{1,0}(z)+\frac{3}{2}\pi^2H_{1,1}(z)-\frac{14}{5}H_{1,3}(z)+\frac{3}{5}H_{2,2}(z)\\
&-5H_{3,0}(z)-\frac{171}{10}H_{3,1}(z)-\frac{17}{5}H_{1,1,2}(z)+\frac{7}{2}H_{1,2,0}(z)-24H_{1,2,1}(z)\\
&-\frac{43}{10}H_{2,0,0}(z)+\frac{191}{10}H_{2,1,0}(z)-64H_{2,1,1}(z)+\frac{203}{5}H_{0,0,0,0}(z)+\frac{8}{5}H_{1,0,0,0}(z)
\end{aligned}$$

$$- \frac{113}{10}H_{1,1,0,0}(z) + \frac{118}{5}H_{1,1,1,0}(z) - 64H_{1,1,1,1}(z) - \frac{31}{3600}\pi^4\Big) + \mathcal{O}(\epsilon^5)\Bigg],$$

$$G_{4B33} = n_3(\epsilon)\Bigg[\epsilon^2\Big(- H_2(z) - H_{1,0}(z)\Big) + \epsilon^3\Big(\frac{1}{3}\pi^2 H_1(z) - H_3(z)$$
$$- 4H_{1,2}(z) + 2H_{2,0}(z) - 8H_{2,1}(z) + 7H_{1,0,0}(z) - 2H_{1,1,0}(z)\Big)$$
$$+ \epsilon^4\Big(- 2S_{2,2}(z) - 2\zeta(3)H_1(z) + \frac{2}{3}\pi^2 H_2(z) - H_4(z) - \frac{2}{3}\pi^2 H_{1,0}(z)$$
$$+ \frac{2}{3}\pi^2 H_{1,1}(z) + 12H_{1,3}(z) - 2H_{2,2}(z) + 2H_{3,0}(z) - 6H_{3,1}(z) - 8H_{1,1,2}(z)$$
$$+ 8H_{1,2,0}(z) - 20H_{1,2,1}(z) - 4H_{2,0,0}(z) + 16H_{2,1,0}(z) - 52H_{2,1,1}(z)$$
$$- 39H_{1,0,0,0}(z) + 14H_{1,1,0,0}(z) - 4H_{1,1,1,0}(z)\Big) + \mathcal{O}(\epsilon^5)\Bigg],$$

$$G_{4B34} = n_3(\epsilon)\Bigg[\epsilon^2\Big(\frac{1}{20}H_2(z) - \frac{\pi^2}{120}\Big) + \epsilon^3\Big(\frac{1}{60}\pi^2 H_0(z) - \frac{1}{60}\pi^2 H_1(z) - \frac{3}{20}H_3(z)$$
$$+ \frac{1}{10}H_{1,2}(z) - \frac{1}{10}H_{2,0}(z) + \frac{3}{10}H_{2,1}(z) - \frac{3}{20}\zeta(3)\Big)$$
$$+ \epsilon^4\Big(- \frac{1}{10}S_{2,2}(z) + \frac{3}{10}\zeta(3)H_0(z) - \frac{3}{10}\zeta(3)H_1(z) + \frac{7}{20}H_4(z)$$
$$- \frac{1}{30}\pi^2 H_{0,0}(z) + \frac{1}{30}\pi^2 H_{1,0}(z) - \frac{1}{30}\pi^2 H_{1,1}(z) - \frac{3}{10}H_{1,3}(z) - \frac{3}{5}H_{2,2}(z)$$
$$+ \frac{3}{10}H_{3,0}(z) - \frac{4}{5}H_{3,1}(z) + \frac{1}{5}H_{1,1,2}(z) - \frac{1}{5}H_{1,2,0}(z) + \frac{3}{5}H_{1,2,1}(z)$$
$$+ \frac{1}{5}H_{2,0,0}(z) - \frac{3}{5}H_{2,1,0}(z) + \frac{7}{5}H_{2,1,1}(z) + \frac{\pi^4}{600}\Big) + \mathcal{O}(\epsilon^5)\Bigg],$$

$$G_{4B35} = n_3(\epsilon)\Bigg[\epsilon^3\Big(3H_3(z) + 3H_{1,2}(z)\Big) + \epsilon^4\Big(6S_{2,2}(z) + 15H_{1,3}(z) + 3H_{2,2}(z)$$
$$- 6H_{3,0}(z) + 18H_{3,1}(z) + 18H_{1,1,2}(z) - 6H_{1,2,0}(z) + 24H_{1,2,1}(z)\Big)$$
$$+ \epsilon^5\Big(- 78S_{2,3}(z) - 114S_{3,2}(z) - 12S_{2,2}(z)H_0(z) + 12S_{2,2}(z)H_1(z)$$
$$- 2\pi^2 H_3(z) + 3H_5(z) - 2\pi^2 H_{1,2}(z) + 3H_{1,4}(z) - 9H_{2,3}(z) + 15H_{3,2}(z)$$

$$+ 150H_{4,1}(z) + 66H_{1,1,3}(z) + 18H_{1,2,2}(z) - 30H_{1,3,0}(z) + 108H_{1,3,1}(z)$$
$$- 6H_{2,1,2}(z) - 6H_{2,2,0}(z) + 12H_{2,2,1}(z) + 12H_{3,0,0}(z) - 36H_{3,1,0}(z)$$
$$+ 198H_{3,1,1}(z) + 84H_{1,1,1,2}(z) - 36H_{1,1,2,0}(z) + 144H_{1,1,2,1}(z)$$
$$+ 12H_{1,2,0,0}(z) - 48H_{1,2,1,0}(z) + 156H_{1,2,1,1}(z)\Big) + \mathcal{O}(\epsilon^6)\Bigg],$$

$$G_{4B36} = n_3(\epsilon)\Bigg[\epsilon^2\frac{3}{2}H_{1,0}(z) + \epsilon^3\Big(-\frac{1}{2}\pi^2H_1(z) + 3H_{1,2}(z) - \frac{21}{2}H_{1,0,0}(z) + 6H_{1,1,0}(z)\Big)$$
$$+ \epsilon^4\Big(24S_{2,2}(z) + 9\zeta(3)H_1(z) + \pi^2H_{1,0}(z) - 2\pi^2H_{1,1}(z) - 33H_{1,3}(z)$$
$$- 24H_{3,1}(z) + 6H_{1,1,2}(z) - 9H_{1,2,0}(z) + \frac{117}{2}H_{1,0,0,0}(z) - 42H_{1,1,0,0}(z)$$
$$+ 18H_{1,1,1,0}(z)\Big) + \mathcal{O}(\epsilon^5)\Bigg],$$

$$G_{4B37} = n_3(\epsilon)\Bigg[-\frac{1}{8} + \epsilon\Big(\frac{5}{8}H_0(z) - \frac{1}{4}H_1(z)\Big)$$
$$+ \epsilon^2\Big(\frac{9}{4}H_2(z) - \frac{25}{8}H_{0,0}(z) + \frac{3}{4}H_{1,0}(z) + \frac{1}{2}H_{1,1}(z)\Big)$$
$$+ \epsilon^3\Big(\frac{1}{6}\pi^2H_1(z) - \frac{37}{4}H_3(z) + \frac{11}{2}H_{1,2}(z) - \frac{27}{4}H_{2,0}(z) + \frac{15}{2}H_{2,1}(z)$$
$$+ \frac{125}{8}H_{0,0,0}(z) - \frac{11}{4}H_{1,0,0}(z) - \frac{3}{2}H_{1,1,0}(z) + 11H_{1,1,1}(z) + 4\zeta(3)\Big)$$
$$+ \epsilon^4\Big(-19S_{2,2}(z) - 20\zeta(3)H_0(z) + 5\zeta(3)H_1(z) - \frac{7}{6}\pi^2H_2(z)$$
$$+ \frac{161}{4}H_4(z) - \frac{1}{3}\pi^2H_{1,0}(z) - \frac{19}{2}H_{1,3}(z) - \frac{43}{2}H_{2,2}(z) + \frac{103}{4}H_{3,0}(z)$$
$$- \frac{17}{2}H_{3,1}(z) + 19H_{1,1,2}(z) - \frac{29}{2}H_{1,2,0}(z) + 29H_{1,2,1}(z) + \frac{99}{4}H_{2,0,0}(z)$$
$$- \frac{33}{2}H_{2,1,0}(z) + 21H_{2,1,1}(z) - \frac{625}{8}H_{0,0,0,0}(z) + \frac{47}{4}H_{1,0,0,0}(z)$$
$$+ \frac{11}{2}H_{1,1,0,0}(z) - 25H_{1,1,1,0}(z) + 98H_{1,1,1,1}(z) + \frac{13}{120}\pi^4\Big) + \mathcal{O}(\epsilon^5)\Bigg],$$

$$
\begin{aligned}
G_{4\mathrm{B}38} = n_2(\epsilon)\Bigg[& \frac{1}{225} + \epsilon\Big(-\frac{2}{225}H_0(z) + \frac{37H_1(z)}{2700} + \frac{i\pi}{180}\Big) \\
& + \epsilon^2\Big(-\frac{1}{90}i\pi H_0(z) + \frac{3}{100}i\pi H_1(z) - \frac{4}{75}H_2(z) + \frac{4}{225}H_{0,0}(z) - \frac{37}{1350}H_{1,0}(z) \\
& + \frac{143}{2700}H_{1,1}(z) - \frac{23}{2700}\pi^2\Big) + \epsilon^3\Big(\frac{23}{1350}\pi^2 H_0(z) - \frac{317}{8100}\pi^2 H_1(z) \\
& + \frac{7}{90}i\pi H_2(z) + \frac{19}{270}H_3(z) + \frac{1}{45}i\pi H_{0,0}(z) - \frac{3}{50}i\pi H_{1,0}(z) + \frac{43}{300}i\pi H_{1,1}(z) \\
& - \frac{7}{50}H_{1,2}(z) + \frac{8}{75}H_{2,0}(z) - \frac{79}{450}H_{2,1}(z) - \frac{8}{225}H_{0,0,0}(z) + \frac{37}{675}H_{1,0,0}(z) \\
& - \frac{143}{1350}H_{1,1,0}(z) + \frac{601}{2700}H_{1,1,1}(z) - \frac{52}{225}\zeta(3) - \frac{7}{270}i\pi^3\Big) \\
& + \epsilon^4\Big(-\frac{4}{675}S_{2,2}(z) + \frac{104}{225}\zeta(3)H_0(z) + \frac{7}{135}i\pi^3 H_0(z) - \frac{118}{225}\zeta(3)H_1(z) \\
& - \frac{1}{25}i\pi^3 H_1(z) - \frac{19}{450}\pi^2 H_2(z) - \frac{7}{45}i\pi H_3(z) - \frac{13}{450}H_4(z) - \frac{23}{675}\pi^2 H_{0,0}(z) \\
& + \frac{317}{4050}\pi^2 H_{1,0}(z) - \frac{1447}{8100}\pi^2 H_{1,1}(z) - \frac{9}{50}i\pi H_{1,2}(z) + \frac{188}{675}H_{1,3}(z) - \frac{7}{45}i\pi H_{2,0}(z) \\
& + \frac{61}{90}i\pi H_{2,1}(z) + \frac{28}{135}H_{2,2}(z) - \frac{19}{135}H_{3,0}(z) + \frac{1}{15}H_{3,1}(z) - \frac{2}{45}i\pi H_{0,0,0}(z) \\
& + \frac{3}{25}i\pi H_{1,0,0}(z) - \frac{43}{150}i\pi H_{1,1,0}(z) + \frac{193}{300}i\pi H_{1,1,1}(z) - \frac{23}{50}H_{1,1,2}(z) \\
& + \frac{7}{25}H_{1,2,0}(z) - \frac{11}{18}H_{1,2,1}(z) - \frac{16}{75}H_{2,0,0}(z) + \frac{79}{225}H_{2,1,0}(z) \\
& - \frac{83}{150}H_{2,1,1}(z) + \frac{16}{225}H_{0,0,0,0}(z) - \frac{74}{675}H_{1,0,0,0}(z) + \frac{143}{675}H_{1,1,0,0}(z) \\
& - \frac{601}{1350}H_{1,1,1,0}(z) + \frac{2507}{2700}H_{1,1,1,1}(z) - \frac{17}{15}i\pi\zeta(3) + \frac{91}{13500}\pi^4\Big) + \mathcal{O}(\epsilon^5)\Bigg].
\end{aligned}
$$

Appendix B

Master Integrals in the Five-Body Case

Here we list the result for all the five-body master integrals that we encountered in the calculation:

$$
\begin{aligned}
F_{5\text{B}1} = n_0(\epsilon) z^2 \bar{z} \Bigg[& \frac{1}{12} + \epsilon\Big(-\frac{1}{4}H_0(z) + \frac{1}{6}H_1(z) + \frac{83}{72}\Big) + \epsilon^2\Big(\frac{83}{36}H_1(z) - \frac{83}{24}H_0(z) \\
& - \frac{1}{2}H_2(z) + \frac{3}{4}H_{0,0}(z) - \frac{1}{2}H_{1,0}(z) + \frac{1}{3}H_{1,1}(z) + \frac{1}{432}\left(4111 - 60\pi^2\right)\Big) \\
& + \epsilon^3\Big(\frac{1}{144}\left(-4111 + 60\pi^2\right) H_0(z) + \frac{1}{216}\left(4111 - 60\pi^2\right) H_1(z) - \frac{83}{12}H_2(z) \\
& + \frac{3H_3(z)}{2} + \frac{83}{8}H_{0,0}(z) - \frac{83}{12}H_{1,0}(z) + \frac{83}{18}H_{1,1}(z) - H_{1,2}(z) + \frac{3}{2}H_{2,0}(z) \\
& - H_{2,1}(z) - \frac{9}{4}H_{0,0,0}(z) + \frac{3}{2}H_{1,0,0}(z) - H_{1,1,0}(z) + \frac{2}{3}H_{1,1,1}(z) + \\
& - \frac{5\left(-31747 + 996\pi^2 + 1296\zeta(3)\right)}{2592}\Big) \\
& + \epsilon^4\Big(\frac{5}{864}\left(-31747 + 996\pi^2 + 1296\zeta(3)\right) H_0(z) + \left(-\frac{4111}{72} + \frac{5\pi^2}{6}\right) H_2(z) \\
& - \frac{5\left(-31747 + 996\pi^2 + 1296\zeta(3)\right) H_1(z)}{1296} + \frac{83H_3(z)}{4} - \frac{9H_4(z)}{2} \\
& + \left(\frac{4111}{48} - \frac{5\pi^2}{4}\right) H_{0,0}(z) + \left(-\frac{4111}{72} + \frac{5\pi^2}{6}\right) H_{1,0}(z) \\
& + \left(\frac{4111}{108} - \frac{5\pi^2}{9}\right) H_{1,1}(z) - \frac{83}{6}H_{1,2}(z) + 3H_{1,3}(z) + \frac{83}{4}H_{2,0}(z) - \frac{83}{6}H_{2,1}(z)
\end{aligned}
$$

$$+3H_{2,2}(z)-\frac{9}{2}H_{3,0}(z)+3H_{3,1}(z)-\frac{249}{8}H_{0,0,0}(z)+\frac{83}{4}H_{1,0,0}(z)-\frac{83}{6}H_{1,1,0}(z)$$
$$+\frac{83}{9}H_{1,1,1}(z)-2H_{1,1,2}(z)+3H_{1,2,0}(z)-2H_{1,2,1}(z)-\frac{9}{2}H_{2,0,0}(z)+3H_{2,1,0}(z)$$
$$-2H_{2,1,1}(z)+\frac{27}{4}H_{0,0,0,0}(z)-\frac{9}{2}H_{1,0,0,0}(z)+3H_{1,1,0,0}(z)-2H_{1,1,1,0}(z)$$
$$+\frac{4}{3}H_{1,1,1,1}(z)+\frac{26456435-1233300\pi^2+2808\pi^4-2689200\zeta(3)}{77760}\Big)+\mathcal{O}(\epsilon^5)\Bigg],$$

$$F_{5B2}=n_0(\epsilon)\bar{z}\Bigg[\frac{1}{2}H_1(z)-\frac{z}{2}+\epsilon\Big(\frac{3}{2}H_0(z)z-\frac{15z}{2}+(6-z)H_1(z)-\frac{3}{2}H_{1,0}(z)+\frac{5}{2}H_{1,1}(z)\Big)$$
$$+\epsilon^2\Big(\frac{45}{2}H_0(z)z+3H_2(z)z-\frac{9}{2}H_{0,0}(z)z+\frac{1}{6}\left(-399+5\pi^2\right)z+3(z-6)H_{1,0}(z)$$
$$+\left(-15z-\frac{5\pi^2}{6}+44\right)H_1(z)+(30-2z)H_{1,1}(z)-3H_{1,2}(z)+\frac{9}{2}H_{1,0,0}(z)$$
$$-\frac{15}{2}H_{1,1,0}(z)+\frac{19}{2}H_{1,1,1}(z)\Big)$$
$$+\epsilon^3\Big(\frac{1}{2}\left(z\left(-911+25\pi^2+30\zeta(3)\right)-9S_{2,2}(z)\right)+\frac{1}{2}\left(399-5\pi^2\right)zH_0(z)$$
$$+\left(\frac{5}{3}\pi^2(z-6)-133z-15\zeta(3)+256\right)H_1(z)+45zH_2(z)-9zH_3(z)$$
$$-\frac{135}{2}zH_{0,0}(z)+\left(45z+\frac{5\pi^2}{2}-132\right)H_{1,0}(z)+\left(-30z-\frac{25\pi^2}{6}+220\right)H_{1,1}(z)$$
$$+6(z-6)H_{1,2}(z)+9H_{1,3}(z)-9zH_{2,0}(z)+6zH_{2,1}(z)+\frac{9}{2}H_{3,1}(z)$$
$$+\frac{27}{2}zH_{0,0,0}(z)-9(z-6)H_{1,0,0}(z)+6(z-15)H_{1,1,0}(z)+(114-4z)H_{1,1,1}(z)$$
$$-15H_{1,1,2}(z)+9H_{1,2,0}(z)-6H_{1,2,1}(z)-\frac{27}{2}H_{1,0,0,0}(z)+\frac{45}{2}H_{1,1,0,0}(z)$$
$$-\frac{57}{2}H_{1,1,1,0}(z)+\frac{65}{2}H_{1,1,1,1}(z)\Big)+\mathcal{O}(\epsilon^4)\Bigg],$$

$$F_{5B3}=\frac{n_0(\epsilon)}{\epsilon}\Bigg[-\frac{1}{6}\pi^2z+zH_2(z)+(z-1)H_{1,1}(z)$$
$$+\epsilon\Big(\frac{1}{2}\pi^2H_0(z)z+11H_2(z)z-3H_3(z)z-3H_{2,0}(z)z+4H_{2,1}(z)z$$

$$
\begin{aligned}
&- 7\zeta(3)z - \frac{11\pi^2 z}{6} - \frac{1}{3}\pi^2(z-1)H_1(z) + 11(z-1)H_{1,1}(z) \\
&+ 2(z-1)H_{1,2}(z) + (3-3z)H_{1,1,0}(z) + 9(z-1)H_{1,1,1}(z)\Big) \\
&+ \epsilon^2\Big(\left(83 - \frac{4}{3}\pi^2\right) H_2(z)z - 33H_3(z)z + 9H_4(z)z - \frac{3}{2}\pi^2 H_{0,0}(z)z \\
&- 33H_{2,0}(z)z + 44H_{2,1}(z)z - 8H_{2,2}(z)z + 9H_{3,0}(z)z + 9H_{2,0,0}(z)z \\
&- 12H_{2,1,0}(z)z + 10H_{2,1,1}(z)z - \frac{1}{18}\left(249\pi^2 + \pi^4 + 1386\zeta(3)\right) z \\
&+ (21-17z)S_{2,2}(z) + \left(21\zeta(3)z + \frac{11}{2}\pi^2 z\right) H_0(z) \\
&- \frac{1}{3}(z-1)\left(11\pi^2 + 42\zeta(3)\right) H_1(z) + \pi^2(z-1)H_{1,0}(z) \\
&- \left(-83 + 3\pi^2\right)(z-1)H_{1,1}(z) + 22(z-1)H_{1,2}(z) + (6-6z)H_{1,3}(z) \\
&+ (5z-21)H_{3,1}(z) - 33(z-1)H_{1,1,0}(z) + 99(z-1)H_{1,1,1}(z) \\
&+ 2(z-1)H_{1,1,2}(z) + (6-6z)H_{1,2,0}(z) + 8(z-1)H_{1,2,1}(z) \\
&+ 9(z-1)H_{1,1,0,0}(z) - 27(z-1)H_{1,1,1,0}(z) + 55(z-1)H_{1,1,1,1}(z)\Big) + \mathcal{O}(\epsilon^3)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
F_{5\mathrm{B}4} = \frac{n_0(\epsilon)}{\epsilon^3}\Bigg[&\frac{1}{6} + \epsilon\Big(-\frac{1}{2}H_0(z) + \frac{2}{3}H_1(z) + \frac{1}{6}\Big) + \epsilon^2\Big(-\frac{1}{2}H_0(z) + \frac{2}{3}H_1(z) \\
&- \frac{H_2(z)}{3} + \frac{3}{2}H_{0,0}(z) - 2H_{1,0}(z) + \frac{8}{3}H_{1,1}(z) + \frac{1}{18}\left(9 - 7\pi^2\right)\Big) \\
&+ \epsilon^3\Big(\frac{1}{6}\left(-9 + 7\pi^2\right) H_0(z) + \left(2 - \frac{11\pi^2}{9}\right) H_1(z) - \frac{1}{3}H_2(z) + H_3(z) + \frac{3}{2}H_{0,0}(z) \\
&- 2H_{1,0}(z) + \frac{8}{3}H_{1,1}(z) - \frac{10}{3}H_{1,2}(z) + H_{2,0}(z) + \frac{2}{3}H_{2,1}(z) - \frac{9}{2}H_{0,0,0}(z) \\
&+ 6H_{1,0,0}(z) - 8H_{1,1,0}(z) + \frac{32}{3}H_{1,1,1}(z) + \frac{1}{18}\left(15 - 7\pi^2 - 174\zeta(3)\right)\Big) \\
&+ \epsilon^4\Big(-\frac{22}{3}S_{2,2}(z) + \left(-\frac{5}{2} + \frac{7\pi^2}{6} + 29\zeta(3)\right) H_0(z) + \left(-1 + \frac{7\pi^2}{9}\right) H_2(z) \\
&+ \frac{1}{9}\left(30 - 11\pi^2 - 222\zeta(3)\right) H_1(z) + H_3(z) - 3H_4(z) + \frac{1}{2}\left(9 - 7\pi^2\right) H_{0,0}(z) \\
&+ \left(-6 + \frac{11\pi^2}{3}\right) H_{1,0}(z) + \left(8 - \frac{44\pi^2}{9}\right) H_{1,1}(z) - \frac{10}{3}H_{1,2}(z) + 10H_{1,3}(z)
\end{aligned}
$$

$$
\begin{aligned}
&+ H_{2,0}(z) + \frac{2}{3}H_{2,1}(z) + \frac{2}{3}H_{2,2}(z) - 3H_{3,0}(z) + \frac{16}{3}H_{3,1}(z) - \frac{9}{2}H_{0,0,0}(z)\\
&+ 6H_{1,0,0}(z) - 8H_{1,1,0}(z) + \frac{32}{3}H_{1,1,1}(z) - \frac{40}{3}H_{1,1,2}(z) + 10H_{1,2,0}(z)\\
&- \frac{16}{3}H_{1,2,1}(z) - 3H_{2,0,0}(z) - 2H_{2,1,0}(z) + \frac{8}{3}H_{2,1,1}(z) + \frac{27}{2}H_{0,0,0,0}(z)\\
&- 18H_{1,0,0,0}(z) + 24H_{1,1,0,0}(z) - 32H_{1,1,1,0}(z) + \frac{128}{3}H_{1,1,1,1}(z)\\
&- \frac{29}{3}\zeta(3) + \frac{19}{540}\pi^4 - \frac{7}{6}\pi^2 + \frac{11}{6}\Big) + \mathcal{O}(\epsilon^5)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
F_{5B5} = \frac{n_0(\epsilon)}{\epsilon^4\bar{z}}\Bigg[&- \frac{10}{3} + \epsilon\Big(10H_0(z) - \frac{20}{3}H_1(z) + \frac{10}{3}\Big) + \epsilon^2\Big(- 10H_0(z) + \frac{20}{3}H_1(z)\\
&+ 20H_2(z) - 30H_{0,0}(z) + 20H_{1,0}(z) - \frac{40}{3}H_{1,1}(z) + \frac{10}{9}\left(-21 + 5\pi^2\right)\Big)\\
&+ \epsilon^3\Big(\left(70 - \frac{50\pi^2}{3}\right)H_0(z) + \frac{20}{9}\left(-21 + 5\pi^2\right)H_1(z) - 20H_2(z) - 60H_3(z)\\
&+ 30H_{0,0}(z) - 20H_{1,0}(z) + \frac{40}{3}H_{1,1}(z) + 40H_{1,2}(z) - 60H_{2,0}(z) + 40H_{2,1}(z)\\
&+ 90H_{0,0,0}(z) - 60H_{1,0,0}(z) + 40H_{1,1,0}(z) - \frac{80}{3}H_{1,1,1}(z)\\
&- \frac{2}{9}\left(-195 + 25\pi^2 - 438\zeta(3)\right)\Big) + \mathcal{O}(\epsilon^4)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
F_{5B6} = n_0(\epsilon)\Bigg[&\frac{1}{3}\pi^2H_0(\bar{z})\bar{z} + 2H_{-2,0}(\bar{z})\bar{z} + 2H_{0,0}(\bar{z})\bar{z} - 2H_{0,0,0}(\bar{z})\bar{z} - \frac{1}{3}\left(\pi^2 - 9\zeta(3)\right)\bar{z}\\
&- 2(\bar{z}+1)H_{-1,0}(\bar{z}) + \epsilon\Big(- \frac{3}{2}\pi^2H_{-2}(\bar{z})\bar{z} + \frac{11}{3}\pi^2H_0(\bar{z})\bar{z} + \frac{1}{2}\pi^2H_2(\bar{z})\bar{z}\\
&+ 6H_3(\bar{z})\bar{z} - 6H_4(\bar{z})\bar{z} - 16H_{-3,0}(\bar{z})\bar{z} + 28H_{-2,0}(\bar{z})\bar{z} + 6H_{-2,2}(\bar{z})\bar{z}\\
&+ \left(34 - \frac{5\pi^2}{3}\right)H_{0,0}(\bar{z})\bar{z} - 6H_{-2,-1,0}(\bar{z})\bar{z} - 6H_{-2,0,0}(\bar{z})\bar{z} - 38H_{0,0,0}(\bar{z})\bar{z}\\
&- 2H_{2,0,0}(\bar{z})\bar{z} + 26H_{0,0,0,0}(\bar{z})\bar{z} + 18\zeta(3)\bar{z} - \frac{3}{20}\pi^4\bar{z} - \frac{17}{3}\pi^2\bar{z}\\
&+ \frac{3}{2}\pi^2(\bar{z}+1)H_{-1}(\bar{z}) - \frac{1}{2}\pi^2(\bar{z}-1)H_1(\bar{z}) - 34(\bar{z}+1)H_{-1,0}(\bar{z})\\
&- 6(\bar{z}+1)H_{-1,2}(\bar{z}) + 6(\bar{z}+1)H_{-1,-1,0}(\bar{z}) + 6(\bar{z}+1)H_{-1,0,0}(\bar{z})
\end{aligned}
$$

$$+ 2(\bar{z}-1)H_{1,0,0}(\bar{z})\Big) + \mathcal{O}(\epsilon^2)\Bigg],$$

$$F_{5\mathrm{B}7} = \frac{n_0(\epsilon)}{\epsilon^2}\Bigg[\frac{1}{6}H_{0,0}(\bar{z}) + \epsilon\Big(- \frac{1}{36}\pi^2 H_0(\bar{z}) + \frac{1}{2}H_3(\bar{z}) + \frac{2}{3}H_{-2,0}(\bar{z}) + \frac{1}{3}H_{0,0}(\bar{z})$$
$$- \frac{11}{6}H_{0,0,0}(\bar{z}) + \frac{1}{2}\zeta(3)\Big) + \epsilon^2\Big(- \frac{1}{2}\pi^2 H_{-2}(\bar{z}) + \left(-\frac{\pi^2}{18} - \zeta(3)\right)H_0(\bar{z})$$
$$+ H_3(\bar{z}) - \frac{11}{2}H_4(\bar{z}) - 4H_{-3,0}(\bar{z}) + \frac{4}{3}H_{-2,0}(\bar{z}) + 2H_{-2,2}(\bar{z})$$
$$+ \frac{1}{36}\left(24+\pi^2\right)H_{0,0}(\bar{z}) - H_{3,0}(\bar{z}) + \frac{3}{2}H_{3,1}(\bar{z}) - 2H_{-2,-1,0}(\bar{z})$$
$$- 2H_{-2,0,0}(\bar{z}) - \frac{11}{3}H_{0,0,0}(\bar{z}) + \frac{27}{2}H_{0,0,0,0}(\bar{z}) + \zeta(3) - \frac{\pi^4}{240}\Big) + \mathcal{O}(\epsilon^3)\Bigg],$$

