Analysis Of “New Physics” In The Flavor Sector Using Effective Field Theory Methods

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Abstract

The enormous amount of data collected by the B factories as well as the perspective concerning the upcoming experiments at the LHC provide us with the opportunity to perform high precision tests of the standard model. This dissertation presents two model independent precision tests of the weak sector of the standard model using extensions of the weak currents which are set up using effective field theory methods in order to control the size of the different contributions. Constraints on the coefficients of the terms can then be obtained by comparison with experimental results.

The first analysis incorporates a test of the left-handedness of the weak interaction in the quark sector. By introducing additional weak couplings the differential and total decay rates of the inclusive semileptonic decay $\bar{B} \to X_c\, \ell\, \bar{\nu}_\ell$ are calculated. The calculation includes the computation of the heavy-quark expansion which describes the interaction of the decaying b quark with respect to the background field of the B meson, up to order $1/m_b^4$. This has been done in a new, systematic way which does not involve the calculation of gluon matrix elements. Furthermore, radiative corrections have been included up to order $\mathcal{O}(\alpha_s)$ including renormalization group running.

The second part deals with the introduction of lepton flavor violating operators in the context of leptonic τ decays. These operators give rise to the effective four-fermion vertex $\tau \to \ell\, \ell'\, \ell''$ as well as the subsequent radiative decay $\tau \to \ell\, \gamma^* \to \ell\, \ell'^+\, \ell'^-$. The resulting Dalitz distributions turn out to predict completely different signatures for the radiative and the effective four-fermion vertices and provide the opportunity to check, whether the decay is induced either by a radiative or by an effective four-fermion vertex. Since different models which contain lepton flavor violation are normally more sensitive to one of those decay modes than to the other, the analysis therefore allows the comparison of the different models within an experimental analysis.

Analyse "Neuer Physik" im Flavor Sektor unter Benutzung von Methoden der effektiven Feldtheorie

Zusammenfassung

Die enorme Menge von Daten, die von den B Fabriken zusammengetragen wurde, sowie die Perspektiven bezüglich der anstehenden Experimente am LHC, geben uns die Möglichkeit, Tests des Standardmodells mit hoher Präzision durchzuführen. Die vorliegende Dissertation beschäftigt sich mit zwei modellunabhängigen Präzisionstests des elektroschwachen Sektors des Standardmodells, in welchen Erweiterungen des schwachen Stroms betrachtet werden. Diese werden zur Kontrolle der Größe der einzelnen Beiträge unter Zuhilfenahme von Methoden der effektiven Feldtheorie eingeführt. Werte bzw. obere Grenzen für die Koeffizienten dieser Terme können dann experimentell ermittelt werden.

Die erste Analyse besteht aus einem Test der Linkshändigkeit der schwachen Wechselwirkung des Standardmodells im Quark-Sektor. Unter Einführung von zusätzlichen schwachen Kopplungen werden dazu die differentiellen und die totale Zerfallsrate des inklusiven semileptonischen Zerfalls $\bar{B} \to X_c\, \ell\, \bar{\nu}_\ell$ berechnet. Dies wird unter Zuhilfenahme der Heavy-Quark-Entwicklung bis zur Ordnung $1/m_b^4$ durchgeführt um die Wechselwirkungen des zerfallenden und gleichzeitig gebundenen b Quarks richtig zu beschreiben. Dazu wurde mit eine neuen, systematische Methode angewandt, mit der sich die Berechnung von Gluonmatrixelementen erübrigt. Zudem wurden Strahlungskorrekturen inklusive des Renormierungsgruppenflusses bis zur Ordnung $\mathcal{O}(\alpha_s)$ durchgeführt.

Der zweite Teil behandelt die Einführung von leptonzahlverletzenden Operatoren im Kontext von leptonischen τ-Zerfällen. Diese Operatoren resultieren in den vier-Fermion-Vertizes $\tau \to \ell\, \ell'\, \ell''$ sowie den Zerfällen $\tau \to \ell\, \gamma^*$ mit anschließendem Zerfall $\gamma^* \to \ell'^+\, \ell'^-$. Die resultierenden Dalitzverteilungen zeigen ein komplett unterschiedliches Verhalten für die radiativen und vier-fermion Vertizes, und können daher benutzt werden, um nachzuprüfen, ob der Zerfall durch einen radiativen oder vier-fermion Vertex induziert wird. Da verschiedene Modelle, die Leptonflavorverletzung vorhersagen, oftmals einen der beiden Zerfallskanäle bevorzugen, kann die vorliegende Analyse dazu verwendet werden die verschiedenen Modelle in einer experimentellen Analyse miteinander zu vergleichen.

Contents

1. Introduction

1.1. Topics and motivation

While the standard model of elementary particle physics [1, 2, 3, 4, 5, 6, 7] provides the possibility of making precise predictions within a vast region of energy scales, it still lacks the potential to explain the deeper origin of the effects we observe in elementary particle interactions. This gives rise to the questions:

- What is the origin of the 27 input parameters of the standard model?
  The standard model contains 27 input parameters which are not explained by the theory, but have to be determined by experiments. These include the 6 quark masses, the 3 masses of the charged leptons, the 3 neutrino masses, 1 Higgs mass, 1 strong and 2 electroweak gauge coupling constants, 3 quark mixing angles and 1 corresponding weak phase as well as 3 lepton mixing angles and 3 corresponding phases. The deeper origin of these parameters remains completely unclear.

- How is the neutrino mass implemented into the standard model?
  There are various ways to implement the neutrino masses, depending on whether the neutrinos are Dirac or Majorana particles. While the other masses can be introduced by the Higgs mechanism there is an additional possibility to generate neutrino masses with Majorana masses.

- What is the origin of the flavor structure described by the standard model?
  We do not know the origin of the masses and the mixings in the quark sector. Furthermore, neutrino oscillation provides a strong evidence that the lepton flavor is violated. Thus the question is, whether the lepton flavor is also violated within the standard model interactions.

- What is the source of the hierarchy concerning the strength of the different forces?
  Up to now we do not know why the gravitational force is 10^{-34} times weaker than the strong interaction.

- How can a grand unification of the strong and the electroweak forces be established?
  At present all attempts to unify the strong force and the electroweak force by using group theoretical methods are not satisfactory. The most prominent structure is the $SU(5)$ suffering from the prediction of the proton decay at a rate beyond the measured boundaries [8]. Even the electroweak unification by the Higgs mechanism has not been proven up to now, since the Higgs particle has not been found. Energies, sufficient enough to create this particle should soon be available at the Large Hadron Collider (LHC) in Geneva.

- How can a theory of everything be set up?
  Besides the fact that we have not found a way to unify the strong and electroweak forces, gravitation has not even been high energy physics implemented into the standard model.

Up to now there is no way found to create a consistent renormalizable quantum field theory for that force. Within high energy physics experiments this force is so weak that we cannot even detect it. Thus the description of the interactions of elementary particles works quite well without regarding this force. However, considering situations like black holes, where the description of gravitational forces is needed as well as the description of quantum mechanical states due to high densities, the standard model fails.

- What is the source of the baryon asymmetry?
  The standard model does not explain why there is much more matter than antimatter in the universe.

- What are dark matter and dark energy?
  Even if we can imagine that dark matter could be produced by some kind of weakly interacting massive particles which have not been found yet, we do not have any idea what dark energy is.

- What is the source of CP violation?
  It is not known why the charge and parity symmetry is violated for the weak interaction and not for the strong interaction.

Besides these rather conceptional problems which are mainly induced by parts missing in the theory, there is a major calculational problem concerning quantum chromodynamics, the part of the standard model describing the strong interactions. There is no way to describe the interactions of the elementary particles below the scale of hadronization Λ_{QCD}, since the strong coupling constant changes with the energy scale and becomes too big to allow a perturbative treatment at such low energies.

All these problems give rise to the construction of various new models which try to explain the problems mentioned above (or at least parts of them). One of the most prominent examples are the supersymmetric theories which give a better access to the unification of the forces within a $SO(10)$ group-theoretical description, as all the coupling constants seem to meet in one point for high energies. Within this theory all particles are provided with supersymmetric partners, for which there is however no evidence nowadays. Other examples are Technicolor or little Higgs models. All these models have in common that they inherit parts of the standard model structure to provide compliance with past experiments and add certain specific features. However, the present analysis shall provide a different access to unknown physics by providing a model independent analysis. The idea is to treat the standard model as an effective theory of some yet unknown more fundamental theory to be able to extend it in a controlled way and yet retain its well tested parts to agree with known data. This gives us the opportunity to test the standard model interactions by comparison with modern experiments. We will perform two different analyses which will feature a similar way in expanding the interaction vertices, yet testing two completely different aspects of the standard model.

In the first analysis we will deal with the inclusive decay $\bar{B} \to X_c\, \bar{\nu}_e\, e^-$ of a B meson to probe the flavor structure of the standard model. This decay is a natural choice as for these inclusive decays a large data set is available and the theoretical description of semileptonic decays has become very precise in the last couple of years and seems to be under good control. Furthermore, there already exists an enormous amount of data generated by the B-factories BaBar (SLAC) and Belle (KEK) which provide results for the lepton spectra with a precision of about one percent, as well as additional perspectives concerning future experiments like for example LHCb. Those B-factories mainly produce the resonance $\Upsilon(4s)$ which decays nearly exclusively to B^+B^- or $B^0\bar{B}^0$. These decay channels are consolidated to the inclusive notation $B\bar{B}$, where the B and $\bar{B}$ denote all possible mesons containing a b quark and one of the lighter quarks. The analysis of the decay $\bar{B} \to X_u\,\bar{\nu}_e\,e^-$ is less suitable since its decay rate is suppressed by the hierarchy of the CKM matrix ($V_{cb} \gg V_{ub}$) by a factor of about 0.01 compared to $\bar{B} \to X_c\,\bar{\nu}_e\,e^-$. Thus there is a huge background of $\bar{B} \to X_c\,\bar{\nu}_e\,e^-$ for these decays which leaves only few experimentally accessible regions within the spectra. The theoretical description of these regions is not as reliable, so that we will not consider $\bar{B} \to X_u\,\bar{\nu}_e\,e^-$ at the moment.

The idea of the analysis is to use effective field theory methods to extend the normally purely left-handed vector current of the standard models' weak decay by a right-handed vector contribution as well as left- and right-handed scalar and tensor currents which will allow us to test the left-handedness of the weak interaction. The calculation of the weak decay of the b quark within the background field of the B meson has been performed utilizing a heavy quark expansion which has been calculated up to order $1/m^4$ in a new systematic way. Since our extension also contains the standard model contributions this higher order nonperturbative correction can also be used in plain standard model analyses. Furthermore, the radiative corrections to the new contributions have been calculated up to one-loop order and attempts have been made to combine the radiative corrections with the nonperturbative ones making use of the reparametrization invariance.

A strong evidence for the lepton flavor violation in nature is already given by the neutrino oscillation. To be able to perform tests whether there are further sources of lepton flavor violation within the scope of weak decays, we perform a second analysis which features an extension of the vertex of τ decays by an effective field theory approach very similar to the one of the first analysis. However, in contrast to the extension of the inclusive $\bar{B}$ decays the vertex is extended in a way that it features a lepton flavor violation, resulting in decays into three charged leptons like $\tau \to \ell\ell'\ell''$ with $\ell = e, \mu$. These decays are realized in two different forms. The first possibility is the direct decay of the tau into three leptons described by an effective four-fermion vertex. As a second possibility we find the subsequent tau decay of the form $\tau \to \ell\gamma \to \ell\ell'^+\ell'^-$ mediated by a radiative current. As the effective four-fermion vertex and the radiative vertex provide results with completely different Dalitz plots this calculation can be used to help experimental studies to classify such rare decays, if the statistics are high enough. Since lepton flavor violating τ decays are realized quite differently in various models this could be used to falsify some models or at least set limits. While for example supersymmetric models prefer the radiative decay variant little Higgs models usually prefer the four-fermion vertex. Additionally, we will add an analysis to compute the constraints from minimal flavor violation on the newly introduced parameters to provide the possibility to gain additional information from experiments how lepton flavor violation could be established.

1.2. Structure

After this introduction we will start with an introduction of the standard model of elementary particle physics in chapter 2. The topics discussed here have more or less become textbook material, and thus this chapter is addressed mainly to readers who are new to the topic. All in all this chapter is far from being a complete summary of all standard model topics, it should rather be understood as an introduction to its structure which we will need in later chapters for the introduction of the extensions mentioned above. For further reading about standard model or gauge theory there exist many books of different styles. The author has mainly used the books [9, 10, 11, 12] as sources during the creation of this chapter. Apart from this, the reader should at least have some basic knowledge about group theory. An introduction for physicists to this topic is given in the book [13].

Chapter 3 presents an introduction to effective field theories and their renormalization. We will specially emphasize the heavy quark effective theory and the heavy quark expansion which will be the main tools for the calculation of the inclusive semileptonic decays discussed in chapter 4. The contents of chapter 3 have become standard in the past years. Thus there are many review papers and also textbooks about this topic available. Some interesting reviews associated with the dissertation at hand are [14, 15, 16].

In chapter 4 we will discuss the calculation of the differential and total decay rates of the inclusive semileptonic process $\bar{B} \to X_c \, \bar{\nu}_e \, e^-$. We will present the extension of the vertex by new right- and left-handed vector, scalar and tensor contributions as well as the nonperturbative $1/m_b$ corrections to order $1/m_b^4$ and the radiative corrections to the currents up to one-loop order. This chapter contains the authors actual work including a new way of treatment for the $1/m_b$ corrections which has led to the possibility to perform calculations up to order $1/m_b^4$ and the calculation of the renormalization group running for the newly introduced currents. The results contained in this chapter have been calculated in collaboration with Prof. Dr. Thomas Mannel, Sascha Turczyk and Robert Feger and were published within the papers [17, 18, 19].

The second part containing new results is presented in chapter 5. Here we perform the analysis of lepton flavor violating tau decays using a similar effective field theory method as in chapter 4. Within this chapter we will generate Dalitz plots to provide indications for experimentalists, how the distributions for these rare decays may look like as well as a possibility to compare models to the results of the measurements. The corresponding publication of the work presented in this section is [20] which has been developed with Prof. Dr. Thomas Mannel, Dr. Thorsten Feldmann and Sascha Turczyk.

The appendix contains some mathematics which is needed throughout this work in chapter A as well as a short introduction to the basics of relativistic quantum field theory and B and two chapters C and D containing some of the results that have been too lengthy to be integrated into the text properly.

1.3. Conventions

In this section we shall list the conventions which we will use in the following chapters. First of all we will work in the so called “God-given” units, where

$$\hbar = c = 1 \,, \tag{1.1}$$

which is most common in elementary particle physics. In this system we have

$$[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1} \tag{1.2}$$

The mass m of a particle is therefore equal to its rest energy mc^2 as well as to its inverse Compton wavelength $mc/\hbar$. The units are naturally given in electron volts (eV), where

$$1\,\text{eV} = 1.602176487(40) \times 10^{-19}\,\text{J}. \tag{1.3}$$

In Fourier transformations the factors of 2π will always appear with the momentum integral, such that the integrals look like

$$f(x) = \int \frac{\mathrm{d}k}{2\pi} \, e^{-ik\cdot x} \, \tilde{f}(k) \tag{1.4}$$

$$\tilde{f}(k) = \int \mathrm{d}x \, e^{ik\cdot x} \, f(x) \,, \tag{1.5}$$

for one dimension.

For our conventions concerning relativity we will follow the notation introduced by Bjorken and Drell [10] which is most common in the field of elementary particle physics. We use the metric tensor

$$g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \tag{1.6}$$

with greek indices running over 0, 1, 2, 3 or t, x, y, z and roman indices running only over the tree spatial components. Following Einstein's summation convention we sum over repeated indices in all cases. Four vectors are denoted in the standard light italic type like ordinary numbers, while three vectors are denoted by boldface type. As an example we have

$$x^\mu = (x^0, \boldsymbol{x}) \tag{1.7}$$

$$x_\mu = g_{\mu\nu} x^\nu = (x^0, -\boldsymbol{x}) \tag{1.8}$$

$$p \cdot x = g_{\mu\nu} p^\mu x^\nu = p^0 x^0 - \boldsymbol{p} \cdot \boldsymbol{x}\,. \tag{1.9}$$

For all on-shell particles we have

$$p^2 = p_\mu p^\mu = E^2 - |\boldsymbol{p}|^2 = m^2\,, \tag{1.10}$$

where m may of course be 0 for massless particles. The partial derivative corresponding to this is defined by

$$\partial_\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial x^0}, \boldsymbol{\nabla}\right)\,. \tag{1.11}$$

Note that the index of the partial derivative is always on the opposite side compared to the vector it acts on. The energy and momentum operators action on wave functions follow the usual conventions

$$E \leftrightarrow i\frac{\partial}{\partial x^0} \qquad \text{and} \qquad \boldsymbol{p} \leftrightarrow -i\boldsymbol{\nabla}\,, \tag{1.12}$$

where E and $\boldsymbol{p}$ are the energy and momentum operators in momentum space, while $i\partial_0$ and $-i\boldsymbol{\nabla}$ denote the according vectors in position space. Transformations between those two spaces are performed by the Fourier transformations (1.4 - 1.5). The relations (1.12) can be combined to $p^\mu \leftrightarrow i\partial^\mu$, where the minus sign is conveniently used with the raised index. For the ϵ-tensor we shall use the convention

$$\epsilon^{0123} = -\epsilon_{0123} = 1. \tag{1.13}$$

2. The standard model of particle physics

This chapter shall be a short introduction to the standard model of particle physics which describes the fundamental microscopic interaction between the elementary particles. These elementary particles can be divided into two groups, namely the 12 fermions (u, d, s, c, b, t, e, μ, τ, ν_e, ν_μ and ν_τ) of spin 1/2 which are the constituents of all observed matter, and the bosons (γ, g, W^-, W^+ and Z^0) of spin 1 which are the force carriers that mediate the interactions between the fermions. The latter particles can be assigned to three different types of interactions, namely the electromagnetic interaction mediated by the photon γ, the strong interaction featuring the gluon g as force carrier and the weak interaction which contains interaction modes moderated by the charged W^- and W^+ as well as the neutral Z^0 particles. The properties of these bosonic particles are listed in Table (2.1). The reader might have noticed that this table contains an additional gauge boson called graviton as a corresponding interaction particle for the gravitational force. This particle is commonly believed to be of spin 2. However, up to now there is no known way, how to implement such a particle into the theory in a renormalizable way, and thus the gravitational force is not implemented into the standard model. Furthermore, we have no experiments so far which would be capable of resolving gravitational effects in particle physics. Thus any attempt to integrate gravitation into the standard model is purely hypothetic. However, due to their tiny relative strength the effects of the gravitational forces can be safely neglected while analyzing the microscopic interaction of the other forces. Therefore gravitation is not integrated into the standard model.

The standard model describes the fundamental interactions using a renormalizable theory which is based on local symmetries. Thus independent symmetry transformations at each space time point are possible. In total the standard model obeys a spontaneously broken $SU(3) \otimes SU(2) \otimes U(1)$ symmetry, where the $SU(3)$ part contains the symmetries of the strong interactions, while the $SU(2) \otimes U(1)$ part describes the symmetries of the electroweak part which consolidates the electromagnetic and weak interactions. The $SU(2) \otimes U(1)$ symmetry in the electroweak sector describes the theory in the massless limit. To introduce masses into the theory one has to perform a spontaneous symmetry breaking which retains the symmetry for higher states but breaks the symmetry of the ground state by an explicit choice. By spontaneous symmetry breaking one also obtains an additional particle with spin 0, namely the Higgs boson H. This particle is introduced by purely theoretical considerations. The current experimental

interaction	force carrier	m	charge	rel. strength
strong	8 gluons g	0	8 strong	1
electromagnetic	photon γ	0	1 electric	10^{-3}
weak	$W^{\pm}$-boson	80,4 GeV	3 weak	10^{-14}
	Z^0-boson	91,2 GeV		
gravitational	graviton?	?	1 mass	10^{-34}

Table 2.1.: The interactions of the standard model and their force carriers

lepton	L_e	L_μ	L_τ	Q	m
ν_e	1	0	0	0	$< 3\,\mathrm{eV}$
e	1	0	0	$-1\mathrm{e}$	$511.00\,\mathrm{keV}$
ν_μ	0	1	0	0	$< 0.19\,\mathrm{MeV}$
μ	0	1	0	$-1\mathrm{e}$	$105.66\,\mathrm{MeV}$
ν_τ	0	0	1	0	$< 18.2\,\mathrm{MeV}$
τ	0	0	1	$-1\mathrm{e}$	$1.777\,\mathrm{GeV}$

Table 2.2.: The leptonic content of the standard model

status only shows that it has a mass bigger than $114,4\,\mathrm{GeV}$, but one believes that it can be discovered with the Large Hadron Collider at CERN which will start to collect data soon. A more precise introduction to the strong and electroweak interactions will be given in the sections 2.3 and 2.1 respectively. In the following we will group the fermionic particles by their behavior under these interactions. The different fermions can be characterized by their quantum numbers and masses. For each fermion an antifermion with the same mass but opposite inner quantum numbers exists.

Leptons

The fundamental fermions that do not interact strongly are called leptons. Their most important properties, such as masses, lifetimes, decay channels etc. can be found in [8]. Table 2.2 lists the masses of the leptons as well as their non-vanishing quantum numbers including the electric charge and the fermion numbers. While all leptons interact weakly, the electric charge can be used to classify the leptons into two classes. On the one hand we have the electrically uncharged neutrinos ν_e, ν_μ and ν_τ which interact exclusively through the weak interaction, while on the other hand we have the electrically charged leptons e, ν and τ which can also interact electromagnetically. A further classification of the leptons can be done using the lepton numbers L_e, L_μ and L_τ. We obtain three lepton pairs, namely (e, ν_e), (μ, ν_μ) and (τ, ν_τ), all built up of a charged and an uncharged lepton. These pairs are often called families. Like the electric charge, the lepton numbers are conserved by the standard model interactions and thus transitions between those families are forbidden. However, the detection of neutrino oscillations[1] provides strong evidence for a lepton flavor violation. The dissertation will present a way to test this lepton number conservation by searching for lepton number violating tau decays in a model independent way in chapter 5. Since the decay of the ν_μ and ν_τ is constricted by the lepton numbers the electron e and the neutrinos are therefore stable particles, while the μ and τ decay by the weak interaction. Note that direct measurements yield only upper limits for the neutrino masses. As neutrino oscillations can be introduced into the theory only for non-vanishing neutrino masses, the neutrino masses are expected to be non-vanishing.

Quarks

In contrast to the leptons, the quarks can also interact strongly. The flavor quantum numbers which are conserved by the strong interaction are the strong isospin (I and its third compo-

[1] The theory of neutrino oscillations has been introduced by Z. Maki, M. Nakagawa and S. Sakata [21, 22] to solve the solar neutrino problem found by the Homestake collaboration [23, 24]

quark	I	I_3	S	C	B	T	Q	m
u	1/2	1/2	0	0	0	0	2/3e	1-5 MeV
d	1/2	$-1/2$	0	0	0	0	$-1/3$e	3-9 MeV
c	0	0	0	+1	0	0	2/3e	1.15-1.35 GeV
s	0	0	-1	0	0	0	$-1/3$e	75-170 MeV
t	0	0	0	0	0	+1	2/3e	174.3±5.1 GeV
b	0	0	0	0	-1	0	$-1/3$e	4.0-4.4 GeV

Table 2.3.: The quark content of the standard model

nent I_3) the strangeness (S), charm (c), bottomness (B) and topness (T). In Table 2.3 these quantum numbers are presented together with the mass and electric charge of the particles. All quarks carry the baryon number 1/3. As for the leptons, we can set up three doublets according to the quantum numbers, namely (u, d), (c, s) and (b, t). Additionally, the quarks can be divided by their electric charge into up-type quarks (u, c, t) with charge 2/3e and down type quarks (d, s, b) with charge $-1/3$e. The charge of the strong interaction is given by an additional degree of freedom called color. Each quark carries one of three colors: red, green and blue (r, g and b). Additionally there exist three anti-colors ($\bar{r}$, $\bar{g}$ and $\bar{b}$) which are carried by the antiquarks. At this point we should mention that the quarks as well as the colors cannot be observed directly. Caused by confinement which will be shortly introduced in section 2.3 the quarks are always found in bound states called hadrons. These bound states are mainly classified into mesons which are built up of a quark and an antiquark and baryons, built up of three quarks. Note that this is not exactly correct, since gluons are exchanged between the quarks bound in a hadron which can decay into virtual quark-antiquark pairs. These are however not used within the classification of the hadron as they are extremely short-lived. Since no free quarks exist, their mass cannot be defined in a conventional way. Instead, various definitions of effective masses are used. The values given in Table 2.3 are those recommended by the Particle Data Group [8]. The only quark where one can eventually define a direct mass is the t quark, as its mass is so high that it can be seen as quasi-free. However, for the other quarks it can be difficult to perform calculations of interactions of quarks bound in a hadronic state. Up to now, there does not even exist a possibility to describe the bound states properly, since the coupling of the strong interaction becomes much too high to perform a perturbation series at small energies. However, for high energies in weak decays of the heavy quarks c and b within hadrons there exists an effective theory which will be used in this work to calculate weak decays of mesons containing a bottom quark.