$$F_{5\mathrm{B}8} = n_0(\epsilon)\Bigg[\frac{1}{2}H_{-2,0}(\bar{z})\bar{z}^2 + \frac{1}{12}(3-2\bar{z})H_{0,0}(\bar{z})\bar{z}^2 - \frac{1}{2}H_{0,0,0}(\bar{z})\bar{z}^2$$
$$+ \frac{1}{36}\left(\pi^2\bar{z}^2 - 3\left(-1+\pi^2-9\zeta(3)\right)\bar{z} - 3\right)\bar{z} + \frac{1}{12}\left(\left(-2+\pi^2\right)\bar{z}+1\right)H_0(\bar{z})\bar{z}$$
$$+ \frac{1}{12}\left(2\bar{z}^3 - 3\bar{z}^2 - 6\bar{z} - 1\right)H_{-1,0}(\bar{z})$$
$$+ \epsilon\Big(- \frac{3}{8}\pi^2 H_{-2}(\bar{z})\bar{z}^2 + \frac{1}{4}(3-2\bar{z})H_3(\bar{z})\bar{z}^2 - \frac{3}{2}H_4(\bar{z})\bar{z}^2 - 4H_{-3,0}(\bar{z})\bar{z}^2$$
$$- \frac{1}{2}(\bar{z}-12)H_{-2,0}(\bar{z})\bar{z}^2 + \frac{3}{2}H_{-2,2}(\bar{z})\bar{z}^2 - \frac{3}{2}H_{-2,-1,0}(\bar{z})\bar{z}^2 - \frac{3}{2}H_{-2,0,0}(\bar{z})\bar{z}^2$$
$$+ \frac{1}{12}(16\bar{z}-81)H_{0,0,0}(\bar{z})\bar{z}^2 - \frac{1}{2}H_{2,0,0}(\bar{z})\bar{z}^2 + \frac{13}{2}H_{0,0,0,0}(\bar{z})\bar{z}^2$$
$$+ \frac{1}{720}\left(-27\pi^4\bar{z} + 20\pi^2\left(11\bar{z}^2 - 45\bar{z} - 3\right) + 30\left(30\zeta(3)\bar{z}^2 + (29+90\zeta(3))\bar{z} - 29\right)\right)\bar{z}$$
$$+ \frac{1}{24}\left(\left(-52+23\pi^2\right)\bar{z}+29\right)H_0(\bar{z})\bar{z} + \frac{1}{8}\left(\left(-4+\pi^2\right)\bar{z}+2\right)H_2(\bar{z})\bar{z}$$
$$- \frac{1}{24}\left(44\bar{z}^2 + \left(-157+10\pi^2\right)\bar{z} + 4\right)H_{0,0}(\bar{z})\bar{z}$$
$$- \frac{1}{16}\pi^2\left(2\bar{z}^3 - 3\bar{z}^2 - 6\bar{z} - 1\right)H_{-1}(\bar{z}) + \frac{1}{48}(\bar{z}-1)\left(12\bar{z} + \pi^2\left(2\bar{z}^2 - 7\bar{z} - 1\right)\right)H_1(\bar{z})$$
$$+ \frac{1}{24}\left(44\bar{z}^3 - 129\bar{z}^2 - 198\bar{z} - 25\right)H_{-1,0}(\bar{z}) + \frac{1}{4}\left(2\bar{z}^3 - 3\bar{z}^2 - 6\bar{z} - 1\right)H_{-1,2}(\bar{z})$$

$$
\begin{aligned}
&+\frac{1}{4}\left(-2\bar{z}^3+3\bar{z}^2+6\bar{z}+1\right)H_{-1,-1,0}(\bar{z})+\frac{1}{4}\left(-2\bar{z}^3+3\bar{z}^2+6\bar{z}+1\right)H_{-1,0,0}(\bar{z})\\
&\left.\left.+\frac{1}{12}\left(-2\bar{z}^3+9\bar{z}^2-6\bar{z}-1\right)H_{1,0,0}(\bar{z})\right)+\mathcal{O}(\epsilon^2)\right],
\end{aligned}
$$

$$
\begin{aligned}
F_{5\text{B}9}=n_0(\epsilon)\bar{z}\Bigg[&1-\bar{z}+H_0(\bar{z})+H_{0,0}(\bar{z})+\epsilon\Big(-3(5+\zeta(3))\bar{z}+\frac{1}{6}\pi^2(4\bar{z}-3)\\
&+\left(2(\bar{z}+5)-\frac{1}{6}\pi^2(2\bar{z}+3)\right)H_0(\bar{z})+(3-3\bar{z})H_1(\bar{z})+3H_2(\bar{z})+3H_3(\bar{z})\\
&+(2-2\bar{z})H_{-2,0}(\bar{z})+4(\bar{z}+1)H_{-1,0}(\bar{z})+(5-4\bar{z})H_{0,0}(\bar{z})+(2\bar{z}-9)H_{0,0,0}(\bar{z})+15\Big)\\
&+\epsilon^2\Big(-((133+9\zeta(3))\bar{z})+\frac{1}{40}\pi^4(6\bar{z}+5)+\frac{1}{3}\pi^2(40\bar{z}-23)+\frac{3}{2}\pi^2(\bar{z}-1)H_{-2}(\bar{z})\\
&-3\pi^2(\bar{z}+1)H_{-1}(\bar{z})+\left(30\bar{z}-\frac{1}{6}\pi^2(26\bar{z}+31)-6\zeta(3)+58\right)H_0(\bar{z})\\
&+\left(-45+\pi^2\right)(\bar{z}-1)H_1(\bar{z})+\left(6(\bar{z}+5)-\frac{1}{2}\pi^2(\bar{z}+1)\right)H_2(\bar{z})\\
&+(15-12\bar{z})H_3(\bar{z})+(6\bar{z}-27)H_4(\bar{z})+(16\bar{z}-8)H_{-3,0}(\bar{z})+(20-38\bar{z})H_{-2,0}(\bar{z})\\
&+(6-6\bar{z})H_{-2,2}(\bar{z})+70(\bar{z}+1)H_{-1,0}(\bar{z})+12(\bar{z}+1)H_{-1,2}(\bar{z})\\
&+\left(-74\bar{z}+\frac{1}{6}\pi^2(10\bar{z}+11)+8\right)H_{0,0}(\bar{z})+6(\bar{z}-1)H_{1,0}(\bar{z})+(9-9\bar{z})H_{1,1}(\bar{z})\\
&-6H_{2,0}(\bar{z})+9H_{2,1}(\bar{z})-6H_{3,0}(\bar{z})+9H_{3,1}(\bar{z})+6(\bar{z}-1)H_{-2,-1,0}(\bar{z})\\
&+6(\bar{z}-1)H_{-2,0,0}(\bar{z})-12(\bar{z}+1)H_{-1,-1,0}(\bar{z})-12(\bar{z}+1)H_{-1,0,0}(\bar{z})\\
&+(58\bar{z}-71)H_{0,0,0}(\bar{z})+(4-4\bar{z})H_{1,0,0}(\bar{z})+2(\bar{z}+1)H_{2,0,0}(\bar{z})\\
&+(55-26\bar{z})H_{0,0,0,0}(\bar{z})-6\zeta(3)+133\Big)+\mathcal{O}(\epsilon^3)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
F_{5\text{B}10}=\frac{n_0(\epsilon)}{\bar{z}\epsilon^2}\Bigg[&\frac{16}{3}H_{-1,0}(\bar{z})-\frac{16}{3}H_{0,0}(\bar{z})+\frac{4\pi^2}{9}+\epsilon\Big(-\frac{32}{9}\pi^2H_{-1}(\bar{z})+\frac{4}{3}\pi^2H_0(\bar{z})\\
&-16H_3(\bar{z})-16H_{-2,0}(\bar{z})-\frac{16}{3}H_{-1,0}(\bar{z})+16H_{-1,2}(\bar{z})+\frac{16}{3}H_{0,0}(\bar{z})\\
&-\frac{32}{3}H_{-1,-1,0}(\bar{z})-\frac{56}{3}H_{-1,0,0}(\bar{z})+\frac{136}{3}H_{0,0,0}(\bar{z})-\frac{4}{9}\left(\pi^2-39\zeta(3)\right)\Big)+\mathcal{O}(\epsilon^2)\Bigg]
\end{aligned}
$$

$$F_{5\mathrm{B}11} = \frac{n_0(\epsilon)}{\epsilon^3}\Bigg[\Big(-\frac{1}{6}H_2(z) - \frac{1}{6}H_{1,1}(z)\Big) + \epsilon\Big(-\frac{1}{3}H_2(z) + \frac{1}{2}H_3(z) - \frac{1}{3}H_{1,1}(z) - \frac{1}{3}H_{1,2}(z) + \frac{1}{2}H_{2,0}(z) - H_{2,1}(z) + \frac{1}{2}H_{1,1,0}(z) - \frac{11}{6}H_{1,1,1}(z)\Big) + \mathcal{O}(\epsilon^2)\Bigg]$$

,

$$F_{5\mathrm{B}12} = \frac{n_0(\epsilon)}{\bar{z}\epsilon^4}\Bigg[-\frac{2}{3} + \epsilon\Big(2H_0(z) - \frac{4}{3}H_1(z) + \frac{2}{3}\Big) + \epsilon^2\Big(-2H_0(z) + \frac{4}{3}H_1(z) + \frac{20}{3}H_2(z) - 6H_{0,0}(z) + 4H_{1,0}(z) + \frac{2}{9}\left(-21 + 5\pi^2\right)\Big) + \epsilon^3\Big(\left(14 - \frac{10\pi^2}{3}\right)H_0(z) + \frac{4}{9}\left(-21 + 5\pi^2\right)H_1(z) - \frac{20}{3}H_2(z) - \frac{52}{3}H_3(z) + 6H_{0,0}(z) - 4H_{1,0}(z) + 16H_{1,2}(z) - 20H_{2,0}(z) + \frac{64}{3}H_{2,1}(z) + 18H_{0,0,0}(z) - 12H_{1,0,0}(z) + \frac{64}{3}H_{1,1,1}(z) - \frac{2}{9}\left(-39 + 5\pi^2 - 96\zeta(3)\right)\Big) + \mathcal{O}(\epsilon^4)\Bigg],$$

$$F_{5\mathrm{B}13} = n_0(\epsilon)z^2\Bigg[\frac{H_0(z)}{4} + \epsilon\Big(\frac{25}{8}H_0(z) + \frac{1}{2}H_2(z) - \frac{7}{4}H_{0,0}(z) - \frac{\pi^2}{12}\Big) + \epsilon^2\Big(\frac{1}{48}\left(1149 - 4\pi^2\right)H_0(z) + \frac{25}{4}H_2(z) - \frac{7}{2}H_3(z) - \frac{175}{8}H_{0,0}(z) - \frac{3}{2}H_{2,0}(z) + H_{2,1}(z) + \frac{37}{4}H_{0,0,0}(z) - \frac{1}{2}\zeta(3) - \frac{25}{24}\pi^2\Big) + \epsilon^3\Big(S_{2,2}(1-z) + \frac{1}{96}\left(13983 - 100\pi^2 - 528\zeta(3)\right)H_0(z) + \zeta(3)H_1(z) + \left(\frac{383}{8} - \frac{5\pi^2}{6}\right)H_2(z) - \frac{175}{4}H_3(z) + \frac{37}{2}H_4(z) + \frac{1}{48}\left(-8043 + 76\pi^2\right)H_{0,0}(z) - \frac{75}{4}H_{2,0}(z) + \frac{25}{2}H_{2,1}(z) - 3H_{2,2}(z) + \frac{21}{2}H_{3,0}(z) - 7H_{3,1}(z) + \frac{925}{8}H_{0,0,0}(z) + \frac{9}{2}H_{2,0,0}(z) - 3H_{2,1,0}(z) + 2H_{2,1,1}(z) - \frac{175}{4}H_{0,0,0,0}(z) - H_{1,1,0,0}(z) - \frac{25}{4}\zeta(3) + \frac{1}{9}\pi^4 - \frac{383}{48}\pi^2\Big) + \mathcal{O}(\epsilon^4)\Bigg],$$

$$
\begin{aligned}
F_{5\mathrm{B}14} = \frac{n_0(\epsilon)}{\epsilon}\Bigg[& -z + zH_0(z) - H_{1,0}(z) + \epsilon\Big(14H_0(z)z + 2H_2(z)z - 7H_{0,0}(z)z \\
& - \frac{1}{3}\left(42+\pi^2\right)z + \frac{1}{3}\left(\pi^2 - 6z\right)H_1(z) - 7H_{1,0}(z) - 2H_{1,2}(z) \\
& + 7H_{1,0,0}(z) - 2H_{1,1,0}(z)\Big) + \epsilon^2\Big(- \frac{1}{3}\left(-357+\pi^2\right)H_0(z)z + 28H_2(z)z \\
& - 14H_3(z)z - 86H_{0,0}(z)z - 6H_{2,0}(z)z + 4H_{2,1}(z)z + 37H_{0,0,0}(z)z \\
& - (119 + 2\zeta(3))z - 2\pi^2 z + 2S_{2,2}(1-z) + \left(4(\zeta(3) - 7z) + \frac{7}{3}\pi^2\right)H_1(z) \\
& + \left(6z + \frac{\pi^2}{3} - 33\right)H_{1,0}(z) + \frac{2}{3}\left(\pi^2 - 6z\right)H_{1,1}(z) - 14H_{1,2}(z) + 14H_{1,3}(z) \\
& + 49H_{1,0,0}(z) - 14H_{1,1,0}(z) - 4H_{1,1,2}(z) + 6H_{1,2,0}(z) - 4H_{1,2,1}(z) \\
& - 37H_{1,0,0,0}(z) + 12H_{1,1,0,0}(z) - 4H_{1,1,1,0}(z) - \frac{\pi^4}{180}\Big) \\
& + \epsilon^3\Big(- \frac{2}{3}\left(-1194 + 13\pi^2 + 33\zeta(3)\right)H_0(z)z + \frac{2}{3}\left(357 - 5\pi^2\right)H_2(z)z \\
& - 172H_3(z)z + 74H_4(z)z + \frac{19}{3}\left(-105+\pi^2\right)H_{0,0}(z)z - 84H_{2,0}(z)z \\
& + 56H_{2,1}(z)z - 12H_{2,2}(z)z + 42H_{3,0}(z)z - 28H_{3,1}(z)z + 434H_{0,0,0}(z)z \\
& + 18H_{2,0,0}(z)z - 12H_{2,1,0}(z)z + 8H_{2,1,1}(z)z - 175H_{0,0,0,0}(z)z + 8\zeta(3)z \\
& + \frac{4}{9}\pi^4 z - \frac{7}{3}\pi^2 z - 796z + 2(2z+7)S_{2,2}(1-z) - 74S_{2,3}(1-z) - 28S_{3,2}(1-z) \\
& + \left(4\zeta(3)z - 238z + \pi^2\left(\frac{10}{3}z + 11\right) + 4S_{2,2}(1-z) + 28\zeta(3) - \frac{41}{30}\pi^4\right)H_1(z) \\
& + \left(84z + 22\zeta(3) + \frac{7}{3}\pi^2 - 131\right)H_{1,0}(z) + \left(-56z + 40\zeta(3) + \frac{14}{3}\pi^2\right)H_{1,1}(z) \\
& + \left(12z + \frac{10}{3}\pi^2 - 66\right)H_{1,2}(z) + 98H_{1,3}(z) - 74H_{1,4}(z) \\
& + \left(-18z - \frac{19}{3}\pi^2 + 231\right)H_{1,0,0}(z) + \frac{2}{3}\left(18z + \pi^2 - 99\right)H_{1,1,0}(z) \\
& + \frac{4}{3}\left(\pi^2 - 6z\right)H_{1,1,1}(z) - 28H_{1,1,2}(z) + 24H_{1,1,3}(z) + 42H_{1,2,0}(z) - 28H_{1,2,1}(z) \\
& + 12H_{1,2,2}(z) - 42H_{1,3,0}(z) + 28H_{1,3,1}(z) - 259H_{1,0,0,0}(z) + (84-4z)H_{1,1,0,0}(z) \\
& - 28H_{1,1,1,0}(z) - 8H_{1,1,1,2}(z) + 8H_{1,1,2,0}(z) - 8H_{1,1,2,1}(z) - 18H_{1,2,0,0}(z) \\
& + 12H_{1,2,1,0}(z) - 8H_{1,2,1,1}(z) + 175H_{1,0,0,0,0}(z) - 148H_{1,1,0,0,0}(z)
\end{aligned}
$$

$$- 12H_{1,1,1,0,0}(z) - 8H_{1,1,1,1,0}(z) + 204\zeta(5) - 17\pi^2\zeta(3) - \frac{7}{180}\pi^4\Big) + \mathcal{O}(\epsilon^4)\Bigg],$$

$$F_{5\text{B}15} = n_0(\epsilon)\Bigg[\frac{1}{2}H_2(z)z^2 - \frac{1}{2}H_{0,0}(z)z^2 - \frac{1}{12}\pi^2 z^2 - \frac{1}{2}H_0(z)z + \frac{1}{2}(1-z)H_1(z)$$
$$+ \frac{1}{2}\left(1-z^2\right)H_{1,0}(z) + \frac{1}{2}\left(z^2-1\right)H_{1,1}(z)$$
$$+ \epsilon\Big(\frac{25}{4}H_2(z)z^2 - \frac{7}{2}H_3(z)z^2 + H_{2,1}(z)z^2 + 6H_{0,0,0}(z)z^2 + \frac{5}{24}\left(-5\pi^2 + 12\zeta(3)\right)z^2$$
$$+ \frac{1}{12}\left(7\pi^2 z - 87\right)H_0(z)z - \frac{5}{4}(5z-2)H_{0,0}(z)z - \frac{29}{4}(z-1)H_1(z)$$
$$+ \frac{1}{4}\left(-25z^2 + 6z + 19\right)H_{1,0}(z) + \frac{5}{4}\left(5z^2 - 2z - 3\right)H_{1,1}(z)$$
$$+ \frac{5}{2}\left(z^2-1\right)H_{1,0,0}(z) - \frac{7}{2}\left(z^2-1\right)H_{1,1,0}(z) + \frac{9}{2}\left(z^2-1\right)H_{1,1,1}(z)\Big) + \mathcal{O}(\epsilon^2)\Bigg],$$

$$F_{5\text{B}16} = \frac{n_0(\epsilon)}{\epsilon^2}\Bigg[-\frac{z}{4} + \epsilon\left(\frac{5}{4}H_0(z)z - \frac{5z}{2} + \frac{1}{4}(-2z-1)H_1(z)\right)$$
$$+ \epsilon^2\Big(\frac{1}{12}z\left(\pi^2(z+2) - 192\right) + 13zH_0(z) + \left(-\frac{9}{2}z - 3\right)H_1(z)$$
$$- \frac{1}{2}(z-6)zH_2(z) + \frac{1}{4}z(2z-25)H_{0,0}(z) + \left(\frac{1}{2}z^2 + z + \frac{3}{4}\right)H_{1,0}(z)$$
$$+ \frac{1}{4}\left(-2z^2 - 2z - 5\right)H_{1,1}(z)\Big)$$
$$+ \epsilon^3\Big(\frac{1}{12}z\left(\pi^2(12z+17) - 6(5\zeta(3)z - 18\zeta(3) + 168)\right)$$
$$+ \left(87z - \frac{1}{12}\pi^2 z(7z+10)\right)H_0(z) + \left(-25z + \frac{5}{12}\pi^2(2z+1) - 23\right)H_1(z)$$
$$+ \frac{3}{2}(21-4z)zH_2(z) + \frac{1}{2}z(7z-30)H_3(z) + \frac{1}{2}z(12z-133)H_{0,0}(z)$$
$$+ \left(6z^2 + 7z + \frac{19}{2}\right)H_{1,0}(z) + \left(-6z^2 - z - \frac{31}{2}\right)H_{1,1}(z)$$
$$+ \left(3z + \frac{3}{2}\right)H_{1,2}(z) - \frac{15}{2}zH_{2,0}(z) - (z-6)zH_{2,1}(z)$$
$$+ \left(\frac{125z}{4} - 6z^2\right)H_{0,0,0}(z) + \left(-\frac{5z^2}{2} - 2z - \frac{9}{4}\right)H_{1,0,0}(z)$$

$$
\begin{aligned}
&+\frac{1}{4}\left(14z^2-2z+15\right)H_{1,1,0}(z)+\frac{1}{4}\left(-18z^2+10z-19\right)H_{1,1,1}(z)\Big)\\
&+\epsilon^4\Big(\frac{1}{24}\pi^2 z(195z+166)-\frac{41}{240}\pi^4 z^2+\frac{41}{144}\pi^4 z+\frac{1}{160}\pi^4\\
&-\frac{1}{4}\left(38z^2-46z+9\right)S_{2,2}(1-z)-\frac{3}{2}z(20\zeta(3)z-61\zeta(3)+264)\\
&-\frac{1}{12}z\left(-138\zeta(3)z+\pi^2(84z+95)+540\zeta(3)-5757\right)H_0(z)\\
&+\frac{1}{4}\left(-62\zeta(3)z^2+(-433+130\zeta(3))z+10\pi^2(3z+2)+21\zeta(3)-575\right)H_1(z)\\
&+\frac{1}{12}\left(10\pi^2(z-6)-585z+2574\right)zH_2(z)+\frac{3}{2}z(28z-107)H_3(z)\\
&+\left(75z-\frac{49z^2}{2}\right)H_4(z)+\frac{1}{12}z\left(585z+\pi^2(33z+50)-5490\right)H_{0,0}(z)\\
&+\frac{1}{12}\left(585z^2+210z-\pi^2\left(16z^2+14z+15\right)+933\right)H_{1,0}(z)\\
&+\frac{1}{12}\left(5\pi^2\left(2z^2+2z+5\right)-9\left(65z^2-38z+165\right)\right)H_{1,1}(z)+9(3z+2)H_{1,2}(z)\\
&+\left(-9z-\frac{9}{2}\right)H_{1,3}(z)-78zH_{2,0}(z)+3(21-4z)zH_{2,1}(z)+3(z-6)zH_{2,2}(z)\\
&+\frac{3}{2}(25-2z)zH_{3,0}(z)+z(7z-30)H_{3,1}(z)+(337-72z)zH_{0,0,0}(z)\\
&+\left(-30z^2-8z-\frac{59}{2}\right)H_{1,0,0}(z)+\left(42z^2-23z+\frac{97}{2}\right)H_{1,1,0}(z)\\
&+\left(-54z^2+48z-\frac{123}{2}\right)H_{1,1,1}(z)+\left(3z^2+3z+\frac{15}{2}\right)H_{1,1,2}(z)\\
&-\frac{3}{2}\left(2z^2+4z+3\right)H_{1,2,0}(z)+(6z+3)H_{1,2,1}(z)+\frac{3}{2}z(2z+13)H_{2,0,0}(z)\\
&-3z(z+4)H_{2,1,0}(z)+2z(2z+3)H_{2,1,1}(z)+\frac{1}{4}z(194z-625)H_{0,0,0,0}(z)\\
&+\left(\frac{13z^2}{2}+7z+\frac{27}{4}\right)H_{1,0,0,0}(z)+\left(-5z^2-6z-9\right)H_{1,1,0,0}(z)\\
&+\frac{1}{4}\left(86z^2-62z+57\right)H_{1,1,1,0}(z)+\frac{1}{4}\left(-110z^2+94z-65\right)H_{1,1,1,1}(z)\Big)+\mathcal{O}(\epsilon^5)\Bigg],
\end{aligned}
$$

$$
F_{5\text{B}17}=\frac{n_0(\epsilon)}{\epsilon^4}\Bigg[-\frac{1}{4}+\epsilon\Big(\frac{5}{4}H_0(z)-\frac{1}{2}H_1(z)-1\Big)+\epsilon^2\Big(5H_0(z)-2H_1(z)
$$

$$+\frac{5}{2}H_2(z)-\frac{25}{4}H_{0,0}(z)+H_{1,0}(z)-H_{1,1}(z)+\frac{\pi^2}{4}-3\Big)$$
$$+\epsilon^3\Big(\left(15-\frac{5}{4}\pi^2\right)H_0(z)+\left(-6+\pi^2\right)H_1(z)+10H_2(z)-\frac{25}{2}H_3(z)$$
$$-25H_{0,0}(z)+4H_{1,0}(z)-4H_{1,1}(z)+2H_{1,2}(z)-6H_{2,0}(z)+5H_{2,1}(z)$$
$$+\frac{125}{4}H_{0,0,0}(z)-H_{1,0,0}(z)+2H_{1,1,0}(z)-2H_{1,1,1}(z)+\frac{21}{2}\zeta(3)+\pi^2-8\Big)+\mathcal{O}(\epsilon^4)\Bigg],$$

$$F_{5B18}=\frac{n_0(\epsilon)}{\epsilon}\Bigg[\frac{19}{360}\pi^4-\frac{1}{3}\pi^2H_{-2}(1-z)+2H_{-2,2}(1-z)+\frac{1}{6}\pi^2H_{1,1}(z)$$
$$-H_{-2,0,0}(1-z)-H_{1,1,2}(z)+H_{1,1,0,0}(z)-H_{1,1,1,0}(z)-2\zeta(3)H_1(z)$$
$$+\epsilon\Big(\frac{7}{3}\pi^2H_{-3}(\bar z)+\left(-\frac{2\pi^2}{3}-7\zeta(3)\right)H_{-2}(\bar z)-\frac{7}{36}\pi^4H_0(\bar z)-2H_4(\bar z)+9H_5(\bar z)$$
$$-14H_{-3,2}(\bar z)+\frac{2}{3}\pi^2H_{-2,-1}(\bar z)+\frac{7}{6}\pi^2H_{-2,0}(\bar z)+4H_{-2,2}(\bar z)-7H_{-2,3}(\bar z)$$
$$+\left(\frac{\pi^2}{3}+9\zeta(3)\right)H_{0,0}(\bar z)-2H_{3,0}(\bar z)+2H_{3,1}(\bar z)+9H_{4,0}(\bar z)-14H_{4,1}(\bar z)$$
$$+7H_{-3,0,0}(\bar z)-4H_{-2,-1,2}(\bar z)-2H_{-2,0,0}(\bar z)+10H_{-2,2,1}(\bar z)-\frac{3}{2}\pi^2H_{0,0,0}(\bar z)$$
$$+2H_{3,0,0}(\bar z)-7H_{3,1,0}(\bar z)+12H_{3,1,1}(\bar z)+2H_{-2,-1,0,0}(\bar z)+9H_{-2,0,0,0}(\bar z)$$
$$+\frac{1}{18}\left(\pi^4+21\pi^2\zeta(3)\right)\Big)+\mathcal{O}(\epsilon^2)\Bigg],$$

$$F_{5B19}=\frac{n_0(\epsilon)}{\epsilon}\Bigg[\frac{1}{3}\pi^2H_{-2}(\bar z)\bar z^2+\frac{1}{3}\pi^2H_{-1}(\bar z)\bar z^2-\frac{1}{24}\left(3\bar z^2-32\bar z+30\right)H_2(\bar z)\bar z^2$$
$$+H_3(\bar z)\bar z^2+H_4(\bar z)\bar z^2-2H_{-2,2}(\bar z)\bar z^2-2H_{-1,2}(\bar z)\bar z^2+H_{2,0}(\bar z)\bar z^2-H_{2,1}(\bar z)\bar z^2$$
$$+\frac{1}{24}\left(3\bar z^2-32\bar z-4\pi^2+30\right)H_{0,0}(\bar z)\bar z^2+H_{3,0}(\bar z)\bar z^2-H_{3,1}(\bar z)\bar z^2$$
$$+H_{-2,0,0}(\bar z)\bar z^2+H_{-1,0,0}(\bar z)\bar z^2+\frac{1}{48}\left(-6\bar z^2+\left(61-8\pi^2\right)\bar z+2\right)H_0(\bar z)\bar z$$
$$+\frac{1}{144}\left(3\pi^2\bar z^3-4\left(9+8\pi^2\right)\bar z^2+\left(51+30\pi^2-4\pi^4\right)\bar z-15\right)\bar z$$
$$+\frac{1}{48}\left(6\bar z^3+53\bar z^2-74\bar z+15\right)H_1(\bar z)+\frac{1}{24}\left(3\bar z^4-32\bar z^3+30\bar z^2-1\right)H_{1,0}(\bar z)$$