Besides the quantum numbers which are introduced by the theoretical description, it is also interesting to know the input parameters of the standard model which have to be taken from experiment. The standard model contains 27 input parameters in total (we count 3 charged lepton masses, 3 neutrino masses, 6 quark masses, 3 gauge coupling constants, 3 quark mixing angles and one corresponding complex phase, 3 lepton mixing angles and 3 complex phases, 1 Higgs mass as well as the QCD vacuum expectation value). If neglecting the neutrino masses, as they were assumed to be zero in earlier definitions of the standard model, leaves 24 parameters. As already mentioned, there is not much known about the masses of the neutrinos, since measurements only provide upper bounds for the neutrino masses. In this work we will mainly neglect the neutrino masses, as they are comparatively small, but within the description of lepton flavor violating τ decays in chapter 5 we will encounter the mixing in the lepton sector which will force us to take the neutrino masses into consideration. A short description regarding how to introduce the neutrino masses into the theory is given in section 2.2.

The following sections shall provide an introduction to the standard model interactions. To understand this introduction the reader should at least have some basic knowledge about group theory (especially Lie algebras) and the basics of relativistic quantum field theory. However, for the readers new to these topics short introductions are given in the appendices A.2 and B respectively.

2.1. Electroweak interactions

Within the standard model the electromagnetic as well as the weak interactions are described by the Weinberg-Salam-Glashow model which is a gauge theory whose input fermionic degrees of freedom are massless spin one-half chiral particles. It has the structure $SU(2)_L \otimes U(1)_Y$, where the $SU(2)_L$ and $U(1)_Y$ represent weak isospin and hypercharge respectively. The subscript 'L' on $SU(2)_L$ indicates that among fermions only left-handed states transform nontrivially under weak isospin. By left- and right-handed we mean the projections of the particles by the chirality projectors

$$P_{L/R} = \frac{1 \mp \gamma_5}{2}\,, \tag{2.1}$$

which fulfill the relations

$$P_{L/R}^2 = P_{L/R}, \quad P_{L/R}P_{R/L} = 0, \quad P_{L/R}\gamma_5 = P_{L/R} \quad \text{and} \qquad P_{L/R}\gamma_\mu = \gamma_\mu P_{R/L}\,, \tag{2.2}$$

where we have used the relations $(\gamma_5)^2 = \gamma_5$ and $\{\gamma_\mu, \gamma_5\} = 0$ respectively. For an arbitrary fermion ψ we will denote the projection by an index L or R, such that we have

$$\psi_{L/R} = P_{L/R}\,\psi\,. \tag{2.3}$$

Due to the fact that the left- and right-handed particles have a different transformation behavior under the gauge group, they have to be treated differently. We will reflect this by writing the left-handed particles as doublets which transform under the $SU(2)_L$ and the right-handed particles as singlets which are not affected by $SU(2)_L$ transformations. Thus we define

$$\begin{aligned}
\boldsymbol{Q} &= \left(\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix} \right) \\
\boldsymbol{q} &= (u, d, s, c, b, t) \\
\boldsymbol{u} &= (u, c, t) \\
\boldsymbol{d} &= (d, s, b)
\end{aligned} \tag{2.4}$$

for the quarks and

$$\begin{aligned}
\boldsymbol{l} &= \left(\begin{pmatrix} \nu_e \\ e \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \right) \\
\boldsymbol{e} &= (e, \mu, \tau) \\
\boldsymbol{\nu} &= (\nu_e, \nu_\mu, \nu_\tau)
\end{aligned} \tag{2.5}$$

for the leptons. The singlets of the up type quarks $\boldsymbol{u}$ and down type quarks $\boldsymbol{d}$ as well as for the electrons e and neutrinos $\boldsymbol{\nu}$ have been taken separately for later purposes. Additionally we will introduce the vectors

$$\begin{aligned}
\boldsymbol{\Psi} &= \left(\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix}, \begin{pmatrix} \nu_e \\ e \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \right) \\
\boldsymbol{\psi} &= (u, d, s, c, b, t, e, \mu, \tau) \\
\boldsymbol{\varphi} &= (u, d, s, c, b, t, e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau)
\end{aligned} \tag{2.6}$$

where we combine the fermion doublets and fermion singlets for easier notation later on. We implicitly assume that the generations of the quarks and leptons behave in the same way under the gauge transformations and thus only the masses of the particles differ. The different behavior of the left and right-handed particles leads to some awkward consequences. For example it is not possible to implement masses directly into the Lagrange density. If we directly write down a mass term m and split the fermions up into left- and right-handed components we get

$$\bar{\psi}^a \, m_a \, \psi^a = (\bar{\psi}^a_L + \bar{\psi}^a_R) \, m_a \, (\psi^a_L + \psi^a_R) = \bar{\psi}^a_L \, m_a \, \psi^a_R + \bar{\psi}^a_R \, m_a \, \psi^a_L. \tag{2.7}$$

We note that only the combinations of left- and right-handed fields survive, as the other ones vanish because of the projectors and the absence of any Dirac structure between the fields. These terms are not invariant under $SU(2)_L$ transformations and thus cannot be implemented in the electroweak Lagrangian which by definition has to be invariant under these transformations.

Since the transformation behavior of the left-handed doublets differs from the one of the right-handed singlets another quantum number has to be introduced to distinguish those objects from each other. This quantum number is known as the weak hypercharge. The weak hypercharge assignment is determined by the requirement that the particles should have the correct charge. It reads

$$Y_W = 2\,(Q - T_3) \tag{2.8}$$

for quarks and leptons, where T_3 is the third component of the weak isospin T of the particle which should not be confused with the isospin I carried by the u and d quarks introduced at the beginning of the chapter. The weak isospin $T = 1/2$ is assigned to all left-handed particles, while it vanishes for the right-handed particles. Furthermore, all down-type quarks $\boldsymbol{d}$ and charged leptons $\boldsymbol{e}$ feature $T_3 = -1/2$ for the third component, while $T_3 = 1/2$ is assigned to the up-type quarks $\boldsymbol{u}$ and the neutrinos.

The other quantum numbers for fermions in the electroweak sector have already been given in the tables 2.2 and 2.3. Having assigned the quantum numbers, we can go on with the analyzation of the electroweak Lagrange density. The electroweak Lagrangian can be separated into three additive parts

$$\mathcal{L}_{\mathrm{EW}} = \mathcal{L}_{\mathrm{G}} + \mathcal{L}_{\mathrm{F}} + \mathcal{L}_{\mathrm{H}}, \tag{2.9}$$

namely the gauge (G), fermion (F) and Higgs (H) part. The gauge and fermion part will again describe the interaction and propagation of the particles like in the QCD, while the Higgs part will generate the masses of the particles. At first we will take a look at the gauge part

$$\mathcal{L}_{\mathrm{G}} = -\frac{1}{4} F_i^{\mu\nu} F^i_{\mu\nu} - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \tag{2.10}$$

of this Lagrangian. Here

$$F^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g_2 \, \epsilon^{ijk} \, W^j_\mu W^k_\nu \tag{2.11}$$

with $i = 1, 2, 3$ denotes the $SU(2)$ field strength and

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{2.12}$$

the $U(1)$ field strength. The $\boldsymbol{W}_\mu = (W_\mu^1, W_\mu^2, W_\mu^3)$ and the B_μ symbolize the weak isospin and weak hypercharge gauge boson fields. Again, the gauge sector describes the propagation of the gauge bosons as well as the interaction between the weak gauge bosons and the electromagnetic ones. The fermionic sector of the Lagrange density contains both, the right-handed as well as the left-handed chiralities. Summing over the left-handed weak isodoublets Q_L and the right-handed weak isosinglets q_R, we obtain

$$\mathcal{L}_{\mathrm{F}} = \sum_a \bar{\boldsymbol{\Psi}}_L^a \, i\not{D} \, \boldsymbol{\Psi}_L^a + \sum_a \bar{\psi}_R^a \, i\not{D} \, \psi_R^a. \tag{2.13}$$

The covariant derivative for the second term which describes the interaction between right-handed particles, takes the simple form

$$D_\mu \psi_R^a = \left(\partial_\mu + i\frac{g_1}{2} Y_W B_\mu\right) \psi_R^a \tag{2.14}$$

since the right-handed chiral fermions do not couple to weak isospin. This expression serves to define the $U(1)$ coupling g_1. Its normalization obviously depends on our convention for the weak hypercharge Y_W. The corresponding covariant derivative for the $SU(2)_L$ doublet Q_L is

$$D_\mu \boldsymbol{\Psi}_L^a = \left(I\left(\partial_\mu + i\frac{g_1}{2} Y_W B_\mu\right) + i g_2 \frac{\boldsymbol{\tau}}{2} \boldsymbol{W}_\mu\right) \boldsymbol{\Psi}_L^a, \tag{2.15}$$

since the left-handed fermions transform under the $SU(2)_L$ as well as under the $U(1)_Y$. In the upper equation we have introduced $SU(2)_L$ gauge coupling constant g_2, the 2×2 unit matrix I and the vector $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ containing the Pauli matrices. For the reason of simplicity we do not display the quark color degree of freedom in this section, thus every time we sum over quark internal degrees of freedom we also have to sum over the colors. The equations given above define a mathematically consistent gauge theory of weak isospin and weak hypercharge, but as we have already mentioned, it does not contain the mass terms that are experimentally observed.

This leads us to the third part of the Lagrange density, namely the Higgs part which is dedicated to add the masses to the theory. Note that in the course of this section we will not introduce neutrino masses within the Weinberg-Salaam part of the standard model, as their description becomes a bit more complicated. However, we will discuss the neutrino masses in section 2.2 alongside the introduction of the Pontecorvo-Maki-Nakagawa-Sakata matrix. The Higgs part $\mathcal{L}_{\mathrm{H}}$ of the Lagrangian can itself be split up into two segments. The first one is $\mathcal{L}_{\mathrm{HG}}$ which contains the Higgs-gauge couplings, and $\mathcal{L}_{\mathrm{HF}}$ which contains the Higgs-fermion couplings. The former can be written as

$$\mathcal{L}_{\mathrm{HG}} = (D^\mu \boldsymbol{\Phi})^* D_\mu \boldsymbol{\Phi} - V(\boldsymbol{\Phi}). \tag{2.16}$$

Here we have introduced a new complex doublet

$$\boldsymbol{\Phi} = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{2.17}$$

of spin 0 Higgs fields. The index of the fields show the electric charge assignments. Each of the quanta of these fields carry one unit of the weak hypercharge. The covariant derivative acting on this doublet is given by

$$D_\mu \mathbf{\Phi} = \left(I \left(\partial_\mu + i\frac{g_1}{2} B_\mu \right) + i g_2 \frac{\boldsymbol{\tau}}{2} \boldsymbol{W}_\mu \right) \mathbf{\Phi}. \tag{2.18}$$

Additionally we introduced the potential

$$V(\mathbf{\Phi}) = -\mu^2 \mathbf{\Phi}^\dagger \mathbf{\Phi} + \lambda (\mathbf{\Phi}^\dagger \mathbf{\Phi})^2, \tag{2.19}$$

which describes the self interaction of the Higgs fields. The parameters μ^2 and λ are positive but otherwise arbitrary. The Lagrange density for the Higgs-fermion couplings reads

$$\mathcal{L}_{\text{HF}} = -\sum_{i,j} y^u_{ij} \boldsymbol{Q}^i_L \widetilde{\mathbf{\Phi}}\, u^i_R - \sum_{i,j} y^d_{ij} \boldsymbol{Q}^i_L \mathbf{\Phi}\, d^i_R - \sum_{i,j} y^e_{ij} \boldsymbol{l}^i_L \mathbf{\Phi}\, e^i_R + \text{h.c.}\,, \tag{2.20}$$

where u^i_R and d^i_R denote the up- and down-type right-handed quarks as introduced in (2.4). Additionally we have introduced the charge conjugate

$$\widetilde{\mathbf{\Phi}} = i\tau_2 \mathbf{\Phi}^* = \begin{pmatrix} \Phi^{0*} \\ -\Phi^- \end{pmatrix} \tag{2.21}$$

to Φ. Note that (2.20) does not contain right-handed neutrinos. These have not been integrated into the theory, since we assume the neutrinos to be massless up to this point. Moreover, note that each of the terms in $\mathcal{L}_\text{H}$ is written in a $SU(2)_L \otimes U(1)_Y$ invariant form. The mass generation for fermions and gauge bosons can now be obtained by the spontaneous breaking of the $SU(2)_L \otimes U(1)_Y$ symmetry. Therefore we first have to get the ground state Higgs configuration by minimizing the potential V which gives the identity

$$\frac{\partial V}{\partial \mathbf{\Phi}^\dagger} = \mathbf{\Phi}(-\mu^2 + 2\lambda \mathbf{\Phi}^\dagger \mathbf{\Phi}) \stackrel{!}{=} 0. \tag{2.22}$$

We interpret this ground state relation in terms of vacuum expectation values which will be labeled with a zero subscript. The ground state equation has two solutions, the trivial solution $\langle \mathbf{\Phi} \rangle_0 = 0$ and the nontrivial solution

$$\langle \mathbf{\Phi}^\dagger \mathbf{\Phi} \rangle_0 = \frac{v^2}{2} \tag{2.23}$$

with

$$v = \sqrt{\frac{\mu^2}{\lambda}} \tag{2.24}$$

The trivial solution does not give any new information as it just eliminates the Higgs Part of the Lagrangian and leaves us with the massless theory. Therefore we will consider the nontrivial solution in the following. Thus the breaking of the original $SU(2)_L \otimes U(1)_Y$ is given by the nontrivial vacuum expectation value

$$\langle \mathbf{\Phi} \rangle_0 = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}. \tag{2.25}$$

As field components we would obtain

$$\mathbf{\Phi} = \begin{pmatrix} \phi^+ \\ (v + H + i\chi)/\sqrt{2} \end{pmatrix}, \tag{2.26}$$

with a physical massive Higgs field of mass $m_H = \sqrt{2}\mu$ and three unphysical Goldstone boson fields $\Phi^\pm$ and χ. However, we will not implement these Higgs fields in the further calculations, since this would lead to Higgs interactions which will be of no interest for the further chapters. We shall rather determine the fermion and gauge boson masses by inserting (2.25) for the Higgs field everywhere in the Lagrange density $\mathcal{L}_\mathrm{H}$. Furthermore, we define the charged boson fields

$$W_\mu^\pm = \sqrt{\frac{1}{2}}\left(W_\mu^1 \mp iW_\mu^2\right) . \tag{2.27}$$

Substituting this into the Lagrange density $\mathcal{L}_\mathrm{EW}$ we obtain the mass contribution

$$\begin{aligned}\mathcal{L}_\mathrm{M} = &-\frac{v}{\sqrt{2}}\sum_{ij} y_{ij}^u\, \bar{u}_L^i u_R^j - \frac{v}{\sqrt{2}}\sum_{ij} y_{ij}^d\, \bar{d}_L^i d_R^j - \frac{v}{\sqrt{2}}\sum_{ij} y_{ij}^e\, \bar{e}_L^i e_R^j \\ &+ \left(\frac{v\, g_2}{2}\right)^2 W_\mu^+ W_-^\mu + \frac{v^2}{8}(W_\mu^3)\begin{pmatrix} g_2^2 & -g_1 g_2 \\ -g_1 g_2 & g_1^2\end{pmatrix}\begin{pmatrix} W_3^\mu \\ B^\mu\end{pmatrix}\end{aligned} \tag{2.28}$$

from the Higgs-fermion part $\mathcal{L}_\mathrm{HF}$ of the Lagrangian, where the sums run over the quark flavors $u^i = u, c, t$ and $d^i = d, s, b$ as well as the charged leptons $e^i = e, \mu, \tau$. We can now define the matrices

$$M_\alpha = \frac{v}{\sqrt{2}} y_\alpha, \tag{2.29}$$

where in this case $\alpha = u, d, e$ denotes all types of charged fermions which we have introduced in (2.4) and (2.5). Note that these matrices are not necessarily diagonal. This means that the states which appear in the original gauge invariant Lagrangian are generally not mass eigenstates. Thus we have to distinguish between the gauge eigenstates which form the gauge basis, and the mass eigenstates which form the mass basis respectively. Therefore the next step is to perform a transformation from the gauge to the mass basis to get a proper description of the masses. If we take only the fermionic part of (2.28) we find

$$\begin{aligned}-\mathcal{L}_\mathrm{M,F} &= \bar{\boldsymbol{u}}_L M_u \boldsymbol{u}_R + \bar{\boldsymbol{d}}_L M_d \boldsymbol{d}_R + \bar{\boldsymbol{e}}_L M_e \boldsymbol{e}_R + \text{h.c.} \\ &= \bar{\boldsymbol{u}}_L S_L^u S_L^{u\dagger} M_u S_L^u S_L^{u\dagger} \boldsymbol{u}_R + \bar{\boldsymbol{d}}_L S_L^d S_L^{d\dagger} M_d S_L^d S_L^{d\dagger} \boldsymbol{d}_R + \bar{\boldsymbol{e}}_L S_L^e S_L^{e\dagger} M_e S_L^e S_L^{e\dagger} \boldsymbol{e}_R + \text{h.c.} \\ &= \bar{\boldsymbol{u}}_L^m M_u^m \boldsymbol{u}_R^m + \bar{\boldsymbol{d}}_L^m M_d^m \boldsymbol{d}_R^m + \bar{\boldsymbol{e}}_L^m M_e^m \boldsymbol{e}_R^m + \text{h.c.} \\ &= \bar{\boldsymbol{u}}^m M_u^m \boldsymbol{u}^m + \bar{\boldsymbol{d}}^m M_d^m \boldsymbol{d}^m + \bar{\boldsymbol{e}}^m M_e^m \boldsymbol{e}^m + \text{h.c.}\,,\end{aligned} \tag{2.30}$$

where the superscript m indicates the states in the mass basis. The 3×3 unitary matrices $S_{L,R}^\alpha$ relate the basis states

$$\begin{aligned}\boldsymbol{u}_L &= S_L^u \boldsymbol{u}_L^m, \quad & \boldsymbol{d}_L &= S_L^d \boldsymbol{d}_L^m, \quad & \boldsymbol{e}_L &= S_L^e \boldsymbol{e}_L^m, \\ \boldsymbol{u}_R &= S_R^u \boldsymbol{u}_R^m, \quad & \boldsymbol{d}_R &= S_R^d \boldsymbol{d}_R^m, \quad & \boldsymbol{e}_R &= S_R^e \boldsymbol{e}_R^m,\end{aligned} \tag{2.31}$$

and induce the biunitary diagonalizations

$$M_u^m = S_L^u\, M_u\, S_R^{u\dagger}, \quad M_d^m = S_L^d\, M_d\, S_R^{d\dagger} \quad \text{and} \quad M_e^m = S_L^e\, M_e\, S_R^{e\dagger}. \tag{2.32}$$

This yields the diagonal quark and lepton mass matrices

$$M_u^m = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_t\end{pmatrix}, \quad M_d^m = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b\end{pmatrix} \quad \text{and} \quad M_e^m = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau\end{pmatrix} \tag{2.33}$$

in the mass basis. Note that this theory does not present a way to calculate the fermion masses, since we do not know the values of the arbitrary Higgs-fermion couplings. Thus all masses have to determined from experimental data. The charged W-boson masses

$$M_W = \frac{v}{2} g_2 \tag{2.34}$$

can be taken directly from (2.28), but the symmetry breaking induces the neutral gauge bosons to undergo mixing. Their mass matrix is not diagonal in the basis of the W^3 and B states. The diagonalization of the basis gives us the new fields

$$Z_\mu = \cos\theta_W W^3_\mu - \sin\theta_W B_\mu \tag{2.35}$$
$$A_\mu = \sin\theta_W W^3_\mu + \cos\theta_W B_\mu \tag{2.36}$$
$$\tag{2.37}$$

where the weak mixing angle θ_W is defined by

$$\tan\theta_W = \frac{g_1}{g_2} \tag{2.38}$$

and the fields A_μ and Z_μ correspond to the massless photon and the massive Z^0-boson respectively. The neutral gauge boson masses are found to be

$$M_\gamma = 0 \quad \text{and} \quad M_z = \frac{v}{2}\sqrt{g_1^2 + g_2^2}. \tag{2.39}$$

Note that we also get the fixed $W^\pm$-to-Z^0 mass ratio

$$\frac{M_W}{M_Z} = \cos\theta_W. \tag{2.40}$$

In the following we will observe how the electroweak currents are described in the standard model. Using the $SU(2)_L \otimes U(1)_Y$ description which has been introduced above, we obtain the interaction Lagrange density

$$\mathcal{L}_\mathrm{I} = -\frac{g_2}{\sqrt{8}}\left(W^+_\mu J^\mu_\mathrm{CH} + W^-_\mu J^{\mu\dagger}_\mathrm{CH}\right) - g_2 W^3_\mu J^\mu_\mathrm{W3} - g_1 B_\mu \left(J^\mu_\mathrm{EM} - J^\mu_\mathrm{W3}\right), \tag{2.41}$$

where

$$J^\mu_\mathrm{EM} = -\bar{\boldsymbol{e}}\gamma^\mu \boldsymbol{e} + \frac{2}{3}\bar{\boldsymbol{u}}\gamma^\mu \boldsymbol{u} - \frac{1}{3}\bar{\boldsymbol{d}}\gamma^\mu \boldsymbol{d} \tag{2.42}$$

is the electroweak current,

$$J^\mu_\mathrm{CH} = \bar{\boldsymbol{\nu}}\gamma^\mu(1-\gamma_5)\boldsymbol{e} + \bar{\boldsymbol{u}}\gamma^\mu(1-\gamma_5)\boldsymbol{d} \tag{2.43}$$

is the charged weak current and J^μ_W3 is the third component of the weak isospin current

$$\boldsymbol{J}^\mu_\mathrm{W} = \sum_a \bar{\boldsymbol{Q}}^a_L \gamma^\mu \frac{\boldsymbol{\tau}}{2} \boldsymbol{Q}^a_L. \tag{2.44}$$

Note that this is still given in the language of interaction eigenvalues. Although the Weinberg-Salam model is initially written down in terms of the gauge basis states, calculations which confront theory with experiment are performed using the mass basis states. Therefore we will rewrite (2.41) in terms of the mass eigenstates. Substitution of A_μ and Z_μ for B_μ and W^3_μ in (2.41) yields

$$\mathcal{L}_\mathrm{I} = -\frac{g_2}{\sqrt{8}}\left(W^+_\mu J^\mu_\mathrm{CH} + W^-_\mu J^{\mu\dagger}_\mathrm{CH}\right) - e\,A_\mu J^\mu_\mathrm{EM} - \frac{g_2}{2\cos\theta_W} Z^\mu \bar{\boldsymbol{\varphi}}\left(g^f_v\gamma_\mu + g^f_a\gamma_\mu\gamma_5\right)\boldsymbol{\varphi} \tag{2.45}$$

with the electromagnetic and weak coupling constants

$$e = g_1\cos\theta_W = g_2\sin\theta_W \tag{2.46}$$
$$g^f_v = T^f_{W3} + 2\sin^2(\theta_W)Q^f_{el} \tag{2.47}$$
$$g^f_a = T^f_{W3}. \tag{2.48}$$

We still have to rewrite the currents in (2.45) into the mass basis. Therefore we will use the equations (2.31) to transform the currents in analogy to the masses. It turns out that the transformation has no effect on the structure of the electromagnetic and weak currents. The reason for this is that each generation is (aside from the mass) a replica of the others, and products of the unitarity transformation matrices always gives rise to the unit matrix in flavor space. Therefore there are, at Lagrangian level, no flavor changing neutral currents in the theory. As an example for that we will consider the leptonic contribution to the electromagnetic current. We obtain

$$\begin{aligned} J^\mu_\mathrm{EM} &= -\bar{\boldsymbol{e}}\gamma^\mu\boldsymbol{e} \\ &= -\bar{\boldsymbol{e}}_L\gamma^\mu\boldsymbol{e}_L - \bar{\boldsymbol{e}}_R\gamma^\mu\boldsymbol{e}_R \\ &= -\bar{\boldsymbol{e}}_L S^{e\dagger}_L\gamma^\mu S^e_L\boldsymbol{e}_L - \bar{\boldsymbol{e}}_R S^{e\dagger}_R\gamma^\mu S^e_R\boldsymbol{e}_R \\ &= -\bar{\boldsymbol{e}}^m_L\gamma^\mu\boldsymbol{e}^m_L - \bar{\boldsymbol{e}}^m_R\gamma^\mu\boldsymbol{e}^m_R \\ &= -\bar{\boldsymbol{e}}^m\gamma^\mu\boldsymbol{e}^m. \end{aligned} \tag{2.49}$$

Note that there is no difficulty in passing the unitarity matrices $S^e_{L,R}$ through the Dirac matrices γ^μ, since the former matrices act in flavor space, whereas the latter matrix acts in spin space. The quark contribution of the electromagnetic current and the charged weak current of the leptons behave analogously.