$$
\begin{aligned}
&+\frac{1}{24}\left(-3\bar{z}^4+32\bar{z}^3-30\bar{z}^2+1\right)H_{1,1}(\bar{z})\\
&+\epsilon\Big(-\frac{7}{3}\pi^2H_{-3}(\bar{z})\bar{z}^2-\frac{1}{3}\left(\pi^2-21\zeta(3)\right)H_{-2}(\bar{z})\bar{z}^2+\frac{1}{3}\left(4\pi^2+21\zeta(3)\right)H_{-1}(\bar{z})\bar{z}^2\\
&-\frac{1}{288}\left(321\bar{z}^2-3584\bar{z}+2052\right)H_2(\bar{z})\bar{z}^2+\frac{1}{24}\left(21\bar{z}^2-224\bar{z}+306\right)H_3(\bar{z})\bar{z}^2\\
&-3H_4(\bar{z})\bar{z}^2-9H_5(\bar{z})\bar{z}^2+14H_{-3,2}(\bar{z})\bar{z}^2-\frac{2}{3}\pi^2H_{-2,-1}(\bar{z})\bar{z}^2-\frac{7}{6}\pi^2H_{-2,0}(\bar{z})\bar{z}^2\\
&+2H_{-2,2}(\bar{z})\bar{z}^2+7H_{-2,3}(\bar{z})\bar{z}^2-\frac{2}{3}\pi^2H_{-1,-1}(\bar{z})\bar{z}^2-\frac{7}{6}\pi^2H_{-1,0}(\bar{z})\bar{z}^2\\
&-8H_{-1,2}(\bar{z})\bar{z}^2+7H_{-1,3}(\bar{z})\bar{z}^2+4H_{2,0}(\bar{z})\bar{z}^2-\frac{1}{24}\left(15\bar{z}^2-160\bar{z}+246\right)H_{2,1}(\bar{z})\bar{z}^2\\
&-3H_{3,0}(\bar{z})\bar{z}^2+8H_{3,1}(\bar{z})\bar{z}^2-9H_{4,0}(\bar{z})\bar{z}^2+14H_{4,1}(\bar{z})\bar{z}^2-7H_{-3,0,0}(\bar{z})\bar{z}^2\\
&+4H_{-2,-1,2}(\bar{z})\bar{z}^2-H_{-2,0,0}(\bar{z})\bar{z}^2-10H_{-2,2,1}(\bar{z})\bar{z}^2+4H_{-1,-1,2}(\bar{z})\bar{z}^2+4H_{-1,0,0}(\bar{z})\bar{z}^2\\
&-10H_{-1,2,1}(\bar{z})\bar{z}^2+\frac{3}{8}\left(-3\bar{z}^2+32\bar{z}+4\pi^2-30\right)H_{0,0,0}(\bar{z})\bar{z}^2-2H_{2,0,0}(\bar{z})\bar{z}^2\\
&+7H_{2,1,0}(\bar{z})\bar{z}^2-12H_{2,1,1}(\bar{z})\bar{z}^2-2H_{3,0,0}(\bar{z})\bar{z}^2+7H_{3,1,0}(\bar{z})\bar{z}^2-12H_{3,1,1}(\bar{z})\bar{z}^2\\
&-2H_{-2,-1,0,0}(\bar{z})\bar{z}^2-9H_{-2,0,0,0}(\bar{z})\bar{z}^2-2H_{-1,-1,0,0}(\bar{z})\bar{z}^2-9H_{-1,0,0,0}(\bar{z})\bar{z}^2\\
&-\frac{1}{12}\zeta(3)\left(3\bar{z}^2-32\bar{z}+30\right)\bar{z}^2-\frac{1}{576}\bar{z}\Big(-\left(\pi^2(684-672\zeta(3))+2519\right)\bar{z}\\
&+1788\bar{z}^2+731\Big)+\frac{\pi^2\left(321\bar{z}-3584\right)\bar{z}^3}{1728}+\frac{1}{36}\pi^4\bar{z}^2\\
&+\frac{1}{576}\Big(-84\pi^2\bar{z}^3+\left(-498+896\pi^2\right)\bar{z}^2+\left(8119-1224\pi^2+112\pi^4-5184\zeta(3)\right)\bar{z}\\
&+350\Big)H_0(\bar{z})\bar{z}+\frac{1}{288}\left(321\bar{z}^3-3512\bar{z}^2+24\left(55+6\pi^2-108\zeta(3)\right)\bar{z}-24\right)H_{0,0}(\bar{z})\bar{z}\\
&+\frac{1}{576}\left(282\bar{z}^3+8387\bar{z}^2-10706\bar{z}+2037\right)H_1(\bar{z})+\frac{1}{288}\Big(321\bar{z}^4-3836\bar{z}^3+3246\bar{z}^2\\
&+564\bar{z}-295\Big)H_{1,0}(\bar{z})+\frac{1}{288}\left(-321\bar{z}^4+4016\bar{z}^3-1656\bar{z}^2-2784\bar{z}+745\right)H_{1,1}(\bar{z})\\
&+\frac{1}{12}\left(-3\bar{z}^4+32\bar{z}^3-30\bar{z}^2+1\right)H_{1,0,0}(\bar{z})+\frac{7}{24}\left(3\bar{z}^4-32\bar{z}^3+30\bar{z}^2-1\right)H_{1,1,0}(\bar{z})\\
&+\frac{1}{2}\left(-3\bar{z}^4+32\bar{z}^3-30\bar{z}^2+1\right)H_{1,1,1}(\bar{z})\Big)+\mathcal{O}(\epsilon^2)\Bigg],
\end{aligned}
$$

$$
F_{5B20}=\frac{n_0(\epsilon)}{\bar{z}\epsilon^4}\Bigg[-\frac{1}{6}H_1(z)+\epsilon\Big(-\frac{1}{3}H_1(z)+\frac{5}{6}H_{1,0}(z)-\frac{2}{3}H_{1,1}(z)\Big)
$$

$$+\epsilon^2\Big(\frac{1}{9}\left(-6+\pi^2\right)H_1(z)+\frac{5}{3}H_{1,0}(z)-\frac{4}{3}H_{1,1}(z)+2H_{1,2}(z)-\frac{25}{6}H_{1,0,0}(z)$$
$$+\frac{10}{3}H_{1,1,0}(z)-\frac{5}{3}H_{1,1,1}(z)\Big)+\mathcal{O}(\epsilon^3)\Bigg],$$

$$F_{5\mathrm{B}21}=\frac{n_0(\epsilon)}{\epsilon^5}\Bigg[\frac{1}{4}+\epsilon\Big(-\frac{1}{2}H_0(\bar z)+\frac{5}{4}H_1(\bar z)+\frac{1}{2}\Big)+\epsilon^2\Big(-H_0(\bar z)+\frac{5}{2}H_1(\bar z)$$
$$-\frac{5}{2}H_2(\bar z)+H_{0,0}(\bar z)-\frac{5}{2}H_{1,0}(\bar z)+\frac{25}{4}H_{1,1}(\bar z)-\frac{\pi^2}{4}+1\Big)$$
$$+\epsilon^3\Big(\frac{1}{2}\left(-4+\pi^2\right)H_0(\bar z)+\left(5-\frac{7}{3}\pi^2\right)H_1(\bar z)-5H_2(\bar z)+5H_3(\bar z)+2H_{0,0}(\bar z)$$
$$-5H_{1,0}(\bar z)+\frac{25}{2}H_{1,1}(\bar z)-6H_{1,2}(\bar z)+5H_{2,0}(\bar z)-\frac{25}{2}H_{2,1}(\bar z)-2H_{0,0,0}(\bar z)$$
$$+5H_{1,0,0}(\bar z)-\frac{25}{2}H_{1,1,0}(\bar z)+\frac{125}{4}H_{1,1,1}(\bar z)+\frac{1}{2}\left(4-\pi^2+5\zeta(3)\right)\Big)+\mathcal{O}(\epsilon^4)\Bigg],$$

$$F_{5\mathrm{B}22}=n_0(\epsilon)\frac{\bar z}{z}\Bigg[\frac{1}{4}(z-2)z+\frac{1}{2}(1-z)H_1(z)+\epsilon\Big(\frac{29}{8}(z-2)z-\frac{3}{4}(z-2)H_0(z)z$$
$$+\frac{1}{4}\left(2z^2-25z+21\right)H_1(z)+\frac{1}{2}(1-z)H_2(z)+\frac{3}{2}(z-1)H_{1,0}(z)+(3-3z)H_{1,1}(z)\Big)$$
$$+\epsilon^2\Big(-\frac{1}{48}\left(-1497+20\pi^2\right)(z-2)z-\frac{87}{8}(z-2)H_0(z)z+\frac{9}{4}(z-2)H_{0,0}(z)z$$
$$+\left(\frac{5}{6}\pi^2(z-1)+\frac{1}{8}\left(58z^2-383z+267\right)\right)H_1(z)-\frac{3}{4}\left(2z^2+3z-7\right)H_2(z)$$
$$+\frac{1}{2}(1-z)H_3(z)-\frac{3}{4}\left(2z^2-25z+21\right)H_{1,0}(z)+\left(z^2-\frac{67z}{2}+\frac{63}{2}\right)H_{1,1}(z)$$
$$+(z-1)H_{1,2}(z)+\frac{3}{2}(z-1)H_{2,0}(z)+(3-3z)H_{2,1}(z)-\frac{9}{2}(z-1)H_{1,0,0}(z)$$
$$+9(z-1)H_{1,1,0}(z)-14(z-1)H_{1,1,1}(z)\Big)+\mathcal{O}(\epsilon^3)\Bigg],$$

$$F_{5\mathrm{B}23}=n_0(\epsilon)\Bigg[\frac{1}{2}(\bar z-1)H_1(\bar z)-\frac{H_2(\bar z)}{2}-\frac{1}{2}H_{1,0}(\bar z)+\epsilon\Big(\left(\frac{25}{4}(\bar z-1)+\frac{\pi^2}{3}\right)H_1(\bar z)$$

$$
\begin{aligned}
&- \frac{17H_2(\bar{z})}{4} + H_3(\bar{z}) + \left(-\bar{z} - \frac{13}{4}\right) H_{1,0}(\bar{z}) + \frac{7}{2}(\bar{z}-1)H_{1,1}(\bar{z}) \\
&- \frac{3}{2}H_{1,2}(\bar{z}) + 2H_{2,0}(\bar{z}) - \frac{7}{2}H_{2,1}(\bar{z}) + 3H_{1,0,0}(\bar{z}) + \frac{1}{2}H_{1,1,0}(\bar{z})\Big) \\
&+ \epsilon^2\Big(\frac{1}{8}\left(-100\bar{z} - 4\pi^2 - 83\right) H_{1,0}(\bar{z}) + \left(\frac{175}{4}(\bar{z}-1) - \frac{\pi^2}{3}\right) H_{1,1}(\bar{z}) \\
&+ \left(-3\bar{z} - \frac{39}{4}\right) H_{1,2}(\bar{z}) + 9H_{1,3}(\bar{z}) + 17H_{2,0}(\bar{z}) - \frac{119}{4}H_{2,1}(\bar{z}) + 6H_{2,2}(\bar{z}) \\
&- 4H_{3,0}(\bar{z}) + 7H_{3,1}(\bar{z}) + \left(2\bar{z} + \frac{47}{2}\right) H_{1,0,0}(\bar{z}) + \left(\frac{45}{4} - 7\bar{z}\right) H_{1,1,0}(\bar{z}) \\
&+ \frac{37}{2}(\bar{z}-1)H_{1,1,1}(\bar{z}) + \frac{3}{2}H_{1,1,2}(\bar{z}) + H_{1,2,0}(\bar{z}) - \frac{9}{2}H_{1,2,1}(\bar{z}) - 8H_{2,0,0}(\bar{z}) \\
&+ 6H_{2,1,0}(\bar{z}) - \frac{37}{2}H_{2,1,1}(\bar{z}) - 14H_{1,0,0,0}(\bar{z}) - 3H_{1,1,0,0}(\bar{z}) - \frac{1}{2}H_{1,1,1,0}(\bar{z}) \\
&+ H_1(\bar{z})\left(\frac{383\bar{z}}{8} - \frac{1}{6}\pi^2(5\bar{z}-22) + 4\zeta(3) - \frac{383}{8}\right) \\
&+ \left(\frac{\pi^2}{6} - \frac{183}{8}\right) H_2(\bar{z}) + \frac{17H_3(\bar{z})}{2} - 2H_4(\bar{z})\Big) \\
&+ \epsilon^3\Big(\frac{1}{48}\left(4\pi^2(20\bar{z}-71) - 3(1532\bar{z} + 16\zeta(3) + 65)\right) H_{1,0}(\bar{z}) \\
&+ \left(\pi^2\left(3 - \frac{35\bar{z}}{6}\right) + \frac{2681\bar{z}}{8} - 4\zeta(3) - \frac{2681}{8}\right) H_{1,1}(\bar{z}) \\
&+ \frac{1}{24}\left(-900\bar{z} + 92\pi^2 - 747\right) H_{1,2}(\bar{z}) + \left(6\bar{z} + \frac{141}{2}\right) H_{1,3}(\bar{z}) \\
&- 42H_{1,4}(\bar{z}) + \frac{1}{6}\left(549 - 4\pi^2\right) H_{2,0}(\bar{z}) + \frac{1}{8}\left(52\pi^2 - 1281\right) H_{2,1}(\bar{z}) \\
&+ 51H_{2,2}(\bar{z}) - 24H_{2,3}(\bar{z}) - 34H_{3,0}(\bar{z}) + \frac{119}{2}H_{3,1}(\bar{z}) - 12H_{3,2}(\bar{z}) + 8H_{4,0}(\bar{z}) \\
&- 14H_{4,1}(\bar{z}) + \left(25\bar{z} + \frac{\pi^2}{3} + \frac{449}{4}\right) H_{1,0,0}(\bar{z}) + \frac{1}{8}\left(-700\bar{z} + 4\pi^2 + 883\right) H_{1,1,0}(\bar{z}) \\
&+ \left(\frac{925(\bar{z}-1)}{4} + \frac{\pi^2}{3}\right) H_{1,1,1}(\bar{z}) + \left(\frac{135}{4} - 21\bar{z}\right) H_{1,1,2}(\bar{z}) - 9H_{1,1,3}(\bar{z}) \\
&+ \left(6\bar{z} + \frac{5}{2}\right) H_{1,2,0}(\bar{z}) - \frac{9}{4}(4\bar{z}+13)H_{1,2,1}(\bar{z}) + 3H_{1,2,2}(\bar{z}) - 10H_{1,3,0}(\bar{z}) \\
&+ 27H_{1,3,1}(\bar{z}) - 68H_{2,0,0}(\bar{z}) + 51H_{2,1,0}(\bar{z}) - \frac{629}{4}H_{2,1,1}(\bar{z}) + 18H_{2,1,2}(\bar{z}) - 8H_{2,2,0}(\bar{z}) \\
&+ 18H_{2,2,1}(\bar{z}) + 16H_{3,0,0}(\bar{z}) - 12H_{3,1,0}(\bar{z}) + 37H_{3,1,1}(\bar{z}) + (-4\bar{z} - 115)H_{1,0,0,0}(\bar{z})
\end{aligned}
$$

$$+\left(14\bar{z}-\frac{79}{2}\right)H_{1,1,0,0}(\bar{z})+\left(\frac{131}{4}-37\bar{z}\right)H_{1,1,1,0}(\bar{z})+\frac{175}{2}(\bar{z}-1)H_{1,1,1,1}(\bar{z})$$
$$-\frac{3}{2}H_{1,1,1,2}(\bar{z})-H_{1,1,2,0}(\bar{z})+\frac{9}{2}H_{1,1,2,1}(\bar{z})+6H_{1,2,0,0}(\bar{z})+11H_{1,2,1,0}(\bar{z})$$
$$-\frac{27}{2}H_{1,2,1,1}(\bar{z})+32H_{2,0,0,0}(\bar{z})-8H_{2,1,0,0}(\bar{z})+38H_{2,1,1,0}(\bar{z})-\frac{175}{2}H_{2,1,1,1}(\bar{z})$$
$$+60H_{1,0,0,0,0}(\bar{z})+14H_{1,1,0,0,0}(\bar{z})+3H_{1,1,1,0,0}(\bar{z})+\frac{1}{2}H_{1,1,1,1,0}(\bar{z})$$
$$+H_1(\bar{z})\left(\bar{z}\left(\frac{4661}{16}-15\zeta(3)\right)+\pi^2\left(\frac{77}{3}-\frac{125\bar{z}}{12}\right)+49\zeta(3)-\frac{7\pi^4}{45}-\frac{4661}{16}\right)$$
$$+\left(7\zeta(3)+\frac{17\pi^2}{12}-\frac{1597}{16}\right)H_2(\bar{z})+\left(\frac{183}{4}-\frac{\pi^2}{3}\right)H_3(\bar{z})-17H_4(\bar{z})+4H_5(\bar{z})\Big)$$
$$+\mathcal{O}(\epsilon^4)\Bigg],$$

$$F_{5\mathrm{B}24}=\frac{n_0(\epsilon)}{\epsilon}\Bigg[H_{1,0}(\bar{z})+\bar{z}-1+\frac{\pi^2}{6}+\epsilon\Big(14\bar{z}+\left(-2\bar{z}-\frac{\pi^2}{3}+2\right)H_0(\bar{z})$$
$$+\left(3\bar{z}-\frac{2\pi^2}{3}-3\right)H_1(\bar{z})+7H_{1,0}(\bar{z})+3H_{1,2}(\bar{z})-2H_{2,0}(\bar{z})-6H_{1,0,0}(\bar{z})$$
$$-H_{1,1,0}(\bar{z})+9\zeta(3)+\frac{7\pi^2}{6}-14\Big)$$
$$+\epsilon^2\Big(\pi^2\left(\frac{43}{6}-\frac{5\bar{z}}{3}\right)+7(17\bar{z}+9\zeta(3)-17)+\left(-6\bar{z}+\frac{4\pi^2}{3}+6\right)H_2(\bar{z})$$
$$+\left(-28\bar{z}-18\zeta(3)-\frac{7\pi^2}{3}+28\right)H_0(\bar{z})+\left(42\bar{z}-8\zeta(3)-\frac{14\pi^2}{3}-42\right)H_1(\bar{z})$$
$$+\frac{2}{3}\left(6\bar{z}+\pi^2-6\right)H_{0,0}(\bar{z})+\left(-6\bar{z}+\pi^2+39\right)H_{1,0}(\bar{z})+\left(9\bar{z}+\frac{2\pi^2}{3}-9\right)H_{1,1}(\bar{z})$$
$$+21H_{1,2}(\bar{z})-18H_{1,3}(\bar{z})-14H_{2,0}(\bar{z})-6H_{2,2}(\bar{z})+4H_{3,0}(\bar{z})-42H_{1,0,0}(\bar{z})$$
$$-7H_{1,1,0}(\bar{z})-3H_{1,1,2}(\bar{z})-2H_{1,2,0}(\bar{z})+9H_{1,2,1}(\bar{z})+12H_{2,0,0}(\bar{z})+2H_{2,1,0}(\bar{z})$$
$$+28H_{1,0,0,0}(\bar{z})+6H_{1,1,0,0}(\bar{z})+H_{1,1,1,0}(\bar{z})+\frac{17\pi^4}{60}\Big)$$
$$+\epsilon^3\Big((796-30\zeta(3))\bar{z}+\pi^2\left(-\frac{70\bar{z}}{3}-17\zeta(3)+\frac{271}{6}\right)+\Big(\frac{1}{3}\pi^2(10\bar{z}-43)$$
$$-14(17\bar{z}+9\zeta(3)-17)-\frac{17\pi^4}{30}\Big)H_0(\bar{z})+\Big(-\pi^2(5\bar{z}+17)+7(51\bar{z}$$

$$
\begin{aligned}
& -8\zeta(3)-51)+\frac{14\pi^4}{45}\Big)H_1(\bar z)+\left(-84\bar z+16\zeta(3)+\frac{28\pi^2}{3}+84\right)H_2(\bar z)\\
& -\frac{4}{3}\left(-9\bar z+2\pi^2+9\right)H_3(\bar z)+\left(56\bar z+36\zeta(3)+\frac{14\pi^2}{3}-56\right)H_{0,0}(\bar z)\\
& +\left(-84\bar z+2\zeta(3)+7\pi^2+215\right)H_{1,0}(\bar z)+\left(126\bar z+8\zeta(3)+\frac{14\pi^2}{3}-126\right)H_{1,1}(\bar z)\\
& +\left(-18\bar z-\frac{23\pi^2}{3}+117\right)H_{1,2}(\bar z)-126H_{1,3}(\bar z)+84H_{1,4}(\bar z)+36H_{2,3}(\bar z)\\
& +28H_{3,0}(\bar z)-2\left(-6\bar z+\pi^2+39\right)H_{2,0}(\bar z)+\left(-18\bar z-\frac{4\pi^2}{3}+18\right)H_{2,1}(\bar z)\\
& -42H_{2,2}(\bar z)+12H_{3,2}(\bar z)-8H_{4,0}(\bar z)-\frac{4}{3}\left(6\bar z+\pi^2-6\right)H_{0,0,0}(\bar z)\\
& -\frac{2}{3}\left(-18\bar z+\pi^2+315\right)H_{1,0,0}(\bar z)+\left(-3(6\bar z+5)-\pi^2\right)H_{1,1,0}(\bar z)\\
& +\left(27(\bar z-1)-\frac{2\pi^2}{3}\right)H_{1,1,1}(\bar z)-21H_{1,1,2}(\bar z)+18H_{1,1,3}(\bar z)\\
& -14H_{1,2,0}(\bar z)+63H_{1,2,1}(\bar z)-6H_{1,2,2}(\bar z)+20H_{1,3,0}(\bar z)-54H_{1,3,1}(\bar z)\\
& +84H_{2,0,0}(\bar z)+14H_{2,1,0}(\bar z)+6H_{2,1,2}(\bar z)+4H_{2,2,0}(\bar z)-18H_{2,2,1}(\bar z)\\
& -24H_{3,0,0}(\bar z)-4H_{3,1,0}(\bar z)+196H_{1,0,0,0}(\bar z)+42H_{1,1,0,0}(\bar z)+7H_{1,1,1,0}(\bar z)\\
& +3H_{1,1,1,2}(\bar z)+2H_{1,1,2,0}(\bar z)-9H_{1,1,2,1}(\bar z)-12H_{1,2,0,0}(\bar z)-22H_{1,2,1,0}(\bar z)\\
& +27H_{1,2,1,1}(\bar z)-56H_{2,0,0,0}(\bar z)-12H_{2,1,0,0}(\bar z)-2H_{2,1,1,0}(\bar z)-120H_{1,0,0,0,0}(\bar z)\\
& -28H_{1,1,0,0,0}(\bar z)-6H_{1,1,1,0,0}(\bar z)-H_{1,1,1,1,0}(\bar z)+207\zeta(5)+327\zeta(3)+\frac{119\pi^4}{60}\\
& -796\Big)+\mathcal{O}(\epsilon^4)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
F_{5\mathrm{B}25} = \frac{n_0(\epsilon)}{z}\Bigg[& -((z-1)zH_2(z))-(z-2)(z-1)H_{1,1}(z)\\
& +\epsilon\Big(-11(z-1)zH_2(z)-(z-1)zH_3(z)-11(z-2)(z-1)H_{1,1}(z)\\
& +\left(-6z^2+8z-2\right)H_{1,2}(z)+3(z-1)zH_{2,0}(z)-2(z-1)(4z-3)H_{2,1}(z)\\
& +3(z-2)(z-1)H_{1,1,0}(z)-(z-1)(13z-22)H_{1,1,1}(z)\Big)+\mathcal{O}(\epsilon^2)\Bigg],
\end{aligned}
$$

$$F_{5\text{B}26} = \frac{n_0(\epsilon)}{\epsilon}\Bigg[\frac{1}{2}H_{1,1}(z) + \epsilon\Big(2H_{1,1}(z) + H_{1,2}(z) + 2H_{2,1}(z) - \frac{3}{2}H_{1,1,0}(z) + \frac{13}{2}H_{1,1,1}(z)\Big) \\ + \mathcal{O}(\epsilon^2)\Bigg],$$

$$F_{5\text{B}27} = \frac{n_0(\epsilon)}{\epsilon^2}\Bigg[- H_3(z) - H_{1,2}(z) + \epsilon\Big(- 2H_3(z) - 2H_{1,2}(z) - 5H_{1,3}(z) - H_{2,2}(z) \\ + 3H_{3,0}(z) - 6H_{3,1}(z) - 6H_{1,1,2}(z) + 3H_{1,2,0}(z) - 6H_{1,2,1}(z)\Big) \\ + \epsilon^2\Big(\left(-4 + \frac{5}{3}\pi^2\right) H_3(z) - H_5(z) + \left(-4 + \frac{5}{3}\pi^2\right) H_{1,2}(z) - 10H_{1,3}(z) \\ - H_{1,4}(z) - 2H_{2,2}(z) + 3H_{2,3}(z) + 6H_{3,0}(z) - 12H_{3,1}(z) + 3H_{3,2}(z) \\ - 12H_{1,1,2}(z) - 22H_{1,1,3}(z) + 6H_{1,2,0}(z) - 12H_{1,2,1}(z) - 2H_{1,2,2}(z) + 15H_{1,3,0}(z) \\ - 30H_{1,3,1}(z) + 2H_{2,1,2}(z) + 3H_{2,2,0}(z) - 6H_{2,2,1}(z) - 9H_{3,0,0}(z) + 18H_{3,1,0}(z) \\ - 28H_{3,1,1}(z) - 28H_{1,1,1,2}(z) + 18H_{1,1,2,0}(z) - 36H_{1,1,2,1}(z) - 9H_{1,2,0,0}(z) \\ + 18H_{1,2,1,0}(z) - 28H_{1,2,1,1}(z)\Big) + \mathcal{O}(\epsilon^3)\Bigg],$$

$$F_{5\text{B}28} = \frac{n_0(\epsilon)}{\epsilon^3\bar{z}}\Bigg[\frac{1}{2}H_{1,0}(z) + \epsilon\Big(- \frac{1}{6}\pi^2 H_1(z) + H_{1,0}(z) + H_{1,2}(z) - \frac{7}{2}H_{1,0,0}(z) + 2H_{1,1,0}(z)\Big) \\ + \epsilon^2\Big(\left(-\frac{\pi^2}{3} - \zeta(3)\right) H_1(z) + \left(2 - \frac{\pi^2}{6}\right) H_{1,0}(z) - \frac{2}{3}\pi^2 H_{1,1}(z) + 2H_{1,2}(z) \\ - 9H_{1,3}(z) - 7H_{1,0,0}(z) + 4H_{1,1,0}(z) + 2H_{1,1,2}(z) - 3H_{1,2,0}(z) + 2H_{1,2,1}(z) \\ + \frac{37}{2}H_{1,0,0,0}(z) - 14H_{1,1,0,0}(z) + 6H_{1,1,1,0}(z)\Big) \\ + \epsilon^3\Big(\left(-\frac{2\pi^2}{3} + \frac{41\pi^4}{180} - 2\zeta(3)\right) H_1(z) + \left(4 - \frac{\pi^2}{3} - 11\zeta(3)\right) H_{1,0}(z) \\ - \frac{4}{3}\left(\pi^2 + 3\zeta(3)\right) H_{1,1}(z) + \left(4 - \frac{5\pi^2}{3}\right) H_{1,2}(z) - 18H_{1,3}(z) + 37H_{1,4}(z) \\ + \left(-14 + \frac{19\pi^2}{6}\right) H_{1,0,0}(z) + \left(8 - \frac{2\pi^2}{3}\right) H_{1,1,0}(z) - 2\pi^2 H_{1,1,1}(z) + 4H_{1,1,2}(z)$$

$$
\begin{aligned}
&- 42H_{1,1,3}(z) - 6H_{1,2,0}(z) + 4H_{1,2,1}(z) - 8H_{1,2,2}(z) + 27H_{1,3,0}(z) - 26H_{1,3,1}(z) \\
&+ 37H_{1,0,0,0}(z) - 28H_{1,1,0,0}(z) + 12H_{1,1,1,0}(z) - 4H_{1,1,1,2}(z) - 6H_{1,1,2,0}(z) \\
&- 4H_{1,1,2,1}(z) + 9H_{1,2,0,0}(z) - 6H_{1,2,1,0}(z) + 4H_{1,2,1,1}(z) - \frac{175}{2}H_{1,0,0,0,0}(z) \\
&+ 74H_{1,1,0,0,0}(z) - 42H_{1,1,1,0,0}(z) + 16H_{1,1,1,1,0}(z)\Big) + \mathcal{O}(\epsilon^4)\Bigg] ,
\end{aligned}
$$

$$
\begin{aligned}
F_{5\mathrm{B}29} = \frac{n_0(\epsilon)}{z^2}\Bigg[&- \frac{1}{2}(z-1)z^2 - \frac{3}{2}(z-1)H_1(z)z^2 + \frac{1}{2}\left(z^2-1\right)H_2(z)z^2 - H_3(z)z^2 \\
&+ H_{2,1}(z)z^2 + \frac{1}{2}\left(z^4 - 4z^3 + z^2 + 4z - 2\right)H_{1,1}(z) + \mathcal{O}(\epsilon)\Bigg] ,
\end{aligned}
$$

$$
\begin{aligned}
F_{5\mathrm{B}30} = \frac{n_0(\epsilon)}{x}\Bigg[&\frac{1}{2}G_{-1,0,-r_3}(x)(x-1)^2 + \frac{1}{2}G_{-1,0,r_3}(x)(x-1)^2 + 2G_{1,-1,-1}(x)(x-1)^2 \\
&- 3G_{1,-1,0}(x)(x-1)^2 + 2G_{1,-1,1}(x)(x-1)^2 - \frac{1}{2}G_{1,-1,-r_3}(x)(x-1)^2 \\
&- \frac{1}{2}G_{1,-1,r_3}(x)(x-1)^2 + 2G_{1,1,-1}(x)(x-1)^2 - 3G_{1,1,0}(x)(x-1)^2 \\
&+ 2G_{1,1,1}(x)(x-1)^2 - \frac{1}{2}G_{1,1,-r_3}(x)(x-1)^2 - \frac{1}{2}G_{1,1,r_3}(x)(x-1)^2 \\
&+ \left(3(x+1)^2H_0(2) - \frac{3}{2}i\pi(x+1)^2\right)G_{-1,-1}(x) \\
&+ \left(4i\pi x - 8xH_0(2)\right)G_{-1,0}(x) + \left(3(x+1)^2H_0(2) - \frac{3}{2}i\pi(x+1)^2\right)G_{-1,1}(x) \\
&+ \left(\frac{1}{2}i\pi\left(3x^2+2x+3\right) + \left(-3x^2-2x-3\right)H_0(2)\right)G_{-1,-r_3}(x) \\
&+ \left(\frac{1}{2}i\pi\left(3x^2+2x+3\right) + \left(-3x^2-2x-3\right)H_0(2)\right)G_{-1,r_3}(x) \\
&+ \left(2i\pi x - 4xH_0(2)\right)G_{0,-1}(x) + \left(2i\pi x - 4xH_0(2)\right)G_{0,1}(x) \\
&+ \left(4xH_0(2) - 2i\pi x\right)G_{0,-r_3}(x) + \left(4xH_0(2) - 2i\pi x\right)G_{0,r_3}(x) \\
&+ \left(\frac{3}{2}i\pi(x-1)^2 - 3(x-1)^2H_0(2)\right)G_{1,-1}(x) + \left(4i\pi x - 8xH_0(2)\right)G_{1,0}(x) \\
&+ \left(\frac{3}{2}i\pi(x-1)^2 - 3(x-1)^2H_0(2)\right)G_{1,1}(x)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\left(3x^2-2x+3\right)H_0(2)-\frac{1}{2}i\pi\left(3x^2-2x+3\right)\right)G_{1,-r_3}(x)\\
&+\left(\left(3x^2-2x+3\right)H_0(2)-\frac{1}{2}i\pi\left(3x^2-2x+3\right)\right)G_{1,r_3}(x)\\
&+\left(2i\pi x-4xH_0(2)\right)G_{-r_3,-1}(x)+\left(8xH_0(2)-4i\pi x\right)G_{-r_3,0}(x)\\
&+\left(2i\pi x-4xH_0(2)\right)G_{-r_3,1}(x)+\left(2i\pi x-4xH_0(2)\right)G_{r_3,-1}(x)\\
&+\left(8xH_0(2)-4i\pi x\right)G_{r_3,0}(x)+\left(2i\pi x-4xH_0(2)\right)G_{r_3,1}(x)\\
&+G_1(x)\left(-\frac{1}{6}\pi^2\left(x^2-6x+1\right)+4i\pi xH_0(2)-8xH_{0,0}(2)\right)\\
&+G_{-1}(x)\left(\frac{1}{6}\pi^2\left(x^2+6x+1\right)+4i\pi xH_0(2)-8xH_{0,0}(2)\right)\\
&+G_{-r_3}(x)\left(-4i\pi H_0(2)x+8H_{0,0}(2)x-\pi^2x\right)\\
&+G_{r_3}(x)\left(-4i\pi H_0(2)x+8H_{0,0}(2)x-\pi^2x\right)-2(x+1)^2G_{-1,-1,-1}(x)\\
&+3(x+1)^2G_{-1,-1,0}(x)-2(x+1)^2G_{-1,-1,1}(x)+\frac{1}{2}(x+1)^2G_{-1,-1,-r_3}(x)\\
&+\frac{1}{2}(x+1)^2G_{-1,-1,r_3}(x)+\left(-\frac{x^2}{2}+5x-\frac{1}{2}\right)G_{-1,0,-1}(x)-8xG_{-1,0,0}(x)\\
&+\left(-\frac{x^2}{2}+5x-\frac{1}{2}\right)G_{-1,0,1}(x)-2(x+1)^2G_{-1,1,-1}(x)+3(x+1)^2G_{-1,1,0}(x)\\
&-2(x+1)^2G_{-1,1,1}(x)+\frac{1}{2}(x+1)^2G_{-1,1,-r_3}(x)+\frac{1}{2}(x+1)^2G_{-1,1,r_3}(x)\\
&+\frac{3}{4}\left(3x^2+2x+3\right)G_{-1,-r_3,-1}(x)+\left(-3x^2-2x-3\right)G_{-1,-r_3,0}(x)\\
&+\frac{3}{4}\left(3x^2+2x+3\right)G_{-1,-r_3,1}(x)+\frac{1}{4}\left(-3x^2-2x-3\right)G_{-1,-r_3,-r_3}(x)\\
&+\frac{1}{4}\left(-3x^2-2x-3\right)G_{-1,-r_3,r_3}(x)+\frac{3}{4}\left(3x^2+2x+3\right)G_{-1,r_3,-1}(x)\\
&+\left(-3x^2-2x-3\right)G_{-1,r_3,0}(x)+\frac{3}{4}\left(3x^2+2x+3\right)G_{-1,r_3,1}(x)\\
&+\frac{1}{4}\left(-3x^2-2x-3\right)G_{-1,r_3,-r_3}(x)+\frac{1}{4}\left(-3x^2-2x-3\right)G_{-1,r_3,r_3}(x)\\
&+4xG_{0,-1,-1}(x)-4xG_{0,-1,0}(x)+4xG_{0,-1,1}(x)-2xG_{0,-1,-r_3}(x)-2xG_{0,-1,r_3}(x)\\
&-2xG_{0,0,-1}(x)-2xG_{0,0,1}(x)+2xG_{0,0,-r_3}(x)+2xG_{0,0,r_3}(x)+4xG_{0,1,-1}(x)\\
&-4xG_{0,1,0}(x)+4xG_{0,1,1}(x)-2xG_{0,1,-r_3}(x)-2xG_{0,1,r_3}(x)-3xG_{0,-r_3,-1}(x)\\
&+4xG_{0,-r_3,0}(x)-3xG_{0,-r_3,1}(x)+xG_{0,-r_3,-r_3}(x)+xG_{0,-r_3,r_3}(x)-3xG_{0,r_3,-1}(x)
\end{aligned}
$$

$$
\begin{aligned}
&+ 4xG_{0,r_3,0}(x) - 3xG_{0,r_3,1}(x) + xG_{0,r_3,-r_3}(x) + xG_{0,r_3,r_3}(x) \\
&+ \frac{1}{2}\left(x^2+10x+1\right)G_{1,0,-1}(x) - 8xG_{1,0,0}(x) + \frac{1}{2}\left(x^2+10x+1\right)G_{1,0,1}(x) \\
&- \frac{1}{2}(x+1)^2 G_{1,0,-r_3}(x) - \frac{1}{2}(x+1)^2 G_{1,0,r_3}(x) - \frac{3}{4}\left(3x^2-2x+3\right)G_{1,-r_3,-1}(x) \\
&+ \left(3x^2-2x+3\right)G_{1,-r_3,0}(x) - \frac{3}{4}\left(3x^2-2x+3\right)G_{1,-r_3,1}(x) \\
&+ \frac{1}{4}\left(3x^2-2x+3\right)G_{1,-r_3,-r_3}(x) + \frac{1}{4}\left(3x^2-2x+3\right)G_{1,-r_3,r_3}(x) \\
&- \frac{3}{4}\left(3x^2-2x+3\right)G_{1,r_3,-1}(x) + \left(3x^2-2x+3\right)G_{1,r_3,0}(x) \\
&- \frac{3}{4}\left(3x^2-2x+3\right)G_{1,r_3,1}(x) + \frac{1}{4}\left(3x^2-2x+3\right)G_{1,r_3,-r_3}(x) \\
&+ \frac{1}{4}\left(3x^2-2x+3\right)G_{1,r_3,r_3}(x) + 2xG_{-r_3,-1,-1}(x) - 4xG_{-r_3,-1,0}(x) \\
&+ 2xG_{-r_3,-1,1}(x) - 4xG_{-r_3,0,-1}(x) + 8xG_{-r_3,0,0}(x) - 4xG_{-r_3,0,1}(x) \\
&+ 2xG_{-r_3,1,-1}(x) - 4xG_{-r_3,1,0}(x) + 2xG_{-r_3,1,1}(x) + 2xG_{r_3,-1,-1}(x) \\
&- 4xG_{r_3,-1,0}(x) + 2xG_{r_3,-1,1}(x) - 4xG_{r_3,0,-1}(x) + 8xG_{r_3,0,0}(x) \\
&\left. - 4xG_{r_3,0,1}(x) + 2xG_{r_3,1,-1}(x) - 4xG_{r_3,1,0}(x) + 2xG_{r_3,1,1}(x) + \mathcal{O}(\epsilon)\right],
\end{aligned}
$$

$$
\begin{aligned}
F_{5B31} = \frac{n_0(\epsilon)}{\epsilon^2\bar{z}}&\left[\frac{1}{6}H_1(z) + \epsilon\Big(\frac{2}{3}H_1(z) - \frac{5}{6}H_{1,0}(z) + \frac{5}{3}H_{1,1}(z)\Big)\right. \\
&+ \epsilon^2\Big(\frac{100}{9}\pi^2 H_0(2) + 2H_1(z) + \left(\frac{1}{3}i\pi(9x+41) + \left(-6x-\frac{82}{3}\right)H_0(2)\right)G_{-1,-1}(x) \\
&+ \left(\frac{152}{3}H_0(2) - \frac{76}{3}i\pi\right)G_{-1,0}(x) + \left((6x+2)H_0(2) - i(3\pi x+\pi)\right)G_{-1,-\mathrm{r3}}(x) \\
&+ \left(\frac{1}{3}i\pi(9x+41) + \left(-6x-\frac{82}{3}\right)H_0(2)\right)G_{-1,1}(x) \\
&+ \left((6x+2)H_0(2) - i(3\pi x+\pi)\right)G_{-1,\mathrm{r3}}(x) + \left(\frac{164}{3}H_0(2) - \frac{82}{3}i\pi\right)G_{0,-1}(x) \\
&+ \left(\frac{152}{3}i\pi - \frac{304}{3}H_0(2)\right)G_{0,0}(x) + \left(\frac{164}{3}H_0(2) - \frac{82}{3}i\pi\right)G_{0,1}(x) \\
&+ \left(2i\pi - 4H_0(2)\right)G_{0,-\mathrm{r3}}(x) + \left(2i\pi - 4H_0(2)\right)G_{0,\mathrm{r3}}(x)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{1}{3}i\pi(41-9x)+\left(6x-\frac{82}{3}\right)H_0(2)\right)G_{1,-1}(x)\\
&+\left(\frac{152}{3}H_0(2)-\frac{76i\pi}{3}\right)G_{1,0}(x)+\left(\frac{1}{3}i\pi(41-9x)+\left(6x-\frac{82}{3}\right)H_0(2)\right)G_{1,1}(x)\\
&+(i\pi(3x-1)+(2-6x)H_0(2))\,G_{1,-\mathrm{r3}}(x)+(i\pi(3x-1)+(2-6x)H_0(2))\,G_{1,\mathrm{r3}}(x)\\
&+G_0(x)\left(\frac{152}{3}i\pi H_0(2)-\frac{304}{3}H_{0,0}(2)+\frac{100\pi^2}{9}\right)+\frac{152}{3}i\pi H_{0,0}(2)\\
&+G_1(x)\left(\frac{1}{9}\pi^2(3x-50)-\frac{76}{3}i\pi H_0(2)+\frac{152}{3}H_{0,0}(2)\right)+\left(4x+\frac{145}{6}\right)G_{-1,-1,-1}(x)\\
&+G_{-1}(x)\left(-\frac{1}{9}\pi^2(3x+50)-\frac{76}{3}i\pi H_0(2)+\frac{152}{3}H_{0,0}(2)\right)-\frac{10}{3}H_{1,0}(z)+\frac{20}{3}H_{1,1}(z)\\
&+\left(-6x-\frac{82}{3}\right)G_{-1,-1,0}(x)+\left(4x+\frac{145}{6}\right)G_{-1,-1,1}(x)+\left(-x-\frac{21}{2}\right)G_{-1,-1,-\mathrm{r3}}(x)\\
&+\left(-x-\frac{21}{2}\right)G_{-1,-1,\mathrm{r3}}(x)+(x-39)G_{-1,0,-1}(x)+\frac{152}{3}G_{-1,0,0}(x)\\
&+(x-39)G_{-1,0,1}(x)+\left(\frac{41}{3}-x\right)G_{-1,0,-\mathrm{r3}}(x)+\left(\frac{41}{3}-x\right)G_{-1,0,\mathrm{r3}}(x)\\
&+\left(4x+\frac{145}{6}\right)G_{-1,1,-1}(x)+\left(-6x-\frac{82}{3}\right)G_{-1,1,0}(x)+\left(4x+\frac{145}{6}\right)G_{-1,1,1}(x)\\
&+\left(-x-\frac{21}{2}\right)G_{-1,1,-\mathrm{r3}}(x)+\left(-x-\frac{21}{2}\right)G_{-1,1,\mathrm{r3}}(x)\\
&+\left(-\frac{9x}{2}-\frac{14}{3}\right)G_{-1,-\mathrm{r3},-1}(x)+(6x+2)G_{-1,-\mathrm{r3},0}(x)+\left(-\frac{9x}{2}-\frac{14}{3}\right)G_{-1,-\mathrm{r3},1}(x)\\
&+\frac{1}{6}(9x+22)G_{-1,-\mathrm{r3},-\mathrm{r3}}(x)+\frac{1}{6}(9x+22)G_{-1,-\mathrm{r3},\mathrm{r3}}(x)+\left(-\frac{9x}{2}-\frac{14}{3}\right)G_{-1,\mathrm{r3},-1}(x)\\
&+(6x+2)G_{-1,\mathrm{r3},0}(x)+\left(-\frac{9x}{2}-\frac{14}{3}\right)G_{-1,\mathrm{r3},1}(x)+\frac{1}{6}(9x+22)G_{-1,\mathrm{r3},-\mathrm{r3}}(x)\\
&+\frac{1}{6}(9x+22)G_{-1,\mathrm{r3},\mathrm{r3}}(x)-\frac{145}{3}G_{0,-1,-1}(x)+\frac{164}{3}G_{0,-1,0}(x)-\frac{145}{3}G_{0,-1,1}(x)\\
&+21G_{0,-1,-\mathrm{r3}}(x)+21G_{0,-1,\mathrm{r3}}(x)+78G_{0,0,-1}(x)-\frac{304}{3}G_{0,0,0}(x)+78G_{0,0,1}(x)\\
&-\frac{82}{3}G_{0,0,-\mathrm{r3}}(x)-\frac{82}{3}G_{0,0,\mathrm{r3}}(x)-\frac{145}{3}G_{0,1,-1}(x)+\frac{164}{3}G_{0,1,0}(x)-\frac{145}{3}G_{0,1,1}(x)\\
&+21G_{0,1,-\mathrm{r3}}(x)+21G_{0,1,\mathrm{r3}}(x)+\frac{28}{3}G_{0,-\mathrm{r3},-1}(x)-4G_{0,-\mathrm{r3},0}(x)+\frac{28}{3}G_{0,-\mathrm{r3},1}(x)\\
&-\frac{22}{3}G_{0,-\mathrm{r3},-\mathrm{r3}}(x)-\frac{22}{3}G_{0,-\mathrm{r3},\mathrm{r3}}(x)+\frac{28}{3}G_{0,\mathrm{r3},-1}(x)-4G_{0,\mathrm{r3},0}(x)+\frac{28}{3}G_{0,\mathrm{r3},1}(x)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{22}{3} G_{0,\mathrm{r3},-\mathrm{r3}}(x) - \frac{22}{3} G_{0,\mathrm{r3},\mathrm{r3}}(x) + \left(\frac{145}{6} - 4x\right) G_{1,-1,-1}(x) \\
&+ \left(6x - \frac{82}{3}\right) G_{1,-1,0}(x) + \left(\frac{145}{6} - 4x\right) G_{1,-1,1}(x) + \left(x - \frac{21}{2}\right) G_{1,-1,-\mathrm{r3}}(x) \\
&+ \left(x - \frac{21}{2}\right) G_{1,-1,\mathrm{r3}}(x) + (-x-39) G_{1,0,-1}(x) + \frac{152}{3} G_{1,0,0}(x) \\
&+ (-x-39) G_{1,0,1}(x) + \left(x + \frac{41}{3}\right) G_{1,0,-\mathrm{r3}}(x) + \left(x + \frac{41}{3}\right) G_{1,0,\mathrm{r3}}(x) \\
&+ \left(\frac{145}{6} - 4x\right) G_{1,1,-1}(x) + \left(6x - \frac{82}{3}\right) G_{1,1,0}(x) + \left(\frac{145}{6} - 4x\right) G_{1,1,1}(x) \\
&+ \left(x - \frac{21}{2}\right) G_{1,1,-\mathrm{r3}}(x) + \left(x - \frac{21}{2}\right) G_{1,1,\mathrm{r3}}(x) + \left(\frac{9x}{2} - \frac{14}{3}\right) G_{1,-\mathrm{r3},-1}(x) \\
&+ (2-6x) G_{1,-\mathrm{r3},0}(x) + \left(\frac{9x}{2} - \frac{14}{3}\right) G_{1,-\mathrm{r3},1}(x) + \frac{1}{6}(22-9x) G_{1,-\mathrm{r3},-\mathrm{r3}}(x) \\
&+ \frac{1}{6}(22-9x) G_{1,-\mathrm{r3},\mathrm{r3}}(x) + \left(\frac{9x}{2} - \frac{14}{3}\right) G_{1,\mathrm{r3},-1}(x) + (2-6x) G_{1,\mathrm{r3},0}(x) \\
&+ \left(\frac{9x}{2} - \frac{14}{3}\right) G_{1,\mathrm{r3},1}(x) + \frac{1}{6}(22-9x) G_{1,\mathrm{r3},-\mathrm{r3}}(x) + \frac{1}{6}(22-9x) G_{1,\mathrm{r3},\mathrm{r3}}(x) \\
&- \frac{304}{3} H_{0,0,0}(2) - \frac{7}{6}\zeta(3) - \frac{4}{3} i\pi^3 \Big) + \mathcal{O}(\epsilon^3) \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
F_{5\mathrm{B}32} = {} & \frac{n_0(\epsilon)}{\epsilon^4 \bar{z}} \Bigg[\frac{1}{12} + \epsilon\Big(- \frac{5}{12} H_0(z) + \frac{1}{6} H_1(z) + \frac{1}{3}\Big) + \epsilon^2\Big(- \frac{5}{3} H_0(z) + \frac{2}{3} H_1(z) \\
&- \frac{3H_2(z)}{2} + \frac{25}{12} H_{0,0}(z) - \frac{1}{3} H_{1,0}(z) - \frac{1}{3} H_{1,1}(z) - \frac{\pi^2}{18} + 1\Big) \\
&+ \epsilon^3\Big(\left(-5 + \frac{5}{18}\pi^2\right) H_0(z) + \left(2 - \frac{5}{18}\pi^2\right) H_1(z) - 6H_2(z) + \frac{13}{2} H_3(z) \\
&+ \frac{25}{3} H_{0,0}(z) - \frac{4}{3} H_{1,0}(z) - \frac{4}{3} H_{1,1}(z) - 3H_{1,2}(z) + 5H_{2,0}(z) - 5H_{2,1}(z) \\
&- \frac{125}{12} H_{0,0,0}(z) + \frac{2}{3} H_{1,0,0}(z) + \frac{8}{3} H_{1,1,0}(z) - \frac{22}{3} H_{1,1,1}(z) - 3\zeta(3) - \frac{2}{9}\pi^2 + \frac{8}{3}\Big) \\
&+ \mathcal{O}(\epsilon^4) \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
F_{5\mathrm{B}33} = n_0(\epsilon)\Bigg[&\frac{1}{12}H_2(z)z^4 - \frac{1}{12}H_{0,0}(z)z^4 + \frac{1}{72}\left(-\pi^2 z^3 + 4z^2 + z - 5\right)z \\
&- \frac{1}{72}\left(2z^2 + 3z + 6\right)H_0(z)z + \frac{1}{72}\left(-6z^3 - 3z^2 - 2z + 11\right)H_1(z) \\
&+ \frac{1}{12}\left(1 - z^4\right)H_{1,0}(z) + \frac{1}{12}\left(z^4 - 1\right)H_{1,1}(z) \\
&+ \epsilon\Big(\frac{161}{144}H_2(z)z^4 - \frac{7}{12}H_3(z)z^4 + \frac{1}{6}H_{2,1}(z)z^4 + H_{0,0,0}(z)z^4 \\
&+ \frac{1}{864}\left(\left(-161\pi^2 + 360\zeta(3)\right)z^3 + 772z^2 + 193z - 965\right)z \\
&+ \frac{1}{864}\left(84\pi^2 z^3 - 578z^2 - 603z - 930\right)H_0(z)z \\
&+ \frac{1}{144}\left(-161z^3 + 20z^2 + 30z + 60\right)H_{0,0}(z)z \\
&+ \frac{1}{864}\left(-1014z^3 - 435z^2 - 446z + 1895\right)H_1(z) \\
&+ \frac{1}{144}\left(-161z^4 + 52z^3 + 18z^2 - 4z + 95\right)H_{1,0}(z) \\
&+ \frac{1}{144}\left(161z^4 - 76z^3 - 30z^2 - 4z - 51\right)H_{1,1}(z) \\
&+ \frac{5}{12}\left(z^4 - 1\right)H_{1,0,0}(z) - \frac{7}{12}\left(z^4 - 1\right)H_{1,1,0}(z) + \frac{3}{4}\left(z^4 - 1\right)H_{1,1,1}(z)\Big) + \mathcal{O}(\epsilon^2)\Bigg],
\end{aligned}
$$

$$
F_{5\mathrm{B}34} = n_0(\epsilon)\Big[0 + \mathcal{O}(\epsilon)\Big],
$$

$$
\begin{aligned}
F_{5\mathrm{B}35} = \frac{n_0(\epsilon)}{\epsilon^2}\Bigg[&\frac{\pi^2}{12} + H_{-1,0}(1-z) - H_{1,1}(z) + \epsilon\Big(-\frac{11}{12}\pi^2 H_{-1}(1-z) \\
&+ 4H_{-1,0}(1-z) + 3H_{-1,2}(1-z) - 4H_{1,1}(z) - 2H_{2,1}(z) - 5H_{-1,-1,0}(1-z) \\
&- 4H_{-1,0,0}(1-z) + 3H_{1,1,0}(z) - 11H_{1,1,1}(z) + \frac{1}{6}\left(2\pi^2 + 33\zeta(3)\right)\Big) + \mathcal{O}(\epsilon^2)\Bigg],
\end{aligned}
$$

$$
F_{5\mathrm{B}36} = \frac{n_0(\epsilon)}{\bar{z}\epsilon^3}\Bigg[-\frac{1}{6}H_1(z) + \epsilon\Big(H_0(2-z)\left(H_0(z-1) - i\pi\right) - \frac{2}{3}H_1(z)
$$

$$
\begin{aligned}
&+ H_{1,0}(2-z) + \frac{5}{6}H_{1,0}(z) - \frac{5}{3}H_{1,1}(z) + \frac{\pi^2}{4}\Big) \\
&+ \epsilon^2\Big(H_0(2-z)\left(-2i\pi H_1(z) + H_0(z-1)\left(2H_1(z)+4\right) - \frac{1}{4}\pi(16i+7\pi)\right) \\
&+ 3H_{-1,2}(1-z) - 5H_0(z-1)H_{0,0}(2-z) + 5i\pi H_{0,0}(2-z) + 4H_{1,0}(2-z) \\
&+ H_1(z)\left(2H_{1,0}(2-z) + \frac{11}{18}\pi^2 - 2\right) + \frac{10}{3}H_{1,0}(z) - \frac{20}{3}H_{1,1}(z) + 2H_{1,2}(z) \\
&- 5H_{2,0}(2-z) - 5H_{1,0,0}(2-z) - \frac{25}{6}H_{1,0,0}(z) + \frac{19}{3}H_{1,1,0}(z) - \frac{38}{3}H_{1,1,1}(z) \\
&- \frac{5}{2}\zeta(3) + \pi^2\Big) + \mathcal{O}(\epsilon^3)\Bigg] .
\end{aligned}
$$

Appendix C

Differential Equation Matrices for the Four-Body Integrals

All necessary matrices for the computation of the four-body integrals are collected in this chapter. They can be split into two parts and are of the form

$$\hat{A}_{i,\epsilon} = \frac{1}{\bar{z}}\hat{A}_{i,z} + \frac{1}{\bar{z}}\hat{A}_{i,\bar{z}}\,.$$

C.1 $\hat{\mathbf{A}}_1$

For the first family, we get:

$$\hat{A}_{1,z} = \begin{pmatrix} -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 3 & \frac{3}{2} & -3 & -3 & 0 & 0 \\ 0 & 2 & 0 & -2 & -\frac{2}{3} & 0 & 0 \\ 0 & 3 & -\frac{9}{2} & -3 & -1 & \frac{5}{2} & -3 \end{pmatrix},$$

$$\hat{A}_{1,\bar{z}} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & -\frac{9}{2} & \frac{15}{4} & \frac{9}{2} & \frac{3}{2} & -\frac{15}{4} & 0 \\ 0 & -3 & \frac{3}{2} & 3 & -1 & \frac{5}{2} & 0 \\ 0 & 3 & -\frac{9}{2} & -3 & -1 & \frac{5}{2} & 2 \end{pmatrix} .$$

C.2 $\hat{\mathbf{A}}_{\mathbf{2}}$

For the second set of equations, we arrive at:

$$\hat{A}_{2,z} = \begin{pmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{11}{3} & 0 & -3 & -\frac{5}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{55}{6} & 5 & -\frac{5}{3} & \frac{20}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{22}{3} & -4 & \frac{4}{3} & -\frac{16}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
54 & 0 & 0 & 0 & 11 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-42 & 0 & 0 & 0 & -10 & -11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
18 & 0 & 0 & 0 & \frac{16}{3} & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{56}{3} & 0 & -\frac{8}{11} & -\frac{10}{11} & -\frac{14}{3} & -\frac{14}{3} & 0 & 4 & 0 & 0 & \frac{20}{3} & 0 & -5 & 0 & 0 & 0 \\
64 & 0 & 0 & 0 & \frac{28}{3} & -4 & 0 & 0 & -8 & 0 & -40 & -4 & 0 & -2 & 0 & 0 \\
56 & 0 & \frac{24}{11} & \frac{30}{11} & 14 & 14 & 0 & -12 & 0 & 0 & -20 & 0 & 15 & 0 & 0 & 0 \\
\frac{4}{3} & 0 & 0 & 0 & \frac{19}{27} & \frac{13}{9} & 0 & 0 & 0 & 0 & -\frac{124}{15} & 0 & 0 & 0 & 0 & -2
\end{pmatrix},$$

$$\hat{A}_{2,\bar{z}} = \begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{55}{3} & 7 & -\frac{1}{3} & \frac{73}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{22}{3} & 2 & -\frac{2}{3} & \frac{7}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{44}{3} & -4 & \frac{4}{3} & -\frac{7}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
36 & 0 & 0 & 0 & 14 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-24 & 0 & 0 & 0 & -8 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-8 & -\frac{6}{11} & \frac{18}{11} & \frac{3}{22} & -2 & -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6 & 0 & 0 & 0 & -2 & -2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
42 & \frac{126}{11} & \frac{102}{11} & \frac{417}{22} & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
-10 & 0 & 0 & 0 & -5 & -5 & -45 & 0 & 0 & 0 & 10 & 5 & 0 & \frac{1}{2} & 0 & 0 \\
-\frac{46}{3} & \frac{60}{11} & \frac{72}{11} & \frac{105}{11} & -8 & -8 & -30 & 4 & 0 & 0 & \frac{40}{3} & 0 & 0 & 0 & 0 & 0 \\
64 & 0 & 0 & 0 & \frac{28}{3} & -4 & 0 & 0 & -8 & 0 & -40 & -4 & 0 & 2 & 0 & 0 \\
128 & 0 & 0 & 0 & \frac{106}{3} & 42 & 0 & 0 & 4 & 0 & -20 & 2 & 0 & 0 & 2 & 0 \\
\frac{164}{45} & 0 & 0 & 0 & \frac{82}{135} & \frac{44}{45} & -\frac{81}{10} & 0 & 0 & 0 & -\frac{16}{5} & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

C.3 $\hat{\mathbf{A}}_3$

For the last of the four-body families, the matrices take the form:

$$\hat{A}_{3,z} = \begin{pmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{11}{3} & 0 & -3 & -\frac{5}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{55}{6} & 5 & -\frac{5}{3} & \frac{20}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{22}{3} & -4 & \frac{4}{3} & -\frac{16}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{36}{35} & 0 & 0 & 0 & -19 & 0 & 0 & 72 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{2}{3} & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{32}{105} & 0 & 0 & 0 & -5 & 0 & 0 & 19 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
24 & 0 & 0 & 0 & -336 & 0 & 0 & 1134 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
72 & \frac{48}{11} & -\frac{12}{11} & \frac{45}{11} & -1008 & 0 & 0 & 4032 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\
40 & 0 & 0 & 0 & -630 & 0 & 0 & 2520 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
72 & 0 & 0 & 0 & -966 & 0 & 0 & 3024 & 4 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-50 & -\frac{60}{11} & 0 & -\frac{75}{11} & 630 & 0 & 0 & -2520 & 0 & 0 & 1 & 0 & 0 & 0 & -3
\end{pmatrix},$$

$$
\hat{A}_{3,\bar{z}} = \begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{55}{3} & 7 & -\frac{1}{3} & \frac{73}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{22}{3} & 2 & -\frac{2}{3} & \frac{7}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{44}{3} & -4 & \frac{4}{3} & -\frac{7}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{4}{7} & 0 & 0 & 0 & -10 & 0 & 0 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{2}{3} & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
42 & \frac{126}{11} & \frac{102}{11} & \frac{417}{22} & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{4}{21} & 0 & 0 & 0 & -4 & 0 & 0 & 18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6 & 0 & 0 & 0 & 126 & 0 & 0 & -504 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
52 & \frac{84}{11} & \frac{52}{11} & \frac{131}{11} & -504 & 0 & 0 & 2016 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{80}{11} & \frac{40}{11} & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
72 & 0 & 0 & 0 & -966 & 0 & 0 & 3024 & 4 & 0 & 0 & 0 & 4 & 0 & 0 \\
-72 & \frac{144}{11} & \frac{96}{11} & \frac{354}{11} & 1722 & 0 & 0 & -6048 & -4 & 6 & -6 & 60 & -2 & 2 & 0 \\
-40 & -\frac{48}{11} & -\frac{32}{11} & -\frac{118}{11} & 1008 & -60 & 0 & -4032 & 0 & -2 & 2 & -20 & 0 & 0 & 2
\end{pmatrix} .
$$

Appendix D

Differential Equation Matrices for the Five-Body Integrals

All matrices that are needed for the computations of the five-body master integrals are collected in this chapter. They split into three parts and are cast into the form:

$$\hat{A}_{i,\epsilon} = \frac{1}{\bar{z}}\hat{A}_{i,\bar{z}} + \frac{1}{1-\bar{z}}\hat{A}_{i,1-\bar{z}} + \frac{1}{1+\bar{z}}\hat{A}_{i,1+\bar{z}} \,.$$