Up to this point we did not encounter any differences between the gauge basis states and the mass eigenstates. However, we will now see, how the mixing between generations manifests in the charged weak interaction currents. By convention we will assign the mixing to the down type quarks by

$$J^\mu_\mathrm{CH} = \bar{\boldsymbol{u}}_L\gamma^\mu\boldsymbol{d}_L = \bar{\boldsymbol{u}}^m_L\gamma^\mu S^{u\dagger}_L S^d_L\boldsymbol{d}^m_L = \bar{\boldsymbol{u}}^m_L\gamma^\mu\, V_\mathrm{CKM}\,\boldsymbol{d}^m_L, \tag{2.50}$$

where we defined the so called Cabbibo Kobayashi Maskawa (CKM) [25] matrix

$$V_\mathrm{CKM} = S^{u\dagger}_L S^d_L = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \tag{2.51}$$

Thus the down type quark states participating in transitions of the weak current are linear combinations of the mass eigenstates. Since the CKM matrix is the product of two unitary matrices, it is itself unitary. Therefore the CKM matrix can be described by three mixing angles θ_{ij} $(i,j=1,2,3)$ and one complex phase δ which means it can be viewed as an Eulerian construction of three rotation matrices and a phase matrix. These matrices read:

$$\begin{aligned} U_{13} &= \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} & U_{12} &= \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \\ U_{23} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} & U_{\delta} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \end{aligned} \tag{2.52}$$

where $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$ $(i, j = 1, 2, 3)$. Combining everything by a product of the three rotations, where U_{13} is transformed by the matrix U_δ according to

$$V_{\text{CKM}} = U_{23} U_\delta^\dagger U_{13} U_\delta U_{12} \tag{2.53}$$

as purposed in [8] we obtain

$$V_{\text{CKM}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{12}\, e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\, e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}\, e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\, e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\, e^{i\delta} & c_{23}c_{13} \end{pmatrix}. \tag{2.54}$$

All CP violation induced by this matrix is done by the phase δ, since the combination of the three rotation matrices U_{12}, U_{13} and U_{23} yields a general unitary matrix.

2.2. Neutrino masses and lepton mixing

Up to now we have not introduced any neutrino mass terms into the standard model. We have not done this for two reasons. The first reason is that in early versions of the standard model the neutrinos have been assumed to be massless. However, there recently has been evidence that neutrinos have masses, and therefore consequently also underly a mixing [26, 27, 28]. The second reason is that there is an additional way of mass creation for right-handed neutrinos which differs from the mass creation in the Weinberg-Salaam model and explains the smallness of the neutrino masses in a natural way.

As we have seen in the past section the masses in the Weinberg-Salaam model are usually created by connecting the left-handed fields to their right-handed partners. Thus the mass term is the term that flips the chirality. Mass terms of this kind are usually called Dirac mass terms and take the form

$$m(\bar{\boldsymbol{\psi}}_R \boldsymbol{\psi}_L + \text{h.c.}). \tag{2.55}$$

The introduction of the Dirac mass terms for neutrinos goes along the same line as the introduction (2.20) of the other fermion masses. In particular we find

$$\mathcal{L}_{\text{DM}} = -\sum_{i,j} y^\nu_{ij} \boldsymbol{l}^i_L \widetilde{\boldsymbol{\Phi}}\, \nu^i_R + \text{h.c.}\,, \tag{2.56}$$

which becomes

$$\mathcal{L}_{\text{DM}} = -\bar{\boldsymbol{\nu}}_L M_\nu \boldsymbol{\nu}_R + \text{h.c.} \tag{2.57}$$

after employing the mechanism of spontaneous symmetry breaking.

Another possibility to introduce masses into the theory is given by the Majorana mass terms which take the form

$$M(\bar{\boldsymbol{\psi}}^c_R \boldsymbol{\psi}_R + \text{h.c.}), \tag{2.58}$$

where $\boldsymbol{\psi}^c = C\gamma^0\boldsymbol{\psi}^*$ with $C = -i\gamma_0\gamma_2$ is the charge-conjugate field of $\boldsymbol{\psi}$ and $\boldsymbol{\psi}_R^c \equiv (\boldsymbol{\psi}_R)^c = P_L\boldsymbol{\psi}^c$ has left-handed chirality. The Majorana mass term violates the lepton number conservation by two units and makes the particle and its antiparticle indistinguishable. This means that we can use it only for right-handed neutrinos which carry neither hypercharge nor weak $SU(2)_L$ charge and can therefore be assumed to be equal to their antiparticles. Extending this for three families we obtain the Lagrange density

$$\mathcal{L}_{MM} = -\frac{1}{2}\bar{\boldsymbol{\nu}}_R\, M\, \boldsymbol{\nu}_R^c + \text{h.c.} \tag{2.59}$$

for the Majorana masses. This gives us the possibility to write the complete mass term as

$$\mathcal{L} = -\frac{1}{2}\left(\bar{\nu}_L^i, \overline{(\nu_R^c)}_L^i\right)\begin{pmatrix} 0 & m_{ij} \\ m_{ij}^T & M_{ij}\end{pmatrix}\begin{pmatrix}(\nu_L^c)_R^j \\ \nu_R^j\end{pmatrix}, \tag{2.60}$$

where we have made use of the relation

$$\overline{(\boldsymbol{\nu}_R^c)}_L(\boldsymbol{\nu}_L^c)_R = \bar{\boldsymbol{\nu}}_L\bar{\boldsymbol{\nu}}_R \tag{2.61}$$

Adding this to the original Lagrangian of the standard model which we have evolved in the earlier sections, we find that the right-handed neutrinos interact with the other particles exclusively through the Lagrangian $\mathcal{L}_{\mathrm{M}}$. Thus the classical experiment by M. Goldhaber, who showed the left-handedness of the neutrino using the weak interaction ${}^{125m}Eu + e^- \rightarrow {}^{152}Sm + \nu_e$ [29], does not preclude the possibility of a Majorana neutrino. The absence of interactions of the right-handed neutrinos offers us an interesting possibility to create small neutrino masses by the so-called seesaw mechanism [30, 31]. Since the Majorana mass term does not evolve from the Higgs mechanism it is not connected to the electroweak vacuum expectation value. This implies that the Majorana masses of the right-handed neutrinos can be large, maybe even as large as the scale of grand unification. This can be used to construct an effective theory by integrating out the right-handed neutrinos. This results in the replacement of the right-handed neutrino fields within their interaction terms

$$\nu_R^i = M_{ij}^{-1}\, m_{jk}\, \nu_L^k \tag{2.62}$$

with all other fields. The upper relation therefore represents an equation for small momenta which means that the kinetic energy of the right-handed neutrino is neglected. From this we get a dimension 5 operator which introduces a Majorana mass term for left-handed neutrinos of the form

$$\mathcal{L}'_{MM} = -\frac{1}{2}\left(\nu_L^{T,i}\,[m^T M^{-1} m]_{ij}\, C\, \nu_L^i + \bar{\nu}_L^i\,[m^T M^{-1} m]_{ij}\, C\, \bar{\nu}_L^{T,i}\right), \tag{2.63}$$

where the Majorana masses of the left-handed neutrinos are small, as they are suppressed by the large Majorana masses of the right-handed neutrinos. This means the seesaw mechanism gives us a natural way to create small Majorana masses for the left-handed neutrinos, especially for the case in which the Majorana masses which have been introduced for the right-handed neutrinos in a most natural way, are expected to be large.

No matter how the mass matrices have been defined, (2.57) as well as (2.63) still have to be diagonalized in the same manner as we did for the quarks and charged leptons in section 2.1. Therefore we again have to perform the biunitary transformations

$$M_\nu^m = S_L^\nu\, M_\nu S_R^{\nu\dagger} \tag{2.64}$$

$$M^m = S_L^\nu\, m^T M^{-1} m\, S_R^{\nu\dagger} \tag{2.65}$$

to diagonalize the mass matrices for the Dirac and Majorana neutrinos respectively. The quantities M_ν^m and M^m denote the diagonal mass matrices in the mass eigenspace with the mass eigenvalues of the (in the case of Majorana masses left-handed) neutrinos. This leads to a mixing matrix similar to the CKM Matrix. The charged current interaction now reads

$$\mathcal{L}_{\text{CC}} = \frac{g}{\sqrt{2}} \bar{\boldsymbol{\nu}}_L \, \gamma^\mu \, V_{\text{PMNS}} \, \boldsymbol{e}_L \, W_\mu^+ \,, \tag{2.66}$$

where e_L denotes again the three left-handed charged leptons and

$$V_{\text{PMNS}} = S_L^{\nu\dagger} S_L^e \tag{2.67}$$

is the unitary Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix which is often also called MNS matrix in the literature. This matrix connects the mass matrices of the neutrinos in the mass eigenbasis (ν_1, ν_2, ν_3) with the ones given in the gauge eigenbasis $(\nu_e, \nu_\mu, \nu_\tau)$ by the relation

$$\nu_\alpha = \sum_i V_{\text{PMNS}}^{\alpha\, i} \nu_i \tag{2.68}$$

with $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$, just as the CKM matrix does for the quarks. Like the CKM matrix the PMNS matrix can be parametrized with three angles, but with three CP violating phases (δ, α_1 and α_2) instead of one. Thus in the standard notation given by [8] it reads

$$V_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{12}\, e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\, e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}\, e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\, e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\, e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} e^{i\alpha_1/2} & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.69}$$

using the same notation conventions as for the CKM matrix introduced at the end of the last section as purposed in [8].

Here we will order the basis of mass eigenstates in such a way that $m_1^2 < m_2^2$ and $\Delta m_{21}^2 < |\Delta m_{31}^2|$, where $\Delta m_{ij}^2 \equiv m_j^2 - m_i^2$. The mass difference Δm_{21}^2 can be obtained experimentally from solar neutrinos. Thus the notation $\Delta m_{\text{sol}}^2 = |\Delta m_{21}^2|$ is often found in the literature. Likewise, the mass difference Δm_{32}^2 can be obtained by the analysis of atmospheric neutrinos, and thus can be written as $\Delta m_{\text{atm}}^2 = |\Delta m_{32}^2|$. Given the current precision of neutrino oscillation experiments and the fact that neutrino oscillations are only sensitive to the mass-squared differences, three possible arrangements of the neutrino masses are allowed:

- Normal hierarchy, i.e. $m_1 < m_2 \ll m_3$. In this case we find $\Delta m_{\text{atm}} \equiv m_3^2 - m_2^2 > 0$ and thus $m_3 \simeq \sqrt{\Delta m_{\text{atm}}^2}$. The solar neutrino oscillation relates the squared masses of the two lighter neutrinos, leaving the mass of the lightest neutrino unconstrained. If we assume $m_1 \ll m_2$ we can find a specific value for m_2.

- Inverted hierarchy, i.e. $m_1 \simeq m_2 \gg m_3$ which implies $m_{1,2} \simeq \sqrt{-\Delta m_{\text{atm}}^2}$ and $\Delta m_{\text{atm}}^2 \equiv m_3^2 - m_2^2 < 0$. Here we have no information about m_3, except that its value is much smaller than the ones of the two other masses.

- Degenerate neutrinos i.e. $m_1 \simeq m_2 \simeq m_3$.

2.3. Quantum chromodynamics

Quantum chromodynamics (in the following sometimes abbreviated with QCD) is the description of the strong interactions through a non-abelian gauge theory. It contains quarks and gluons instead of leptons and photons as its basic degrees of freedom. The theory of quantum chromodynamics is described by a renormalizable gauge theory which is based on the symmetry group $SU(3)$. In the following we will denote quark fields with q_j^f adopting the notation from the last section defined in equation (2.4). The index $j = 1, 2, 3$ represents the color in the fundamental representation of the $SU(3)$. The gauge bosons are the gluons which will be symbolized as A_μ^a. The gluons also carry color, but denote that the index arrises from the adjoint representation which implies $a = 1, ..., 8$. In this section we will denote the octet color indices by the letters at the beginning of the alphabet (e.g. $a, b, c, ... = 1, ..., 8$) and the triplet color indices by the ones in the middle (e.g. $j, k, l, ... = 1, 2, 3$). Using this notation the Lagrangian of quantum chromodynamics takes the form

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a + \bar{q}_j^f \left(i\not{D}_{jk} - m^f \delta_{jk} \right) q_k^f, \tag{2.70}$$

where the repeated indices are summed over. The gauge field strength tensor is given by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_s\, f_{abc}\, A_\mu^b\, A_\nu^c. \tag{2.71}$$

Here g_s denotes the $SU(3)$ gauge coupling parameter and f_{abc} the antisymmetric structure constant of the fundamental representation of the $SU(3)$. The gauge covariant derivative is defined by

$$iD_\mu = i\partial_\mu + g_s\, t^k A_\mu^k. \tag{2.72}$$

Note that this is independent of the representation and the t^k symbolize any set of $SU(3)$ generators for an arbitrary representation matching the representation of symbol the derivative acts on. When acting on a quark field, like in the QCD Lagrangian (2.70) it is used in its fundamental representation, and the matrices take the form of the Gell-Mann matrices presented in appendix A.2. The first term in (2.70) contains, like in QED, the propagators of the gauge fields. But since the gluons carry color themselves, it also gives rise to gluon-gluon interactions. This makes the structure of the QCD more difficult than the one of QED, calculations up to one-loop level and beyond are considered. The second term describes the interactions of the quarks with the gluon fields. Remember that the covariant derivative contains the gauge field which induces the interaction terms. In this way the gauge covariance is directly associated with the interaction. Additionally, this term contains the mass terms for the quarks. In spite of its putative simple structure, the QCD Lagrangian has a very abundant dynamical content.

Because of its huge relative strength regarding the other forces contained in the standard model, radiative corrections are performed exclusively with respect to the QCD within this work. In the later sections we will see that the effects of QCD play a crucial role in the phenomenology of weak decays of hadrons - especially since all decays of single quarks take place in the background field of the hadron because of confinement. In the present subsection we will shortly review the basic formalism of perturbative QCD and its renormalization, where we will concentrate specially on the aspects which we will need for the present work. The starting point shall again be the Lagrangian density of QCD. Expressing it in a more verbose way than in equation (2.70) and adding a gauge fixing term to eliminate the superfluous gauge degrees of freedom which can give rise to specific divergencies, we obtain

$$\begin{aligned} \mathcal{L}_{\mathrm{QCD}} = &-\frac{1}{4}(\partial_\mu A_\mu^a - \partial_\nu A_\mu^a) - \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 + g_s f^{abc}(\partial^\mu \chi^{a*} \chi^b A_\mu^c) + \chi^{a*} \partial^\mu \partial_\mu \chi^a \\ &- \frac{g_s}{2} f^{abc}(\partial_\mu A_\mu^a - \partial_\nu A_\mu^a) A^{b\mu} A^{c\nu} - \frac{g_s^2}{4} f^{abe} f^{cde} A_\mu^a A_\nu^b A^{c\mu} A^{d\nu} \\ &+ g_s\, \bar{q}_i^f T_{ij}^a \gamma^\mu q_j^f\, A_\mu^a + \bar{q}_i^f (i\not{\partial}_{ij} - m^f \delta_{ij}) q_j^f\ , \end{aligned} \tag{2.73}$$

where χ^a denotes the ghost field and ξ the gauge parameter. The ghost fields are Grassmann variables which couple to gluons exclusively. They occur only within loops and never appear as asymptotic states (which is, why we have not introduced them in equation (2.70)), but only remove unphysical fields introduced by the gauge fixation which appear in non-abelian gauge theories.

To deal with the divergencies that appear during the calculation for loop corrections to the green functions two steps have to be performed. The first one consists of the regularization which means to explicitly parametrize the singularities. A common method for this is the dimensional regularization, where a continuation of the space-time dimensions to $D = 4 - 2\epsilon$ [32, 33, 34, 35, 36] is performed. The reason for this is that integrals which are UV divergent for $D = 4$ dimensions, are convergent for a sufficiently small D. In the next step the analytic continuation of D results in the transformation of the divergencies to singularities in the complex D plane. Therefore the loop integrals are replaced by the description

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \longrightarrow \mu^{4-D} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \,. \tag{2.74}$$

The arbitrary reference mass scale μ has been introduced to obtain a dimensionless coupling g_s in $D = 4 - 2\epsilon$ dimensions. This approach has the advantage that Lorentz invariance and gauge invariance are retained. The only problem arrising is the fact that the Dirac matrix γ_5 has no trivial continuation to D dimensions. However, there are solutions to this problem by either installing correction terms to restore the chiral symmetry or modifying γ_5 algebra.

The second step consists of the renormalization to obtain finite Greens functions. This is often done by the subtraction of the divergences in the minimal subtraction scheme MS [37] or the modified minimal subtraction scheme ($\overline{MS}$) [38]. While the MS scheme only consists of the subtraction of the $1/\epsilon$ poles (where $\epsilon = 4 - D$) which have been separated form the physical contributions, the modified minimal subtraction scheme $\overline{MS}$ also removes additional terms of $\ln(4\pi)$ and γ_E (the Euler constant) which normally accompany the divergencies. To eliminate the divergencies the fields and parameters in the Lagrangian have to be renormalized. This is commonly done by multiplying the parameters by renormalization constants Z. These read

$$\begin{array}{lll} A^a_{0\mu} = Z_3^{1/2} A^a_\mu & q^f_0 = Z_q^{1/2} q^f & \chi^a_0 = \tilde{Z}_3^{1/2} \chi^a \\ g_{s0} = Z_g \, g_s \, \mu^\epsilon & \xi_0 = Z_3 \, \xi & m_0 = Z_m \, m \,, \end{array} \tag{2.75}$$

where the index 0 marks the unrenormalized quantities. Since we will not consider Greens Functions with external ghost fields, it is not necessary to apply a ghost field renormalization. Furthermore, we do not need the gauge parameter renormalization when we are dealing with gauge independent quantities like Wilson coefficient functions (which we will encounter in later sections). Considering the parameters and fields in the original Lagrange density (2.73) as unrenormalized (bare) quantities, we may now re-express (2.73) in terms of (2.75). This results in a Lagrange density where only renormalized quantities appear. As an example we obtain

$$\mathcal{L}_\mathrm{F} = \bar{q}^f_0 \, i\partial\!\!\!/ \, q^f_0 - m \, \bar{q}^f_0 \, q^f_0 = \bar{q}^f \, i\partial\!\!\!/ \, q^f - m \, \bar{q}^f \, q^f + (Z_q - 1) \, \bar{q}^f \, i\partial\!\!\!/ \, q^f - (Z_q \, Z_m - 1) \, m \, \bar{q}^f \, q^f \tag{2.76}$$

for the quark kinetic term. The counterterms that appear additionally to the terms we had for the bare quantities can now formally been treated as interaction terms contributing to the Green functions calculated in pertubation theory. The Feynman rule for the counterterms contained in (2.76), for example, reads

$$i(Z_q - 1)p\!\!\!/ - i(Z_q Z_m - 1)m \,. \tag{2.77}$$

The constants are then chosen in such a way that they cancel the divergencies in the Green functions according to the chosen renormalization scheme. This way of implementation of the renormalization is called the counterterm method, since additional interaction vertices are provided to cancel the divergencies.

In the following lines we will deal with the renormalization group equations which play an important role in the study of perturbative QCD effects. These differential equations describe the dependence of the renormalized parameters on the renormalization scale μ. Te renormalization group equations can be easily derived from the definition (2.75) by using the fact that the bare quantities are μ independent. Thus the renormalized coupling $g_s(\mu)$ obeys

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu} g_s(\mu) = \beta(\epsilon, g_s(\mu))\,, \tag{2.78}$$

where

$$\beta(\epsilon, g_s) = -\epsilon g_s - g_s \frac{1}{Z_g}\frac{\mathrm{d}Z_g}{\mathrm{d}\ln\mu} = -\epsilon g_s + \beta(g_s)\,, \tag{2.79}$$

which serves as a definition of the β function[2]. The upper equations are valid in arbitrary dimensions. For four dimensions $\beta(\epsilon, g_s)$ reduces to $\beta(g_s)$ with $\epsilon = 0$. Similarly, the anomalous dimension γ_m of the mass can be defined by

$$\frac{\mathrm{d}m(\mu)}{\mathrm{d}\ln(\mu)} = -\gamma_m(g_s)m(\mu) \tag{2.80}$$

and

$$\gamma_m(g_s) = \frac{1}{Z_m}\frac{\mathrm{d}Z_m}{\mathrm{d}\ln\mu}\,. \tag{2.81}$$

In the MS and $\overline{MS}$ scheme the renormalization constants contain just the pole terms in ϵ. Therefore they can be expanded by

$$Z_i = 1 + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k} Z_{i,k}(g_s)\,. \tag{2.82}$$

Applying the equations (2.78) and (2.79) results in

$$\frac{1}{Z_i}\frac{\mathrm{d}Z_i}{\mathrm{d}\ln(\mu)} = -2g_s^2 \frac{\partial Z_{i,1}(g_s)}{\partial g_s^2}\,, \tag{2.83}$$

which can be used to perform a direct calculation of the renormalization group functions from the $1/\epsilon$-pole part of the renormalization constants. Up to two-loop level the beta function reads [14]

$$\beta(g) = -\beta_0 \frac{g_s^3}{16\pi^2} - \beta_1 \frac{g_s^5}{(16\pi^2)^2}\,. \tag{2.84}$$

The second term becomes relevant only for next-to-leading order corrections, and can be neglected for the calculations which we will perform in chapter 4. However, we include this term for convenience. The upper equation can be rewritten in terms of $\alpha_s = g_s^2/4\pi$ which yields

$$\frac{\mathrm{d}\alpha_s}{\mathrm{d}\ln(\mu)} = -2\beta_0 \frac{\alpha_s^2}{4\pi} - 2\beta_1 \frac{\alpha_s^3}{(4\pi)^2}\,. \tag{2.85}$$

[2]The beta function is also called Callan-Symanzik function, as it was introduced by C. G. Callan Jr. [39] and K. Symanzik [40]

In the same way we can write the two-loop expression for the anomalous quark mass dimension which reads

$$\gamma_m(\alpha_s) = \gamma_{m0}\frac{\alpha_s}{4\pi} + \gamma_{m1}\left(\frac{\alpha_s}{4\pi}\right)^2 . \tag{2.86}$$

Furthermore, we shall include the $1/\epsilon$ pole part

$$Z_{q,1} = a_1\frac{\alpha_s}{4\pi} + a_2\left(\frac{\alpha_s}{4\pi}\right)^2 \tag{2.87}$$

of the quark field renormalization constant Z_1 to $\mathcal{O}(\alpha_s^2)$. The coefficients contained in the upper equations calculated in the MS ($\overline{MS}$) scheme read:

$$\begin{gathered}
\beta_0 = \frac{11N - 2\,n_f}{3} \qquad \beta_1 = \frac{34}{3}N^2 - \frac{10}{3}\,N\,n_f - 2\,C_F\,n_f \qquad C_F = \frac{N^2-1}{2N} \\
\gamma_{m0} = 6\,C_F \qquad \gamma_{m1} = C_F\left(3\,C_F + \frac{97}{3}N - \frac{10}{3}n_f\right) \\
a_1 = -C_F \qquad a_2 = C_F\left(\frac{3}{4}C_F - \frac{17}{4}N + \frac{1}{2}n_f\right) ,
\end{gathered} \tag{2.88}$$

where N is the number of colors and n_f is the number of quark flavors. The expressions for a_1 and a_2 are valid in Feynman gauge ($\xi = 1$). At two-loop order the solution of the renormalization group equation 2.85 provides the running coupling constant

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln\frac{\mu^2}{\Lambda^2_{\text{QCD}}}}\left[1 - \frac{\beta_1}{\beta_0^2}\frac{\ln\left(\ln\frac{\mu^2}{\Lambda^2_{\text{QCD}}}\right)}{\ln\frac{\mu^2}{\Lambda^2_{\text{QCD}}}}\right] . \tag{2.89}$$

at NLO. This running coupling constant vanishes for $\mu/\Lambda_{\text{QCD}} \to \infty$ due to asymptotic freedom. To retain the leading order corrections β_1 simply has to be set 0. Finally, we write down the two-loop expression for the running quark mass in the MS ($\overline{MS}$) scheme

$$m(\mu) = m(m)\left[\frac{\alpha_s(\mu)}{\alpha_s(m)}\right]^{\frac{\gamma_{m0}}{2\beta_0}}\left[1 + \left(\frac{\gamma_{m1}}{2\beta_0} - \frac{\beta_1\gamma_{m0}}{2\beta_0^2}\right)\frac{\alpha_s(\mu)-\alpha_s(m)}{4\pi}\right] , \tag{2.90}$$

which is obtained by integrating equation (2.80). Up to one-loop order the equations (2.84) and (2.88) above provides us with

$$\beta = -\left(11 - \frac{2}{3}n_f\right)\frac{g_s^3}{16\pi^2} \tag{2.91}$$

for three colors. The sign of this function is obviously negative, if the number of flavors n_f does not exceed 16. For the observed case of $n_f = 6$ flavors this has interesting consequences which distinguish the QCD from the QED. The QED vacuum acts like a medium with the dielectric constant $\epsilon_{\text{QED}} > 1$ which comes from the vacuum polarization (the spontaneous creation of virtual fermion-antifermion pairs). This results in a screening of the electric charges which leads to a weakening of the electric force for long distances. The magnetic susceptibility of the QED vacuum is therefore $\mu_{\text{QED}} < 1$ and hence the QED vacuum is a diamagnetic medium. In QCD the analog effect is given by the creation of virtual quark-antiquark pairs. But as the gluons also contain a color charge one obtains additional contributions from the creation of virtual gluons which even exceed the effects of the quark-antiquark pairs in the case of $n_f \leq 16$.