D.1 $\hat{\mathbf{A}}_{101}$

For the first family these three matrices look as follows:

$$\hat{A}_{101,\bar{z}} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 \\ 24 & -27 & 12 & -48 & 0 \\ 6 & -6 & 2 & -8 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} ,$$

$$\hat{A}_{101,1-\bar{z}} = \begin{pmatrix} -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 9 & -27 & 3 & -27 & 0 \\ 1 & -6 & \frac{2}{3} & -6 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix},$$

$$\hat{A}_{101,1+\bar{z}} = \left(\hat{\mathbf{0}}_{\mathbf{5\times5}}\right).$$

D.2 $\hat{\mathbf{A}}_{\mathbf{102}}$

For the second family, we get the following coefficient matrices:

$$\hat{A}_{102,\bar{z}} = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{52}{5} & 2 & -\frac{121}{10} & 6 & -\frac{54}{5} & 0 & 0 & -\frac{29}{2} & 0 \\ -\frac{1415}{4} & -\frac{741}{8} & \frac{6557}{16} & -\frac{1523}{4} & -\frac{4401}{4} & 0 & 0 & \frac{14365}{16} & 0 \\ \frac{199}{270} & \frac{11}{36} & -\frac{1069}{1080} & \frac{55}{54} & \frac{11}{10} & 0 & 0 & -\frac{521}{216} & 0 \\ \frac{1}{2} & -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 & 0 & 12 & -4 & 0 & 0 \\ -151 & -\frac{81}{2} & \frac{703}{4} & -161 & -459 & 0 & 0 & \frac{1519}{4} & 0 \\ -18 & 20 & -6 & -7 & -\frac{81}{2} & 0 & 0 & 16 & -2 \end{pmatrix},$$

$$\hat{A}_{102,1-\bar{z}} = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{267}{20} & \frac{9}{10} & -\frac{73}{40} & \frac{5}{4} & -\frac{279}{40} & 0 & 0 & -\frac{367}{120} & 0 \\ -\frac{11769}{8} & -\frac{189}{2} & \frac{1533}{8} & -\frac{525}{4} & \frac{5859}{8} & 0 & 0 & \frac{2569}{8} & 0 \\ \frac{799}{180} & \frac{1}{5} & -\frac{73}{180} & \frac{5}{18} & -\frac{31}{20} & 0 & 0 & -\frac{367}{540} & 0 \\ -2 & -\frac{3}{2} & 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ -5 & -6 & 0 & 0 & 0 & 4 & 2 & 0 & 0 \\ -\frac{2481}{4} & -\frac{81}{2} & \frac{657}{8} & -\frac{225}{4} & \frac{2511}{8} & 0 & 0 & \frac{1101}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{A}_{102,1+\bar{z}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{171}{20} & -\frac{21}{10} & -\frac{3}{8} & -\frac{3}{5} & -\frac{135}{4} & 0 & 0 & \frac{49}{40} & 0 \\ -\frac{8949}{8} & \frac{1099}{4} & \frac{785}{16} & \frac{157}{2} & \frac{35325}{8} & 0 & 0 & -\frac{7693}{48} & 0 \\ \frac{209}{60} & -\frac{77}{90} & -\frac{11}{72} & -\frac{11}{45} & -\frac{55}{4} & 0 & 0 & \frac{539}{1080} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1881}{4} & \frac{231}{2} & \frac{165}{8} & 33 & \frac{7425}{4} & 0 & 0 & -\frac{539}{8} & 0 \\ 18 & -20 & 6 & 7 & \frac{81}{2} & 0 & 0 & -16 & -3 \end{pmatrix}.$$

D.3 $\hat{\mathbf{A}}_{\mathbf{103}}$

For F_{103} the matrices take the form:

$$\hat{A}_{103,\bar{z}} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & -27 & 12 & -48 & 0 & 0 & 0 & 0 & 0 \\ 6 & -6 & 2 & -8 & 0 & 0 & 0 & 0 & 0 \\ \frac{8}{5} & \frac{1}{5} & -\frac{2}{5} & \frac{8}{5} & 6 & -\frac{14}{15} & -\frac{1}{5} & -\frac{4}{5} & 0 \\ 12 & -1 & -\frac{2}{3} & 2 & 20 & \frac{2}{3} & -2 & -2 & 0 \\ -4 & \frac{14}{3} & -\frac{32}{9} & \frac{44}{3} & \frac{20}{3} & -\frac{64}{9} & \frac{10}{3} & -\frac{20}{3} & 0 \\ -2 & 3 & -\frac{4}{3} & 6 & 0 & -2 & 0 & 2 & 0 \\ 0 & 8 & -\frac{8}{3} & 12 & 0 & -\frac{4}{3} & -2 & 8 & 2 \end{pmatrix},$$

$$\hat{A}_{103,1-\bar{z}} = \begin{pmatrix} -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & -27 & 3 & -27 & 0 & 0 & 0 & 0 & 0 \\ 1 & -6 & \frac{2}{3} & -6 & 0 & 0 & 0 & 0 & 0 \\ \frac{4}{5} & \frac{3}{5} & -\frac{11}{30} & \frac{3}{2} & 0 & -\frac{3}{5} & 0 & -\frac{9}{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 6 & -\frac{8}{3} & 12 & 0 & -4 & 0 & -6 & 0 \\ -2 & 3 & -\frac{4}{3} & 6 & 0 & -2 & 0 & -3 & 0 \\ 0 & 8 & -\frac{8}{3} & 12 & 0 & -\frac{4}{3} & -2 & 8 & -3 \end{pmatrix},$$

$$\hat{A}_{103,1+\bar{z}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{8}{5} & \frac{1}{5} & -\frac{1}{10} & \frac{3}{10} & 3 & -\frac{8}{15} & -\frac{1}{5} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

D.4 $\hat{\mathbf{A}}_{104}$

$\hat{A}_{104}$ has the following components:

$$\hat{A}_{104,\bar{z}} = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & 2 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{16}{15} & -\frac{8}{5} & 0 & 0 & -\frac{8}{5} & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 \\ -\frac{12}{5} & \frac{4}{5} & 0 & 0 & -\frac{8}{5} & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & -10 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{163}{30} & -\frac{34}{5} & 0 & 0 & \frac{259}{10} & 2 & \frac{43}{8} & -\frac{17}{2} & 0 & -3 & 0 & 0 & 0 \\ \frac{27}{5} & -\frac{33}{5} & 0 & 0 & \frac{141}{5} & 2 & \frac{27}{4} & -\frac{27}{4} & 0 & 0 & -4 & 0 & 0 \\ -\frac{8}{3} & 0 & 0 & 0 & -\frac{16}{3} & 2 & 0 & -\frac{5}{2} & 0 & -2 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix},$$

$$\hat{A}_{104,1-\bar{z}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{61}{15} & \frac{32}{5} & 0 & 0 & -\frac{57}{5} & -2 & -\frac{19}{4} & \frac{7}{2} & 0 & -2 & 0 & 0 & 0 \\ -\frac{61}{15} & \frac{32}{5} & 0 & 0 & -\frac{57}{5} & -2 & -\frac{19}{4} & \frac{7}{2} & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{A}_{104,1+\bar{z}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{8}{5} & \frac{1}{5} & -\frac{1}{10} & \frac{3}{10} & 3 & -\frac{8}{15} & -\frac{1}{5} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

D.5 $\hat{\mathbf{A}}_{105}$

For $\hat{A}_{105}$ we get:

$$\hat{A}_{105,\bar{z}} = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{16}{15} & -\frac{8}{5} & -\frac{8}{5} & 0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{12}{5} & \frac{4}{5} & -\frac{8}{5} & 0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ -3 & -3 & 0 & -2 & 0 & 0 & 0 & 0 & -4 & -2 & 0 & 0 & 0 \\ -3 & -3 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -4 & 0 & 0 \\ -2 & 0 & -2 & 0 & -2 & 0 & 0 & 2 & 0 & 0 & -2 & -2 & 0 \\ 9 & 7 & 0 & \frac{14}{3} & 0 & 0 & 0 & 0 & -8 & 10 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{A}_{105,1-\bar{z}} = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{112}{15} & 0 & -\frac{32}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{12}{5} & \frac{4}{5} & -\frac{24}{5} & 0 & 0 & -2 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & -5 & 0 & -2 & 0 & 0 & 0 & 0 & 5 & -\frac{20}{3} & 0 & 0 & 0 \\ -6 & -6 & 0 & -2 & 0 & 0 & 0 & 0 & 6 & -8 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 12 & 0 & \frac{14}{3} & 0 & 0 & 0 & 0 & -18 & \frac{59}{3} & 0 & 0 & 1 \end{pmatrix},$$

$$\hat{A}_{105,1+\bar{z}} = \left(\hat{\mathbf{0}}_{\mathbf{13}\times\mathbf{13}}\right) .$$

D.6 $\hat{\mathbf{A}}_{\mathbf{106}}$

For the last family, the system of equations has the coefficients:

$$\hat{A}_{106,\bar{z}} = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & -2 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2}{9} & \frac{2}{9} & \frac{1}{9} & 0 & 0 & 0 & 0 & 1 & -4 & 0 & 0 & 0 & 0 \\ -3 & -3 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & -4 & -2 & 0 & 0 \\ -3 & -3 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 \\ -\frac{7}{2} & -\frac{3}{2} & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 8 & -2 & 0 & 0 \\ 10 & 12 & 0 & 6 & 0 & 0 & 2 & 0 & 0 & -12 & 18 & 0 & 0 \end{pmatrix},$$

$$\hat{A}_{106,1-\bar{z}} = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 8 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 1 & 0 & 0 & 0 & 0 & -1 & 18 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{9} & \frac{1}{9} & 0 & 0 & 0 & 0 & -\frac{1}{3} & 6 & 0 & 0 & 0 & 0 \\ -5 & -5 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 5 & -\frac{20}{3} & 0 & 0 \\ -6 & -6 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 6 & -8 & 0 & 0 \\ \frac{9}{2} & \frac{11}{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -8 & \frac{31}{3} & 1 & 0 \\ 6 & 6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -6 & 8 & 0 & 0 \end{pmatrix},$$

$$\hat{A}_{106,1+\bar{z}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -6 & 0 & -4 & 0 & 0 & -2 & 0 & 0 & 6 & -10 & 0 & -5 \end{pmatrix}.$$

Appendix E

Differential Equations for the Last Four Five-Body Integrals

Some of the five-body master integrals were excluded from the transformation to the ϵ-base. We give here their differential equations in the original base, which we used to solve these:

$$\begin{aligned}
\partial_z F_{5\text{B}30} = & \frac{(3\epsilon-2)(4\epsilon-3)\, c_{30,1}(\epsilon,z)}{3z(z+1)(z+3)\bar{z}^3\epsilon^3(3\epsilon-1)} F_{5\text{B}1} + \frac{(2\epsilon-1)^2\, c_{30,2}(\epsilon,z)}{3(z+1)(z+3)\bar{z}^3\epsilon^2(3\epsilon-1)} F_{5\text{B}2} \\
& + \frac{2(3\epsilon-2)(2\epsilon-1)(9z\epsilon-2z-9\epsilon)}{3z(z+3)\bar{z}^2\epsilon^2} F_{5\text{B}13} - \frac{2z(3\epsilon-2)(2\epsilon-1)}{(z+1)(z+3)\bar{z}^2\epsilon} F_{5\text{B}22} \\
& - \frac{(3\epsilon-2)}{(z+3)\bar{z}} F_{5\text{B}23} + \frac{(35z\epsilon^2-24z\epsilon+4z-35\epsilon^2+28\epsilon-4)}{3(z+3)\bar{z}^2\epsilon} F_{5\text{B}15} \\
& + \frac{2(2\epsilon-1)(7z\epsilon-2z-7\epsilon)}{3(z+3)\bar{z}^2\epsilon} F_{5\text{B}16} - \frac{2(z\epsilon+z-7\epsilon+1)}{(z+1)(z+3)\bar{z}(3\epsilon-1)} F_{5\text{B}26} \\
& - \frac{(z^2+3)\,(4\epsilon-1)}{(z+1)(z+3)\bar{z}^2} F_{5\text{B}25} - \frac{2(2z\epsilon+2\epsilon+1)}{(z+3)\bar{z}} F_{5\text{B}30} - \frac{2}{z+3} F_{5\text{B}31}\,,
\end{aligned}$$

$$\begin{aligned}
\partial_z F_{5\text{B}31} = & -\frac{(3\epsilon-2)(4\epsilon-3)c_{31,1}(\epsilon,z)}{3z(z+1)(z+3)\bar{z}^4\epsilon^3} F_{5\text{B}1} - \frac{(2\epsilon-1)^2 c_{31,2}(\epsilon,z)}{3z(z+1)(z+3)\bar{z}^4\epsilon^2} F_{5\text{B}2} \\
& - \frac{2(3\epsilon-2)(3\epsilon-1)(2\epsilon-1)c_{31,13}(\epsilon,z)}{3z(z+3)\bar{z}^3\epsilon^2} F_{5\text{B}13} - \frac{c_{31,15}(\epsilon,z)}{3z(z+3)\bar{z}^3\epsilon} F_{5\text{B}15} \\
& - \frac{2(2\epsilon-1)c_{31,16}(\epsilon,z)}{3z(z+3)\bar{z}^3\epsilon} F_{5\text{B}16} + \frac{2z(3z+1)(3\epsilon-2)(3\epsilon-1)(2\epsilon-1)}{(z+1)(z+3)\bar{z}^4\epsilon} F_{5\text{B}22}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{2(3\epsilon-2)(3\epsilon-1)}{(z+3)\bar{z}^2} F_{5\mathrm{B}23} - \frac{(2z^2\epsilon - z^2 + 20z\epsilon + 2\epsilon + 1)}{(z+1)(z+3)\bar{z}^2} F_{5\mathrm{B}26} \\
&+ \frac{2z(3\epsilon-1)(4\epsilon-1)}{(z+1)(z+3)\bar{z}^2} F_{5\mathrm{B}25} + \frac{(3\epsilon-1)(4\epsilon-1)}{(z+3)\bar{z}} F_{5\mathrm{B}30} - \frac{2(z+6\epsilon+1)}{(z+3)\bar{z}} F_{5\mathrm{B}31} \,,
\end{aligned}
$$

$$
\begin{aligned}
\partial_z F_{5\mathrm{B}32} = &- \frac{(3\epsilon-2)(4\epsilon-3)\, c_{32,1}(\epsilon,z)}{3z^2(z+1)\bar{z}^4\epsilon^3} F_{5\mathrm{B}1} + \frac{(2\epsilon-1)^2\, c_{32,1}(\epsilon,z)}{3z^2(z+1)\bar{z}^3\epsilon^2} F_{5\mathrm{B}2} \\
&+ \frac{(3\epsilon-2)(3\epsilon-1)(2\epsilon-1)}{3z^2\bar{z}^2\epsilon^2} F_{5\mathrm{B}13} - \frac{(2z+1)(3\epsilon-2)(3\epsilon-1)(2\epsilon-1)}{(z+1)\bar{z}^4\epsilon^2} F_{5\mathrm{B}2} \\
&+ \frac{(3\epsilon-2)(3\epsilon-1)}{z\bar{z}^2\epsilon} F_{5\mathrm{B}23} + \frac{2(5z\epsilon - 2z - 2\epsilon + 1)}{3z^2\bar{z}^2} F_{5\mathrm{B}15} \\
&+ \frac{2(2\epsilon-1)(2z\epsilon - 5\epsilon + 1)}{3z^2\bar{z}^2\epsilon} F_{5\mathrm{B}16} + \frac{z(3\epsilon-1)(4\epsilon-1)}{(z+1)\bar{z}^3\epsilon} F_{5\mathrm{B}25} \\
&+ \frac{(5z+1)}{z(z+1)\bar{z}^2} F_{5\mathrm{B}26} + \frac{(3\epsilon-1)}{z\bar{z}} F_{5\mathrm{B}30} + \frac{1}{z\bar{z}} F_{5\mathrm{B}31} - \frac{(5z\epsilon + z - 3\epsilon)}{z\bar{z}} F_{5\mathrm{B}32} \,,
\end{aligned}
$$

$$
\begin{aligned}
\partial_z F_{5\mathrm{B}36} = &- \frac{(3\epsilon-2)\, c_{36,1}(\epsilon,z)}{3(z-2)z^2(z+1)\bar{z}^4\epsilon^3(5\epsilon-3)} F_{5\mathrm{B}1} - \frac{c_{36,24}(\epsilon,z)}{3z^2(z+1)\bar{z}^3\epsilon(5\epsilon-3)} F_{5\mathrm{B}24} \\
&+ \frac{\mathrm{c3613}(3\epsilon-2)}{3z^2(z+1)\bar{z}^3\epsilon^2(4\epsilon-3)(5\epsilon-3)} F_{5\mathrm{B}13} + \frac{z(3\epsilon-1)(4\epsilon-1)}{(z-2)(z+1)\bar{z}^3\epsilon} F_{5\mathrm{B}25} \\
&- \frac{(2\epsilon-1)\, c_{36,2}(\epsilon,z)}{3(z-2)z^2(z+1)\bar{z}^3\epsilon^2(4\epsilon-3)(5\epsilon-3)} F_{5\mathrm{B}2} + \frac{(3\epsilon-2)(3\epsilon-1)}{\bar{z}^2\epsilon} F_{5\mathrm{B}23} \\
&+ \frac{(3\epsilon-1)}{(z-2)\bar{z}} F_{5\mathrm{B}15} + \frac{z(2z+1)(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{(z-2)(z+1)\bar{z}^4\epsilon^2} F_{5\mathrm{B}22} \\
&+ \frac{(3\epsilon-2)(7\epsilon-4)(7\epsilon-3)\, c_{36,33}(\epsilon,z)}{z^2(z+1)\bar{z}^3(\epsilon-2)\epsilon(5\epsilon-3)} F_{5\mathrm{B}33} \\
&- \frac{z(z+5)}{(z-2)(z+1)\bar{z}^2} F_{5\mathrm{B}26} + \frac{1}{\bar{z}} F_{5\mathrm{B}31} + \frac{5\epsilon}{(z-2)\bar{z}} F_{5\mathrm{B}35} - \frac{(2\epsilon+1)}{\bar{z}} F_{5\mathrm{B}36} \,.
\end{aligned}
$$

In the equations, we have used the abbreviations:

$$
\begin{aligned}
c_{30,1}(\epsilon,z) &= z^2\left(96\epsilon^3 - 63\epsilon^2 + 19\epsilon - 2\right) - 2z\left(21\epsilon^3 + 3\epsilon^2 - 5\epsilon + 1\right) \\
&\quad - 9\epsilon\left(6\epsilon^2 - 5\epsilon + 1\right) \,, \\
c_{30,2}(\epsilon,z) &= 3z^2\epsilon(17\epsilon-3) + z\left(-96\epsilon^2 + 30\epsilon - 2\right) + 45\epsilon^2 - 9\epsilon - 2 \,,
\end{aligned}
$$

$$
\begin{aligned}
c_{31,1}(\epsilon,z) &= z^3\left(48\epsilon^3-34\epsilon^2+10\epsilon-1\right)+z^2\epsilon\left(-72\epsilon^2+19\epsilon-2\right)\\
&\quad+z\left(-18\epsilon^3+20\epsilon^2-6\epsilon+1\right)+3\epsilon\left(6\epsilon^2-7\epsilon+2\right)\,,\\
c_{31,2}(\epsilon,z) &= 3z^4\epsilon(6\epsilon-1)+z^3\left(-102\epsilon^2+26\epsilon-1\right)+z^2\epsilon(57\epsilon-14)\\
&\quad+z\left(-6\epsilon^2+2\epsilon+1\right)+3\epsilon(3\epsilon-1)\,,\\
c_{31,13}(\epsilon,z) &= \left(5z^2\epsilon-z^2-7z\epsilon+z+6\epsilon\right)\,,\\
c_{31,15}(\epsilon,z) &= z^3\left(50\epsilon^3-55\epsilon^2+19\epsilon-2\right)-2z^2\left(56\epsilon^3-55\epsilon^2+18\epsilon-2\right)\\
&\quad+z\left(50\epsilon^3-53\epsilon^2+17\epsilon-2\right)+6(1-2\epsilon)\epsilon^2\,,\\
c_{31,16}(\epsilon,z) &= z^3\left(10\epsilon^2-7\epsilon+1\right)+z^2\left(-14\epsilon^2+7\epsilon-1\right)\\
&\quad+z\epsilon(31\epsilon-7)+3(1-5\epsilon)\epsilon\,,
\end{aligned}
$$

$$
\begin{aligned}
c_{32,1}(\epsilon,z) &= z^2\left(12\epsilon^2+2\epsilon-1\right)+3z(\epsilon-1)\epsilon+12\epsilon^2-8\epsilon+1\,,\\
c_{32,2}(\epsilon,z) &= 9z^2(4\epsilon-1)+z(15\epsilon-4)+3\epsilon-1\,,
\end{aligned}
$$

$$
\begin{aligned}
c_{36,1}(\epsilon,z) &= 3z^4\epsilon\left(65\epsilon^3-86\epsilon^2+34\epsilon-3\right)+z^3\left(-253\epsilon^4+140\epsilon^3+92\epsilon^2-72\epsilon+9\right)\\
&\quad+z^2\left(-431\epsilon^4+1095\epsilon^3-796\epsilon^2+216\epsilon-18\right)+z\big(-521\epsilon^4+796\epsilon^3\\
&\quad-488\epsilon^2+126\epsilon-9\big)+2\left(235\epsilon^4-432\epsilon^3+302\epsilon^2-90\epsilon+9\right)\,,\\
c_{36,2}(\epsilon,z) &= z^4\left(20\epsilon^4+33\epsilon^3-92\epsilon^2+54\epsilon-9\right)+z^3\big(1045\epsilon^4-2225\epsilon^3+1723\epsilon^2\\
&\quad-564\epsilon+63\big)-3z^2\left(78\epsilon^4-61\epsilon^3-24\epsilon^2+29\epsilon-6\right)+z\epsilon\big(173\epsilon^3-85\epsilon^2\\
&\quad-79\epsilon+39\big)+2\epsilon\left(-205\epsilon^3+291\epsilon^2-125\epsilon+15\right)\,,\\
c_{36,13}(\epsilon,z) &= 2z^3\left(95\epsilon^3-186\epsilon^2+112\epsilon-21\right)\epsilon+z^2\big(-324\epsilon^4+721\epsilon^3-550\epsilon^2\\
&\quad+171\epsilon-18\big)+2z\left(-105\epsilon^3+164\epsilon^2-86\epsilon+15\right)\epsilon+500\epsilon^4-975\epsilon^3\\
&\quad+670\epsilon^2-189\epsilon+18\,,\\
c_{36,24}(\epsilon,z) &= z^4\left(125\epsilon^3-150\epsilon^2+55\epsilon-6\right)-4z^3\epsilon\left(15\epsilon^2-16\epsilon+4\right)\\
&\quad-4z^2\epsilon\left(19\epsilon^2-16\epsilon+3\right)-4z\epsilon\left(15\epsilon^2-16\epsilon+4\right)\\
&\quad+125\epsilon^3-150\epsilon^2+55\epsilon-6\,,\\
c_{36,33}(\epsilon,z) &= z^2(5\epsilon-1)-4z\epsilon+5\epsilon-1\,.
\end{aligned}
$$

Appendix F

Transformations Matrices for the Four-Body Bases

F.1 $\hat{\mathbf{T}}_1$

Here we give all the transformation matrices that bring the original base of the four-body integrals into the ϵ-base via

$$\vec{G}_i = \hat{T}_i^{-1} \vec{F}_i \,. \tag{F.1}$$

The matrix $\hat{T}_1$ takes the following form:

$$\hat{T}_1 = \left(\begin{array}{cc} \hat{T}_{1,a_1} & \hat{T}_{1,a_2} \end{array} \right) ,$$

with the entries:

$$\hat{T}_{1,a_1} = \begin{pmatrix} -\frac{5z\epsilon-2z-3\epsilon+1}{z\bar{z}} & 0 & 0 & 0 \\ 0 & -\frac{5z\epsilon-2z-3\epsilon+1}{z\bar{z}} & \frac{\epsilon(3\epsilon-1)}{2z(2\epsilon-1)^2} & 0 \\ 0 & \frac{2(2\epsilon-1)^2}{z\bar{z}} & -\frac{3z\epsilon-z+\epsilon}{z\bar{z}} & 0 \\ 0 & 0 & 0 & -\frac{5z\epsilon-2z-2\epsilon+1}{z\bar{z}} \\ 0 & \frac{2(2\epsilon-1)^3}{z(2z-1)\bar{z}\epsilon^2} & -\frac{2\epsilon-1}{z(2z-1)\bar{z}\epsilon} & -\frac{2(2\epsilon-1)^3}{z(2z-1)\bar{z}\epsilon^2} \\ 0 & \frac{20(2\epsilon-1)^3(4\epsilon-1)}{3z(2z-1)\bar{z}\epsilon(\epsilon+1)} & \frac{4(z-3)(2\epsilon-1)(4\epsilon-1)}{3z(2z-1)\bar{z}(\epsilon+1)} & -\frac{20(2\epsilon-1)^3(4\epsilon-1)}{3z(2z-1)\bar{z}\epsilon(\epsilon+1)} \\ 0 & -\frac{8(2\epsilon-1)^3(4\epsilon-1)}{z(2z-1)\bar{z}^3\epsilon^2(3\epsilon+1)} & \frac{8(2\epsilon-1)(4\epsilon-1)}{(2z-1)\bar{z}^3\epsilon(3\epsilon+1)} & \frac{8(2\epsilon-1)^3(4\epsilon-1)}{z(2z-1)\bar{z}^3\epsilon^2(3\epsilon+1)} \end{pmatrix},$$

$$\hat{T}_{1,a_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{2(5\epsilon-1)}{2z-1} & \frac{3(\epsilon+1)}{(2z-1)(4\epsilon-1)} & 0 \\ \frac{2(z+2)\epsilon(4\epsilon-1)(5\epsilon-1)}{3z(2z-1)\bar{z}(\epsilon+1)} & -\frac{\left(4z^2-3z+3\right)\epsilon}{z(2z-1)\bar{z}} & 0 \\ \frac{4(4\epsilon-1)(5\epsilon-1)}{z(2z-1)\bar{z}^2(3\epsilon+1)} & -\frac{4(z+1)(\epsilon+1)}{z(2z-1)\bar{z}^2(3\epsilon+1)} & -\frac{5z\epsilon+z-3\epsilon}{z\bar{z}} \end{pmatrix}.$$

F.2 $\hat{\mathbf{T}}_2$

The matrix $\hat{T}_2$ takes the following form:

$$\hat{T}_2 = \begin{pmatrix} \hat{T}_{2,a_1} & \hat{T}_{2,a_2} & \hat{T}_{2,a_3} \\ \hat{T}_{2,b_1} & \hat{T}_{2,b_2} & \hat{T}_{2,b_3} \end{pmatrix},$$

the components being:

$$\hat{T}_{2,a_1} = \begin{pmatrix} -\frac{120z\bar{z}\epsilon^4}{(\epsilon-1)(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)} & 0 & 0 \\ 0 & \frac{180z\bar{z}\epsilon^4}{11(2\epsilon-1)^3(4\epsilon-1)} & -\frac{60\epsilon^3\left(6z^2\epsilon^2-2z^2\epsilon-27z\epsilon^2+15z\epsilon-2z+25\epsilon^2-15\epsilon+2\right)}{11(2\epsilon-1)^3(3\epsilon-1)(4\epsilon-1)} \\ 0 & 0 & -\frac{120(z+1)\epsilon^3}{11(2\epsilon-1)(4\epsilon-1)} \\ 0 & 0 & -\frac{120\epsilon^3(3z\epsilon-z-5\epsilon+1)}{11(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} \\ -\frac{30\bar{z}\epsilon^3\left(10z\epsilon^2-7z\epsilon+z+6\epsilon^2-2\epsilon\right)}{z(2\epsilon-1)^3(3\epsilon-1)(4\epsilon-1)} & 0 & 0 \\ -\frac{30\bar{z}\epsilon^3\left(4z\epsilon^2+3z\epsilon-z-12\epsilon^2+4\epsilon\right)}{z(2\epsilon-1)^2(3\epsilon-1)(4\epsilon-1)} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{18(\epsilon+1)}{3\epsilon-1} & \frac{360z\epsilon^3}{11(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} & \frac{24\left(10z\epsilon^3-25\epsilon^3-11\epsilon^2+12\epsilon-2\right)}{11(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} \\ -\frac{6\epsilon(\epsilon+1)(16\epsilon-3)}{(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} & 0 & 0 \\ -\frac{24\epsilon^2(\epsilon+1)}{(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} & -\frac{360z\epsilon^3}{11(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} & -\frac{24\epsilon^2(10z\epsilon-\epsilon-1)}{11(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} \end{pmatrix},$$

$$\hat{T}_{2,a_2} = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{15\epsilon^3\left(66z^2\epsilon^2-22z^2\epsilon-183z\epsilon^2+93z\epsilon+10\bar{z}+125\epsilon^2-75\epsilon\right)}{11(2\epsilon-1)^3(3\epsilon-1)(4\epsilon-1)} & 0 & 0 \\ \frac{30(z-5)\epsilon^3}{11(2\epsilon-1)(4\epsilon-1)} & 0 & 0 \\ -\frac{30\epsilon^3(21z\epsilon-5z-25\epsilon+5)}{11(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} & 0 & 0 \\ 0 & \frac{5\bar{z}\epsilon^3\left(3z^2\epsilon-8z\epsilon+2z-3\epsilon\right)}{z(2\epsilon-1)^3(4\epsilon-1)} & \frac{15\bar{z}\epsilon^3\left(z^2\epsilon-4z\epsilon+z-\epsilon\right)}{z(2\epsilon-1)^3(4\epsilon-1)} \\ 0 & \frac{10\bar{z}\epsilon^3(z\epsilon-z+3\epsilon)}{z(2\epsilon-1)^2(4\epsilon-1)} & \frac{15\bar{z}\epsilon^3(2z\epsilon-z+2\epsilon)}{z(2\epsilon-1)^2(4\epsilon-1)} \\ 0 & 0 & 0 \\ \frac{30\left(22z\epsilon^3-25\epsilon^3-11\epsilon^2+12\epsilon-2\right)}{11(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} & \frac{16(\epsilon+1)}{3(3\epsilon-1)} & \frac{7(\epsilon+1)}{3\epsilon-1} \\ 0 & \frac{2\epsilon\left(45z\epsilon^2-41\epsilon^2-33\epsilon+8\right)}{3(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} & \frac{\epsilon\left(30z\epsilon^2-34\epsilon^2-27\epsilon+7\right)}{(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} \\ -\frac{30\epsilon^2(22z\epsilon-\epsilon-1)}{11(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} & \frac{6\epsilon^2(5z\epsilon-\epsilon-1)}{(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} & \frac{6\epsilon^2(5z\epsilon-\epsilon-1)}{(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} \end{pmatrix},$$

$$\hat{T}_{2,a_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{135 z\bar{z}\epsilon^4}{(2\epsilon-1)^3(4\epsilon-1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3(\epsilon+1)}{3\epsilon-1} & \frac{\epsilon+1}{3\epsilon-1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4\epsilon}{3\epsilon-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12\epsilon^2}{(2\epsilon-1)(3\epsilon-1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{T}_{2,b_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{88\epsilon^2}{\bar{z}} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{t_{2,1}}{(2\epsilon-1)^3(3\epsilon-2)(3\epsilon-1)(4\epsilon-1)} & 0 & 0 \end{pmatrix},$$

$$\hat{T}_{2,b_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{76\epsilon^2}{3(z-1)} & \frac{32\epsilon^2}{\bar{z}} \\ 0 & 0 & 0 \\ 0 & \frac{t_{2,2}}{(2\epsilon-1)^3(3\epsilon-2)(3\epsilon-1)(4\epsilon-1)} & \end{pmatrix},$$

$$
\hat{T}_{2,b_3} = \begin{pmatrix}
0 & 0 & 0 & 0 & -\frac{60\bar{z}\epsilon^3}{(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{3\epsilon}{2\epsilon-1} & 0 & 0 & 0 \\
0 & 0 & \frac{4\epsilon^2}{\bar{z}} & 0 & -\frac{40\epsilon^2}{\bar{z}} & -\frac{4\epsilon^2}{\bar{z}} & 0 & -\frac{z\epsilon^2}{\bar{z}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\bar{z}} & 0 \\
\frac{t_{2,4}}{(2\epsilon-1)^3(3\epsilon-2)(3\epsilon-1)(4\epsilon-1)} & 0 & 0 & 0 & -\frac{t_{2,5}}{(2\epsilon-1)^3(3\epsilon-2)(3\epsilon-1)(4\epsilon-1)} & 0 & 0 & 0 & 0 & \frac{9z\epsilon^2}{(2\epsilon-1)^2}
\end{pmatrix} .
$$

For better readability we used the abbreviations:

$$
\begin{aligned}
t_{2,1} &= -60z^2(\epsilon-1)(4\epsilon-1)\epsilon^4 + \frac{2}{5}z(\epsilon+1)\left(1278\epsilon^3 - 1377\epsilon^2 + 448\epsilon - 52\right)\epsilon^2 \\
&\quad - 30\left(28\epsilon^3 - 27\epsilon^2 + 11\epsilon - 2\right)\epsilon^3 - \frac{60(\epsilon-1)(3\epsilon-1)\epsilon^4}{z}\,, \\
t_{2,2} &= -15z^2\left(3\epsilon^2 - 1\right)\epsilon^4 + \frac{1}{15}z(\epsilon+1)\left(2691\epsilon^3 - 2619\epsilon^2 + 781\epsilon - 94\right)\epsilon^2 \\
&\quad - 5\left(63\epsilon^3 - 66\epsilon^2 + 25\epsilon - 4\right)\epsilon^3 - \frac{15(\epsilon-1)(3\epsilon-1)\epsilon^4}{z}\,, \\
t_{2,3} &= -15z^2\left(3\epsilon^2 - 1\right)\epsilon^4 + \frac{3}{5}z(\epsilon+1)\left(399\epsilon^3 - 416\epsilon^2 + 134\epsilon - 16\right)\epsilon^2 \\
&\quad - 15\left(33\epsilon^3 - 37\epsilon^2 + 14\epsilon - 2\right)\epsilon^3 - \frac{15(\epsilon-1)(3\epsilon-1)\epsilon^4}{z}\,, \\
t_{2,4} &= 270z^2\epsilon^4(2\epsilon-1)(3\epsilon-2) - \frac{27}{2}z\epsilon^2(\epsilon+1)(2\epsilon-1)(3\epsilon-2)(8\epsilon-1)\,, \\
t_{2,5} &= -30z^2(\epsilon-1)(2\epsilon-1)(3\epsilon-1)\epsilon^3 - \frac{24}{5}z(\epsilon+1)(2\epsilon-1)(3\epsilon-1)(7\epsilon-3)\epsilon^2 \\
&\quad + 30(2\epsilon-1)(3\epsilon-1)(5\epsilon-3)\epsilon^3\,.
\end{aligned}
$$

F.3 $\hat{\mathbf{T}}_3$

The matrix $\hat{T}_3$ takes the following form:

$$\hat{T}_3 = \begin{pmatrix} \hat{T}_{3,a_1} & \hat{T}_{3,a_2} & \hat{T}_{3,a_3} \\ \hat{T}_{3,b_1} & \hat{T}_{3,b_2} & \hat{T}_{3,b_3} \end{pmatrix},$$

the components being:

$$\hat{T}_{3,a_1} = \begin{pmatrix} \frac{40(z-1)z\epsilon^3}{(\epsilon-1)(3\epsilon-2)(3\epsilon-1)} & 0 & 0 \\ 0 & -\frac{60(z-1)z\epsilon^3}{11(2\epsilon-1)^2(4\epsilon-1)} & -\frac{t_{3,2}}{11(2\epsilon-1)^2(3\epsilon-1)(4\epsilon-1)} \\ 0 & 0 & -\frac{40(z+1)\epsilon^2}{11(4\epsilon-1)} \\ 0 & 0 & -\frac{40\epsilon^2(3z\epsilon-z-5\epsilon+1)}{11(3\epsilon-1)(4\epsilon-1)} \\ \frac{4(z-1)\epsilon^2\left(43z\epsilon^2-28z\epsilon+4z+15\epsilon^2-5\epsilon\right)}{z(2\epsilon-1)^2(3\epsilon-1)(4\epsilon-1)} & 0 & 0 \\ \frac{40(z-1)\epsilon^3}{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)} & 0 & 0 \\ 0 & \frac{120z\epsilon^2}{11(3\epsilon-1)(4\epsilon-1)} & \frac{8\left(10z\epsilon^2-9\epsilon^2-7\epsilon+2\right)}{11(3\epsilon-1)(4\epsilon-1)} \\ -\frac{t_{3,1}}{z^2(2\epsilon-1)^2(3\epsilon-2)(3\epsilon-1)(4\epsilon-3)(4\epsilon-1)} & 0 & 0 \\ -\frac{8(\epsilon+1)(5\epsilon-1)}{(3\epsilon-1)(4\epsilon-1)} & 0 & 0 \\ \frac{80\epsilon^3}{(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} & \frac{48z\epsilon^2(\epsilon+1)}{11(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} & \frac{8\epsilon^2(4z\epsilon+4z-30\epsilon+5)}{11(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} \end{pmatrix},$$

$$\hat{T}_{3,a_2} = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{t_{3,3}}{11(2\epsilon-1)^2(3\epsilon-1)(4\epsilon-1)} & 0 & 0 \\ \frac{10(z-5)\epsilon^2}{11(4\epsilon-1)} & 0 & 0 \\ -\frac{10\epsilon^2(21z\epsilon-5z-25\epsilon+5)}{11(3\epsilon-1)(4\epsilon-1)} & 0 & 0 \\ 0 & \frac{105(z-1)\epsilon^2\left(3z^2\epsilon-8z\epsilon+2z-3\epsilon\right)}{z(2\epsilon-1)^2(4\epsilon-1)} & 0 \\ 0 & 0 & \frac{60(z-1)^2\epsilon^3}{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)} \\ \frac{10\left(22z\epsilon^2-9\epsilon^2-7\epsilon+2\right)}{11(3\epsilon-1)(4\epsilon-1)} & 0 & 0 \\ 0 & -\frac{t_{3,4}}{2z^2(2\epsilon-1)^2(4\epsilon-3)(4\epsilon-1)} & 0 \\ 0 & -\frac{14\left(45z\epsilon^2-41\epsilon^2-33\epsilon+8\right)}{(3\epsilon-1)(4\epsilon-1)} & 0 \\ \frac{2\epsilon^2(44z\epsilon+44z-150\epsilon+25)}{11(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} & \frac{252\epsilon^2(z\epsilon+z-5\epsilon)}{(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} & 0 \end{pmatrix},$$

$$\hat{T}_{3,a_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{630(z-1)\epsilon^2\left(2z^2\epsilon-4z\epsilon+z-2\epsilon\right)}{z(2\epsilon-1)^2(4\epsilon-1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{4(2\epsilon-1)}{3(3\epsilon-1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{t_{3,5}}{z^2(2\epsilon-1)^2(4\epsilon-3)(4\epsilon-1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{126\left(20z\epsilon^2-16\epsilon^2-13\epsilon+3\right)}{(3\epsilon-1)(4\epsilon-1)} & \frac{4(2\epsilon-1)}{3(3\epsilon-1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1008\epsilon^2(z\epsilon+z-5\epsilon)}{(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} & 0 & -\frac{2z\epsilon^2}{(2\epsilon-1)(3\epsilon-1)} & -\frac{\bar{z}\epsilon^2}{(2\epsilon-1)(3\epsilon-1)} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{T}_{3,b_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{T}_{3,b_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

$$\hat{T}_{3,b_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{20z\epsilon^2}{(3\epsilon-2)(3\epsilon-1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2\epsilon-1}{3\epsilon} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-2\epsilon}{3\bar{z}\epsilon} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\bar{z}} \end{pmatrix} .$$

For better readability, we used the abbreviations:

$$t_{3,1} = 2\bar{z}^2\epsilon^2\Big(146z^2\epsilon^4 - 562z^2\epsilon^3 + 562z^2\epsilon^2 - 206z^2\epsilon + 24z^2 + 480z\epsilon^4 - 755z\epsilon^3 + 380z\epsilon^2 - 60z\epsilon - 90\epsilon^4 + 135\epsilon^3 - 65\epsilon^2 + 10\epsilon\Big) ,$$

$$t_{3,2} = 20\epsilon^2\left(6z^2\epsilon^2 - 2z^2\epsilon - 27z\epsilon^2 + 15z\epsilon - 2z + 25\epsilon^2 - 15\epsilon + 2\right) ,$$

$$t_{3,3} = 5\epsilon^2\left(66z^2\epsilon^2 - 22z^2\epsilon - 183z\epsilon^2 + 93z\epsilon - 10z + 125\epsilon^2 - 75\epsilon + 10\right) ,$$

$$t_{3,4} = 105\bar{z}^2\epsilon^2\left(6z^3\epsilon^2 - 6z^3\epsilon - 14z^2\epsilon^2 + 23z^2\epsilon - 6z^2 - 30z\epsilon^2 + 18z\epsilon + 6\epsilon^2 - 3\epsilon\right) ,$$

$$t_{3,5} = 315\bar{z}^2\epsilon^2\left(4z^3\epsilon^2 - 4z^3\epsilon - 4z^2\epsilon^2 + 10z^2\epsilon - 3z^2 - 20z\epsilon^2 + 12z\epsilon + 4\epsilon^2 - 2\epsilon\right) .$$

Appendix G

Transformations Matrices for the Five-Body Bases

Here we give all six transformation matrices that bring the original base of the four-body integrals into the ϵ-base via

$$\vec{G}_i = \hat{T}_i^{-1} \vec{F}_i \,. \tag{G.1}$$

G.1 $\hat{\mathbf{T}}_{\mathbf{101}}$

The matrix $\hat{T}_{101}$ takes the following form:

$$\hat{T}_{101} = \begin{pmatrix} -\frac{3z^2\bar{z}\epsilon^3(\epsilon+1)}{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)(4\epsilon-3)(4\epsilon-1)} & 0 & 0 & 0 & 0 \\ \frac{3z\bar{z}\epsilon^3(\epsilon+1)}{(2\epsilon-1)^3(3\epsilon-1)(4\epsilon-1)} & -\frac{3\bar{z}\epsilon^2(\epsilon+1)}{2(2\epsilon-1)^3(4\epsilon-1)} & 0 & 0 & 0 \\ -\frac{3(\epsilon+1)}{(4\epsilon-1)(5\epsilon-1)} & \frac{3(\epsilon+1)}{(4\epsilon-1)(5\epsilon-1)} & \frac{(z-2)(\epsilon+1)}{(4\epsilon-1)(5\epsilon-1)} & -\frac{3(z-3)(\epsilon+1)}{(4\epsilon-1)(5\epsilon-1)} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\bar{z}} \end{pmatrix} .$$

G.2 $\hat{\mathbf{T}}_{102}$

The matrix $\hat{T}_{102}$ reads:

$$\hat{T}_{102} = \begin{pmatrix} \hat{T}_{102,a_1} & \hat{T}_{102,a_2} & \hat{T}_{102,a_3} \end{pmatrix} ,$$

the components being:

$$\hat{T}_{102,a_1} = \begin{pmatrix} \frac{3z^2\bar{z}\epsilon^4(3\epsilon+1)}{2(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)(4\epsilon-3)(4\epsilon-1)} & 0 & 0 \\ \frac{3\bar{z}\epsilon^3(3\epsilon+1)(2\bar{z}\epsilon+\epsilon-1)}{4(2\epsilon-1)^3(3\epsilon-1)(4\epsilon-1)} & \frac{3\bar{z}\epsilon^3(3\epsilon+1)}{4(2\epsilon-1)^3(4\epsilon-1)} & 0 \\ \frac{3\epsilon(3\epsilon+1)(8\bar{z}\epsilon+\bar{z}+4\epsilon)}{2(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & -\frac{9\epsilon^2(3\epsilon+1)}{(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & \frac{3\epsilon(3\epsilon+1)(31\bar{z}\epsilon-9\bar{z}+22\epsilon)}{8(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} \\ 0 & 0 & \frac{1}{2}(3\epsilon+1) \\ -\frac{\epsilon(3\epsilon+1)\,t_{102,1}}{8(2\epsilon-1)^3(4\epsilon-1)(5\epsilon-2)(5\epsilon-1)(6\epsilon-1)} & \frac{\epsilon^2(3\epsilon+1)\,t_{102,2}}{4(2\epsilon-1)^3(4\epsilon-1)(5\epsilon-2)(5\epsilon-1)(6\epsilon-1)} & \frac{\epsilon(3\epsilon-1)(3\epsilon+1)\,t_{102,3}}{32(2\epsilon-1)^3(4\epsilon-1)(5\epsilon-2)(5\epsilon-1)(6\epsilon-1)} \\ 0 & -\frac{3\epsilon(3\epsilon+1)}{2(4\epsilon-1)(5\epsilon-1)} & 0 \\ 0 & 0 & 0 \\ \frac{3\bar{z}\epsilon^2(3\epsilon+1)\,t_{102,7}}{2(2\epsilon-1)^3(3\epsilon-1)(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & -\frac{3\bar{z}\epsilon^3(3\epsilon-1)(3\epsilon+1)(38\epsilon-13)}{2(2\epsilon-1)^3(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & -\frac{3\bar{z}\epsilon^2(3\epsilon+1)\,t_{102,8}}{8(2\epsilon-1)^2(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} \\ 0 & 0 & 0 \end{pmatrix} ,$$

$$\hat{T}_{102,a_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{3\epsilon(3\epsilon+1)(5\bar{z}\epsilon-3\bar{z}+24\epsilon)}{8(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & -\frac{27\epsilon(3\epsilon+1)(111\bar{z}\epsilon-9\bar{z}+32\epsilon)}{16(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & 0 \\ -\frac{5}{8}(3\epsilon+1) & -\frac{171}{16}(3\epsilon+1) & 0 \\ -\frac{t_{102,4}\,\epsilon(3\epsilon-1)(3\epsilon+1)}{32(2\epsilon-1)^3(4\epsilon-1)(5\epsilon-2)(5\epsilon-1)(6\epsilon-1)} & \frac{9\epsilon(3\epsilon-1)(3\epsilon+1)\,t_{102,5}}{64(2\epsilon-1)^3(4\epsilon-1)(5\epsilon-2)(5\epsilon-1)(6\epsilon-1)} & 0 \\ 0 & 0 & \frac{3(\bar{z}+1)\epsilon(3\epsilon+1)}{(4\epsilon-1)(5\epsilon-1)} \\ 0 & 0 & 0 \\ \frac{3\bar{z}\epsilon^2(3\epsilon+1)\,t_{102,9}}{8(2\epsilon-1)^2(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & -\frac{27\bar{z}\epsilon^2(3\epsilon+1)\,t_{102,10}}{16(2\epsilon-1)^2(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

$$\hat{T}_{102,a_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{\epsilon(3\epsilon+1)(27\bar{z}\epsilon-21\bar{z}+170\epsilon)}{8(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & 0 \\ 0 & \frac{17}{12}(3\epsilon+1) & 0 \\ 0 & \frac{\epsilon(3\epsilon-1)(3\epsilon+1)\,t_{102,6}}{96(2\epsilon-1)^3(4\epsilon-1)(5\epsilon-2)(5\epsilon-1)(6\epsilon-1)} & 0 \\ -\frac{3\bar{z}\epsilon(3\epsilon+1)}{2(4\epsilon-1)(5\epsilon-1)} & 0 & 0 \\ \frac{\epsilon(3\epsilon+1)}{2(\epsilon+1)} & 0 & 0 \\ 0 & -\frac{\bar{z}\epsilon^2(3\epsilon+1)\,t_{102,11}}{8(2\epsilon-1)^2(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & 0 \\ 0 & 0 & \frac{1}{\bar{z}} \end{pmatrix},$$

with the functions:

$$\begin{aligned}
t_{102,1} &= -6\bar{z}^2 + \left(28\bar{z}^3 + 15\bar{z}^2 - 24\bar{z} - 4\right)\epsilon + \left(-406\bar{z}^3 + 177\bar{z}^2 + 183\bar{z} + 52\right)\epsilon^2 \\
&\quad + \left(2156\bar{z}^3 - 948\bar{z}^2 - 357\bar{z} - 248\right)\epsilon^3 + \left(-4970\bar{z}^3 + 1728\bar{z}^2 - 168\bar{z} + 512\right)\epsilon^4 \\
&\quad + 12\left(350\bar{z}^3 - 100\bar{z}^2 + 61\bar{z} - 32\right)\epsilon^5\,, \\
t_{102,2} &= -3 - 18\bar{z} + 9\bar{z}^2 + \left(-137\bar{z}^2 + 172\bar{z} + 39\right)\epsilon + \left(762\bar{z}^2 - 616\bar{z} - 186\right)\epsilon^2 \\
&\quad + \left(-1828\bar{z}^2 + 988\bar{z} + 384\right)\epsilon^3 + 6\left(265\bar{z}^2 - 100\bar{z} - 48\right)\epsilon^4\,, \\
t_{102,3} &= 54\bar{z}^2 + \left(-46\bar{z}^3 - 465\bar{z}^2 - 132\bar{z} - 22\right)\epsilon + \left(529\bar{z}^3 + 1093\bar{z}^2 + 936\bar{z} + 220\right)\epsilon^2 \\
&\quad + \left(-1955\bar{z}^3 + 204\bar{z}^2 - 2380\bar{z} - 704\right)\epsilon^3 + 4\left(575\bar{z}^3 - 520\bar{z}^2 + 518\bar{z} + 176\right)\epsilon^4\,, \\
t_{102,4} &= 18\bar{z}^2, \left(-26\bar{z}^3 - 117\bar{z}^2 - 144\bar{z} - 24\right)\epsilon + \left(299\bar{z}^3 + 143\bar{z}^2 + 978\bar{z} + 240\right)\epsilon^2 \\
&\quad + \left(-1105\bar{z}^3 + 350\bar{z}^2 - 2300\bar{z} - 768\right)\epsilon^3 + 4\left(325\bar{z}^3 - 150\bar{z}^2 + 460\bar{z} + 192\right)\epsilon^4\,, \\
t_{102,5} &= -54\bar{z}^2 + \left(-114\bar{z}^3 + 1095\bar{z}^2 + 192\bar{z} + 32\right)\epsilon + \Big(1311\bar{z}^3 - 6253\bar{z}^2 \\
&\quad - 1086\bar{z} - 320\Big)\epsilon^2 + \left(-4845\bar{z}^3 + 15166\bar{z}^2 + 1620\bar{z} + 1024\right)\epsilon^3 \\
&\quad + 4\left(1425\bar{z}^3 - 3430\bar{z}^2 - 108\bar{z} - 256\right)\epsilon^4\,, \\
t_{102,6} &= 126\bar{z}^2 - \left(198\bar{z}^3 + 753\bar{z}^2 + 1020\bar{z} + 170\right)\epsilon + \Big(2277\bar{z}^3 + 457\bar{z}^2 \\
&\quad + 6948\bar{z} + 1700\Big)\epsilon^2 + \left(-8415\bar{z}^3 + 4064\bar{z}^2 - 16428\bar{z} - 5440\right)\epsilon^3 \\
&\quad + \left(9900\bar{z}^3 - 5840\bar{z}^2 + 13224\bar{z} + 5440\right)\epsilon^4\,,
\end{aligned}$$

$$
\begin{aligned}
t_{102,7} &= -1 - \bar{z} + (9 - 15\bar{z})\epsilon + (178\bar{z} - 35)\epsilon^2 + (69 - 528\bar{z})\epsilon^3 + 6(76\bar{z} - 7)\epsilon^4\,, \\
t_{102,8} &= -1 - 9\bar{z} + (26\bar{z} + 55)\epsilon + (53\bar{z} - 184)\epsilon^2\,, \\
t_{102,9} &= 7 - 3\bar{z}(-2\bar{z} - 29)\epsilon + (55\bar{z} + 42)\epsilon^2\,, \\
t_{102,10} &= -29 + 9\bar{z} + (255 - 186\bar{z})\epsilon + (507\bar{z} - 646)\epsilon^2\,, \\
t_{102,11} &= 47 - 21\bar{z} - 3(10\bar{z} + 59)\epsilon + (441\bar{z} + 220)\epsilon^2\,.
\end{aligned}
$$

G.3 $\hat{\mathbf{T}}_{103}$

For the third family the transformation matrix looks as follows:

$$
\hat{T}_{103} = \begin{pmatrix} \hat{T}_{103,a_1} & \hat{T}_{103,a_2} & \hat{T}_{103,a_3} \end{pmatrix}\,,
$$

the components being:

$$
\hat{T}_{103,a_1} = \begin{pmatrix}
-\frac{3z^2\bar{z}\epsilon^4(3\epsilon+1)}{2(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)(4\epsilon-3)(4\epsilon-1)} & 0 & 0 \\
\frac{3z\bar{z}\epsilon^4(3\epsilon+1)}{2(2\epsilon-1)^3(3\epsilon-1)(4\epsilon-1)} & -\frac{3\bar{z}\epsilon^3(3\epsilon+1)}{4(2\epsilon-1)^3(4\epsilon-1)} & 0 \\
-\frac{3\epsilon(3\epsilon+1)}{2(4\epsilon-1)(5\epsilon-1)} & \frac{3\epsilon(3\epsilon+1)}{2(4\epsilon-1)(5\epsilon-1)} & \frac{(z-2)\epsilon(3\epsilon+1)}{2(4\epsilon-1)(5\epsilon-1)} \\
0 & 0 & 0 \\
-\frac{3\epsilon(3\epsilon+1)(z\epsilon+\bar{z}+5\epsilon)}{2(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & 0 & \frac{\epsilon^2(3\epsilon+1)}{(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} \\
0 & 0 & 0 \\
\frac{3\bar{z}\epsilon^2(3\epsilon+1)\,t_{103,1}}{2(2\epsilon-1)^3(3\epsilon-1)(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & \frac{3\bar{z}\epsilon^3(3\epsilon+1)}{2(2\epsilon-1)^3(4\epsilon-1)} & \frac{\bar{z}\epsilon^3(3\epsilon+1)(7\epsilon-2)}{(2\epsilon-1)^2(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}\,,
$$

$$\hat{T}_{103,a_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{3(z-3)\epsilon(3\epsilon+1)}{2(4\epsilon-1)(5\epsilon-1)} & 0 & 0 \\ \frac{\epsilon(3\epsilon+1)}{2(\epsilon+1)} & 0 & 0 \\ -\frac{3\epsilon^2(3\epsilon+1)}{(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & \frac{15(z-2)\epsilon^2(3\epsilon+1)}{(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & -\frac{\epsilon(3\epsilon+1)(8z\epsilon+\bar{z}-12\epsilon)}{(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} \\ 0 & 0 & \frac{1}{4}(3\epsilon+1) \\ -\frac{3\bar{z}\epsilon^3(3\epsilon+1)(7\epsilon-2)}{(2\epsilon-1)^2(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & \frac{15(z-2)\bar{z}\epsilon^3(3\epsilon+1)(7\epsilon-2)}{(2\epsilon-1)^2(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & -\frac{\bar{z}\epsilon^2(3\epsilon+1)\,t_{103,2}}{2(2\epsilon-1)^2(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{T}_{103,a_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{3\epsilon(3\epsilon+1)(5z\epsilon+\bar{z}-3\epsilon)}{4(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & -\frac{3\epsilon(3\epsilon+1)(7z\epsilon+\bar{z}-9\epsilon)}{2(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & 0 \\ 0 & 0 & 0 \\ \frac{3\bar{z}\epsilon^2(3\epsilon+1)\,t_{103,3}}{4(2\epsilon-1)^2(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & -\frac{3\bar{z}\epsilon^2(3\epsilon+1)\,t_{103,4}}{2(2\epsilon-1)^2(4\epsilon-1)(5\epsilon-1)(6\epsilon-1)} & 0 \\ 0 & \frac{3\epsilon+1}{4\epsilon} & 0 \\ 0 & 0 & -\frac{1}{\bar{z}} \end{pmatrix}.$$

The functions $t_{103,i}$ are defined as:

$$t_{103,1} = 2 - z + (3z-7)\epsilon + (25z-38)\epsilon^2 + (193-105z)\epsilon^3 + (78z-210)\epsilon^4\,,$$
$$t_{103,2} = -1 + 2z + (29-24z)\epsilon + (52z-78)\epsilon^2\,,$$
$$t_{103,3} = -2 + z + (13-6z)\epsilon + (5z-21)\epsilon^2\,,$$
$$t_{103,4} = -2 + z + (25-10z)\epsilon + (19z-63)\epsilon^2\,.$$

G.4 $\hat{\mathbf{T}}_{104}$

For the fourth family the transformation matrix looks as follows:

$$\hat{T}_{104} = \left(\begin{array}{ccc} \hat{T}_{4,a_1} & \hat{T}_{4,a_2} & \hat{T}_{4,a_3} \end{array} \right),$$

the components being:

$$\hat{T}_{104,a_1} = \left(\begin{array}{ccc}
\frac{6z^2\bar{z}\epsilon^5}{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)(4\epsilon-3)(4\epsilon-1)} & 0 & 0 \\
\frac{3\bar{z}\epsilon^4(2\bar{z}\epsilon+\epsilon-1)}{(2\epsilon-1)^3(3\epsilon-1)(4\epsilon-1)} & \frac{3\bar{z}\epsilon^4}{(2\epsilon-1)^3(4\epsilon-1)} & 0 \\
0 & -\frac{6\epsilon^2}{(4\epsilon-1)(5\epsilon-1)} & \frac{2(\bar{z}+1)\epsilon^2}{(4\epsilon-1)(5\epsilon-1)} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-\frac{2\epsilon^2\left(6\bar{z}\epsilon^2+6\epsilon^2-7\epsilon+1\right)}{(2\epsilon-1)(3\epsilon-1)^2(4\epsilon-1)} & 0 & 0 \\
\frac{2\epsilon^3\, t_{104,1}}{(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)(7\epsilon-2)(7\epsilon-1)} & -\frac{6\epsilon^3\left(\bar{z}\epsilon^2-\bar{z}\epsilon+12\epsilon^2-7\epsilon+1\right)}{(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)(7\epsilon-2)(7\epsilon-1)} & 0 \\
-\frac{\epsilon^3\, t_{104,3}}{2(2\epsilon-1)^2(3\epsilon-1)(4\epsilon-1)(7\epsilon-2)(7\epsilon-1)} & \frac{3\epsilon^3\, t_{104,4}}{2(2\epsilon-1)^2(3\epsilon-1)(4\epsilon-1)(7\epsilon-2)(7\epsilon-1)} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-\frac{t_{104,5}\epsilon\, e_{104,1}}{30(4\epsilon-1)} & \frac{\epsilon\, e_{104,1}\, t_{104,7}}{5(4\epsilon-1)} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array} \right),$$

$$\hat{T}_{104,a_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{6\bar{z}\epsilon^2}{(4\epsilon-1)(5\epsilon-1)} & 0 & 0 \\ \frac{2\epsilon^2}{\epsilon+1} & 0 & 0 \\ 0 & \frac{6z^2\epsilon^4}{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)(4\epsilon-1)} & 0 \\ 0 & \frac{4\epsilon^2(3\bar{z}\epsilon-7\epsilon+1)}{(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} & \frac{2\epsilon^2}{(2\epsilon-1)(3\epsilon-1)} \\ 0 & \frac{4\epsilon^3 t_{104,2}}{(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)(7\epsilon-2)(7\epsilon-1)} & 0 \\ 0 & -\frac{2\epsilon^3 t_{104,6}}{(2\epsilon-1)^2(3\epsilon-1)(4\epsilon-1)(7\epsilon-2)(7\epsilon-1)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{\epsilon\, e_{104,1}\, t_{104,8}}{10(4\epsilon-1)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{T}_{104,a_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{15\epsilon^3(2\bar{z}\epsilon+3\epsilon-1)}{2(2\epsilon-1)(3\epsilon-1)(7\epsilon-2)(7\epsilon-1)} & \frac{15z\epsilon^3(5\bar{z}\epsilon-\bar{z}-3\epsilon+1)}{2(2\epsilon-1)(3\epsilon-1)(7\epsilon-2)(7\epsilon-1)} & 0 & 0 & 0 & 0 & 0 \\ -\frac{15\epsilon^4(5\bar{z}\epsilon-2\bar{z}-3\epsilon+1)}{2(2\epsilon-1)^2(3\epsilon-1)(7\epsilon-2)(7\epsilon-1)} & -\frac{15z\epsilon^3\left(25\bar{z}\epsilon^2-15\bar{z}\epsilon+2\bar{z}+6\epsilon^2-2\epsilon\right)}{4(2\epsilon-1)^2(3\epsilon-1)(7\epsilon-2)(7\epsilon-1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\epsilon}{2\epsilon-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ -\frac{1}{8}\epsilon\, e_{104,1}\, t_{104,9} & \frac{\epsilon\, e_{104,1}\, t_{104,10}}{4} & 0 & -\frac{\bar{z}^2(\epsilon-1)}{2\epsilon-1} & \bar{z}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\bar{z}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\bar{z}} \end{pmatrix}.$$