Therefore the QCD vacuum is a paramagnetic medium with $\mu_{\text{QCD}} > 1$ which antiscreens the color charges because of $\epsilon_{\text{QCD}} < 1$. Thus the color charges become even stronger with growing distance. This implies that the coupling constant g_s decreases and vanishes asymptotically for increasing energies. However, for small energy scales the coupling constant gets too strong for an expansion in its orders. This happens for about $\mu < \lambda_{\text{QCD}}$. Thus it is not possible to describe hadrons in low energy processes by using QCD up to now.

Another well known effect in QCD is the confinement. It describes the fact that quarks can appear only in bound states - in contrast to leptons which can be measured separately. The confinement results from the potential between two color charges which is phenomenologically often taken in the form

$$V_s = -\frac{4}{3}\frac{\alpha_s}{r} + k\,r, \tag{2.92}$$

where $k > 0$ is a constant. For short distances r a potential proportional to $-1/r$ like the Coulomb potential is obtained, while for long distances the second term in (2.92) provides a strong growth to the forces at high distances of the color sources. This makes it impossible to separate the quarks, since more energy has to be expend for the separation than for the creation of a new quark-antiquark pair. Thus every attempt to separate the quarks results in the creation of two new hadrons instead of the separation of the quarks. Color charged particles like quarks and gluons can therefore only be found in bound color neutral states as mesons or baryons. Experimentally this can be shown in reactions like $e^+e^- \to q\,\bar{q}$ with high center of mass energies. The color charged quark-antiquark pair in this reaction is separated with high velocities which results in the creation of additional quark-antiquark pairs. The resulting particles are observed as jets of color neutral hadrons. In this work we are specially interested in the semileptonic decays of B-mesons. The weak interactions describe only the interactions between quarks and leptons, since we have the quarks in a bound state caused by confinement we have to implement the strong interactions between the b-quark and the light spectator quark. This is done by the heavy quark effective theory which is described in section 3.3.

2.4. The S-matrix and the optical theorem

In this section we will introduce how to calculate a differential decay rate. Therefore we will analyze the differential cross section and show how it is connected to the differential decay rate by the optical theorem. The starting point is to consider two wave packets ϕ_A and ϕ_B in the initial state in the distant past $T \to -\infty$ and final state of n wave packets $\phi_1\phi_2\ldots\phi_n$ in the distant future $T \to \infty$. How the interaction at $T = 0$ really works is unknown and furthermore not measurable. Therefore we will avoid to be forced to do calculations in this point. A wave packet can be described by the state

$$|\phi\rangle = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{k}}}}\phi(\boldsymbol{k})|\boldsymbol{k}\rangle, \tag{2.93}$$

where $\phi(\boldsymbol{k})$ is the Fourier transform of the spatial wave function and $|\boldsymbol{k}\rangle$ is a one-particle state of momentum $\boldsymbol{k}$ in the interacting theory. In a free theory we would have $|\boldsymbol{k}\rangle = \sqrt{2E_{\boldsymbol{k}}}a^\dagger_{\boldsymbol{k}}|0\rangle$. The factor of $1/\sqrt{2\,E_{\boldsymbol{k}}}$ is chosen to convert the relativistic normalization of $|\boldsymbol{k}\rangle$ to the conventional normalization in which the sum of all probabilities adds up to 1:

$$\langle\phi|\phi\rangle = 1 \qquad \text{if} \qquad \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}|\phi(\boldsymbol{k})|^2 = 1. \tag{2.94}$$

The probability for transition from the initial state $|\phi_A\phi_B\rangle$ to the final state $|\phi_1,\phi_2,\ldots\rangle$ is now given by

$$\mathcal{P} = |\langle \underbrace{\phi_1\phi_2\ldots}_{\text{future}} | \underbrace{\phi_A\phi_B}_{\text{past}} \rangle|^2 \,. \tag{2.95}$$

The state $|\phi_A\phi_B\rangle$ of two wave packets is constructed in the far past, while the state $|\phi_1,\phi_2,\ldots\rangle$ is a state for several wave packets in the far future. In the following we will mark this with the subscripts *in* and *out* respectively. Even though these states are constructed in the Heisenberg picture (and therefore are time independent) they are set up for two different times, and therefore have an nontrivial overlap.

The *in* state $|\boldsymbol{p}_A\boldsymbol{p}_B\rangle_{in}$ can be defined by setting up $|\phi_A\phi_B\rangle$ in the remote past and taking the limit in which the wave packets $\phi_i(\boldsymbol{k}_i)$ become concentrated around definite momenta $\boldsymbol{p}_i$. It is useful to view $|\phi_A\phi_B\rangle$ as a superposition of such states. Furthermore, it is important to take into account the transverse displacement of the wave packets ϕ_B relative to ϕ_A in position space. Although this could be left implicit in the form $\phi(\boldsymbol{k}_B)$ we prefer to write $\phi(\boldsymbol{k}_B)$ with an explicit factor of $\exp(-i\boldsymbol{b}\cdot\boldsymbol{k}_B)$ for the spatial translation with an impact parameter $\boldsymbol{b}$ to get collinear wave functions in the reference momentum-space. The initial state can then be written by

$$|\phi_A\phi_B\rangle_{\text{in}} = \int \frac{\mathrm{d}^3\boldsymbol{k}_A}{(2\pi)^3} \int \frac{\mathrm{d}^3\boldsymbol{k}_B}{(2\pi)^3} \frac{\phi_A(\boldsymbol{k}_A)\,\phi_B(\boldsymbol{k}_B)\,e^{-i\boldsymbol{b}\cdot\boldsymbol{k}_B}}{\sqrt{(2E_A)(2E_B)}} |\boldsymbol{k}_A\boldsymbol{k}_B\rangle_{\text{in}}. \tag{2.96}$$

Analogously we define

$$_{\text{out}}\langle\phi_1\phi_2\ldots| = \left(\prod_f \int \frac{\mathrm{d}^3\boldsymbol{p}_f}{(2\pi^3)} \frac{\phi_f(\boldsymbol{p}_f)}{\sqrt{2E_f}}\right) {}_{\text{out}}\langle\boldsymbol{p}_2\boldsymbol{p}_2\cdots| \tag{2.97}$$

for the *out* states. To combine these states we first have to calculate the matrix element $_{\text{out}}\langle\boldsymbol{p}_1\boldsymbol{p}_2\ldots|\boldsymbol{k}_A\boldsymbol{k}_B\rangle_{\text{in}}$ which gives the probability that the initial state $|\boldsymbol{k}_A\boldsymbol{k}_B\rangle_{\text{in}}$ is migrated to the final state $_{\text{out}}\langle\boldsymbol{p}_1\boldsymbol{p}_2\ldots|$. Therefore we take $|\boldsymbol{k}_A\boldsymbol{k}_B\rangle_{\text{in}}$ at the time $-T$ while we take $_{\text{out}}\langle\boldsymbol{p}_1\boldsymbol{p}_2\ldots|$ at the time $+T$ for $T\to\infty$. The two pieces can then be connected by the time evolution operator $e^{-iH(2T)}$, with which we transform the *out* state to an *in* state. The matrix element is then given by

$$\begin{aligned} _{\text{out}}\langle p_1p_2\ldots|k_1k_2\rangle_{\text{in}} &= \lim_{T\to\infty}\langle\underbrace{p_1p_2\ldots}_{T}|\underbrace{k_1k_2}_{-T}\rangle \\ &= \lim_{T\to\infty}\langle p_1p_2\ldots|e^{-iH(2T)}|k_1k_2\rangle \\ &\equiv \langle p_1p_2\ldots|S|k_1k_2\rangle\,. \end{aligned} \tag{2.98}$$

After this any reference time can be chosen, since the matrix element only depends on the time difference of the states. Thus the S-matrix is defined by the time evolution operator and describes how the initial and final states which live in different times, are connected to each other by the scattering. Independent of the initial and final states, we always have four momentum conservation. Therefore we always get a factor of $\delta^{(4)}(k_1+k_2-\sum p_f)$ which can be extracted from the matrix element to get the invariant matrix element $\mathcal{M}$ defined by

$$\langle p_1p_2...|iT|k_1k_2\rangle = (2\pi)^4\,\delta^{(4)}\left(k_1+k_2-\sum p_f\right)\cdot i\mathcal{M}\left(k_1,k_2\to p_1p_2...\right). \tag{2.99}$$

Here we additionally used that the S-matrix can be written as

$$S = \mathbb{1} + iT, \tag{2.100}$$

where the $\mathbb{1}$ reflects the part of the operator belonging to the probability that the particle does not interact, while the iT reflects the interaction part of the S-matrix. In the following we will ignore the $\mathbb{1}$ -part of the S-matrix, since we are not interested in the trivial case of forward scattering, where no interaction takes place. For the case of forward scattering equation 2.99 simplifies to

$$\langle p_1 p_2 \ldots | iT | k_1 k_2 \rangle = (2\pi)^4 \, i\mathcal{M} \left(k_1 k_2 \to k_1 k_2 \right), \tag{2.101}$$

where we have defined the invariant matrix element $\mathcal{M}$. The probability for the initial state $|\phi_A \phi_B\rangle$ to scatter and become the final state $|p_1 p_2 \ldots\rangle$ of n particles is given by

$$\mathcal{P}(A\,B \to 1\,2\ldots) = \left(\prod_{f=1}^{n} \int \frac{\mathrm{d}^3 \boldsymbol{p}_f}{(2\pi)^3} \frac{1}{2E_f} \right) |_{out}\langle p_1 p_2 \ldots | \phi_A \phi_B \rangle_{in}|^2 \,, \tag{2.102}$$

where we used that the momenta of the final particles lie in the small region $d^3p_1 \ldots d^3p_n$. Furthermore, we used the normalization (2.94). In the following we will assume a single target (A) and many incident particles (B) with different impact parameters $\boldsymbol{b}$. Thus the number of scattering events is given by

$$N = \sum_i \mathcal{P}_i = \int \mathrm{d}^2 b \, n_B \, \mathcal{P}(\boldsymbol{b}) \,, \tag{2.103}$$

where we sum over all incident particles i. Here we additionally introduced the particle number density n_B of the incident particles B which describes the number of particles per unit area within the plane defined by the impact parameters $\boldsymbol{b}$. Here we assume that the number density n_B is constant over the range of the interaction. Therefore it can be taken out of the integral contained in N providing us with the cross section

$$\sigma = \frac{N}{n_B N_A} = \frac{N}{n_B \cdot 1} = \int \mathrm{d}^2 \mathcal{P}(\boldsymbol{b}) \,. \tag{2.104}$$

Combining everything the formula for the differential cross section turns out to be

$$\begin{aligned} \mathrm{d}\sigma = \left(\prod_{f=1}^{n} \frac{\mathrm{d}^3 \boldsymbol{p}_f}{(2\pi)^3} \frac{1}{2E_f} \right) \int \mathrm{d}^2 b \left(\prod_{i=A,B} \int \frac{\mathrm{d}^3 \boldsymbol{k}_i}{(2\pi)^3} \frac{\phi_i(\boldsymbol{k}_1)}{\sqrt{2E_i}} \int \frac{\mathrm{d}^3 \bar{\boldsymbol{k}}_i}{(2\pi)^3} \frac{\phi_i^*(\bar{\boldsymbol{k}}_1)}{\sqrt{2\bar{E}_i}} \right) \\ \times e^{i\boldsymbol{b}\cdot(\bar{\boldsymbol{k}}_B - \boldsymbol{k}_B)} \left(_{out}\langle \{\boldsymbol{p}_f\} | \{\boldsymbol{k}_i\} \rangle_{in} \right) \left(_{out}\langle \{\boldsymbol{p}_f\} | \{\bar{\boldsymbol{k}}_i\} \rangle_{in} \right)^* \,, \end{aligned} \tag{2.105}$$

where we have used $\bar{k}_A$ and $\bar{k}_B$ as dummy integration variables in the second half of the squared amplitude. The $\mathrm{d}^2 b$ integral can be performed to give a factor $(2\pi)^2 \delta^{(2)}(k_B^\perp - \bar{k}_B^\perp)$. Additionally using (2.101) to replace the matrix elements by $\mathcal{M}$ leads us to

$$\left(_{out}\langle \{\boldsymbol{p}_f\} | \{\boldsymbol{k}_i\} \rangle_{in} \right) = i\mathcal{M}(\{k_i\} \to \{p_f\})(2\pi)^4 \delta^{(4)}\Big(\sum k_i - \sum P_f \Big) \tag{2.106}$$

$$\left(_{out}\langle \{\boldsymbol{p}_f\} | \{\bar{\boldsymbol{k}}_i\} \rangle_{in} \right)^* = -i\mathcal{M}(\{\bar{k}_i\} \to \{p_f\})(2\pi)^4 \delta^{(4)}\Big(\sum \bar{k}_i - \sum P_f \Big) \,. \tag{2.107}$$

The delta function in the second relation can be used together with $\delta^{(2)}(k_B^\perp - \bar{k}_B^\perp)$ to perform all of the $\bar{k}$ integrals in (2.105). Of the six integrals only those over $\bar{k}_A^z$ and $\bar{k}_B^z$ are nontrivial.

We find

$$\begin{aligned}
&\int \mathrm{d}\bar{k}_A^z \mathrm{d}\bar{k}_B^z \delta\left(\bar{k}_A^z + \bar{k}_B^z - \sum p_f^z\right) \delta\left(\bar{E}_A + \bar{E}_B - \sum E_f\right) \\
&\qquad = \int \mathrm{d}\bar{k}_A^z \delta\left(\sqrt{\bar{k}_A^2 + m_A^2} + \sqrt{\bar{k}_B^2 + m_B^2} - \sum E_f\right)\Bigg|_{\bar{k}_B^z = \sum p_f^z - \bar{k}_A^z} \\
&\qquad = \frac{1}{\left|\frac{\bar{k}_A^z}{\bar{E}_A} - \frac{\bar{k}_B^z}{\bar{E}_B}\right|} = \frac{1}{|v_A - v_B|}
\end{aligned} \tag{2.108}$$

The integrations deliver the constraints $\bar{k}_A^z + \bar{k}_B^z = \sum p_f^z$ and $\bar{E}_A + \bar{E}_B = \sum E_f$ coming from the z integral above as well as $\bar{k}_A^\perp = k_A^\perp$ and $\bar{k}_B^\perp = k_B^\perp$ implied by the other four integrals. $|v_A - v_B|$ denotes the relative velocity of the particles. As the initial wave packets are localized in momentum space and centered on $\boldsymbol{p}_A$ and $\boldsymbol{p}_B$, we can evaluate all factors that are smooth functions of $\boldsymbol{k}_A$ and $\boldsymbol{k}_B$ at $\boldsymbol{p}_A$ and $\boldsymbol{p}_B$ and pull them out of the integral. Besides the delta functions nearly all quantities including $E_A, E_B, |v_A - v_B|$ and $\mathcal{M}$ can be treated this way. Thus we obtain the expression

$$\begin{aligned}
\mathrm{d}\sigma = &\left(\prod_f \frac{\mathrm{d}^3\boldsymbol{p}_f}{(2\pi)^3}\frac{1}{E_f}\right) \frac{|\mathcal{M}(p_A, p_B \to \{p_f\})|^2}{2E_A\, 2E_B |v_A - v_B|} \\
&\times \int \frac{\mathrm{d}^3\boldsymbol{k}_A}{(2\pi)^3} \int \frac{\mathrm{d}^3\boldsymbol{k}_B}{(2\pi)^3} |\phi_A(\boldsymbol{k}_A)|^2 |\phi_B(\boldsymbol{k}_B)|^2 (2\pi)^2 \delta^{(4)}\left(k_A + k_B - \sum p_f\right).
\end{aligned} \tag{2.109}$$

Up to now this formula does not include any inputs resulting of the properties of a real detector. All detectors have a finite resolution. Even if one would consider an ideal detector, we would have to use a certain binning to prevent the detector from always giving a flat distribution[3]. Thus detectors sum incoherently over momentum bites of finite size. Normally the measurement of the final-state momentum in not of such high quality that it can resolve the small variation of the momentum that results from the momentum spread of the initial wave packets ϕ_A and ϕ_B. In this case even the momentum vector $k_A + k_B$ inside the delta function can be well approximated by its central value $p_A + p_B$. With this approximation we can perform the integrals over k_A and k_B using the normalization condition (2.94). Finally we get the relation

$$\mathrm{d}\sigma = \frac{1}{2E_A E_B |v_A - v_B|} \left(\prod_f \frac{\mathrm{d}^3\boldsymbol{p}_f}{(2\pi)^3}\frac{1}{E_f}\right) |\mathcal{M}(p_A, p_B \to \{p_f\})|^2 (2\pi)^4 \delta^{(4)}\left(p_A + p_B - \sum p_f\right) \tag{2.110}$$

between the differential cross section and the invariant matrix element $\mathcal{M}$. In this work we are interested in differential decay rates rather than cross sections. To obtain a differential decay rate we cannot simply set the number of particles in the initial state to one, as it would make no sense to regard an unstable particle for $T \to \pm\infty$. However, it is possible to connect the differential cross section of a forward scattering to a decay rate by the LSZ Formula. Since the construction of the LSZ formula is beyond the scope of this work, we will just present the formula

$$\mathrm{d}\Gamma = \frac{1}{2E_A} \left(\prod_f \frac{\mathrm{d}^3\boldsymbol{p}_f}{(2\pi)^3}\frac{1}{E_f}\right) |\mathcal{M}(p_A \to \{p_f\})|^2 (2\pi)^4 \delta^{(4)}\left(p_A - \sum p_f\right), \tag{2.111}$$

[3] This would of course happen since it is very unlikely to get exactly the same energy for two particles twice, if one has to consider the full spectrum of real numbers, not to mention a distribution of particles at all.

for the differential decay rate here. A more detailed description of the LSZ formalism and the derivation of the differential decay rate can be found in chapter 7 of [9]. The optical theorem is a straightforward consequence of the unitarity

$$S^\dagger S = e^{iH^\dagger 2T} e^{-iH2T} = 1 \tag{2.112}$$

of the S-matrix. The combination of the equations (2.100) and (2.112) leads to the relation

$$-\, i \left(T - T^\dagger \right) = T^\dagger T. \tag{2.113}$$

Taking the matrix element between the initial state $|k_1 k_2\rangle$ on both sides of this equation and inserting a complete set of intermediate states

$$\sum_n \left(\prod_{i=1}^{n} \int \frac{\mathrm{d}^3 \boldsymbol{p}_i}{(2\pi)^3} \frac{1}{2E_i} \right) |\{p_f\}\rangle\langle\{p_f\}| \tag{2.114}$$

on its right-hand side we obtain

$$-\, i \left(\langle k_1 k_2 | T | k_1 k_2 \rangle - \langle k_1 k_2 | T^\dagger | k_1 k_2 \rangle \right) = \sum_f \int \mathrm{d}\Pi_f \langle k_1 k_2 | T^\dagger | \{p_f\}\rangle\langle\{p_f\}| T | k_1 k_2 \rangle, \tag{2.115}$$

where $\int \mathrm{d}\Pi_f$ denotes the phase space of the intermediate states f. Under usage of the definition (2.101) of the invariant matrix element this can be rewritten to

$$\begin{aligned} &-i \left(\mathcal{M}(k_1 k_2 \to k_1 k_2) - \mathcal{M}^*(k_1 k_2 \to k_1 k_2) \right) \\ &\quad = \sum_f \int \mathrm{d}\Pi_f \mathcal{M}^*(\{p_f\} \to k_1 k_2) \mathcal{M}(k_1 k_2 \to \{p_f\}) \times \delta^{(4)} \left(k_1 k_2 - \sum_f p_f \right). \end{aligned} \tag{2.116}$$

Since, for the case of forward scattering, no new particles can be created, we can set $k_i = p_i$. The left hand side is obviously the imaginary part of the invariant matrix element, while the right hand side is proportional to the total cross section. Introducing the energy E_{cm} and momentum p_{cm} of either particle in the center-of-mass frame we obtain the relation

$$\operatorname{Im} \mathcal{M}(k_1 k_2 \to k_1 k_2) = 2\, E_{cm}\, p_{cm}\, \sigma_{tot}(k_1 k_2 \to \text{anything}), \tag{2.117}$$

which is called the optical theorem. It relates the forward scattering amplitude to the total cross section for production of all final states.

3. Effective field theories

3.1. What are effective field theories?

While describing a physical system we normally focus on the degrees of freedom which are relevant at the distance scales under considerations. As an example imagine the description of the solar system by classical mechanics. Instead of calculating all inertia tensors for all objects and considering every small satellite of every planet the planets are usually approximated by point masses and every item that is to small to matter is ignored. In this way the planets orbits can be calculated with relative ease, as all the uninteresting degrees of freedom that needlessly complicate the calculations do not have to be handeled. Going one step further the calculations could even be performed in a general relativistic context or, even worse, by quantum mechanical considerations. In this way classical mechanics are indeed an effective theory for quantum mechanics (when $\hbar \to 0$) and general relativity (when $c \to \infty$).

In particle physics the way of constructing an effective field theory [41, 42, 43, 44] depends on distance or energy scales. Analog to the picture of the solar system we can observe the nucleus of an atom. Even if we know that the nuclei are composed of a rather complicated structure of quarks and additional virtual particles the appropriate degrees of freedom in nuclear physics are those of the nucleons, while the quark structure becomes relevant only at much smaller distances or higher energies respectively. To construct an effective field theory in elementary particle physics, we always need disparate mass scales, where some of the degrees of freedom become relevant only at much larger energy scales. As an example we can take a heavy particle which cannot be created at an energy scale smaller than its mass. For the decays of bottom quarks which are analyzed in this work, this means that for example a top quark is a heavy degree of freedom that can be neglected since the top mass is larger than the energy scale of the decay process. This is of course also valid for the other heavy degrees of freedom which appear in the standard model at much higher scales (e.g. the weak bosons at $\mathcal{O}(100\,\mathrm{GeV})$) as the scale of the b quark which is of order $m_b \approx 5\,\mathrm{GeV}$. Consequently the Lagrangian for the description of the b-decay does not have to contain these degrees of freedom. Commonly the fact that it is possible to create a Lagrange density for low energy scales which does not contain degrees of freedom needed for higher energy scales only, is ensured by the decoupling theorem proved by Appelquist and Carazzone [45]. This theorem declares that the heavy degrees of freedom decouple at energy scales much lower than their mass with very few exceptions. In this case decoupling means that any effect of these degrees of freedom is (up to logarithmic contributions) suppressed by inverse powers of the heavy scale.

So, how can we construct an effective field theory for decays at quark level? The starting point is, as discussed above, the presence of a large scale Λ which usually presents the mass of a heavy particle. In the case of weak interactions of hadrons this scale is usually given by the mass M_W of the weak charged boson. The decay is described by a transition matrix element from an initial state $|i\rangle$ and a final state $|f\rangle$ in a theory containing the full set of degrees of freedom. If the states involve only energies below the heavy scale Λ, we can introduce a local effective Hamiltonian, since all effects of interactions above the scale Λ appear local in the typical scales of the states $|i\rangle$ and $|f\rangle$. The matrix element of the local effective Hamiltonian can be written as a series [46]

$$\langle f|\mathcal{H}_{\text{eff}}|i\rangle = \sum_k C_k(\Lambda)\,\langle f|\hat{O}_k|i\rangle\Big|_\Lambda \tag{3.1}$$

of matrix elements of local operators $\langle f|\hat{O}_k|i\rangle|_\Lambda$ and the coefficients $C_k(\Lambda)$. The local operators contain the long distance contributions at scales below Λ, while the coefficients contain the short distance contributions above Λ. This seems natural as the light degrees of freedom are still fully contained in the Hamiltonian, while the heavy degrees of freedom are not contained explicitly in the theory any more. This means that the heavy degrees of freedom are hidden in the coefficients $C_k(\Lambda)$ which are of course only valid at the scale Λ. However, we will see later that the coefficients can be rescaled to fit calculations at other scales, but of course only up to a point where additional degrees of freedom appear. The calculation of the operators is usually performed by a procedure called *matching*. This can be done by calculating the amplitude $A = \langle f|\mathcal{H}_{\text{eff}}|i\rangle$ to the same order in α_s as the matrix elements $\langle f|\hat{O}_k|i\rangle$ of the operators. The coefficients $C_k(\Lambda)$ can then be obtained by comparison. Since the sum (3.1) in general runs over an infinite set of operators, it is only useful if we can truncate the sum. However, for practical reasons it may be appropriate to just leave the operators coefficients uncalculated and use them as input parameters to the effective theory. In this way the parameters have to be extracted from experiment.

Since the effective Hamiltonian is a density it has the mass dimension 4. This also means that every term in (3.1) has to be of dimension 4. Because of the fact that long and short distances factorize, the short distance coefficients $C_k(\Lambda)$ do not depend on the long distance scale. Therefore the mass dimension of these terms depends only on powers of the large scale Λ. In order to simplify the power counting of powers in $1/\Lambda$ it is convenient to factor out the appropriate powers of $1/\Lambda$ and make the coefficient dimensionless. Thus the effective Hamilton density can be written as

$$\langle f|\mathcal{H}_{\text{eff}}|i\rangle = \sum_k \frac{1}{\Lambda^k} \sum_j c_{k,j}\,\langle f|\hat{O}_{k,j}|i\rangle\Big|_\Lambda, \tag{3.2}$$

where k denotes the dimension. For a fixed dimension k we have to take into account that more than one operator can contribute. Hence we still have to sum over j which runs from 0 to the number of operators for a certain dimension k. Considering naive dimensional arguments only, we find that the coefficients $c_{k,j}$ are dimensionless and therefore cannot depend on Λ. However, since in a renormalizable theory the coupling constant, although dimensionless, depends on a dimensional quantity, namely the scale of the observed process, the constants also depend on the scale Λ. In this work we will use the expansion of the effective Hamiltonian in multiple ways. In chapter 4 we will use it to extend the standard model by new operators to perform a test on the standard model currents. Additionally we use it to expand the charged weak propagator with which we are confronted in this section in terms of local operators which are much easier to handle. The third part where we use the expansion is described in chapter 5, in which we describe the decay of a bottom quark to a charm quark within the background field of a B meson. In this case the expansion describes the nonperturbative interactions with the other constituents contained in the meson.

3.2. Renormalization of effective theories

In this section we will describe how to renormalize an effective field theory. The starting point is the relation (3.2) which shows the expansion of the effective Hamilton density for a certain energy scale Λ which separates the short distance parts in the Wilson coefficients $c_{k,j}$ from the long distance parts contained in the matrix elements $\langle f|\hat{O}_{k,j}|i\rangle$. The whole expansion would be extremely unpractical if we had to perform it for every value of Λ on its own. Fortunately there is the possibility to rescale the expansion to another energy scale μ, at least in the region where we are not confronted with new degrees of freedom which appear at higher scales in the full theory. The scale parameter μ is arbitrary, but always smaller than the original heavy scale Λ where a new degree of freedom appears. Hence for our new scale $\mu \leq \Lambda$ all contributions to a matrix element above μ can be called short-distance pieces, while anything below μ belongs to the long-distance part. Applying the same arguments for the new scale μ ads for Λ before, equation (3.2) becomes

$$\langle f|\mathcal{H}_{\text{eff}}|i\rangle = \sum_j \frac{1}{\Lambda^j} \sum_k c_{k,j}(\Lambda/\mu)\langle f|\hat{O}_{k,j}|i\rangle\Big|_\mu , \tag{3.3}$$

where again the dimensionless coefficients $c_{k,j}(\Lambda/\mu)$ contain the short-distance contributions and the matrix elements $\langle f|\hat{O}_{k,j}|i\rangle|_\mu$ contain the long-distance contributions. Note that these quantities now both depend on the scale parameter μ. This implies that we can move contributions from the matrix element to the coefficients and vice versa by changing the scale μ. The renormalizability of the dimension 4 part of the effective theory ensures that no power corrections of order $1/\mu$ can appear. However, since we have introduced an arbitrary scale which we can change as desired for our calculations, the physical quantity $\langle f|\mathcal{H}_{\text{eff}}|i\rangle$ may not depend on it. This leads us to the renormalization group equations which describe the behavior of the effective theory under renormalization. Besides the fact that the change of the scale from μ to μ' shifts parts from the matrix elements to the operators or the other way round, it also induces a mixing between the operators. This means that the contribution of one matrix element $O_{k,j}$ turns under the change of the scale from μ to μ' into a sum of contributions from all the matrix elements which have the same quantum numbers as the original operator. For a description of this behavior the starting point is the derivative of the matrix element of the effective Hamiltonian by the scale μ

$$0 = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\langle f|\mathcal{H}_{\text{eff}}|i\rangle, \tag{3.4}$$

which has to vanish, as this matrix element must be μ independent. Now we can use (3.3) to expand the matrix element in (3.4). Since the relations we will obtain are valid seperately for each order of the $1/\Lambda$ expansion we can omit the sum over those orderings in the following. The fact that mixing occurs only with operators of the same mass dimension is again caused by dimensional regularization [34, 35]. Thus operators with the same mass dimension form a basis closed under renormalization. The expansion of (3.4) gives

$$0 = \sum_k \left[\left(\mu\frac{\mathrm{d}}{\mathrm{d}\mu} c_k(\Lambda/\mu)\right) \langle f|\hat{O}_k|i\rangle\Big|_\mu + c_k(\Lambda/\mu) \left(\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\langle f|\hat{O}_k|i\rangle\Big|_\mu\right)\right]. \tag{3.5}$$

Because of mixing the operator $O_{k,j}$ turns into a linear combination of operators of the same mass dimension

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\langle f|\hat{O}_k|i\rangle\Big|_\mu = \sum_j \gamma_{kj}(\mu)\langle f|\hat{O}_j|i\rangle\Big|_\mu . \tag{3.6}$$

for an infinitesimal change in $\log\mu$. The matrix $\gamma(\mu)$ is called the anomalous-dimension matrix which describes the mixing of the operators in dependence of the scale μ. Inserting this into equation (3.5) we get

$$\begin{aligned} 0 &= \sum_k \left(\mu \frac{\mathrm{d}}{\mathrm{d}\mu} c_k(\Lambda/\mu) \right) \langle f|\hat{O}_k|i\rangle \Big|_\mu + \sum_k \sum_j c_k(\Lambda/\mu) \gamma_{kj}(\mu) \langle f|\hat{O}_j|i\rangle \Big|_\mu \\ &= \sum_k \left(\delta_{kj}\, \mu \frac{\mathrm{d}}{\mathrm{d}\mu} c_j(\Lambda/\mu) \right) \langle f|\hat{O}_k|i\rangle \Big|_\mu + \sum_k \sum_j c_j(\Lambda/\mu) \gamma_{jk}(\mu) \langle f|\hat{O}_k|i\rangle \Big|_\mu \\ &= \sum_k \sum_j \left[\delta_{kj}\, \mu \frac{\mathrm{d}}{\mathrm{d}\mu} + \gamma^T_{kj} \right] c_j(\Lambda/\mu) \langle f|\hat{O}_k|i\rangle \Big|_\mu . \end{aligned} \tag{3.7}$$

Since the operators O_i form a basis for a fixed scale μ, none of the operators may be written as a linear combination of the others when a certain value of μ is chosen. Hence (3.7) is equivalent to the set of equations

$$\sum_j \left[\delta_{kj}\, \mu \frac{\mathrm{d}}{\mathrm{d}\mu} + \gamma^T_{kj}(\mu) \right] c_j(\Lambda/\mu) = 0 \tag{3.8}$$

for the coefficients. Although the anomalous dimension matrix is in a naive point of view a dimensionless quantity and therefore should not depend on the mass scale μ, in a renormalizable theory there is a hidden scale given by the scale dependence of the coupling constant. This implies that for observable quantities a change in the scale may be compensated by an appropriate change in the masses and coupling constants of the theory. In the following we shall consider the case of massless QCD which means that we will consider only the renormalization group flow induced by strong interactions. In this case only the strong coupling constant α_s changes with scale, while the running of the other coupling constants with scale is disregarded. In the following we will rewrite all our quantities in terms of α_s to be able to perform a perturbative expansion in powers of this quantity. The anomalous dimension depends on the scale exclusively through the strong coupling:

$$\gamma_{ij}(\mu) = \gamma_{ij}(\alpha_s(\mu)). \tag{3.9}$$

For the Wilson coefficients the case is a bit different, since they also have a direct dependence on the scale [14]:

$$c_j(\Lambda/\mu) = c_j(\Lambda/\mu, \alpha_s(\mu)). \tag{3.10}$$

The only quantity which still has to be rewritten in terms of the strong coupling is the total derivative by the scale. Therefore we introduce the beta function

$$\beta(\alpha_s(\mu)) = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \alpha_s(\mu), \tag{3.11}$$

which describes the change of the coupling constant with the scale. Using this we can rewrite the total derivative to

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} = \mu \frac{\partial}{\partial\mu} \beta(\alpha_s) + \frac{\partial}{\partial\alpha_s}. \tag{3.12}$$

All in all (3.8) takes the form

$$\sum_i \left[\delta_{ij} \left(\mu \frac{\partial}{\partial\mu} \beta(\alpha_s) + \frac{\partial}{\partial\alpha_s} \right) + \gamma^T_{ij}(\alpha_s) \right] c_j(\Lambda/\mu, \alpha_s) = 0. \tag{3.13}$$

The original idea was to move the effects above the scale Λ into the short-distance coefficients c_i. These coefficients are determined by a matching calculation which means that the effective theory is compared with the full theory at the scale Λ. The initial condition for the coefficients' renormalization group flow is therefore given at $\mu = \Lambda$, or in other words $c_j(\Lambda/\mu = 1, \alpha_s)$. For any practical application it is suitable to compute the coefficients c_j as well as the beta function in perturbation theory as a series in α_s. The expansion of the coefficients, the anomalous dimension and the beta function gives us

$$c_i(\lambda/\mu = 1, \alpha_s) = \sum_n a_i^{(n)} \left(\frac{\alpha_s}{4\pi}\right)^n \tag{3.14}$$

$$\beta(\alpha_s) = \alpha_s \sum_{n=0} \beta^{(n)} \left(\frac{\alpha_s}{4\pi}\right)^{n+1} \tag{3.15}$$

$$\gamma_{ij}(\alpha_s) = \sum_{n=0} \gamma_{ij}^{(n)} \left(\frac{\alpha_s}{4\pi}\right)^{n+1} . \tag{3.16}$$

Here we have taken into account that the first nonvanishing terms are of second order for the beta function and usually of first order for the anomalous dimension. Its origin has already been discussed in section 2.3. It reads

$$\beta_0 = \frac{1}{3}(33 - 2n_f), \tag{3.17}$$

where n_f denotes the number of active flavors which means the number of quarks with masses below the scale μ. Taking the perturbative expansion as an input for the renormalization group equations, the coefficients c_j may be computed at the lower scale $\mu \leq \Lambda$. From this we get the expansion

$$\begin{aligned} c_i(\Lambda/\mu, \alpha_s) = b_i^{(00)} & \\ & + b_i^{(11)} \left(\frac{\alpha_s}{4\pi}\right) \ln\frac{\Lambda}{\mu} + b_i^{10} \left(\frac{\alpha_s}{4\pi}\right) \\ & + b_i^{(22)} \left(\frac{\alpha_s}{4\pi}\right)^2 \ln^2\frac{\Lambda}{\mu} + b_i^{(21)} \left(\frac{\alpha_s}{4\pi}\right)^2 \ln\frac{\Lambda}{\mu} + b_i^{(20)} \left(\frac{\alpha_s}{4\pi}\right)^2 \\ & + b_i^{(33)} \left(\frac{\alpha_s}{4\pi}\right)^3 \ln^3\frac{\Lambda}{\mu} + b_i^{(32)} \left(\frac{\alpha_s}{4\pi}\right)^3 \ln^2\frac{\Lambda}{\mu} + b_i^{(31)} \left(\frac{\alpha_s}{4\pi}\right)^3 \ln\frac{\Lambda}{\mu} + \ldots , \end{aligned} \tag{3.18}$$

where the superscripts of the coefficients b_i denote the power of α_s and the power of the logarithm $\ln(\Lambda/\mu)$ respectively. For the special case of $\Lambda = \mu$ all the logarithms vanish and we obtain $b_i^{(n0)} = a_i^{(n)}$. As the perturbative expansion of the renormalization group functions β and γ_{ij} are known, we can resum the columns of 3.18 using the renormalization group equations. For the first nonvanishing terms $\beta^{(0)}$ and $\gamma_{ij}^{(0)}$, the solution of the renormalization group equations contains all orders of α_s. The resummation is done over all contributions with coefficients $b_{(k,i)}^{nn}$, i.e. all contributions where the power of the logarithm is equal to the power of α_s. This is called leading logarithmic approximation. For the case where only a single operator of dimension k appears, and therefore no mixing occurs, we can solve the renormalization group equations to obtain

$$c_i(\Lambda/\mu, \alpha_s(\Lambda)) = b_i^{(00)} \left(\frac{\alpha_s(\Lambda)}{\alpha_s(\mu)}\right)^{\frac{\gamma^{(0)}}{\beta^{(0)}}} . \tag{3.19}$$

In this case the matching has to be considered only at tree level which implies that only the coefficient $b_i^{(00)}$ is needed.

To be able to perform a renormalization of the effective theory we additionally have to regard it from the point of QCD radiative corrections which we have already discussed in section 2.3. Therefore we will again start from the expansion (3.3). In addition to the normal renormalization constants (2.75) which we obtain from the standard QCD radiative corrections we need an additional multiplicative operator renormalization

$$\hat{O}_i^{(0)} = Z_{ij}\hat{O}_j \tag{3.20}$$

to eliminate divergencies which are introduced by the effective theory. Again we have marked the unrenormalized quantities with the superscript 0. Installing additionally the renormalization of the quark fields we obtain

$$\hat{O}_i^{(0)} = Z_q^{-2} Z_{ij}\hat{O}_j \, . \tag{3.21}$$

Viewing the renormalization in a sightly different, but equivalent way, we can define the renormalization of the Wilson coefficients which would provide us with

$$C_i^{(0)} = Z_{ij}^c C_j \, . \tag{3.22}$$

Regarding the renormalization of one coefficient-operator pair we obtain

$$C_i^{(0)}\hat{O}_i(q^{(0)}) = Z_q^2 Z_{ij}^c C_j \hat{O}_i = C_i\hat{O}_i + (Z_q^2 Z_{ij}^c - \delta_{ij})C_j\hat{O}_i \, . \tag{3.23}$$

This means that the effective Hamiltonian 3.3 can be rewritten in terms renormalized couplings and fields ($C_i\hat{O}_i$) and some additional counterterms. The argument $q^{(0)}$ in the first term of the upper equation indicates that the interaction term $\hat{O}_i$ is composed of bare fields. Calculating the amplitude using the Hamiltonian 3.3 including the renormalization of the coefficients 3.23 we obtain the finite renormalized result

$$Z_q^2 Z_{ij}^c C_j \langle f|\hat{O}_i|i\rangle^{(0)} = C_j \langle f|\hat{O}_i|i\rangle \, . \tag{3.24}$$

Comparing with (3.21) we find

$$Z_{ij}^c = Z_{ij}^{-1} \, . \tag{3.25}$$

Thus the renormalization of operators is equivalent to the renormalization of the coupling constants. Like the anomalous dimension which we have introduced for the mass matrices in equation 2.81, we define the anomalous dimension introduced by the operator renormalization by

$$\gamma = Z^{-1}\frac{\mathrm{d}}{\mathrm{d}\ln(\mu)} Z \, , \tag{3.26}$$

which again reflects the fact that the unrenormalized Wilson coefficients $\boldsymbol{C}^{(0)} = Z_c\boldsymbol{C}$ are μ-independent. Using (3.25) we find the renormalization group equation

$$\frac{\mathrm{d}}{\mathrm{d}\ln(\mu)}\boldsymbol{C}(\mu) = \gamma^T(\alpha_s)\boldsymbol{C}(\mu) \, . \tag{3.27}$$

The solution of this equation can be written in terms of a μ-evolution matrix U as

$$\boldsymbol{C}(\mu) = U(\mu, M_W)\,\boldsymbol{C}(M_W) \, . \tag{3.28}$$

At this point we should mention that the μ-dependence of the couplings $\boldsymbol{C}$ completely vanishes if we completely neglect QCD loop corrections. This means that the anomalous scaling behavior for the dimensionless coefficients is a genuine quantum effect which distinguishes the quantum theory from classical theories. This is the reason, why the factor γ is called anomalous scale dimension (Compare (3.26) to an n-dimensional μ-dependent term μ^n which behaves like $\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\mu^n = n\mu^n$). For further purposes we define

$$\langle f|\hat{\boldsymbol{O}}|i\rangle^{(0)} = Z_q^{-2} Z \langle f|\hat{\boldsymbol{O}}|i\rangle = Z_{\mathrm{GF}}\langle f|\hat{\boldsymbol{O}}|i\rangle \tag{3.29}$$

as the renormalization constant for the Green functions $\langle f|\hat{\boldsymbol{O}}|i\rangle$.

In the MS ($\overline{MS}$) scheme the renormalization constants are chosen in a way that the pure pole divergencies $1/\epsilon^k$ ($D = 4 - 2\epsilon$) are subtracted (plus the Euler constant γ_E and the term $\ln(4\pi)$ which accompany the divergences, for the $\overline{MS}$ scheme). Since finite parts are not subtracted we can expand Z in powers of $1/\epsilon$ as follows:

$$Z = 1 + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k} Z_k(g_s)\,. \tag{3.30}$$

Using the definition (2.79) of the beta function we can derive the useful relation [47]

$$\gamma(g_s) = -2g_s^2 \frac{\partial Z_1(g_s)}{\partial g_s^2} = -2\alpha_s \frac{\partial Z_1(g_s)}{\partial \alpha_s} \tag{3.31}$$

Similar to (3.30) we can expand the renormalization constants for the quark fields Z_q and the greens functions Z_{GF}:

$$Z_q = 1 + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k} Z_{q,k}(g_s) \tag{3.32}$$

$$Z_{\mathrm{GF}} = 1 + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k} Z_{\mathrm{GF},k}(g_s)\,. \tag{3.33}$$

The renormalization constant Z_{GF} can be obtained immediately form the calculation of the unrenormalized Green functions (3.29). To be able to compute $\gamma(g)$ we still need $Z_1(g_s)$. Combining the upper relations (3.29) - (3.33), we obtain

$$Z_1 = 2Z_{q,1} + Z_{\mathrm{GF},1} \tag{3.34}$$

For next to leading order the renormalization constant for the Green functions reads

$$Z_{\mathrm{GF},1} = b_1 \frac{\alpha_s}{4\pi} + b_2 \frac{\alpha_s}{4\pi}^2\,. \tag{3.35}$$

Using the well known expression (2.87) as well as the relations (3.31), (3.34) and 3.35 we finally obtain the one-loop and two-loop anomalous dimension matrices

$$\gamma_{ij}^{(0)} = -2[2a_1\delta_{ij} + (b_1)_{ij}] \tag{3.36}$$

$$\gamma_{ij}^{(1)} = -2[2a_2\delta_{ij} + (b_2)_{ij}]\,. \tag{3.37}$$

These equations may be used as a receipt to extract the anomalous dimensions from the divergent parts of the unrenormalized Green functions.

3.3. Heavy quark effective theory

In this section we will shortly review the heavy quark effective theory (HQET) which describes decays of heavy quarks confined in the background field of a hadron with light spectator quarks. Here we will concentrate on the aspects of the HQET which are of importance for the further chapters, while a good review of it is given in [15]. As a starting point we shall remind that the Lagrange density (2.9) of the electroweak interaction describes only the decays of separate quarks rather than the one of hadrons. This arises from the fact that the quarks also underly the strong interaction. Because of this, the quarks couple to a complicated cloud of quarks, antiquarks and gluons in the hadron. All in all it would be very difficult to perform calculations in full QCD alongside the weak interactions with the full set of degrees of freedom. At the moment there is no known way to calculate such a system from first principles, at least not in a perturbative way. Therefore we will perform the calculations within an effective theory. As we already mentioned in section 2.3, in QCD the antiscreening mechanism dominates. Thus the effective coupling constant (2.89) which reads

$$\alpha_s(\Lambda^2) = \frac{12\pi}{(33 - 2n_f)\ln(\Lambda^2/\Lambda^2_{\mathrm{QCD}})} \tag{3.38}$$

at one-loop level, decreases at large scales Λ and the strong interactions therefore become weak at short distances. We used n_f to mark the number of flavors and introduced the scale parameter $\Lambda_{\mathrm{QCD}} \approx 0.2\,\mathrm{GeV}$ which separates the regions of large and small coupling. A sufficiently large quark mass m_Q allows us to perform a perturbative expansion in $\alpha_s(m_Q^2)$ because of the asymptotic freedom. Therefore it is natural to divide the quarks into two classes. When the mass of a quark m_Q is much larger than the scale Λ_{QCD}, it is called a heavy quark. Thus we classify u, d and s as light quarks and c, b and t as heavy quarks. The HQET is normally used to describe the decay of charm or bottom quarks in the context of a hadron since their masses are high enough. Of course the decay of top quarkscould also be described with it, but since the top quark normally decays before the process of hadronization occurs, it is perfectly described by the partonic decay contained in the electroweak Lagrangian. From $m_Q \gg \Lambda_{\mathrm{QCD}}$ we can also conclude that the Compton wavelength λ_Q is much smaller than the length scale $R_{\mathrm{had}} \sim 1/\Lambda_{\mathrm{QCD}} \sim 1\,\mathrm{fm}$ which determines the typical size of a hadron. As the soft gluons which couple to the background field, can only resolve distances much larger than λ_Q, the light degrees of freedom are therefore blind to the flavor and spin orientation of the heavy quark. They are affected only by the color field which extends over large distances because of confinement. Considering the rest frame of the heavy quark, only the electric color field has in fact to be considered, as relativistic effects like color magnetism vanish for $m_Q \to \infty$. Thus also the heavy quark spin decouples from from the interactions, as it participates only through such relativistic effects. In addition to that, the heavy quark mass becomes irrelevant. The reason for that is the fact that the hadron and the heavy quark contained by it have the same velocity in the limit $m_Q \to \infty$. Thus in the hadron's rest frame the heavy quark is at rest, too. The description of the wave function of the background field follows from a solution of the field equations of QCD, where a static triplet color source at the heavy quark's location has been assumed as a boundary condition. This boundary condition is - since it is static - independent of m_Q, and so is the solution for the configuration of the light degrees of freedom. As all degrees of freedom except for the color decouple in the $m_Q \to \infty$ limit it actually does not matter which heavy quark is actually contained in the hadron. This fact is called "heavy quark symmetry".

The heavy quark symmetry is an approximate symmetry, since the heavy quark mass is actually not infinite. Therefore corrections of the order Λ_{QCD}/m_Q arise. The condition $m_Q \gg \Lambda_{\text{QCD}}$ is necessary and sufficient for a system containing a heavy quark to be close to the symmetry limit. This condition is complementary to the condition for the chiral symmetry which arises in the opposite limit $m_Q \ll \Lambda_{\text{QCD}}$ of small quark masses. However, there is a huge difference between the sources of the heavy quark and the chiral symmetry. While the chiral symmetry is a symmetry of the QCD Lagrangian in the limit of vanishing quark masses, the heavy quark symmetry is not even an approximate symmetry of the Lagrangian, but rather a good approximation to QCD in a certain kinematic region. It is realized only in systems in which a single heavy quark interacts by the exchange of soft gluons, and is therefore much less general than the chiral symmetry. While the heavy quark is static in the heavy quark limit $\Lambda_{\text{QCD}}/m_Q \to 0$, the corrections of order Λ_{QCD}/m_Q describe small fluctuations of the heavy quark around its mass shell. In the following we will show how these corrections are implemented into the theory by a small residual momentum describing the motion of the heavy quark inside of the hadron due to its interactions.