With the functions:

$$
\begin{aligned}
e_{104,1} &= \frac{1}{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)(7\epsilon-4)(7\epsilon-3)(7\epsilon-2)(7\epsilon-1)}\,,\\
t_{104,1} &= 5-4\bar{z}+2\bar{z}^2+\left(-18\bar{z}^2+23\bar{z}-35\right)\epsilon+\left(40\bar{z}^2-19\bar{z}+60\right)\epsilon^2\,,\\
t_{104,2} &= 7-4\bar{z}+2\bar{z}^2+\left(-18\bar{z}^2+19\bar{z}-46\right)\epsilon+\left(40\bar{z}^2-3\bar{z}+63\right)\epsilon^2\,,\\
t_{104,3} &= -6+20\bar{z}-8\bar{z}^2+\left(92\bar{z}^2-210\bar{z}+61\right)\epsilon+\left(-340\bar{z}^2+686\bar{z}-196\right)\epsilon^2\\
&\quad+\left(400\bar{z}^2-736\bar{z}+201\right)\epsilon^3\,,\\
t_{104,4} &= -2+(4\bar{z}+23)\epsilon-14(\bar{z}+6)\epsilon^2+(10\bar{z}+99)\epsilon^3\,,\\
t_{104,5} &= 48\bar{z}^2+\left(720\bar{z}^4-7680\bar{z}^3+6092\bar{z}^2-600\right)\epsilon-20\Big(654\bar{z}^4-6474\bar{z}^3+3698\bar{z}^2\\
&\quad+846\bar{z}-525\Big)\epsilon^2+3\left(32040\bar{z}^4-293620\bar{z}^3+95567\bar{z}^2+70660\bar{z}-25000\right)\epsilon^3\\
&\quad+\left(-365880\bar{z}^4+3119100\bar{z}^3-340493\bar{z}^2-1019580\bar{z}+279300\right)\epsilon^4\\
&\quad+\Big(761400\bar{z}^4-6112740\bar{z}^3-349067\bar{z}^2+2346960\bar{z}-570600\Big)\epsilon^5\\
&\quad-15\Big(54800\bar{z}^4-421732\bar{z}^3-68323\bar{z}^2+171780\bar{z}-40320\Big)\epsilon^6\\
&\quad+18\left(20000\bar{z}^4-150400\bar{z}^3-30787\bar{z}^2+59770\bar{z}-14400\right)\epsilon^7\,,\\
t_{104,6} &= -6+10\bar{z}-4\bar{z}^2+\left(46\bar{z}^2-97\bar{z}+61\right)\epsilon+\left(-170\bar{z}^2+291\bar{z}-181\right)\epsilon^2\\
&\quad+8\Big(25\bar{z}^2-36\bar{z}+21\Big)\epsilon^3\,,\\
t_{104,7} &= 48\bar{z}^2-4\left(277\bar{z}^2+15\right)\epsilon-10\left(18\bar{z}^3-1229\bar{z}^2+186\bar{z}-105\right)\epsilon^2\\
&\quad+6\left(305\bar{z}^3-11604\bar{z}^2+4005\bar{z}-1250\right)\epsilon^3+\Big(-7230\bar{z}^3+213667\bar{z}^2\\
&\quad-120600\bar{z}+27930\Big)\epsilon^4+\left(13830\bar{z}^3-359237\bar{z}^2+293910\bar{z}-57060\right)\epsilon^5\\
&\quad-15\left(850\bar{z}^3-20593\bar{z}^2+23184\bar{z}-4032\right)\epsilon^6\\
&\quad+9\left(500\bar{z}^3-11659\bar{z}^2+17760\bar{z}-2880\right)\epsilon^7\,,\\
t_{104,8} &= -1104\bar{z}^2+\left(480\bar{z}^4-5120\bar{z}^3+30284\bar{z}^2-560\right)\epsilon-20\Big(436\bar{z}^4-4340\bar{z}^3\\
&\quad+14403\bar{z}^2+1260\bar{z}-580\Big)\epsilon^2+\Big(64080\bar{z}^4-595720\bar{z}^3+1330447\bar{z}^2+343240\bar{z}\\
&\quad-98380\Big)\epsilon^3+\left(-243920\bar{z}^4+2118720\bar{z}^3-3338881\bar{z}^2-1759040\bar{z}+420520\right)\epsilon^4\\
&\quad+\left(507600\bar{z}^4-4122120\bar{z}^3+4639861\bar{z}^2+4284520\bar{z}-943800\right)\epsilon^5\\
&\quad-5\left(109600\bar{z}^4-832864\bar{z}^3+668679\bar{z}^2+998400\bar{z}-210960\right)\epsilon^6
\end{aligned}
$$

$$
\begin{aligned}
&+ 12\left(20000\bar{z}^4 - 142400\bar{z}^3 + 80904\bar{z}^2 + 186360\bar{z} - 38395\right)\epsilon^7\,, \\
t_{104,9} &= 528\bar{z}^2 + \left(120 - 10076\bar{z}^2\right)\epsilon + 4\left(180\bar{z}^3 + 16859\bar{z}^2 + 900\bar{z} - 405\right)\epsilon^2 \\
&- 5\left(1320\bar{z}^3 + 43739\bar{z}^2 + 6312\bar{z} - 1704\right)\epsilon^3 + \Big(22320\bar{z}^3 + 372337\bar{z}^2 + 101400\bar{z} \\
&- 21780\Big)\epsilon^4 - 3\left(11000\bar{z}^3 + 106183\bar{z}^2 + 47160\bar{z} - 9000\right)\epsilon^5 \\
&+ 9\left(2000\bar{z}^3 + 11891\bar{z}^2 + 8040\bar{z} - 1440\right)\epsilon^6\,, \\
t_{104,10} &= -336\bar{z}^2 + \left(-180\bar{z}^4 + 1920\bar{z}^3 + 4612\bar{z}^2 + 60\right)\epsilon + \Big(2550\bar{z}^4 - 25500\bar{z}^3 - 25432\bar{z}^2 \\
&+ 1620\bar{z} - 810\Big)\epsilon^2 - 5\left(2766\bar{z}^4 - 25722\bar{z}^3 - 14629\bar{z}^2 + 2814\bar{z} - 852\right)\epsilon^3 \\
&+ \left(36150\bar{z}^4 - 311730\bar{z}^3 - 116039\bar{z}^2 + 44760\bar{z} - 10890\right)\epsilon^4 \\
&+ \left(-45750\bar{z}^4 + 365730\bar{z}^3 + 95703\bar{z}^2 - 61830\bar{z} + 13500\right)\epsilon^5 \\
&+ 9\left(2500\bar{z}^4 - 18550\bar{z}^3 - 3517\bar{z}^2 + 3480\bar{z} - 720\right)\epsilon^6\,.
\end{aligned}
$$

G.5 $\hat{\mathbf{T}}_{\mathbf{105}}$

The transformation matrix for family 105 contains the following entries:

$$
\hat{T}_{105} = \begin{pmatrix} \hat{T}_{105,a_1} & \hat{T}_{105,a_2} & \hat{T}_{105,a_3} \end{pmatrix}\,,
$$

the components being:

$$\hat{T}_{105,a_1} = \begin{pmatrix} \frac{6z^2\bar{z}\epsilon^3(2\epsilon-1)}{(3\epsilon-2)(3\epsilon-1)(4\epsilon-3)(4\epsilon-1)} & 0 & 0 \\ \frac{3\bar{z}\epsilon^2(2\bar{z}\epsilon+\epsilon-1)}{(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} & \frac{3\bar{z}\epsilon^2}{8\epsilon^2-6\epsilon+1} & 0 \\ 0 & 0 & \frac{6z^2\epsilon^2(2\epsilon-1)}{(3\epsilon-2)(3\epsilon-1)(4\epsilon-1)} \\ -\frac{6\bar{z}(\bar{z}+1)\epsilon^3}{(3\epsilon-2)(3\epsilon-1)(4\epsilon-1)} & 0 & 0 \\ 0 & 0 & -\frac{4\epsilon(2\epsilon-1)(3\bar{z}\epsilon+\epsilon-1)}{(3\epsilon-2)(3\epsilon-1)(4\epsilon-1)} \\ \frac{2\epsilon(2\epsilon-1)\,t_{104,1}}{(3\epsilon-1)(4\epsilon-1)(7\epsilon-2)(7\epsilon-1)} & -\frac{6\epsilon(2\epsilon-1)\left((\bar{z}+12)\epsilon^2-(\bar{z}+7)\epsilon+1\right)}{(3\epsilon-1)(4\epsilon-1)(7\epsilon-2)(7\epsilon-1)} & \frac{4t_{105,2}\epsilon(2\epsilon-1)}{(3\epsilon-1)(4\epsilon-1)(7\epsilon-2)(7\epsilon-1)} \\ -\frac{\epsilon\,t_{104,3}}{2(3\epsilon-1)(4\epsilon-1)(7\epsilon-2)(7\epsilon-1)} & \frac{3\epsilon\,t_{104,4}}{2(3\epsilon-1)(4\epsilon-1)(7\epsilon-2)(7\epsilon-1)} & -\frac{2\epsilon\,t_{104,6}}{(3\epsilon-1)(4\epsilon-1)(7\epsilon-2)(7\epsilon-1)} \\ -\frac{2(2\epsilon-1)\left(6(\bar{z}+1)\epsilon^2-7\epsilon+1\right)}{(1-3\epsilon)^2(4\epsilon-1)} & 0 & 0 \\ \frac{6\bar{z}(1-2\epsilon)\epsilon}{z(12\epsilon^2-7\epsilon+1)} & \frac{6\bar{z}(1-2\epsilon)\epsilon}{z(12\epsilon^2-7\epsilon+1)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{t_{105,1}}{z^2(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} & \frac{t_{105,2}}{z^2(2\epsilon-1)(3\epsilon-1)(4\epsilon-1)} & 0 \end{pmatrix},$$

$$\hat{T}_{105,a_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{2\bar{z}^2\epsilon^2}{z(9\epsilon^2-9\epsilon+2)} & 0 & 0 \\ 0 & \frac{2\epsilon(2\epsilon-1)}{9\epsilon^2-9\epsilon+2} & 0 \\ 0 & 0 & \frac{15\epsilon(2\epsilon-1)((2\bar{z}+3)\epsilon-1)}{2(3\epsilon-1)(7\epsilon-2)(7\epsilon-1)} \\ 0 & 0 & -\frac{15\epsilon^2((5\bar{z}-3)\epsilon-2\bar{z}+1)}{2(3\epsilon-1)(7\epsilon-2)(7\epsilon-1)} \\ 0 & 0 & 0 \\ \frac{2\bar{z}(1-2\epsilon)\epsilon}{z(12\epsilon^2-7\epsilon+1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{t_{105,3}}{z^2(8\epsilon^2-6\epsilon+1)} & 0 & 0 \end{pmatrix},$$

$$\hat{T}_{105,a_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{15z\epsilon(2\epsilon-1)(z+(5\bar{z}-3)\epsilon)}{2(3\epsilon-1)(7\epsilon-2)(7\epsilon-1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{15z\epsilon\left((25\bar{z}+6)\epsilon^2-(15\bar{z}+2)\epsilon+2\bar{z}\right)}{4(3\epsilon-1)(7\epsilon-2)(7\epsilon-1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2-4\epsilon}{1-3\epsilon} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{6(\bar{z}-2)\bar{z}\epsilon(2\epsilon-1)}{z(3\epsilon-1)(4\epsilon-1)} & \frac{4(\bar{z}-3)\bar{z}\epsilon(2\epsilon-1)}{z(3\epsilon-1)(4\epsilon-1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\epsilon-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\epsilon)^2}{\epsilon^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\epsilon)^2}{\bar{z}\epsilon^2} & 0 \\ 0 & 0 & \frac{t_{105,4}}{z^2(12\epsilon^2-7\epsilon+1)} & \frac{t_{105,5}}{3z^2(12\epsilon^2-7\epsilon+1)} & 0 & 0 & 1 \end{pmatrix} .$$

Some of the functions were already defined for family 104 in the last section, the new ones are:

$$\begin{aligned} t_{105,1} &= -2z^2 + 3\left(2\bar{z}^3 + \bar{z}^2 - 10\bar{z} + 6\right)\epsilon + \left(-33\bar{z}^3 + 35\bar{z}^2 + 65\bar{z} - 52\right)\epsilon^2 \\ &\quad + \left(51\bar{z}^3 - 84\bar{z}^2 - 33\bar{z} + 48\right)\epsilon^3 , \\ t_{105,2} &= -2z^2 + 3\left(2\bar{z}^3 + \bar{z}^2 - 10\bar{z} + 6\right)\epsilon + \left(-33\bar{z}^3 + 35\bar{z}^2 + 65\bar{z} - 52\right)\epsilon^2 \\ &\quad + \left(45\bar{z}^3 - 72\bar{z}^2 - 39\bar{z} + 48\right)\epsilon^3 , \\ t_{105,3} &= z^2 + \left(-2\bar{z}^3 - \bar{z}^2 + 10\bar{z} - 6\right)\epsilon + 2\left(\bar{z}^3 - 6\bar{z} + 4\right)\epsilon^2 , \\ t_{105,4} &= -2z^2 + \left(-3\bar{z}^4 + 18\bar{z}^3 - 16\bar{z}^2 - 16\bar{z} + 14\right)\epsilon + 3\left(\bar{z}^4 - 12\bar{z}^3 + 18\bar{z}^2 + 4\bar{z} - 8\right)\epsilon^2 , \\ t_{105,5} &= 7z^2 + \left(6\bar{z}^4 - 48\bar{z}^3 + 41\bar{z}^2 + 62\bar{z} - 49\right)\epsilon + -6\left(\bar{z}^4 - 17\bar{z}^3 + 26\bar{z}^2 + 10\bar{z} - 14\right)\epsilon^2 . \end{aligned}$$

G.6 $\hat{\mathbf{T}}_{106}$

For the last family, we have the following transformation:

$$\hat{T}_{106} = \begin{pmatrix} \hat{T}_{106,a_1} & \hat{T}_{106,a_2} & \hat{T}_{106,a_3} \end{pmatrix} ,$$

with the entries:

$$\hat{T}_{106,a_1} = \begin{pmatrix}
\frac{6z^2\bar{z}\epsilon^4}{(3\epsilon-2)(3\epsilon-1)(4\epsilon-3)(4\epsilon-1)} & 0 & 0 \\
\frac{3\bar{z}\epsilon^3(2\bar{z}\epsilon+\epsilon-1)}{(2\epsilon-1)^2(3\epsilon-1)(4\epsilon-1)} & \frac{3\bar{z}\epsilon^3}{(2\epsilon-1)^2(4\epsilon-1)} & 0 \\
0 & 0 & \frac{6z^2\epsilon^3}{(3\epsilon-2)(3\epsilon-1)(4\epsilon-1)} \\
-\frac{6\bar{z}(\bar{z}+1)\epsilon^4}{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)(4\epsilon-1)} & 0 & 0 \\
0 & 0 & -\frac{4\epsilon^2(3\bar{z}\epsilon+\epsilon-1)}{(3\epsilon-2)(3\epsilon-1)(4\epsilon-1)} \\
-\frac{2\epsilon\left(6\bar{z}\epsilon^2+6\epsilon^2-7\epsilon+1\right)}{(3\epsilon-1)^2(4\epsilon-1)} & 0 & 0 \\
-\frac{2\epsilon\left(6\bar{z}\epsilon^2+6\epsilon^2-7\epsilon+1\right)}{(3\epsilon-1)^2(4\epsilon-1)} & 0 & \frac{4\epsilon(3\bar{z}\epsilon+\epsilon-1)}{(3\epsilon-1)(4\epsilon-1)} \\
\frac{6\epsilon^2\, t_{106,1}}{(3\epsilon-1)(4\epsilon-1)(7\epsilon-2)(7\epsilon-1)} & -\frac{6\epsilon^2\, t_{106,2}}{(3\epsilon-1)(4\epsilon-1)(7\epsilon-2)(7\epsilon-1)} & \frac{6\epsilon^2(5\epsilon-1)\, t_{106,3}}{(3\epsilon-1)(4\epsilon-1)(7\epsilon-2)(7\epsilon-1)} \\
\frac{(\epsilon-2)\epsilon^2\, e_{106,1}\, t_{106,5}}{(2\epsilon-1)(4\epsilon-3)(4\epsilon-1)} & -\frac{(\epsilon-2)\epsilon^2\, e_{106,1}\, t_{106,6}}{(2\epsilon-1)(4\epsilon-3)(4\epsilon-1)} & \frac{2(\epsilon-2)\epsilon^2\, e_{106,1}\, t_{106,7}}{(4\epsilon-3)(4\epsilon-1)} \\
-\frac{6\bar{z}\epsilon^2}{z(3\epsilon-1)(4\epsilon-1)} & -\frac{6\bar{z}\epsilon^2}{z(3\epsilon-1)(4\epsilon-1)} & 0 \\
0 & 0 & 0 \\
-\frac{\epsilon\, e_{106,2}\, t_{106,10}}{4z^3} & -\frac{3\epsilon\, e_{106,2}\, t_{106,11}}{4z^3} & 0 \\
0 & 0 & 0
\end{pmatrix} ,$$

$$\hat{T}_{106,a_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{2\bar{z}^2\epsilon^3}{z(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)} & 0 & 0 & 0 & 0 \\ \frac{2\epsilon^2}{(3\epsilon-2)(3\epsilon-1)} & \frac{2\epsilon^2}{(3\epsilon-2)(3\epsilon-1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2\epsilon}{3\epsilon-1} & 0 \\ 0 & 0 & \frac{2\epsilon}{3\epsilon-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{18\epsilon^2\, t_{106,4}}{(3\epsilon-1)(7\epsilon-2)(7\epsilon-1)} \\ 0 & 0 & 0 & 0 & e_{106,1}\, t_{106,8} \\ -\frac{2\bar{z}\epsilon^2}{z(3\epsilon-1)(4\epsilon-1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\epsilon\, e_{106,2}\, t_{106,12}}{2z^3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\hat{T}_{106,a_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{108(\bar{z}-2)\bar{z}\epsilon^2(5\epsilon-1)}{(3\epsilon-1)(7\epsilon-2)(7\epsilon-1)} & 0 & 0 & 0 & 0 \\ e_{106,1}\, t_{106,9} & 0 & 0 & 0 & 0 \\ 0 & -\frac{6(\bar{z}-2)\bar{z}\epsilon^2}{z(3\epsilon-1)(4\epsilon-1)} & \frac{4(\bar{z}-3)\bar{z}\epsilon^2}{z(3\epsilon-1)(4\epsilon-1)} & 0 & 0 \\ 0 & 0 & \epsilon & 0 & 0 \\ 0 & -\,e_{106,2}\, t_{106,13} & e_{106,2}\, t_{106,14} & \frac{\epsilon}{2\epsilon-1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} .$$

The functions that are used are defined as:

$$e_{106,1} = \frac{1}{(3\epsilon-2)(3\epsilon-1)(7\epsilon-4)(7\epsilon-3)(7\epsilon-2)(7\epsilon-1)}\,,$$

$$e_{106,2} = \frac{1}{(2\epsilon-1)^2(3\epsilon-2)(3\epsilon-1)(4\epsilon-3)(4\epsilon-1)}\,,$$

$$t_{106,1} = z^2 + \left(-9\bar{z}^2 + 15\bar{z} - 7\right)\epsilon + \left(20\bar{z}^2 - 25\bar{z} + 12\right)\epsilon^2\,,$$

$$t_{106,2} = 1 + (-\bar{z} - 7)\epsilon + (\bar{z} + 12)\epsilon^2\,,$$

$$t_{106,3} = -2 + 2\bar{z} - \bar{z}^2 + \left(4\bar{z}^2 - 2\bar{z} + 2\right)\epsilon\,,$$

$$t_{106,4} = 1 + 2\bar{z} - \bar{z}^2 + \left(5\bar{z}^2 - 12\bar{z} - 3\right)\epsilon\,,$$

$$t_{106,5} = 36z^4 - 6\left(99\bar{z}^4 - 386\bar{z}^3 + 555\bar{z}^2 - 334\bar{z} + 77\right)\epsilon + \Big(3896\bar{z}^4 - 14850\bar{z}^3$$

$$
\begin{aligned}
&+ 20421\bar{z}^2 - 10792\bar{z} + 2352\Big)\epsilon^2 + \Big(- 13014\bar{z}^4 + 49024\bar{z}^3 - 65017\bar{z}^2 + 29349\bar{z} \\
&- 6090\Big)\epsilon^3 + \left(23420\bar{z}^4 - 88510\bar{z}^3 + 115177\bar{z}^2 - 43537\bar{z} + 8484\right)\epsilon^4 \\
&+ \left(-21600\bar{z}^4 + 83192\bar{z}^3 - 108497\bar{z}^2 + 34311\bar{z} - 6048\right)\epsilon^5 \\
&+ \left(8000\bar{z}^4 - 31892\bar{z}^3 + 42542\bar{z}^2 - 11551\bar{z} + 1728\right)\epsilon^6 .
\end{aligned}
$$

$$
\begin{aligned}
t_{106,6} &= 36 - 6\left(6\bar{z}^3 - 21\bar{z}^2 + 26\bar{z} + 77\right)\epsilon + \left(306\bar{z}^3 - 1065\bar{z}^2 + 1276\bar{z} + 2352\right)\epsilon^2 \\
&- \left(1016\bar{z}^3 - 3335\bar{z}^2 + 3495\bar{z} + 6090\right)\epsilon^3 + \left(1646\bar{z}^3 - 4619\bar{z}^2 + 2875\bar{z} + 8484\right)\epsilon^4 \\
&+ \left(-1300\bar{z}^3 + 2539\bar{z}^2 + 1851\bar{z} - 6048\right)\epsilon^5 + \left(400\bar{z}^3 - 244\bar{z}^2 - 2711\bar{z} + 1728\right)\epsilon^6 , \\
t_{106,7} &= -18\left(\bar{z}^4 - 4\bar{z}^3 + 6\bar{z}^2 - 4\bar{z} + 2\right) + 3\left(87\bar{z}^4 - 328\bar{z}^3 + 444\bar{z}^2 - 224\bar{z} + 108\right)\epsilon \\
&+ \left(-1426\bar{z}^4 + 5138\bar{z}^3 - 6255\bar{z}^2 + 1902\bar{z} - 785\right)\epsilon^2 + \Big(3655\bar{z}^4 - 12920\bar{z}^3 \\
&+ 14574\bar{z}^2 - 1812\bar{z} + 158\Big)\epsilon^3 - \left(4400\bar{z}^4 - 15640\bar{z}^3 + 17061\bar{z}^2 - 168\bar{z} - 1253\right)\epsilon^4 \\
&+ \left(2000\bar{z}^4 - 7306\bar{z}^3 + 8058\bar{z}^2 + 162\bar{z} - 914\right)\epsilon^5 , \\
t_{106,8} &= 6(\epsilon - 2)\epsilon^2 \Big[- 6\left(\bar{z}^4 - 4\bar{z}^3 + 6\bar{z}^2 - 4\bar{z} - 1\right) + \Big(55\bar{z}^4 - 240\bar{z}^3 + 396\bar{z}^2 \\
&- 288\bar{z} - 33\Big)\epsilon + \left(-150\bar{z}^4 + 686\bar{z}^3 - 1190\bar{z}^2 + 900\bar{z} + 54\right)\epsilon^2 \\
&+ \left(125\bar{z}^4 - 590\bar{z}^3 + 1058\bar{z}^2 - 816\bar{z} - 27\right)\epsilon^3\Big] , \\
t_{106,9} &= -36(\bar{z} - 2)\bar{z}(\epsilon - 2)\epsilon^2(5\epsilon - 2)\Big[3\left(\bar{z}^2 - 2\bar{z} + 2\right) + \left(-20\bar{z}^2 + 44\bar{z} - 44\right)\epsilon \\
&+ \Big(25\bar{z}^2 - 58\bar{z} + 58\Big)\epsilon^2\Big] , \\
t_{106,10} &= -24z^3 + \left(54\bar{z}^4 - 494\bar{z}^3 + 1068\bar{z}^2 - 924\bar{z} + 284\right)\epsilon - 6\Big(2\bar{z}^5 + 77\bar{z}^4 \\
&- 524\bar{z}^3 + 964\bar{z}^2 - 750\bar{z} + 214\Big)\epsilon^2 + \Big(36\bar{z}^6 - 111\bar{z}^5 + 1797\bar{z}^4 - 9079\bar{z}^3 \\
&+ 14727\bar{z}^2 - 10464\bar{z} + 2776\Big)\epsilon^3 \\
&- 3\left(32\bar{z}^6 - 121\bar{z}^5 + 954\bar{z}^4 - 3952\bar{z}^3 + 5850\bar{z}^2 - 3867\bar{z} + 960\right)\epsilon^4 \\
&+ 3\left(20\bar{z}^6 - 82\bar{z}^5 + 507\bar{z}^4 - 1877\bar{z}^3 + 2613\bar{z}^2 - 1637\bar{z} + 384\right)\epsilon^5 , \\
t_{106,11} &= -12z^3 + 2\left(9\bar{z}^4 - 106\bar{z}^3 + 249\bar{z}^2 - 225\bar{z} + 71\right)\epsilon - 2\Big(2\bar{z}^5 + 77\bar{z}^4 - 631\bar{z}^3 \\
&+ 1285\bar{z}^2 - 1071\bar{z} + 321\Big)\epsilon^2 + \Big(27\bar{z}^5 + 447\bar{z}^4 - 3289\bar{z}^3 + 6157\bar{z}^2 - 4836\bar{z} \\
&+ 1388\Big)\epsilon^3 + \left(-53\bar{z}^5 - 542\bar{z}^4 + 3892\bar{z}^3 - 6910\bar{z}^2 + 5197\bar{z} - 1440\right)\epsilon^4 \\
&+ \left(32\bar{z}^5 + 235\bar{z}^4 - 1713\bar{z}^3 + 2937\bar{z}^2 - 2139\bar{z} + 576\right)\epsilon^5 ,
\end{aligned}
$$

$$
\begin{aligned}
t_{106,12} &= 12z^3 + \left(-9\bar{z}^4 + 177\bar{z}^3 - 462\bar{z}^2 + 438\bar{z} - 142\right)\epsilon + \Big(-2\bar{z}^5 + 74\bar{z}^4 - 916\bar{z}^3 \\
&\quad + 2205\bar{z}^2 - 2020\bar{z} + 642\Big)\epsilon^2 + \Big(12\bar{z}^5 - 217\bar{z}^4 + 2153\bar{z}^3 - 4937\bar{z}^2 + 4430\bar{z} \\
&\quad - 1388\Big)\epsilon^3 + \left(-22\bar{z}^5 + 276\bar{z}^4 - 2362\bar{z}^3 + 5240\bar{z}^2 - 4644\bar{z} + 1440\right)\epsilon^4 \\
&\quad + 4\left(3\bar{z}^5 - 32\bar{z}^4 + 246\bar{z}^3 - 532\bar{z}^2 + 468\bar{z} - 144\right)\epsilon^5\,, \\
t_{106,13} &= -\frac{\epsilon(2\epsilon-1)}{2z^3}\Big[-24z^3 + \left(-3\bar{z}^6 + 9\bar{z}^5 + 54\bar{z}^4 - 446\bar{z}^3 + 924\bar{z}^2 - 780\bar{z} + 236\right)\epsilon \\
&\quad + \Big(18\bar{z}^6 - 72\bar{z}^5 - 255\bar{z}^4 + 1982\bar{z}^3 - 3678\bar{z}^2 + 2856\bar{z} - 812\Big)\epsilon^2 \\
&\quad + \Big(-33\bar{z}^6 + 153\bar{z}^5 + 399\bar{z}^4 - 3222\bar{z}^3 + 5682\bar{z}^2 - 4212\bar{z} + 1152\Big)\epsilon^3 \\
&\quad + 6\left(3\bar{z}^6 - 16\bar{z}^5 - 34\bar{z}^4 + 292\bar{z}^3 - 500\bar{z}^2 + 360\bar{z} - 96\right)\epsilon^4\Big]\,, \\
t_{106,14} &= \frac{\epsilon(2\epsilon-1)}{z^3}\Big[-6z^3 + \left(-\bar{z}^6 + 3\bar{z}^5 + 27\bar{z}^4 - 164\bar{z}^3 + 285\bar{z}^2 - 213\bar{z} + 59\right)\epsilon \\
&\quad + \Big(6\bar{z}^6 - 23\bar{z}^5 - 143\bar{z}^4 + 806\bar{z}^3 - 1236\bar{z}^2 + 819\bar{z} - 203\Big)\epsilon^2 \\
&\quad + \Big(-11\bar{z}^6 + 48\bar{z}^5 + 246\bar{z}^4 - 1377\bar{z}^3 + 1994\bar{z}^2 - 1242\bar{z} + 288\Big)\epsilon^3 \\
&\quad + 2\left(3\bar{z}^6 - 15\bar{z}^5 - 68\bar{z}^4 + 386\bar{z}^3 - 540\bar{z}^2 + 324\bar{z} - 72\right)\epsilon^4\Big]\,.
\end{aligned}
$$

Appendix H

Five-Particle Phase Space

A parametrization for the five-particle phase space has been derived in Ref. [72]. For completeness we give a brief overview of the derivation here, for the detailled calculation we refer to the original paper.
The starting point of the derivation is, as for the lower dimensional measures, the formula Eq. (3.23).
This is first rewritten in terms of energies and relative angles of the particle trajectories, which can then in turn be replaced by the momentum invariants s_{ij}.
Introducing the shorthand notation

$$\begin{aligned} &y_1 = s_{12}/q^2,\, y_2 = s_{13}/q^2,\, y_3 = s_{23}/q^2,\, y_4 = s_{14}/q^2,\, y_5 = s_{24}/q^2 \\ &y_6 = s_{34}/q^2,\, y_7 = s_{15}/q^2,\, y_8 = s_{25}/q^2,\, y_9 = s_{35}/q^2,\, y_{10} = s_{45}/q^2\,, \end{aligned} \tag{H.1}$$

the phase space integrations for a decaying b-quark can be written as

$$\begin{aligned} \int dPS_5 =& (2\pi)^{5-4D} 2^{-2-2D} V(D-1)V(D-2)V(D-3)V(D-4)(m_b^2)^{2D-5} \\ & \int \prod_{j=1}^{10} \delta(1 - \sum_{i=1}^{10} y_1)(-\Delta_5)^{\frac{D}{2}-3}\Theta(-\Delta_5)\,. \end{aligned} \tag{H.2}$$