The starting point for our considerations is the QCD Lagrangian for heavy quarks

$$\mathcal{L}_h = \bar{h}\left(i\not{D} - m_h\right)h, \tag{3.39}$$

where h denotes a heavy quark singlet. We will now describe how the interactions with the background field are contained in this Lagrange density. Since the interactions of the heavy quark with the background field are small in comparison to the heavy quark mass m_h it is nearly on-shell and therefore moves in good approximation with hadron's velocity v. Hence we can split the heavy quark momentum p_h up into two parts

$$p_h^\mu = m_h v^\mu + k^\mu, \tag{3.40}$$

where $m_h v^\mu$ describes the motion of the heavy quark with the hadron, while the momentum k^μ is the residual momentum mentioned above describing the movement of the heavy quark inside the hadron. The velocity of the hadron satisfies

$$v = \frac{p_H}{m_H}; \qquad v^2 = 1, \qquad v_0 > 0, \tag{3.41}$$

where p_H and m_H are the hadrons momentum and velocity respectively. This means that the residual momentum is a measure for the off-shellness of the heavy quark and therefore also for the interactions with the background field. Note that this is a quite naive description. Later on it will turn out that we often have to see k as an operator rather than a velocity. As we are not interested in the movement of the hadron, but rather the movement of the heavy quark in the background field of the hadron, we will redefine the field h by the transformation

$$h_v = e^{im_b\, v\cdot x} h. \tag{3.42}$$

The phase factor $e^{im_b\, v\cdot x}$ just removes the momentum $m_h\, v$, but leaves the residual momentum k untouched. The result is a field h_v which only depends on the long distance modes with momenta of order Λ_{QCD} because of the small binding effects of k. Therefore all derivatives acting on h_v are of order Λ_{QCD} respectively. Another common step for HQET is the definition of the projectors

$$P_\pm = \frac{1 \pm \not{v}}{2}, \tag{3.43}$$

which projects the fields on their “large” and “small” components

$$h_v^+ = e^{im_h\, v\cdot x} P_v^+ h \tag{3.44}$$
$$h_v^- = e^{im_h\, v\cdot x} P_v^- h. \tag{3.45}$$

Note that the large and small components are marked by the superscripts + and − instead of the ordinary big and small h to avoid confusion with other quantities. Of course these projectors fulfill $P_\pm^2 = P_\pm$ and $P_\pm P_\mp = 0$. In the restframe of the B-meson, where $v = (1, 0, 0, 0)$ is valid, h_v^- corresponds to the upper and h_v^- to the lower components to the h quarks Dirac spinor. In this case h_v^+ represents a static color source of the heavy quark. In the following we will see that the antiquark modes h_v^- will appear only in higher order corrections. All in all the original heavy quark field looks like

$$h = e^{-im_h\, v\cdot x} \left(h_v^+ + h_v^-\right). \tag{3.46}$$

To describe antiquarks as a static color source in a hadron the corresponding relations are

$$\tilde{h}_v^+ = e^{im_h\, v\cdot x} P_v^- h \tag{3.47}$$
$$\tilde{h}_v^- = e^{im_h\, v\cdot x} P_v^+ h, \tag{3.48}$$

where just the projection operators are exchanged. The corresponding original quark field then looks like

$$\tilde{h} = e^{im_b\, v\cdot x} \left(\tilde{h}_v^+ + \tilde{h}_v^-\right) = e^{im_b\, v\cdot x} \left(h_v^- + h_v^+\right). \tag{3.49}$$

Thus only the sign in the velocity transformation and the role of h_v^+ and h_v^- have been interchanged. In the following we will concentrate on heavy quarks rather than heavy antiquarks. Anyway, the calculations are exactly the same and can be retained by $v \to -v$ and $h_v^- \leftrightarrow h_v^+$. Using the redefinition (3.46) on the Lagrange density for the heavy quark (3.39) we obtain

$$\mathcal{L}_h = \left(\bar{h}_v^+ + \bar{h}_v^-\right) \left[i\not{D} - m_h(1 - \not{v})\right] \left(h_v^+ + h_v^-\right). \tag{3.50}$$

Since every consideration up to now has been done with the movement of the heavy quark inside the meson moving with velocity v, it is reasonable to split up the covariant derivatives into a longitudinal and transversal component to v:

$$D_\mu = v_\mu (v\cdot D) + D_\mu^\perp, \qquad D_\mu^\perp = (g_{\mu\nu} - v_\mu v_\nu) D^\nu, \qquad \{\not{D}^\perp, \not{v}\} = 0. \tag{3.51}$$

If we additionally consider the relations

$$P_+\gamma_\mu P_+ = v_\mu P_+ \quad \text{and} \quad P_-\gamma_\mu P_+ = (\gamma_\mu - v_\mu \not{v})P_+ \tag{3.52}$$

we obtain

$$\mathcal{L}_h = \bar{h}_v^+(iv\cdot D)h_v^+ + \bar{h}_v^-(-iv\cdot D - 2m_h)h_v^- + \bar{h}_v^- i\not{D}_\perp h_v^+ + \bar{h}_v^+ i\not{D}_\perp h_v^- \tag{3.53}$$

for the heavy quark Lagrangian. From this we can immediately extract that h_v^+ describes massless degrees of freedom, while h_v^- describes a state with mass $2m_h$ which is just the energy needed to create a quark-antiquark pair. Thus the h_v^- correspond to higher order corrections of h_v. This can be explicitly seen if we use the classical equations of motion to eliminate the h_v^- in the heavy quark Lagrangian. At first we solve

$$\frac{\partial}{\partial h_v^-} = (-iv\cdot D - 2m_h)h_v^- + i\not{D}_\perp h_v^- = 0 \tag{3.54}$$

to h_v^- and get the dependence

$$h_v^- = \frac{1}{iv\cdot D + 2m_h} i\not{D}_\perp h_v^+ \tag{3.55}$$

of the small and large components. Using this in expression (3.53) we obtain

$$\mathcal{L}_h = h_v^+ (iv\cdot D) h_v^+ + h_v^+ i\not{D}_\perp \frac{1}{iv\cdot D + 2m_h} i\not{D}_\perp h_v^+ . \tag{3.56}$$

This expression is still exact, but it is not local anymore. The nonlocality is based on the operator $1/(2m_h + iv\cdot D)$ which describes the propagator of a quark antiquark pair. Without the Compton wavelength $2m_h$ in the denominator of the propagator the inverse derivatives could be ascribed to local delta functions and their derivatives by Fourier transformation. Thus it seems obvious to perform an expansion according to the derivatives $iv\cdot D$, to present the propagator as a series of local operators. The initial quark fields therefore take the form

$$\begin{aligned} h\,(x) &= e^{-im_Q v\cdot x}\left[1 + \left(\frac{1}{2m_Q + iv\cdot D}\right) i\not{D}_\perp\right] h_v^+ \\ &= e^{-im_Q v\cdot x}\left[1 + \frac{1}{2m_Q}\not{D}_\perp + \left(\frac{1}{2m_Q}\right)^2 (-ivD)\,\not{D}_\perp + \dots\right] h_v^+ , \end{aligned} \tag{3.57}$$

while the Lagrange density looks like

$$\begin{aligned} \mathcal{L} &= \bar{h}_v^+ (ivD)\, h_v^+ + \bar{h}_v^+ i\not{D}_\perp \left(\frac{1}{2m_Q + ivD}\right) i\not{D}_\perp h_v^+ \\ &= \bar{h}_v^+ (ivD)\, h_v^+ + \frac{1}{2m_Q}\bar{h}_v^+ \left(i\not{D}_\perp\right)^2 h_v^+ + \left(\frac{1}{2m_Q}\right)^2 \bar{h}_v^+ \left(i\not{D}_\perp\right)(-ivD)\left(i\not{D}_\perp\right) h_v^+ + \dots . \end{aligned} \tag{3.58}$$

Thus we have an expansion where every higher order is suppressed by $1/m_h$ in comparison with its predecessor. The best results for the expansion in $m_h \gg \Lambda_{\mathrm{QCD}}$ obviously can be obtained for $m_h \to \infty$ which is called the heavy quark limit. In this limit only the first term of the expansion survives. As it is completely mass independent two further symmetries which are not present in QCD show up in this limit. These are the heavy-flavor symmetry which implies the flavor independence of quark-gluon interactions, and the heavy-flavor spin symmetry which indicates that both spin degrees of freedom couple to the gluons in the same way leading to a symmetry of the Lagrangian under the rotations of the heavy-quark spin. However, the discussion of these symmetries is beyond the scope of this work, especially since the higher orders of the expansion break them. Rather we shall show considerations to the dependence of the heavy quark mass and the meson mass up to order $1/m_h$. Therefore we have to rewrite the second order terms of the expansion (3.58) by the relation

$$\begin{aligned} \bar{h}_v^+ i\not{D}_\perp i\not{D}_\perp h_v^+ =& \bar{h}_v^+ iD_\perp^\mu iD_\perp^\nu \gamma_\mu\gamma_\nu h_v^+ \\ =& \bar{h}_v^+ iD_\perp^\mu iD_\perp^\nu \,\frac{1}{2}\Big(\{\gamma_\mu,\gamma_\nu\} + [\gamma_\mu,\gamma_\nu]\Big) h_v^+ \\ =& \bar{h}_v^+ iD_\perp^\mu iD_\perp^\nu \,\frac{1}{2}\Big(g_{\mu\nu} - i\sigma_{\mu\nu}\Big) h_v^+ \\ =& \bar{h}_v^+ \left[(iD_\perp)^2 - i\sigma_{\mu\nu}\, iD_\perp^\mu iD_\perp^\nu h_v^+\right] h_v^+ \\ =& \bar{h}_v^+ \left[(iD_\perp)^2 + \frac{g}{2}\sigma_{\mu\nu}G^{\mu\nu}\right] h_v^+ , \end{aligned} \tag{3.59}$$

where $G^{\mu\nu} = -ig[iD_\mu, iD_\nu]/2$ is the QCD field strength tensor that we have already defined in section 2.3. Note that we can replace the $iD^\mu_\perp$ in the second term by the standard covariant derivatives iD^μ since $\bar{h}_v^+ \sigma_{\mu\nu} v^\nu h_v^+ = 0$. Using 3.59 in 3.58 and neglecting the higher order terms gives us the Lagrange density

$$\mathcal{L} = \bar{h}_v^+ (ivD) h_v^+ + \frac{1}{2m_h} \bar{h}_v^+ (iD_\perp)^2 h_v^+ + \frac{g}{4m_h} \bar{h}_v^+ \sigma_{\mu\nu} G^{\mu\nu} h_v^+ \tag{3.60}$$

The first term of the $1/m_h$ corrections can be interpreted as the kinetic energy of the heavy quark with respect to the rest frame of the hadron. Since it explicitly contains the quark mass it breaks the flavor symmetry of the heavy quarks contained in the lowest order of the expansion. The second term is the chromomagnetic moment of the heavy quark. Besides its explicit mass dependence it also depends on the quark spin and therefore breaks the heavy-quark spin symmetry as well as the heavy-quark flavor symmetry. In the following we will discuss the expectation values of the terms contained in (3.60). Therefore we have, of course, to introduce the HQET states at first. In QCD a hadronic state is usually normalized by

$$\langle H(p')|H(p)\rangle = 2E_{\mathbf{p}} \, (2\pi)^3 \, \delta^3 \left(\mathbf{p} - \mathbf{p}'\right) \tag{3.61}$$

The relativistic energy $E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m_H^2}$ thereby implements the Lorentz invariance which is sacrificed in HQET since one explicitly uses the rest frame of the hadron. In the heavy-quark limit we additionally encounter the problem of infinite normalization caused by the no longer finite relativistic energy $E_{\boldsymbol{p}} = m_H \, v_0$. This problem is solved by eliminating the hadron mass from the normalization. Thus we get

$$\langle H_{v'}(k')|H_v(k)\rangle = 2 \, v_0 \, \delta_{v'v} \, \delta \left(\mathbf{k} - \mathbf{k}'\right) \tag{3.62}$$

for the heavy quark, where v_0 is the zeroth component of the hadrons velocity v. As we perform our calculations in the rest frame of the hadron we consider only the changes in the residual momentum $\mathbf{k}$, while the velocity $\mathbf{v}$ is removed from our considerations. Thus only the residual momentum remains in the delta distribution. To ensure that only hadron states in one rest frame appear we added an additional $\delta_{v'v}$. Altogether the normalization of the HQET differs from the normalization of the standard QCD mainly in a factor $\sqrt{m_H}$:

$$|H(p)\rangle = \sqrt{m_H} \Big[|H_v(k)\rangle + \mathcal{O}\left(1/m_Q\right) \Big]. \tag{3.63}$$

Now the states obviously have a different mass dimension which has to be compensated somehow. Therefore we take a look at the Dirac spinors of the full QCD which fulfill the condition

$$\bar{u}\left(p, s'\right) \gamma_\mu u\left(p, s\right) = 2p_\mu \delta_{s's} \, , \tag{3.64}$$

while the ones of HQET obey

$$\bar{u}\left(v, s'\right) \gamma_\mu u\left(v, s\right) = 2v_\mu \delta_{s's}. \tag{3.65}$$

respectively. We immediately see that the Dirac spinors also differ from the ones of full QCD by a factor of $\sqrt{m_H}$:

$$u\left(p, s\right) = \sqrt{m_H} \, u\left(v, s\right). \tag{3.66}$$

This compensates the different masses of the states. Having defined the normalization of the HQET states, we can now continue with the derivation of a relation between the hadron masses and the quark masses. The hadron mass in the HQET is given by $\bar{\Lambda} = m_H - m_h$, as the heavy quark mass and kinetic energy have been subtracted from all energies in the field redefinition (3.46). At order m_h all hadrons containing a certain heavy quark h are degenerate and have the same mass m_h. The mass differences are induced by the terms proportional to $1/m_b$ contained in (3.60) which introduce the dependence on the spin and the heavy quark mass to the Lagrangian. Thus we can derive the formula

$$m_H = m_h + \bar{\Lambda} + \frac{3\mu_\pi^2 + d_H \mu_G^2}{6m_b} \tag{3.67}$$

from (3.60) describing the dependence of the meson mass and the heavy-quark mass. Here we introduced the nonperturbative HQET parameters

$$\bar{\Lambda} = -\frac{1}{2}\langle H_v|\bar{h}_v^+\,(ivD)\,h_v^+|H_v\rangle \tag{3.68}$$

$$\mu_\pi^2 = -\frac{\langle H_v|\bar{h}_v^+(iD_\perp)^2 h_v^+|H_v\rangle}{2\,m_H} \tag{3.69}$$

$$d_H\,\mu_G^2 = \frac{3\,\langle H_v|\bar{h}_v^+\sigma_{\mu\nu}\,iD^\mu\,iD^\nu h_v^+|H_v\rangle}{2\,m_H}. \tag{3.70}$$

whose values have to be extracted from experiment. $\bar{\Lambda}$ contains the static energy of the heavy-quark field as well as the light degrees of freedom. The factor $1/2$ arrises from the normalization introduced above. The dependence on the heavy quark mass enters by the parameter μ_π^2, while the parameter μ_G^2 additionally introduces the spin dependence to the Lagrangian. This spin dependence is described by the factor $d_H = (\boldsymbol{S}_h \cdot \boldsymbol{S}_l)$, where $\boldsymbol{S}_h$ and $\boldsymbol{S}_l$ are the spins of the heavy quark and the light degrees of freedom respectively. They have been introduced as the chromomagnetic operator is not invariant under spin transformations. Using the total spin of the meson $\boldsymbol{J} = \boldsymbol{S}_h + \boldsymbol{S}_l$, we can express the product of the spins by

$$\mathbf{S}_h \cdot \mathbf{S}_l = \frac{1}{2}\left(\mathbf{J}^2 - \mathbf{S}_h^2 - \mathbf{S}_l^2\right) \tag{3.71}$$

with

$$\mathbf{J}^2 = j\,(j+1)\,, \qquad \mathbf{S}_h^2 = s_q\,(s_q+1) \qquad \text{and} \qquad \mathbf{S}_l^2 = s_l\,(s_l+1)\,. \tag{3.72}$$

From the quark mass and the nonperturbative parameters we obtain the masses of the B-meson ($J^P = 0^-$) and the B^*-meson ($J^P = 1^-$) up to the order $1/m_b$:

$$m_B = m_b + \bar{\Lambda} + \frac{\mu_\pi^2 - \mu_G^2}{2m_b} \tag{3.73}$$

$$m_B^* = m_b + \bar{\Lambda} + \frac{3\mu_\pi^2 + \mu_G^2}{6m_b} \tag{3.74}$$

The parameters of HQET cannot be calculated from first principles, but have to be extracted by experiment. Anyhow, the parameters can be determined by model independent measurements. As an example we can easily calculate the parameter λ_2 when considering the mass difference

$$m_B^* - m_B = \frac{2\mu_\pi^2}{3m_b}. \tag{3.75}$$

of B and B^*. Replacing the b-quark mass by $2m_b \approx m_B^* + m_B$ obtained from the lowest order in the expansion we get

$$(m_B^* + m_B)(m_B^* - m_B) \approx (m_B^*)^2 - m_B^2 = \frac{4}{3}\mu_G^2, \tag{3.76}$$

which we can solve to

$$\mu_G^2 \approx \frac{3((m_B^*)^2 - m_B^2)}{4}. \tag{3.77}$$

Thus we obtain

$$\mu_G^2 \approx 0.37\,\mathrm{GeV}^2 \tag{3.78}$$

using the values $m_B = 5279.0 \pm 0.5\,\mathrm{MeV}$ and $m_B^* = 5325.0 \pm 0.6\,\mathrm{MeV}$ from [8].

3.4. Heavy quark expansion for inclusive decays

In this work we will deal with the inclusive decay $\bar{B} \to X_c\, e\, \bar{\nu}_e$. Inclusive decays of heavy hadrons can also be described using effective field theory methods [48, 49, 50, 51, 52, 53, 54, 55, 56, 57]. The method is set up in close analogy to deep inelastic scattering and relies on the operator product expansion [46]. The assumptions for the heavy quark expansion are the same as we have made for the heavy quark effective theory which has been discussed in the last section. However, we will not have to split the heavy quark up into small and large components. The starting point is the effective Hamilton density

$$\mathcal{H}_{\mathrm{eff}} = R\,h \tag{3.79}$$

for a transition in which the flavor changes by one unit. It contains a heavy quark field h and other field operators (e.g. light quarks, photons or leptons) included in R. The inclusive decay rate for a heavy hadron H containing the quark h can be related to a forward matrix element by the optical theorem. Summing over all final states $|X\rangle$ the inclusive decay rate is given by

$$\begin{aligned}\Gamma &\propto \sum_X (2\pi)^4 \delta^4(P_B - P_X) |\langle X|\mathcal{H}_{\mathrm{eff}}|H(v)\rangle|^2 \\ &= \int \mathrm{d}^4x \langle H(v)|\mathcal{H}_{\mathrm{eff}}^\dagger(x)\mathcal{H}_{\mathrm{eff}}(0)|H(v)\rangle \\ &= 2\,\mathrm{Im} \int \mathrm{d}^4x\, \langle H(v)|T[\mathcal{H}_{\mathrm{eff}}^\dagger(x)\mathcal{H}_{\mathrm{eff}}(0)]|H(v)\rangle.\end{aligned} \tag{3.80}$$

Like for HQET we will perform a field redefinition

$$h_v = e^{im_b\, v\cdot x} h \tag{3.81}$$

to use that $m_h \gg \Lambda_{\mathrm{QCD}}$ and retain just the residual momentum k inside the effective Hamiltonian. This leads to

$$\Gamma \propto 2\,\mathrm{Im} \int \mathrm{d}^4x\, e^{-im_h\, v\cdot x} \langle H(v)|T[\mathcal{H}_{\mathrm{eff}}^{v\dagger}(x)\mathcal{H}_{\mathrm{eff}}^v(0)]|H(v)\rangle, \tag{3.82}$$

where $H_{\mathrm{eff}}^v = R\,h_v$. The remaining matrix element does not involve large momenta of the order of the heavy quark mass any more which means that the short distance expansion becomes useful if the mass m_h is large compared to the scale $\bar{\Lambda}$ of the matrix element. If this is the case, it makes sense to perform an operator product expansion which has the general form

$$\int \mathrm{d}^4x\, e^{-im_h\, v\cdot x}\, T[\mathcal{H}_{\mathrm{eff}}^{v\dagger}(x)\mathcal{H}_{\mathrm{eff}}^v(0)] = \sum_{n=0}^{\infty} \left(\frac{1}{2m_h}\right)^n C_{n+3}(\mu)\hat{O}_{n+3}(\mu). \tag{3.83}$$

Here the $\hat{O}_n(\mu)$ denote operators of dimension n, with their matrix elements renormalized at the scale μ, and $C_n(\mu)$ are the corresponding Wilson coefficients. Taking again the forward matrix element of this we obtain

$$\Gamma \propto 2\,\mathrm{Im}\sum_{n=0}^{\infty}\left(\frac{1}{2m_h}\right)^n C_{n+3}(\mu)\langle H(v)|\hat{O}_{n+3}|H(v)\rangle. \tag{3.84}$$

Note that we have used the states $|H(v)\rangle$ of the full theory rather than the ones of the HQET which we defined in the last section. Thus (3.84) still does not contain the full expansion in inverse powers of the heavy mass. However, since it has been found that it is in fact advantageous to omit the HQET expansion and treat the matrix elements as phenomenological parameters [58], we will not install the further HQET expansion here. This, of course, leads to different normalizations and different definition of parameters. Since the parameters we define will still contain the full states it can be shown that the operators of lower orders contain parts of the operators of higher orders in the definition of the HQET. We will approach this subject in a further section, where we will analyze the physical meaning of the parameters we define for the inclusive $B \to X_c\, e\, \bar{\nu}_e$ decay.

The next step is to define the normalization. Therefore we will consider the lowest order terms in the operator product expansion. These are obviously terms of dimension 3. Considering Lorentz invariance and parity we get only the structures $\bar{h}_v \not{v} h_v$ and $\bar{h}_v\, h_v$. As the heavy quark states h_v differ from the full QCD h operators only by the phase redefinition, the operators turn out to be the same as in full QCD. Thus we have the relations $\bar{h}_v \not{v} h_v = \bar{h} \not{v} h$ and $\bar{h}_v\, h_v = \bar{h}\, h$ between the effective and the full QCD operators. Furthermore, the two operators differ from each other only in terms of order $1/m_h^2$:

$$\bar{h}_v\, h_v = v^\mu\, \bar{h}_v \gamma_\mu h_v + \frac{1}{2m_h^2}\bar{h}_v\left[(iD)^2 - (ivD)^2 + \frac{i}{2}\sigma_{\mu\nu}G^{\mu\nu}\right] h_v + \mathcal{O}\left(1/m_h^3\right), \tag{3.85}$$

where $G_{\mu\nu}$ is the gluon field strength. As we have already implied in the upper equation the operator $\bar{h}_v \not{v} h_v$ is proportional to the current $\bar{h}\gamma^\mu h$ which is even normalized in full QCD. Thus the matrix elements of the dimension 3 contributions are known. The standard normalization is then given by

$$\langle H(v)|\bar{h}_v \not{v} h_v|H(v)\rangle = 2m_H, \tag{3.86}$$

where m_H is the mass of the heavy hadron.

4. The semileptonic process $\bar{B} \to X_c\, l\, \bar{\nu}_l$

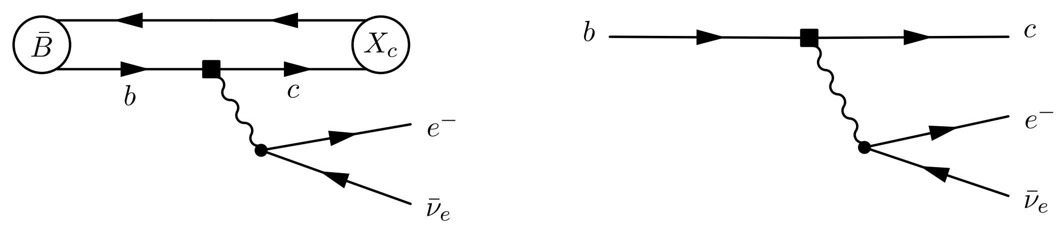


Figure 4.1.: Feynman diagrams for the B-meson decay (left) and the corresponding decay of a free b-quark (right) with extended vertices (marked as boxes)

The enormous amount of data which has recently been produced by the B factories as well as the perspectives of future experiments concerning b quarks together with reliable theoretical methods will allow us to perform precision tests of the standard model. Besides the precise determination of the standard model parameters V_{ub} and V_{cb} which are known to a relative precision of roughly 2% and 10% [59], this gives us the opportunity to look for possible effects beyond the standard model. The sensitivity to such "new physics" effects is expected to be largest in channels which do not have a big standard model contribution. In particular the loop induced flavor changing neutral currents such as $b \to s$ transitions which are suppressed by the GIM mechanism [60], are expected to be a good place to search for "new physics". The analysis which we will perform here, is well known in the context of leptonic μ and τ decays and is usually expressed as values or limits for the so-called "Michel parameters" [61, 62]. This means that we will modify the vertex for quarks and W-Bosons of the standard model by additional contributions from right-handed vector couplings as well as left and right-handed scalar and tensor couplings. Thus in a fit we could test the contributions from the new couplings, deriving upper bounds or even find a value for them. All this shall be done for inclusive $B \to X_c\, e^-\, \bar{\nu}_e$ decays, whose tree-level Feynman diagrams are presented in fig. 4.1 (left). The square at the b-W^--c-vertex denotes the nonstandard contributions for this vertex. The leptonic part is left completely untouched, since it has already been discussed separately within the μ and τ decays [8]. There is an extensive literature on possible non-standard model contributions to semileptonic B decays [63, 64, 65, 66, 67]. In contrast to these analyses, our analysis is completely model independent. However, we neglect the lepton masses and hence our analysis would need to be extended to include discussions like e.g. on charged Higgs contributions as in [64, 65, 67]. Furthermore, we consider different observables (i.e. the moments of the spectra) which have become available only recently through the precise data of the B factories. In this way we expect a much better sensitivity to nonstandard contributions. Experimental limits on nonstandard contributions come from various sources. Besides the already mentioned limits on the leptonic vertex, the neutrino data yields limits on right-handed admixtures, such that the data in the leptonic sector indicates that a right-handed admixture is well below 1% [8]. Limits on a right-handed contribution can also be inferred from high-energy processes [8]. The global analysis of the LEP data does not give any indication for a right-handed admixture which would show up in inconsistencies in the determination of the weak mixing angle. From the electroweak fit one obtains a mass limit for a right-handed W^- boson of $M_{W_R} > 715\,\mathrm{GeV}$ which yields a limit for a right-handed admixture in the charged currents of $c_R < 1.3\%$. Finally we can consider the data on the exclusive decay $B \to D^{(*)} \, \ell \, \bar{\nu}_\ell$ to obtain limits on a right-handed contribution, as it has been done in [63]. This gives a limit of $(14-18)\%$ on the right-handed admixture which is not as stringent as the ones mentioned above, but refers only to the specific transition $b \to c$.

During the calculation of the inclusive semileptonic $\bar{B} \to X_c \, e^- \, \bar{\nu}_e$ transition we of course encounter the problem that the decaying b quark is confined in the B meson. The decay of the b quark (see fig. 4.1) therefore underlies interactions with the additional content of the B meson. The HQE [51, 49, 52, 53] which has already been discussed in section 3.4, has become a very reliable tool to describe such effects in decays of heavy quarks, in particular for semileptonic decays. According to our considerations in section 3.4 the HQE yields an expansion of the total rate which reads

$$\Gamma = \Gamma^{(0,0)} + \frac{1}{m_b^2}\Gamma^{(0,2)} + \frac{1}{m_b^3}\Gamma^{(0,3)} + \frac{1}{m_b^4}\Gamma^{(0,4)} + \dots \,, \tag{4.1}$$

where the second index of the rate Γ denotes the order of the HQE, while the first index marks the order of the QCD radiative corrections which we will discuss later. Note that there are no nonperturbative corrections of order $1/m_b$. The reasons for that will become clear during the calculations performed within this chapter. The $\Gamma^{(0,x)}$ all contain a certain amount of nontrivial parameters which have to be extracted form experiment. Usually the kinetic energy parameter μ_π and the chromomagnetic moment μ_G at order $1/m_b^2$ and the Darwin term ρ_D as well as the spin orbit term ρ_{LS} at order $1/m_b^3$are used as independent parameters. Here we will introduce another set of parameters due to practical reasons, but we shall provide a conversion from our parameters to the standard ones.

Within the HQE the order of the moments is related to the order in the $1/m_b$ expansion [68, 69]. For example, the moments $\langle (m_b - 2E_l)^n \rangle$ for the case of charmless semi-leptonic B decays are determined by the contributions of the order $1/m_b^n$. Thus in order to exploit precise measurements of the lepton energy spectrum, like for example measurements of higher moments, it is mandatory to perform the theoretical calculation up to a sufficient order in the $1/m_b$ expansion.

There are two different mass schemes for the HQE, namely the kinetic scheme and the $1S$ scheme. In this dissertation we will stick to the kinetic scheme. The alternative $1S$ scheme [56, 57] yields similar precision.

Besides the HQE we are also confronted with QCD radiative corrections. This means that we additionally have to consider the expansion in α_s which reads

$$\Gamma = \Gamma^{(0,0)} + \frac{\alpha_s}{\pi}\Gamma^{(1,0)} + \frac{\alpha_s^2}{\pi^2}\Gamma^{(2,0)} + \frac{\alpha_s^3}{\pi^3}\Gamma^{(3,0)} + \dots \,, \tag{4.2}$$

where the first index of Γ marks the order in the α_s expansion. The current state of the art features the $\mathcal{O}(\alpha_s^2)$ corrections for the partonic rate [70] and the $1/m_b^3$ results at tree level [71]. Furthermore, also the combined corrections of HQE and QCD radiative corrections should be calculated, such that we obtain the expansion

$$\Gamma = \Gamma^{(0,0)} + \frac{\alpha_s}{\pi}\Gamma^{(1,0)} + \frac{1}{m_b^2}\Gamma^{(0,2)} + \frac{\alpha_s}{\pi \, m_b^2}\Gamma^{(1,2)} + \dots \,. \tag{4.3}$$

These combined corrections of order α_s concerning the QCD corrections and order $1/m_b^2$ for the HQE have been calculated lately for the kinetic energy parameter μ_π [72].

In this chapter we will present a summary and deeper explanation of the publications [18, 17, 19]. Besides the calculation of the differential and total decay rates including Michel parameters, we will present the calculation of the nonperturbative corrections up to order $1/m_b^4$ in a new systematic way which retains the ordering of the operators. This prevents us from calculating one- or even more-gluon matrix elements which normally have to be calculated to retain the ordering. Furthermore, we will discuss the calculation of radiative corrections for the Michel parameter analysis and include a discussion of the approaches and problems of the combined corrections in α_s and $1/m_b$. Among the radiative corrections we will also include a section about reparametrization invariance which provides an easier way of the calculation of the one-loop correction to $1/m_b^2$ for μ_π than presented in [72].

4.1. Effective interactions for anomalous quark-couplings

As a starting point we will consider the extension of the standard model by the Michel parameter analysis. The terms which we calculate within this analysis explicitly contain the standard model terms so that all further calculations which we perform will also apply to the pure standard model. In this way one can retain e.g. the standard model $1/m_b^4$ corrections from the results without the extensions in case the new physics contributions are of no interest. In the course of the extensions by the Michel parameters we will retain the particle content of the standard model as well as all its other properties apart from the weak interaction vertex. The analysis will furthermore be done in a systematic way to ensure that we have control of the size of the additional terms. Redeeming these requirements is a bit tricky, since the standard model is the most generic renormalizable theory which reflects the observed $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ and the observed particle spectrum. However, it is still possible to extend the standard model in a model independent way without introducing new particles by going to higher mass dimensions in the Lagrange density. The dimension of the action

$$S = \int \mathrm{d}^4x\, \mathcal{L} \tag{4.4}$$

is always zero in natural dimensions. The mass dimension -4 of d^4x therefore has to be compensated by a Lagrangian $\mathcal{L}$ of dimension 4. If we reconsider the Lagrange densities of the standard model which have been introduced in the last sections, we obtain the dimensions

$$\dim[q] = \frac{3}{2}, \quad \dim[iD_\mu] = 1, \quad \dim[\Phi] = 1 \quad \text{and} \quad \dim[B_{\mu\nu}] = \dim[W_{\mu\nu}] = 2 \tag{4.5}$$

for the various components contained in them. This directly leads us to the problem that we cannot introduce terms of higher orders in the Lagrangian without violating the dimension of the action. This problem is solved by the introduction of a Lagrange density $\mathcal{L}$ from which we assume that it describes the interactions correctly. Even if the exact form of this Lagrange density is unknown we know its expansion

$$\mathcal{L} = \mathcal{L}_{4D} + \frac{1}{\Lambda}\mathcal{L}_{5D} + \frac{1}{\Lambda^2}\mathcal{L}_{6D} + \dots\,, \tag{4.6}$$

in the parameter Λ of dimension 1 which and represents the scale at which we suspect new phenomena. Note that we have not mentioned how the Lagrangian $\mathcal{L}$ looks. We just introduced it to get a series of effective Lagrange densities, where the Lagrangian $\mathcal{L}_{4D}$ with four dimensions obviously has to be the standard model Lagrangian we know to be consistent with the experimental data that has already been taken. In this sense the standard model Lagrange density presents an effective theory of a higher-level theory which we do not know at the time. The terms of higher dimensions than 4 are suppressed by orders of $1/\Lambda$ and therefore small. Thus it will be sufficient to analyze the lowest nonvanishing correction to the standard model. Additionally we will retain the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ symmetry of the original standard model Lagrangian. Furthermore, we are interested only in operators which contain two quark fields, since these are the terms which are needed to describe the considered $\bar{B} \to X_c\, \bar{\nu}_\ell\, \ell$ decay. As we construct the higher order Lagrange densities before spontaneous symmetry breaking, the operators contained in the specific orders can be classified by the helicity of the two quark fields. Hence we define the term O_{XY} to denote an operator with the quark helicities X and Y. Further operators with other configurations of quark fields can be taken from [73]. An equivalent list of operators can be found in [74] which however uses a different notation in terms of a custodial symmetry. The task is now to parametrize the lowest nonvanishing operators and compare the results with the experiment to observe whether corrections to the standard model in this sense are necessary. However, it turns out that it is impossible to parametrize terms of dimension 5 which are consistent with the $SU(2)_L \otimes U(1)_Y$ symmetry[1]. Therefore the terms which have to be considered are

$$\mathcal{L} = \mathcal{L}_{SM} + \frac{1}{\Lambda^2}\sum_i O_{LL}^{(i)} + \frac{1}{\Lambda^2}\sum_i O_{LR}^{(i)} + \frac{1}{\Lambda^2}\sum_i O_{RR}^{(i)}. \tag{4.7}$$

Note that the operators O_{LR} always contain both possibilities of the quark helicities with the left-handed initial and final state. While arranging the operators we additionally have to consider the equations of motion

$$(i\not{D})\boldsymbol{Q}_L^i = \boldsymbol{\Phi}^\dagger y_d\, d_R^i + \widetilde{\boldsymbol{\Phi}}^\dagger y_u\, u_R^i \tag{4.8}$$

$$(i\not{D})u_R^i = \widetilde{\boldsymbol{\Phi}}^\dagger\, y_u^\dagger\, \boldsymbol{Q}_L^i \tag{4.9}$$

$$(i\not{D})d_R^i = \boldsymbol{\Phi}^\dagger\, y_d^\dagger\, \boldsymbol{Q}_L^i \tag{4.10}$$

$$(D_\nu W^{\mu\nu})^I = -g_2\left(\boldsymbol{\Phi}^\dagger i\overleftrightarrow{D}^\mu \tau^I \boldsymbol{\Phi} + \sum_i \bar{\boldsymbol{Q}}_L^i\gamma^\mu\tau^I\boldsymbol{Q}_L^i + \sum_i \bar{\boldsymbol{l}}_L^i\gamma^\mu\tau^I\boldsymbol{l}_L^i\right) \tag{4.11}$$

$$\partial_\nu B^{\mu\nu} = -g_1\left(-\frac{1}{2}\sum_i \bar{\boldsymbol{l}}_L^i\gamma^\mu\boldsymbol{l}_L^i - \bar{\boldsymbol{e}}_L\gamma^\mu\boldsymbol{e}_L + \frac{1}{6}\sum_i \bar{\boldsymbol{Q}}_L^i\gamma^\mu\boldsymbol{Q}_L^i + \frac{2}{3}\bar{\boldsymbol{u}}_R\gamma^\mu\boldsymbol{u}_R - \frac{1}{3}\bar{\boldsymbol{d}}_R\gamma^\mu\boldsymbol{d}_R\right) \tag{4.12}$$

from the electroweak Lagrangian which connect the covariant derivatives of the left-handed doublets to combinations of right-handed singlets and Higgs doublets. Thus we have to consider only the terms, where no derivatives act on the external state fields to get a set of independent operators. Our dimension six operators require two fermions and three other powers of momentum. Since all vectors have hypercharge zero and all fermion fields have different hypercharge, the fermions are always a field and its conjugate. Therefore we have to provide a Dirac matrix γ_μ, as the operators have a defined chirality. The gauge fields can be either introduced by three covariant derivatives on the fermions or one covariant derivative and one field strength. In this case we will choose to use the second alternative, since this will alleviate the further calculations. Furthermore, we can omit dual field strengths, since for fermions of defined chirality one

[1] This is of course only true if we use the standard model quark fields exclusively. Using Majorana neutrinos (or newly introduced fields with properties differing from the standard model) the creation dimension 5 operators would indeed be possible.

has the identity

$$\widetilde{F}^{\mu\nu}\gamma_\mu D_\nu \boldsymbol{\Psi}_{L/R} = \pm\left(iF^{\mu\nu}\gamma_\mu D_\nu - \frac{1}{2}\sigma_{\mu\nu}F^{\mu\nu}\not{D}\right)\boldsymbol{\Psi}_{L/R} \tag{4.13}$$

The remaining independent operators are:

$$O_{LL}^{(1)} = \sum_i \bar{\boldsymbol{Q}}_L^i \gamma_\mu W^{\mu\nu} D_\nu \,\boldsymbol{Q}_L^i \tag{4.14}$$

$$O_{LL}^{(2)} = \sum_i \bar{\boldsymbol{Q}}_L^i \gamma_\mu B^{\mu\nu} D_\nu \,\boldsymbol{Q}_L^i \tag{4.15}$$

$$O_{RR}^{(1)} = \bar{\boldsymbol{u}}_R \gamma_\mu B^{\mu\nu} D_\nu \boldsymbol{u}_R \tag{4.16}$$

$$O_{RR}^{(2)} = \bar{\boldsymbol{d}}_R \gamma_\mu B^{\mu\nu} D_\nu \boldsymbol{d}_R \tag{4.17}$$

The next class of operators which we will consider consists of fermions and scalars. Again we have to implement three powers of momentum, this time by Higgs fields or covariant derivatives. Since the covariant derivatives have to act on a gauge invariant quantity and no field is a singlet it is not possible to create combinations containing two derivatives. Operators with one derivative must contain a Dirac matrix to be a Lorentz scalar. Thus we would obtain operators of the form $(\bar{\boldsymbol{q}}\gamma_\mu \boldsymbol{q})\partial_\mu(\boldsymbol{\Phi}^\dagger\boldsymbol{\Phi})$ which however can be rewritten into other operators by partial integration and the equations of motion. Thus we are left with two operators

$$O_{LR}^{(1)} = \sum_i \bar{\boldsymbol{Q}}_L^i (\boldsymbol{\Phi}^\dagger\boldsymbol{\Phi})\widetilde{\boldsymbol{\Phi}}\, u_R^i + \text{h.c.} \tag{4.18}$$

$$O_{LR}^{(2)} = \sum_i \bar{\boldsymbol{Q}}_L^i (\boldsymbol{\Phi}^\dagger\boldsymbol{\Phi})\boldsymbol{\Phi}\, d_R^i + \text{h.c.} \tag{4.19}$$

containing no derivatives. Note that we also used $\boldsymbol{\Phi}^\dagger\boldsymbol{\Phi} = -\widetilde{\boldsymbol{\Phi}}^\dagger\widetilde{\boldsymbol{\Phi}}$ because of $(\tau_2)^2 = 1$. Therefore we do not get any further combinations of $\boldsymbol{\Phi}$ and $\widetilde{\boldsymbol{\Phi}}$. The next class of operators contains fermions, scalars and vectors. We can either have one or two scalars. Let us consider the latter alternative first. We get the following list of operators

$$O_{LL}^{(3)} = \sum_i (\boldsymbol{\Phi}^\dagger iD_\mu\boldsymbol{\Phi})(\bar{\boldsymbol{Q}}_L^i\gamma^\mu\boldsymbol{Q}_L^i) \tag{4.20}$$

$$O_{LL}^{(4)} = \sum_i (\boldsymbol{\Phi}^\dagger\tau^I iD_\mu\boldsymbol{\Phi})(\bar{\boldsymbol{Q}}_L^i\tau^I\gamma^\mu\boldsymbol{Q}_L^i) \tag{4.21}$$

$$O_{RR}^{(3)} = (\boldsymbol{\Phi}^\dagger iD_\mu\boldsymbol{\Phi})(\bar{\boldsymbol{u}}_R\gamma^\mu\boldsymbol{u}_R) \tag{4.22}$$

$$O_{RR}^{(4)} = (\boldsymbol{\Phi}^\dagger iD_\mu\boldsymbol{\Phi})(\bar{\boldsymbol{d}}_R\gamma^\mu\boldsymbol{d}_R) \tag{4.23}$$

$$O_{RR}^{(5)} = (\boldsymbol{\Phi}^T iD_\mu\boldsymbol{\Phi})(\bar{\boldsymbol{u}}_R\gamma^\mu\boldsymbol{d}_R) \tag{4.24}$$

Combinations where the covariant derivative acts on one of the quark fields can be eliminated by the equations of motion. Because of the hypercharge assignments the last operator is the only operator containing two $\boldsymbol{\Phi}$ instead of $\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}^\dagger$. Now we will regard the operators with one scalar. Here we are confronted with three possibilities for the covariant derivatives: The two derivatives both act on the scalar, one acts on the scalar while the other acts on one fermion or both derivatives act on the fermions. Let us first consider the case where both derivatives act on the scalar. We have two alternatives to contract the derivatives, namely $(D_\mu D_\nu\boldsymbol{\Phi})(\bar{\psi}g_{\mu\nu}\psi)$ and $(D_\mu D_\nu\boldsymbol{\Phi})(\bar{\psi}\sigma_{\mu\nu}\psi)$. ψ and $\bar{\psi}$ denote the fermion fields which are allowed by the $SU(2) \otimes U(1)$ invariance. The first one does not need to be considered because of the equations of motion. The second one is equivalent to the operator $\Phi(\bar{\psi}\sigma_{\mu\nu}\psi)F_{\mu\nu}$, where $F_{\mu\nu}$ is the field strength corresponding to the quarks. For the operators in which the derivatives act on one fermion and on one scalar we get the two possibilities $(D_\mu\Phi)(\bar{\psi}D^\mu\psi)$ and $(D_\mu\Phi)(\bar{\psi}\sigma^{\mu\nu}D_\nu\psi)$ plus analog terms in which the derivative acts on the $\bar{\psi}$. However, it can be shown that the second alternative is equivalent to the first one alongside a term without vectors by the use of the equations of motion. The last case we have to discuss is the one with two derivatives acting on the fermions. We observe three possible combinations as the derivatives can both act on either the $\bar{\psi}$ or the ψ or one acts on the $\bar{\psi}$ and one on the ψ respectively. If they both act on one fermion we can use the relation

$$\not{D}\not{D} = D_\mu D_\nu g^{\mu\nu} - iD_\mu D_\nu \sigma^{\mu\nu} \tag{4.25}$$

to show that the corresponding operator is equivalent to an operator with a field strength. For the mixed case the general form $\Phi D_\mu\bar{\Psi}(a\, g^{\mu\nu} + b\,\sigma^{\nu\nu})D_\nu\Psi$ reduces by means of partial integration to operators we have already discussed. We end up with the following list of operators with fermions scalars and vectors:

$$O_{LR}^{(4)} = \sum_i (\bar{\boldsymbol{Q}}_L^i\, iD_\mu\, u_R^i)\, iD^\mu\, \widetilde{\boldsymbol{\Phi}} + \text{h.c.} \tag{4.26}$$

$$O_{LR}^{(5)} = \sum_i (\bar{\boldsymbol{Q}}_L^i\, i\overleftarrow{D}_\mu\, u_R^i)\, iD^\mu\, \widetilde{\boldsymbol{\Phi}} + \text{h.c.} \tag{4.27}$$

$$O_{LR}^{(6)} = \sum_i (\bar{\boldsymbol{Q}}_L^i\, iD_\mu\, d_R^i)\, iD^\mu\, \boldsymbol{\Phi} + \text{h.c.} \tag{4.28}$$

$$O_{LR}^{(7)} = \sum_i (\bar{\boldsymbol{Q}}_L^i\, i\overleftarrow{D}_\mu\, d_R^i)\, iD^\mu\, \boldsymbol{\Phi} + \text{h.c.} \tag{4.29}$$

$$O_{LR}^{(8)} = \sum_i (\bar{\boldsymbol{Q}}_L^i \sigma_{\mu\nu} W^{\mu\nu} u_R^i)\, \widetilde{\boldsymbol{\Phi}} + \text{h.c.} \tag{4.30}$$

$$O_{LR}^{(9)} = \sum_i (\bar{\boldsymbol{Q}}_L^i \sigma_{\mu\nu} B^{\mu\nu} u_R^i)\, \widetilde{\boldsymbol{\Phi}} + \text{h.c.} \tag{4.31}$$

$$O_{LR}^{(10)} = \sum_i (\bar{\boldsymbol{Q}}_L^i \sigma_{\mu\nu} W^{\mu\nu} d_R^i)\, \boldsymbol{\Phi} + \text{h.c.} \tag{4.32}$$

$$O_{LR}^{(11)} = \sum_i (\bar{\boldsymbol{Q}}_L^i \sigma_{\mu\nu} B^{\mu\nu} d_R^i)\, \boldsymbol{\Phi} + \text{h.c.} \tag{4.33}$$

In the following considerations we only want to consider the parts of the operators which we need for our calculation of the $B \to X_c\, \bar{\nu}_e\, e$ decays. Thus only the contributions

$$\bar{\boldsymbol{Q}} = \begin{pmatrix} \bar{c} \\ 0 \end{pmatrix}, \quad \boldsymbol{Q} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \quad \bar{q} = \bar{c}, \quad \text{and} \quad q = b \tag{4.34}$$

survive. Additionally we still have to perform spontaneous symmetry breaking. Therefore we have to set

$$\boldsymbol{\Phi} = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \quad \text{and} \quad \widetilde{\boldsymbol{\Phi}} = \begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix}, \tag{4.35}$$

since we are again not interested in quark-Higgs-couplings. Considering again equation (2.18) as well as equation (2.27) we get

$$D_\mu \boldsymbol{\Phi} = I\, \partial_\mu \boldsymbol{\Phi} + \frac{ig_2}{\sqrt{2}} \begin{pmatrix} 0 & W_\mu^+ \\ W_\mu^- & 0 \end{pmatrix} \boldsymbol{\Phi} \tag{4.36}$$

for the covariant derivative of the Higgs field which contains just the expressions for the charged current we are interested in. Using (4.35) we end up with

$$D_\mu \mathbf{\Phi} = \frac{i g_2}{\sqrt{2}} W_\mu^+ \begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix} \quad \text{and} \quad D_\mu \widetilde{\mathbf{\Phi}} = \frac{i g_2}{\sqrt{2}} W_\mu^- \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}. \tag{4.37}$$