Δ_5 is related to the determinant of the Gram matrix (defined as $G_{ij} = p_i \cdot p_j$) via

$$-\Delta_5 = -\frac{1}{2}\det G/(q^2)^5\,. \tag{H.3}$$

Explicitly written, this looks as follows:

$$\begin{aligned}
-\Delta_5 = {} & y_{10}^2 y_1 y_2 y_3 + y_9^2 y_1 y_4 y_5 + y_8^2 y_2 y_4 y_6 + y_7^2 y_3 y_5 y_6 + y_6^2 y_1 y_7 y_8 \\
& + y_5^2 y_2 y_7 y_9 + y_4^2 y_3 y_8 y_9 + y_3^2 y_4 y_7 y_{10} + y_1^2 y_6 y_9 y_{10} \\
& + y_{10}[y_2 y_3 y_5 y_7 + y_1 y_3 y_6 y_7 + y_2 y_3 y_4 y_8 + y_1 y_2 y_6 y_8 + y_1 y_3 y_4 y_9 + y_1 y_2 y_5 y_9] \\
& + y_9[y_4 y_5(y_3 y_7 + y_2 y_8) + y_2 y_6(y_5 y_7 + y_4 y_8)] + y_6 y_7 y_8(y_3 y_4 + y_2 y_5)\,. \qquad \text{(H.4)}
\end{aligned}$$

It is now of great help to use a parametrization that factorizes the Gram determinant. One example for this is given in Ref. [71]. For this, y_1 is eliminated by momentum conservation and the double invariants y_6 and y_7 are traded for a triple and quadruple invariant, respectively.
The theta function constraint is solved for y_5, which gives the solutions

$$y_5^{\pm} = y_5^0 \pm \sqrt{R_5}\,, \qquad \text{(H.5)}$$

then $\sqrt{R_5} \geq 0$ is solved for y_8 and finally $y_8^+ - y_8^-$ is solved for y_{10}.
New variables t_i are introduced that are integrated from zero to one:

$$\begin{aligned}
s_{1345}/q^2 &= t_7\,, & s_{34}/q^2 &= t_2 t_6 t_7 \bar{t}_4\,, \\
s_{134}/q^2 &= t_6 t_7\,, & s_{15}/q^2 &= t_7 \bar{t}_6[1 - t_9(1 - t_2 t_2)] - y_{10}\,, \\
s_{13}/q^2 &= t_6 t_7 \bar{t}_2\,, & s_{25}/q^2 &= y_8^- + (y_8^+ - y_8^-) t_8 \\
s_{23}/q^2 &= t_3 \bar{t}_7 (1 - t_2 t_4)(t_6 \bar{t}_9 + t_9)\,, & s_{35}/q^2 &= t_7 t_9 \bar{t}_6 (1 - t_2 t_4)\,, \\
s_{14}/q^2 &= t_2 t_4 t_6 t_7 & s_{45}/q^2\,, &= y_{10}^- + (y_{10}^+ - y_{10}^-) t_{10}\,, \\
s_{24}/q^2 &= y_5^- + (y_5^+ - y_5^-) t_5)\,.
\end{aligned}$$

With these new variables, the phase space now finally factorizes:

$$\begin{aligned}
\int d\Phi^D_{1\to 5} = {} & \mathcal{K}_\Gamma^{(5)} (q^2)^{2D-5} \int_0^1 \prod_{j=2}^{10} dt_j [t_5 \bar{t}_5]^{-1-\epsilon} [t_8 \bar{t}_8 t_{10} \bar{t}_{10}]^{-\frac{1}{2}-\epsilon} \\
& \times [t_2 t_6 \bar{t}_6 \bar{t}_7]^{1-2\epsilon} [(\bar{t}_2 t_3 \bar{t}_3 t_4 \bar{t}_4 t_9 \bar{t}_9)]^{-\epsilon} t_7^{2-3\epsilon}\,. \qquad \text{(H.6)}
\end{aligned}$$

Here, we collected the prefactors from the phase space in $\mathcal{K}_\Gamma^{(5)}$, which reads:

$$\begin{aligned}\mathcal{K}_\Gamma^{(5)} &= (2\pi)^{5-4D}2^{-2-2D}2^{-8\epsilon}V(D-1)V(D-2)V(D-3)V(D-4)\\ &= \frac{4^{4\epsilon}}{2^{17}\pi^9\Gamma(-2\epsilon)\Gamma(2-2\epsilon)}\,.\end{aligned} \tag{H.7}$$

Note that this is the phase space without a cut included, which means that we have not yet assigned p_4 to the photon. We thus still have the power to rename all the momenta to best fit the kernels at hand in our calculation.

Bibliography

[1] S. L. Glashow, *Partial Symmetries of Weak Interactions*, *Nucl. Phys.* **B 22**, 579 (1961).

[2] S. Weinberg, *A Model of Leptons*, *Phys. Rev. Lett.* **19**, 1264 (1967).

[3] A. Salam, *Gauge Unification of Fundamental Forces*, *Rev. Mod. Phys.* **52**, 525 (1980).

[4] **CMS** Collaboration, *Observation of a New Boson at a Mass of 125 GeV with the CMS Experiment at the LHC*, *Phys. Lett.* **B 716**, 30 (2012), arXiv:1207.7235 [hep-ex].

[5] **ATLAS** Collaboration, *Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC*, *Phys. Lett.* **B 716**, 1 (2012), arXiv:1207.7214 [hep-ex].

[6] **BaBar** Collaboration, *Precision Measurement of the $B \to X_s\gamma$ Photon Energy Spectrum, Branching Fraction, and Direct CP Asymmetry $A_{CP}(B \to X_{s+d}\gamma)$*, *Phys. Rev. Lett.* **109**, 191801 (2012), arXiv:1207.2690 [hep-ex].

[7] **Belle** Collaboration, *Measurement of Inclusive Radiative B-meson Decays with a Photon Energy Threshold of 1.7 GeV*, *Phys. Rev. Lett.* **103**, 241801 (2009), arXiv:0907.1384 [hep-ex].

[8] **CLEO** Collaboration, *Branching fraction and photon energy spectrum for $b \to s\gamma$*, *Phys. Rev. Lett.* **87**, 251807 (2001), hep-ex/0108032.

[9] **HFLAV** Collaboration, *Averages of b-hadron, c-hadron, and τ-lepton properties as of 2021*, , arXiv:2206.07501 [hep-ex].

[10] M. Kaminski, M. Misiak, and M. Poradziński, *Tree-level contributions to $B \to X_s\gamma$*, *Phys. Rev.* **D 86**, 094004 (2012), `arXiv:1209.0965 [hep-ph]`.

[11] M. Misiak, A, Rehman, and M. Steinhauser, *Towards $\bar{B} \to X_s\gamma$ at the NNLO in QCD without interpolation in m_c*, *JHEP* **06**, 175 (2020), `arXiv:2002.01548 [hep-ph]`.

[12] A. J. Buras et al., *Completing the NLO QCD calculation of $\bar{B} \to X_s\gamma$*, *Nucl. Phys.* **B 631**, 219 (2002), `hep-ph/0203135`.

[13] M. Misiak et al., *Estimate of $\mathcal{B}(\bar{B} \to X_s\gamma)$ at $O(\alpha_s^2)$*, *Phys. Rev. Lett.* **98**, 022002 (2007), `hep-ph/0609232`.

[14] M. Misiak and M. Poradziński, *Completing the Calculation of BLM corrections to $\bar{B} \to X_s\gamma$*, *Phys. Rev.* **D 83**, 014024 (2011), `arXiv:1009.5685 [hep-ph]`.

[15] C. Bobeth, M. Misiak, and J. Urban, *Photonic penguins at two loops and m_t dependence of $BR[B \to X_s l^+ l^-]$*, *Nucl. Phys.* **B 574**, 291 (2000), `hep-ph/9910220`.

[16] M. Misiak and M. Steinhauser, *Three loop matching of the dipole operators for $b \to s\gamma$ and $b \to sg$*, *Nucl. Phys.* **B 683**, 277 (2004), `hep-ph/0401041`.

[17] M. Gorbahn and U. Haisch, *Effective Hamiltonian for non-leptonic $|\Delta F| = 1$ decays at NNLO in QCD*, *Nucl. Phys.* **B 713**, 291 (2005), `hep-ph/0411071`.

[18] M. Gorbahn, U. Haisch, and M. Misiak, *Three-loop mixing of dipole operators*, *Phys. Rev. Lett.* **95**, 102004 (2005), `hep-ph/0504194`.

[19] M. Czakon, U. Haisch, and M. Misiak, *Four-Loop Anomalous Dimensions for Radiative Flavour-Changing Decays*, *JHEP* **03**, 008 (2007), `hep-ph/0612329`.

[20] I. R. Blokland et al., *The Electromagnetic dipole operator effect on $\bar{B} \to X_s\gamma$ at $\mathcal{O}(\alpha_s^2)$*, *Phys. Rev.* **D 72**, 033014 (2005), `hep-ph/0506055`.

[21] K. Melnikov and A. Mitov, *The Photon energy spectrum in $\bar{B} \to X_s\gamma$ in perturbative QCD through $\mathcal{O}(\alpha_s^2)$*, *Phys. Lett.* **B 620**, 69 (2005), `hep-ph/0505097`.

[22] H. M. Asatrian et al., *NNLL QCD contribution of the electromagnetic dipole operator to* $\Gamma(\bar{B} \to X_s\gamma)$, *Nucl. Phys.* **B 749**, 325 (2006), `hep-ph/0605009`.

[23] K. Bieri, C. Greub, and M. Steinhauser, *Fermionic NNLL corrections to* $b \to s\gamma$, *Phys. Rev.* **D 67**, 114019 (2003), `hep-ph/0302051`.

[24] Z. Ligeti et al., *The* $\Gamma(\bar{B} \to X_s\gamma)$ *photon spectrum*, *Phys. Rev.* **D 60**, 034019 (1999), `hep-ph/9903305`.

[25] R. Boughezal, M. Czakon, and T. Schutzmeier, *NNLO fermionic corrections to the charm quark mass dependent matrix elements in* $\bar{B} \to X_s\gamma$, *JHEP* **09**, 072 (2007), `arXiv:0707.3090 [hep-ph]`.

[26] M. Misiak and M. Steinhauser, *NNLO QCD corrections to the* $\bar{B} \to X_s\gamma$ *matrix elements using interpolation in* m_c, *Nucl. Phys.* **B 764**, 62 (2007), `hep-ph/0609241`.

[27] M. Misiak and M. Steinhauser, *Large-*m_c *Asymptotic Behaviour of* $O(\alpha_s^2)$ *Corrections to* $B \to X_s\gamma$, *Nucl. Phys.* **B 840**, 271 (2010), `arXiv:1005.1173 [hep-ph]`.

[28] H. M. Asatrian et al., *Charm quark mass dependence of the electromagnetic dipole operator contribution to* $\bar{B} \to X_s\gamma$ *at* $O(\alpha_s^2)$, *Phys. Lett.* **B 647**, 173 (2007), `hep-ph/0611123`.

[29] T. Ewerth, *Fermionic corrections to the interference of the electro- and chromomagnetic dipole operators in* $\bar{B} \to X_s\gamma$ *at* $O(\alpha_s^2)$, *Phys. Lett.* **B 669**, 167 (2008), `arXiv:0805.3911 [hep-ph]`.

[30] H. M. Asatrian et al., *Complete* (O_7, O_8) *contribution to* $B \to X_s\gamma$ *at order* α_s^2, *Phys. Rev.* **D 82**, 074006 (2008), `arXiv:1005.5587 [hep-ph]`.

[31] A. Ferroglia and U. Haisch, *Chromomagnetic Dipole-Operator Corrections in* $\bar{B} \to X_s\gamma$ *at* $O(\beta_0\alpha_s^2)$, *Phys. Rev.* **D 82**, 094012 (2010), `arXiv:1009.2144 [hep-ph]`.

[32] M. Benzke et al., *Factorization at Subleading Power and Irreducible Uncertainties in* $\bar{B} \to X_s\gamma$ *Decay*, *JHEP* **08**, 099 (2010), `arXiv:1003.5012 [hep-ph]`.

[33] T. Ewerth, P. Gambino, and S. Nandi, *Power suppressed effects in $\bar{B} \to X_s\gamma$ at $O(\beta_0\alpha_s)$*, *Nucl. Phys.* **B 830**, 278 (2010), arXiv:0911.2175 [hep-ph].

[34] A. Alberti, P. Gambino, and S. Nandi, *Perturbative corrections to power suppressed effects in semileptonic B decays*, *JHEP* **01**, 147 (2014), arXiv:1311.7381 [hep-ph].

[35] C. Greub et al., *Specific three-loop contributions to $b \to s\gamma$ associated with the current-current operators*, *JHEP* **05**, 201 (2023), arXiv:2303.01714 [hep-ph].

[36] C. Greub et al., *Three-loop contributions to $b \to s\gamma$ associated with the current-current operators*, , arXiv:2407.17270 [hep-ph].

[37] M. Czaja et al., *The $Q_{1,2}$–Q_7 interference contributions to $b \to s\gamma$ at $\mathcal{O}(\alpha_s^2)$ for the physical value of m_c*, *Eur. Phys. J.* **C 83**, 1108 (2023), arXiv:2309.14707 [hep-ph].

[38] M. Fael et al., *Three-loop $b \to s\gamma$ vertex with current-current operators*, *JHEP* **11**, 166 (2023), arXiv:2309.14706 [hep-ph].

[39] **Belle** Collaboration, *The Belle II Physics Book*, *PTEP* **12**, 123C01 (2019), arXiv:1808.10567 [hep-ex].

[40] **BaBar** and **Belle** Collaborations, *The Physics of the B Factories*, *Eur. Phys. J.* **C 74**, 3026 (2014), arXiv:1406.6311 [hep-ex].

[41] **Particle Data Group**, *Review of Particle Physics*, *PTEP* **2022**, 083C01 (2022).

[42] M. D. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press (2014).

[43] M. E. Peskin D. V. and Schroeder, *An Introduction to quantum field theory*, Westview press (1995).

[44] T. Slatyer, *Indirect Detection of Dark Matter*, arXiv:1710.05137 [hep-ph].

[45] **Particle Data Group**, *Review of Particle Physics*, *PTEP* **2020**, 083C01 (2020).

[46] **HFLAV**, *Averages of b-hadron, c-hadron, and τ-lepton properties as of 2018*, *Eur. Phys. J.* **C 81**, 226 (2021), arXiv:1909.12524 [hep-ex].

[47] **LHCb**, *Measurement of lepton universality parameters in $B^+ \to K^+\ell^+\ell^-$ and $B^0 \to K^{*0}\ell^+\ell^-$ decays*, *Phys. Rev.* **D 108**, 032002 (2023), arXiv:2212.09153 [hep-ex].

[48] **LHCb**, *Measurement of the ratios of branching fractions $\mathcal{R}(D^*)$ and $\mathcal{R}(D^0)$*, *Phys. Rev.* **D 131**, 111802 (2023), arXiv:2302.02886 [hep-ex].

[49] **LHCb**, *Angular Analysis of the $B^+ \to K^{*+}\mu^+\mu^-$ Decay*, *Phys. Rev. Lett.* **126**, 161802 (2021), arXiv:2012.13241 [hep-ex].

[50] **CMS**, *Measurement of angular parameters from the decay* $\mathrm{B}^0 \to \mathrm{K}^{*0}\mu^+\mu^-$ *in proton-proton collisions at $\sqrt{s} = 8$ TeV*, *Phys. Lett.* **B 781**, 517 (2018), arXiv:1710.02846 [hep-ex].

[51] **Belle II**, *Evidence for $B+ \to K+\nu\nu^-$ decays*, *Phys. Rev.* **D 109**, 112006 (2024), arXiv:2311.14647 [hep-ex].

[52] A. Lenz, *High precision in CKM unitarity tests in b and c decays*, arXiv:2207.11055 [hep-ex].

[53] T. Mannel, *Flavour physics and CP violation*, *Eur. Phys. J.* **C 33**, 167 (2004).

[54] A. L. Kagan and M. Neubert, *QCD anatomy of $B \to X_s\gamma$ decays*, *Eur. Phys. J.* **C 7**, 5 (1999), hep-ph/9805303.

[55] C. W. Bauer, M. E. Luke, and T. Mannel, *Light cone distribution functions for B decays at subleading order in $1/m_b$*, *Phys. Rev.* **D 68**, 094001 (2003), hep-ph/0102089.

[56] S. L. Glashow, J. Iliopoulos, and L. Maiani, *Weak Interactions with Lepton-Hadron Symmetry*, *Phys. Rev.* **D 2**, 1285 (1970).

[57] S. Dimopoulos and H. Georgi, *Softly broken supersymmetry and SU(5)*, *Nucl. Phys.* **B 193**, 150 (1981).

[58] M. Misiak and P. Gambino, *Quark mass effects in $\bar{B} \to X_s\gamma$*, *Nucl. Phys.* **B 611**, 338 (2001), `hep-ph/0104034`.

[59] S. Catani et al., *Resummation of large logarithms in $e^+ e^-$ event shape distributions*, *Nucl. Phys.* **B 407**, 3 (1993).

[60] M. Ciuchini et al., *The $\Delta S = 1$ effective Hamiltonian including next-to-leading order QCD and QED corrections*, *Nucl. Phys.* **B 415**, 403 (1994), `hep-ph/9304257`.

[61] K. G. Chetyrkin, M. Misiak, and M. Münz, *Weak radiative B meson decay beyond leading logarithms*, *Phys. Lett.* **B 400**, 206 (1997), `hep-ph/9612313`.

[62] T. Huber, M. Poradziński, and J. Virto, *Four-body contributions to $\bar{B} \to X_s\gamma$ at NLO*, *JHEP* **01**, 115 (2015), `arXiv:1411.7677 [hep-ph]`.

[63] G. Passarino and M. Veltman, *One-loop corrections for e^+e^- annihilation into $\mu^+\mu^-$ in the Weinberg model*, *Nucl. Phys.* **B 160**, 151 (1979).

[64] K. G. Chetyrkin and F. V. Tkachov, *Integration by parts: The algorithm to calculate β-functions in 4 loops*, *Nucl. Phys.* **B 192**, 159 (1981).

[65] S. Laporta, *High precision calculation of multiloop Feynman integrals by difference equations*, *Int. J. Mod. Phys.* **A 15**, 5087 (2000), `hep-ph/0102033`.

[66] A. V. Smirnov and A. V. Petukhov, *The Number of Master Integrals is Finite*, *Lett. Math. Phys.* **A 97**, 37 (2011), `arXiv:1004.4199 [hep-th]`.

[67] T. van Ritbergen, J. A. M. Vermaseren, and S. A. Larin, *The Four loop beta function in quantum chromodynamics*, *Phys. Lett.* **B 400**, 379 (1997), `hep-ph/9701390`.

[68] R. N. Lee et al., *Quark and Gluon Form Factors in Four-Loop QCD*, *Phys. Rev. Lett.* **128**, 212002 (2022), `arXiv:2202.04660 [hep-ph]`.

[69] S. Abreu et al., *Four-loop two-mass tadpoles and the ρ parameter*, `arXiv:2407.21700 [hep-ph]`.

[70] A. Maier, P. Marquard, and Y. Schröder, *Towards QCD at Five Loops*, arXiv:2407.16385 [hep-ph].

[71] A. Gehrmann-De Ridder, T. Gehrmann, and G. Heinrich, *Four particle phase space integrals in massless QCD*, *Nucl. Phys.* **B 682**, 265 (2004), hep-ph/0311276.

[72] G. Heinrich, *Towards $e^+e^- \to 3$ jets at NNLO by sector decomposition*, *Eur. Phys. J.* **C 48**, 25 (2006), hep-ph/0601062.

[73] M. Czakon, *Automatized analytic continuation of Mellin-Barnes integrals*, *Comp. Phys. Commun.* **175**, 559 (2006), hep-ph/0511200.

[74] A. V. Smirnov and V. A. Smirnov, *On the Resolution of Singularities of Multiple Mellin-Barnes Integrals*, *Eur. Phys. J.* **C 62**, 445 (2009), arXiv:0901.0386 [hep-ph].

[75] B. Jantzen, *New proofs for the two Barnes lemmas and an additional lemma*, *J. Math. Phys.* **54**, 012304 (2013), math-ph/0104034.

[76] D. Kosower, *barnesroutines.m*, https://mbtools.hepforge.org/.

[77] A. V. Kotikov, *Differential equations method: New technique for massive Feynman diagrams calculation*, *Phys. Lett.* **B 254**, 158 (1991).

[78] E. Remiddi, *Differential equations for Feynman graph amplitudes*, *Nuovo Cim.* **110A**, 1435 (1997), hep-ph/9711188.

[79] M. Argeri and P. Mastrolia, *Feynman Diagrams and Differential Equations*, *Int. J. Mod. Phys.* **A 22**, 4375 (2007), arXiv:0707.4037 [hep-ph].

[80] J. Moser, *The Order of a Singularity in Fuchs' Theory*, *Mathematische Zeitschrift* **72** (1959), no. 1 379–398.

[81] R. N. Lee and A. A. Pomeransky, *Normalized Fuchsian form on Riemann sphere and differential equations for multiloop integrals*, arXiv:1707.07856 [hep-ph].

[82] J. M. Henn, *Multiloop integrals in dimensional regularization made simple*, *Phys. Rev. Lett.* **110**, 251601 (2013), arXiv:1304.1806 [hep-ph].

[83] R. N. Lee, *Reducing differential equations for multiloop master integrals*, *JHEP* **04**, 108 (2015), `arXiv:1411.0911 [hep-ph]`.

[84] M. Prausa, *epsilon: A tool to find a canonical basis of master integrals*, *Comp. Phys. Commun.* **219**, 361 (2017), `arXiv:1701.00725 [hep-ph]`.

[85] O. Gituliar and V. Magerya, *Fuchsia: a tool for reducing differential equations for Feynman master integrals to epsilon form*, *Comp. Phys. Commun.* **219**, 329 (2017), `arXiv:1701.04269 [hep-ph]`.

[86] C. Meyer, *Algorithmic transformation of multi-loop master integrals to a canonical basis with CANONICA*, *Comp. Phys. Commun.* **222**, 295 (2018), `arXiv:1705.06252 [hep-ph]`.

[87] Wolfram Research, Inc., *Wolfram Functions*, `https://functions.wolfram.com/`.

[88] T. Huber and D. Maître, *HypExp 2, Expanding Hypergeometric Functions about Half-Integer Parameters*, *Comp. Phys. Commun.* **178**, 755 (2008), `arXiv:0708.2443 [hep-ph]`.

[89] J. A. M. Vermaseren, *Harmonic sums, Mellin transforms and integrals*, *Int. J. Mod. Phys.* **A 14**, 2037 (1999), `hep-ph/9806280`.

[90] D. Maître, *HPL, a mathematica implementation of the harmonic polylogarithms*, *Comp. Phys. Commun.* **174**, 222 (2006), `hep-ph/0507152`.

[91] A. B. Goncharov, *Multiple zeta-values, Galois groups, and geometry of modular varieties*, `math/0005069`.

[92] C. Duhr and F. Dulat, *PolyLogTools - polylogs for the masses*, *JHEP* **08**, 135 (2019), `arXiv:1904.07279 [hep-th]`.

[93] A. B. Goncharov, *Über Whittakersche bzw. Besselsche Funktionen und deren Produkte*, *Nieuw Arch. Wiskd.* **II. Ser. 18**, 10 (1936).

[94] P. Nogueira, *Automatic Feynman graph generation*, *J. Comp. Phys.* **105**, 279 (1993).

[95] B. Ruijl, T. Ueda, and J. A. M. Vermaseren, *FORM version 4.2*, arXiv:1707.06453 [hep-ph].

[96] J. C. Collins, *Renormalization: An Introduction to Renormalization, The Renormalization Group, and the Operator Product Expansion*, Cambridge University Press (1986).

[97] G. 't Hooft and M. J. G. Veltman, *Regularization and Renormalization of Gauge Fields*, *Nucl. Phys.* **B 44**, 189 (1972).

[98] S. A. Larin, *The Renormalization of the axial anomaly in dimensional regularization*, *Phys. Lett.* **B 303**, 113 (1993), hep-ph/9302240.

[99] M. S. Chanowitz, M. Furman, and I. Hinchliffe, *The Axial Current in Dimensional Regularization*, *Nucl. Phys.* **B 159**, 225 (1979).

[100] J. G. Körner, D. Kreimer, and K. Schilcher, *A Practicable γ_5 scheme in dimensional regularization*, *Z. Phys.* **C 54**, 503 (1992).

[101] D. Kreimer, *The γ_5 Problem and Anomalies: A Clifford Algebra Approach*, *Phys. Lett.* **B 237**, 59 (1990).

[102] L. Chen, *An observation on Feynman diagrams with axial anomalous subgraphs in dimensional regularization with an anticommuting γ_5*, *JHEP* **11**, 30 (2023), arXiv:2304.13814 [hep-ph].

[103] C. Anastasiou and K. Melnikov, *Higgs boson production at hadron colliders in NNLO QCD*, *Nucl. Phys.* **B 646**, 220 (2002), hep-ph/0207004.

[104] R. E. Cutkosky, *Singularities and discontinuities of Feynman amplitudes*, *J. Math. Phys.* **1**, 429 (1960).

[105] A. V. Smirnov and F. S. Chukharev, *FIRE6: Feynman Integral REduction with Modular Arithmetic*, *Comp. Phys. Commun.* **247**, 106877 (2020), arXiv:1901.07808 [hep-ph].

[106] Siegen University, *High Performance Computing Cluster OMNI*, https://cluster.uni-siegen.de/omni/system-information/.

[107] A. V. Smirnov and V. A. Smirnov, *How to choose master integrals*, *Nucl. Phys.* **B 960**, 960 (2020), arXiv:2002.08042 [hep-ph].

[108] J. Usovitsch, *How to choose master integrals*, arXiv:2002.08173 [hep-ph].

[109] J. Klappert et al., *Integral reduction with Kira 2.0 and finite field methods*, *Comp. Phys. Commun.* **266**, 108024 (2021), arXiv:2008.06494 [hep-ph].

[110] M. Czakon, *MBasymptotics.m*, https://mbtools.hepforge.org/.

[111] Wolfram Research, Inc., *Mathematica, Version 13.1*, https://www.wolfram.com/mathematica.

[112] J. Ablinger, *A Computer Algebra Toolbox for Harmonic Sums Related to Particle Physics*, arXiv:1011.1176 [math-ph].

[113] Lars-Thorben Moos, *GitHub Repository for $B \to X_s\gamma$ results*, https://github.com/LarsTMoos/BSGammaFourBody.

[114] O. Gituliar, *Master integrals for splitting functions from differential equations in QCD*, *JHEP* **02**, 017 (2016), arXiv:1512.02045 [hep-ph].

[115] S. Moch and V. Magerya, *Calculating Four-Loop Corrections in QCD*, arXiv:2104.07994 [hep-ph].

[116] O. Gituliar, V. Magerya, and A. Pikelner, *Five-Particle Phase-Space Integrals in QCD*, *JHEP* **06**, 099 (2018), arXiv:1803.09084 [hep-ph].

[117] P. Gambino, M. Gorbahn, and U. Haisch, *Anomalous dimension matrix for radiative and rare semileptonic B decays up to three loops*, *Nucl. Phys.* **B 673**, 238 (2003), hep-ph/0306079.

[118] S. Catani and M. Grazzini, *Infrared factorization of tree level QCD amplitudes at the next-to-next-to-leading order and beyond*, *Nucl. Phys.* **B 570**, 287 (2000), hep-ph/9908523.

[119] G. F. R. Sborlini, D. de Florian and G. Rodrigo, *Triple collinear splitting functions at NLO for scattering processes with photons*, *JHEP* **10**, 161 (2014), arXiv:1408.4821 [hep-ph].

[120] P. K. Dhani, G. Rodrigo, and G. F. R. Sborlini, *Triple-collinear splittings with massive particles*, *JHEP* **12**, 188 (2023), arXiv:2310.05803 [hep-ph].

[121] T. Becher and M. Neubert, *Toward a NNLO calculation of the $\bar{B} \to X_s\gamma$ decay rate with a cut on photon energy: II. Two-loop result for the jet function*, *Phys. Lett.* **B 637**, 4 (2006).

[122] R. Brüser, Z. Liu, and M. Stahlhofen, *Three-Loop Quark Jet Function*, *Phys. Rev. Lett.* **121**, 7 (2018), arXiv:1804.09722 [hep-ph].

[123] W. T. Giele and E. W. Nigel Glover, *Higher order corrections to jet cross-sections in e+ e- annihilation*, *Phys. Rev.* **D 46**, 1980 (1992).

[124] K. Brune et al., *Calculation of NLO Splitting Functions*, *Work in Progress*.

Acknowledgements

First, I would like to thank Prof. Dr. Tobias Huber for giving me the opportunity of writing my PhD thesis under his supervision. I am grateful for his patience and understanding and the hands-on support in cross-checking results and answering questions.
I would like to thank Prof. Dr. Guido Bell for useful input during our discussions on splitting functions and for being the second referee of this thesis. I would also like to thank PD Dr. Matthias Kleinmann and PD Dr. Carmen Diez Pardos for joining my thesis committee.
I would like to thank Dr. Kevin Brune for his contributions to the splitting function part of the calculation and his support via discussions, calculations and proofreading.
I would like to thank Dr. Vitaly Magerya for fruitful discussions about differential equations and their boundary conditions.
Being part of the Theoretical Particle Physics Institute at University Siegen was a great pleasure and I want to express my gratitude to all present and former members for creating such a welcoming atmosphere. I especially want to thank Dr. Kevin Brune, Dr. Kevin Olschewsky, Dr. Jan Piclum, Dr. Simon Braß and Prof. Dr. Wolfgang Kilian for the many fun game nights during our time together.

Finally, I want to thank my family and close friends for their constant support and understanding, especially my parents Birgit and Hartmut, my siblings Jörn-Malte and Lina-Sophie and my wonderful girlfriend Anna. Your support means the world to me and without it, this would not have been possible.