At this point we will not discuss how the covariant derivative acts on the fermion fields, since we want to retain them in our current. Thus we will just use the relations (4.34-4.37) to evaluate the operators for the case of $B \to X_c \, \bar{\nu}_e \, e$ decays. Since we like to couple the hadronic current to the ordinary leptonic current of the standard model

$$J_{l,\mu} = \frac{g_2}{\sqrt{2}} \bar{\nu}_e \, \not{W}^- P_L \, e \tag{4.38}$$

we have to sandwich the operators between the states $\langle c \, W^- |$ and $|b\rangle$. Thus we can eliminate all operators which do not contain a W_μ^+. This causes the operators (4.18) and (4.19) to vanish. From the operators (4.20-4.24) only the term

$$O_{RR}^{(5)} = i \, g_2 \, \frac{v^2}{\sqrt{2}} \, \bar{c} \, W_\mu^+ \, \gamma_\mu \, P_L \, b \tag{4.39}$$

survives which is a new right-handed vector current. The other operators vanish because of non matching quark or Higgs fields. Regarding the equations (4.26-4.29) we obtain the nonvanishing operators

$$O_{LR}^{(4)} = 2 \, i \, g_2 \frac{v}{\sqrt{2}} \, \bar{c} \, W_\mu^+ \, P_L \, D_\mu \, b \tag{4.40}$$

$$O_{LR}^{(5)} = 2 \, i \, g_2 \, \frac{v}{\sqrt{2}} \, \bar{c} \, \overleftarrow{D}_\mu \, W_\mu^+ \, P_L \, b \tag{4.41}$$

$$O_{LR}^{(6)} = 2 \, i \, g_2 \frac{v}{\sqrt{2}} \, \bar{c} \, W_\mu^+ \, P_R \, D_\mu \, b \tag{4.42}$$

$$O_{LR}^{(7)} = 2 \, i \, g_2 \, \frac{v}{\sqrt{2}} \, \bar{c} \, \overleftarrow{D}_\mu \, W_\mu^+ \, P_R \, b \tag{4.43}$$

which describe a left- and a right-handed scalar current. Using (2.27) we can rewrite the $W^{\mu\nu}$ fields in (4.30-4.33) to

$$W_{\mu\nu} = \sqrt{2} \, \tau^+ (\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+) + \dots \, , \tag{4.44}$$

where the dots represent terms containing multiple $W^\pm$, W^- and the parts to the field strength describing the neutral weak current which vanish in our case. The operators which contain the field strength $B^{\mu\nu}$ vanish for the same reason. Performing a partial integration and using the antisymmetry of the $\sigma^{\mu\nu}$ to rewrite (4.44) to $2\sqrt{2} \, \tau^+ \partial_\mu W_\nu^+$, we obtain the left- and right-handed tensor operators

$$O_{LR}^{(8)} = 2 \, g_2 \, v \left(\bar{c} \, \overleftarrow{\partial}_\mu \, \sigma^{\mu\nu} \, W_\mu^+ \, P_R \, b + \bar{c} \, \sigma^{\mu\nu} \, W_\mu^+ \, \partial_\mu \, P_R \, b \right) \tag{4.45}$$

$$O_{LR}^{(10)} = 2 \, g_2 \, v \left(\bar{c} \, \overleftarrow{\partial}_\mu \, \sigma^{\mu\nu} \, W_\mu^+ \, P_L \, b + \bar{c} \, \sigma^{\mu\nu} \, W_\mu^+ \, \partial_\mu \, P_L \, b \right) \tag{4.46}$$

This concludes our list of operators, since the operators (4.15-4.17) vanish as they contain neutral current parts only and (4.14) vanishes because of (4.34). The Lagrange density for the hadronic part can now be obtained by simply summing up the operators by means of (4.7) and multiplying with the leptonic current (4.38) which has been inherited unchanged from the standard model. Here we notice that the Lagrange density still contains a W boson propagator. As this propagator contains degrees of freedom higher than the scale of the $B \to X_c\, e\, \bar{\nu}_e$ decay, we will consider it in an effective field theory content. Thus we take the full propagator

$$P^W_{\lambda\nu}(0,x) = T\left[W^+_\lambda(x) W^-_\nu(0)\right] = \frac{1}{q^2 - M_W^2 + i\epsilon}\left(g_{\lambda\nu} - \frac{(1-\xi)\, q_\lambda q_\nu}{\left(q^2 - \xi M_W^2 + i\epsilon\right)}\right), \tag{4.47}$$

where $q = p_b - p_c$ denotes the momentum of the leptonic system and therefore the propagators momentum. The time ordered product is in this case only written pro forma, since the terms are already time ordered because of $x_0 > 0$. As we have $(p_b - p_c)^2 < m_b^2 \ll M_W^2$ we can expand the denominator in powers of $1/M_W^2$. This gives us

$$\frac{1}{q^2 - M_W^2} = \frac{1}{M_W^2}\left(1 + \frac{q^2}{M_W^2} + \dots\right), \tag{4.48}$$

where we can safely neglect the terms in the brackets as the expansion converges very rapidly because of $(m_b/M_W)^2 \approx 0.003$. This leads us to the propagator

$$P^W_{\lambda\nu}(0,x) = \frac{g_{\lambda\nu}}{M_W^2} \tag{4.49}$$

in Feynman gauge ($\xi = 1$). Thus the effective theory gives a local four-fermion vertex located at the space time point x. The effective Hamilton density can now be calculated by the relation $\mathcal{H} = -i\mathcal{L}$. Using additionally the definition

$$\frac{G_F}{\sqrt{2}} = \frac{g_2^2}{8M_W^2}, \tag{4.50}$$

of the Fermi constant, we obtain the Hamilton density

$$\mathcal{H}_{\text{eff}} = \frac{4G_F V_{cb}}{\sqrt{2}}\, J_{h,\mu}\, J_l^\mu, \tag{4.51}$$

with the extended hadronic current

$$\begin{aligned} J_{h,\mu} = {} & c_L\ \bar{c}\gamma_\mu P_L b + c_R\ \bar{c}\gamma_\mu P_R b + g_L\ \bar{c}\, i\overleftrightarrow{D}_\mu P_L b + g_R\ \bar{c}\, i\overleftrightarrow{D}_\mu P_R b \\ & + d_L\ i\partial^\nu(\bar{c}\, i\sigma_{\mu\nu} P_L b) + d_R\ i\partial^\nu(\bar{c}\, i\sigma_{\mu\nu} P_R b) \end{aligned} \tag{4.52}$$

and the leptonic current

$$J_l^\mu = e\, \gamma^\mu P_L\, \nu_e\,. \tag{4.53}$$

The coefficient c_L in front of the standard model term has to be of the order $c_L \propto 1$ (as we know that it describes the phenomenology correctly up to the current accuracy of the experiments), while $c_R \propto v^2/\Lambda^2$ and $g_{L/R} \propto v/\Lambda^2$ as well as $d_{L/R} \propto v/\Lambda^2$. Thus the right-handed vector current is suppressed by a factor of v^2/Λ^2 and the scalar and tensor currents even by a factor of v/Λ^2. Therefore we do not expect the corrections to the standard model to be very large. The major contribution of new terms will surely come from the right-handed vector admixture, since $v^2/\Lambda^2 \gg (v\, m_b)/\Lambda^2$, where the m_b has been added to be consistent with the units. However, since we have additional constants whose order of magnitudes we do not know, it is reasonable to perform the calculation for all additional terms in the current.

4.2. Kinematics

The aim of the following sections is to calculate the total decay rate as well as moments of the hadronic and leptonic energies. The form and impact of those moments will be discussed at a later stage and it will be straight forward to implement their calculation later on. Thus we consider the generic formula for the total inclusive decay rate of the B-meson in its rest system

$$\begin{aligned}\Gamma &= \sum_{X_c} \sum_{\substack{\text{spins}\\\text{leptons}}} \frac{1}{2m_B} \left(\prod_{f=e,\bar{\nu}_e} \int \frac{\mathrm{d}^3\boldsymbol{p}_f}{(2\pi)^3} \frac{1}{2E_f} \right) |\mathcal{M}(m_B \to p_f)|^2 (2\pi)^4 \delta^4 \left(p_B - \sum p_f \right) \\ &= \sum_{X_c} \sum_{\substack{\text{spins}\\\text{leptons}}} \frac{1}{2m_B} \left(\int \frac{\mathrm{d}^3\boldsymbol{p}_e}{(2\pi)^3} \frac{1}{2E_e} \right) \left(\int \frac{\mathrm{d}^3\boldsymbol{p}_\nu}{(2\pi)^3} \frac{1}{2E_\nu} \right) |\mathcal{M}(m_B \to p_e, p_\nu, p_{X_c})|^2 \\ &\qquad \times (2\pi)^4 \delta^4 \left(p_B - (p_e + p_\nu + p_{X_c}) \right) ,\end{aligned} \tag{4.54}$$

which has already been discussed in section 2.4. Here we encounter a sum over all final states X_c containing a c-quark which can be created by the decay. This means that the phase space is implicitly contained in the sum. As we are considering an inclusive decay, we only have to integrate over the phase space of the leptons as the integration over the hadron X_c is contained in the sum over the various X_c. The kinematics of the leptonic phase space integration are given by the delta function $\delta^4 \left(p_B - (p_e + p_\nu + p_{X_c}) \right)$. To simplify (4.54), we have to examine the matrix element $\mathcal{M}$. Therefore we need the Hamilton density

$$\mathcal{H}_{\text{eff}} = \frac{4G_F V_{cb}}{\sqrt{2}} J_{h,\mu} J_l^\mu \tag{4.55}$$

which has been introduced in the last subsection. Using this Hamiltonian the square of the absolute value of the matrix element takes the form

$$\begin{aligned}|\mathcal{M}(\bar{B} \to X_c\, e^- \bar{\nu}_e)|^2 &= \left| \langle X_c\, e^- \bar{\nu}_e | \mathcal{H}_{\text{eff}} | \bar{B} \rangle \right|^2 \\ &= 8G_F^2 |V_{cb}|^2 \left| \langle X_c e^- \bar{\nu}_e | J_l^\mu J_{h,\mu} | \bar{B} \rangle \right|^2 \\ &= 8G_F^2 |V_{cb}|^2 \left| \langle X_c | J_{h,\mu} | \bar{B} \rangle \langle e^- \bar{\nu}_e | J_l^\mu | 0 \rangle \right|^2 \\ &= 8G_F^2 |V_{cb}|^2 \langle \bar{B} | J_{h,\mu}^\dagger | X_c \rangle \langle X_c | J_{h,\nu} | \bar{B} \rangle \langle 0 | J_l^{\dagger\mu} | e^- \bar{\nu}_e \rangle \langle e^- \bar{\nu}_e | J_l^\nu | 0 \rangle .\end{aligned} \tag{4.56}$$

The matrix element has been splitt into a hadronic and a leptonic part in the last step. Those parts can be regarded separately which appreciably simplifies the following calculations. Thus it is convenient to define a leptonic tensor

$$L^{\mu\nu} = \sum_{\substack{\text{spins}\\\text{leptons}}} \langle 0 | J_l^{\dagger\mu} | e^- \bar{\nu}_e \rangle \langle e^- \bar{\nu}_e | J_l^\nu | 0 \rangle \tag{4.57}$$

and a hadronic tensor

$$W_{\mu\nu} = \frac{1}{2m_B} \sum_{X_c} \langle \bar{B} | J_{h,\mu}^\dagger | X_c \rangle \langle X_c | J_{h,\nu} | \bar{B} \rangle (2\pi)^3 \delta^4 (p_B - p_e - p_\nu - p_{X_c}). \tag{4.58}$$

Using those two tensors the decay rate takes the form

$$\Gamma = 16\pi\, G_F^2\, |V_{cb}|^2 \left(\int \frac{\mathrm{d}^3\boldsymbol{p}_e}{(2\pi)^3} \frac{1}{2E_e} \right) \left(\int \frac{\mathrm{d}^3\boldsymbol{p}_\nu}{(2\pi)^3} \frac{1}{2E_\nu} \right) W_{\mu\nu} L^{\mu\nu}. \tag{4.59}$$

The next step is to eliminate all trivial integrations from (4.54). As we are considering a three body decay, we need three variables for the description of the kinematics. Thus it is convenient to calculate the triple differential decay rate in the first instance. As variables we choose the energy of the electron $E_e = |\boldsymbol{p}_e|$, the neutrino energy $E_\nu = |\boldsymbol{p}_\nu|$ and the invariant mass of the leptonic system q^2, where $q = p_e + p_\nu$. To keep things as easy as possible, we won't extract the integrals belonging to this quantities by hand, but just add the delta functions

$$\delta\left(E_e - |\boldsymbol{p}_e|\right), \quad \delta\left(E_\nu - |\boldsymbol{p}_\nu|\right) \quad \text{and} \quad \delta\left(q^2 - (p_e + p_\nu)^2\right). \tag{4.60}$$

Inserting this into (4.59) we get the triple differential rate

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2 \mathrm{d}E_e \mathrm{d}E_\nu} = \frac{2\, G_F^2\, |V_{cb}|^2}{(2\pi)^5} \iint \mathrm{d}^3\boldsymbol{p}_e \mathrm{d}^3\boldsymbol{p}_\nu W_{\mu\nu} L^{\mu\nu} \delta\left(E_e - |\boldsymbol{p}_e|\right) \delta\left(E_\nu - |\boldsymbol{p}_\nu|\right) \delta\left(q^2 - (p_e + p_\nu)^2\right). \tag{4.61}$$

To evaluate this, it is convenient to use spherical coordinates

$$\begin{aligned} \mathrm{d}^3\boldsymbol{p}_e &= |\boldsymbol{p}_e|^2 \mathrm{d}|\boldsymbol{p}_e|\, \mathrm{d}\phi_e\, \mathrm{d}\cos\theta_e \\ \mathrm{d}^3\boldsymbol{p}_\nu &= |\boldsymbol{p}_\nu|^2 \mathrm{d}|\boldsymbol{p}_\nu|\, \mathrm{d}\phi_\nu\, \mathrm{d}\cos\theta_\nu \end{aligned} \tag{4.62}$$

for the phase space integrations of the leptons. Nothing depends on the direction of the neutrino, and integrating over it gives a factor of 4π. The z axis for the electron can be chosen to be aligned with the neutrino direction. Integrating over the electron azimuthal angle gives a factor of 2π. Consequently the lepton phase space is

$$\mathrm{d}^3\boldsymbol{p}_e\, \mathrm{d}^3\boldsymbol{p}_\nu = 8\pi^2 |\boldsymbol{p}_e|^2 |\boldsymbol{p}_\nu|^2\, \mathrm{d}|\boldsymbol{p}_e|\, \mathrm{d}|\boldsymbol{p}_\nu|\, \mathrm{d}\cos\theta, \tag{4.63}$$

where $\theta = \theta_e$ is the angle between the electron and neutrino directions. To be able to perform the integrations we also have to rewrite the delta function concerning q^2. As the lepton masses relative to the masses of the quarks which can occur in the B meson, we will calculate the phase space in the massless limit of the leptons. Therefore we get $p_e^2 = 0$ and $p_\nu^2 = 0$ which is equal to $|\boldsymbol{p}_e|^2 = E_e^2$ and $|\boldsymbol{p}_\nu|^2 = E_\nu^2$. Thus the square of the electron and neutrino momentum in the delta function can be evaluated as follows:

$$\begin{aligned} (p_e + p_\nu)^2 &= p_e^2 + p_\nu^2 + 2 p_e \cdot p_\nu \\ &= 2E_e E_\nu - 2|\boldsymbol{p}_\nu|\, |\boldsymbol{p}_e|\, \cos\theta \\ &= 2E_e E_\nu (1 - \cos\theta) \end{aligned} \tag{4.64}$$

As the integration is performed over $\cos\theta$ we have to use the formula

$$\delta\left(g(x)\right) = \sum_{i=1}^{n} \frac{1}{|g'(x_i)|} \delta(x - x_i) \qquad \text{with} \qquad g(x_i) = 0, \quad g'(x_i) \neq 0 \tag{4.65}$$

from [75] to rewrite the delta function. We get

$$\delta\left(q^2 - 2E_e E_\nu (1 - \cos\theta)\right) = \frac{1}{2E_e E_\nu} \delta\left(\cos\theta - \left(1 - \frac{q^2}{2E_e E_\nu}\right)\right) \tag{4.66}$$

Using all this in (4.61) we end up with

$$\begin{aligned} \frac{\mathrm{d}\Gamma}{\mathrm{d}q^2 \mathrm{d}E_e \mathrm{d}E_\nu} = \frac{2\, G_F^2\, |V_{cb}|^2}{(2\pi)^3} &\iint \mathrm{d}|\boldsymbol{p}_e|\, \mathrm{d}|\boldsymbol{p}_\nu|\, \mathrm{d}\cos\theta \frac{|\boldsymbol{p}_e|^2 |\boldsymbol{p}_\nu|^2}{E_e^2\, E_\nu^2} W_{\mu\nu} L^{\mu\nu} \\ &\cdot \delta\left(E_e - |\boldsymbol{p}_e|\right) \delta\left(E_\nu - |\boldsymbol{p}_\nu|\right) \delta\left(\cos\theta - \left(1 - \frac{q^2}{2E_e E_\nu}\right)\right), \end{aligned} \tag{4.67}$$

which simplifies to

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2\mathrm{d}E_e\mathrm{d}E_\nu} = \frac{2\,G_F^2\,|V_{cb}|^2}{(2\pi)^3} W_{\mu\nu}L^{\mu\nu} \tag{4.68}$$

after evaluating the integrations. Since the leptonic tensor has, in contrast to the hadronic one, not been extended in section 4.1 the calculation is straight forward. We just need the fermion spin sums

$$\sum_{\text{spins}} \bar{u}(p_e)u(p_e) = \not{p}_e + m_e \quad \text{and} \quad \sum_{\text{spins}} \bar{v}(p_\nu)v(p_\nu) = \not{p}_\nu - m_\nu. \tag{4.69}$$

Remember that we chose the massless limit for the leptonic phase space. Therefore the masses in the upper equation vanish. Using the standard model leptonic current current $J_l^\mu = \bar{e}\,\gamma^\mu P_L\,\bar{\nu}_e$ from section 2.1 we get

$$\begin{aligned}
L^{\mu\nu} &= \sum_{\substack{\text{spins}\\ \text{leptons}}} \langle 0|J_l^{\dagger\mu}|e^-\bar{\nu}_e\rangle\langle e^-\bar{\nu}_e|J_l^\nu|0\rangle \\
&= \sum_{\substack{\text{spins}\\ \text{leptons}}} \bar{u}(p_e)\,\gamma_\mu\,P_L\,v(p_\nu)\bar{v}(p_\nu)\,\gamma^\nu\,P_L\,u(p_e) \\
&= \mathrm{Tr}\{\not{p}_e\,\gamma_\mu\,P_L\,\not{p}_\nu\,\gamma^\nu\,P_L\} \\
&= 2\left(p_e^\mu\,p_\nu^\nu + p_e^\nu\,p_\nu^\mu + g^{\mu\nu}\,p_e\,p_\nu - i\epsilon^{\eta\mu\lambda\nu}p_{e,\eta}\,p_{\bar{\nu}_e,\lambda}\right).
\end{aligned} \tag{4.70}$$

The calculation of the hadronic tensor will be far more complicated, as it contains the interactions of the b quark with the background field as well as the extended current (4.52) which has been introduced in section 4.1. At this point we will confine ourselves to decompose it into Lorentz scalar structure functions according to

$$W_{\mu\nu} = -g_{\mu\nu}W_1 + v_\mu v_\nu W_2 - i\epsilon_{\mu\nu\alpha\beta}v^\alpha q^\beta W_3 + q_\mu q_\nu W_4 + (q_\mu v_\nu + v_\mu q_\nu)\,W_5\,, \tag{4.71}$$

where the scalar structure functions W_i are only functions of the Lorentz invariant quantities q^2 and $v \cdot q$. Using (4.71) together with (4.70) and (4.68) we obtain

$$\frac{\mathrm{d}^3\Gamma}{\mathrm{d}q^2\mathrm{d}E_e\mathrm{d}E_\nu} = \frac{4\,G_F^2\,|V_{cb}|^2}{(2\pi)^3}\left[W_1\,q^2 + W_2\left(2E_eE_\nu - \frac{q^2}{2}\right) + W_3\,q^2\,(E_e - E_\nu)\right]. \tag{4.72}$$

Note that the structure functions W_4 and W_5 do not contribute to the decay rate. This comes from the massless limit in the leptonic sector which implies $q_\alpha L^{\alpha\beta} = q_\beta L^{\alpha\beta} = 0$.

4.3. Calculation of the hadronic tensor

The next step consists of calculating the hadronic tensor which is obviously the last missing piece for the calculation of the triple differential decay rate. The calculation of the hadronic tensor is obviously not as easy as the calculation of the leptonic one, as the hadronic tensor incorporates the existence of a gluonic background field. In section 3 we have already discussed how to approach systems like that and we will perform the calculation utilizing the heavy quark expansion which has been discussed in the subsection 3.4. Thus we first have to observe how the optical theorem works in the case of inclusive $\bar{B} \to X_c\,e\,\bar{\nu}_e$ decays.

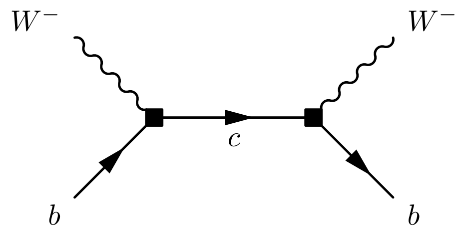


Figure 4.2.: Feynman diagram for the time-ordered product on quark-level

4.3.1. The optical theorem for B-decays

The aim is to describe the hadronic tensor $W_{\mu\nu}$ which has been introduced in section 4.2, by a forward scattering amplitude to be able to use the method given in subsection 3.4. Therefore we will at first consider the matrix element

$$T_{\mu\nu} = -\frac{i}{2m_B} \int \mathrm{d}^4x\, e^{-iqx} \langle \bar{B}|\mathrm{T}[J^\dagger_{h,\mu}(x) J_{h,\nu}(0)]|\bar{B}\rangle \tag{4.73}$$

of the time ordered product concerning the arbitrary current $J_{h,\mu}$ at the space time points x and 0 which describes the forward scattering amplitude of a B-meson. In our case the current $J_{h,\mu}$ is given by the extended hadronic current (4.52). Using the definition of the time ordered product

$$T[\phi(x)\phi(y)] = \Theta(x^0 - y^0)\phi(x)\phi(y) + \Theta(y^0 - x^0)\phi(y)\phi(x) \tag{4.74}$$

we obtain

$$T_{\mu\nu} = \frac{-i}{2m_B} \int \mathrm{d}^4x e^{-iqx} \Big(\langle \bar{B}|J^\dagger_{h,\mu}(x) J_{h,\nu}(0)|\bar{B}\rangle \theta(x_0) + \langle \bar{B}|J_{h,\nu}(0) J^\dagger_{h,\mu}(x)|\bar{B}\rangle \theta(-x_0) \Big) . \tag{4.75}$$

In the following we will introduce intermediate states between the two currents to rewrite the scattering amplitude $T_{\mu\nu}$ into the hadronic tensor $W_{\mu\nu}$ according to the optical theorem. At first, we therefore have to introduce intermediate states between the currents $J_{h,\nu}(0)$ and $J_{h,\mu}(x)$. Hence we have two time orderings with two different intermediate states. Since $J_{h,\nu}(0) = \bar{c}(0)\Gamma_\nu b(0)$ contains a creator for a b-quark and an annihilator for a c-quark (and vice versa for the antiquarks of course), while the conjugated current $J^\dagger_{h,\mu}(x) = \bar{c}(x)\Gamma_\mu b(x)$ contains an annihilator for a b-quark and a creator for the c-quark respectively, the combination $J^\dagger_{h,\mu}(x) J_{h,\nu}(0)$ contains a single c quark and is therefore marked with X_c while the other one contains an anti-c quark and two b quarks and will be marked with $X_{\bar{c}bb}$. Obviously the intermediate state X_c is the one we need for our decay. We will later see that the other combination is eliminated by the phase space. Introducing the intermediate states, (4.75) turns into

$$\begin{aligned} T_{\mu\nu} = \frac{-i}{2m_B} \int \mathrm{d}^4x e^{-iqx} \Bigg(& \sum_{X_c} \langle \bar{B}|J^\dagger_{h,\mu}(x)|X_c\rangle \langle X_c|J_{h,\nu}(0)|\bar{B}\rangle \theta(x_0) \\ & + \sum_{X_{\bar{c}bb}} \langle \bar{B}|J_{h,\nu}(0)|X_{\bar{c}bb}\rangle \langle X_{\bar{c}bb}|J^\dagger_{h,\mu}(x)|\bar{B}\rangle \theta(-x_0) \Bigg). \end{aligned} \tag{4.76}$$

Here the matrix elements between the B meson and the intermediate states are still at two different space time points x and 0. The translation invariance allows us to rewrite this by

$$\begin{aligned} T_{\mu\nu} = \frac{-i}{2m_B} \int \mathrm{d}^4 x \Bigg(& \sum_{X_c} e^{-i(q-p_B+p_{X_c})x} \langle \bar{B} | J^\dagger_{h,\mu}(0) | X_c \rangle \langle X_c | J_{h,\nu}(0) | \bar{B} \rangle \theta(x_0) \\ & + \sum_{X_{\bar{c}bb}} e^{-i(q-p_{X_{\bar{c}bb}}+p_B)x} \langle \bar{B} | J_{h,\nu}(0) | X_{\bar{c}bb} \rangle \langle X_{\bar{c}bb} | J^\dagger_{h,\mu}(0) | \bar{B} \rangle \theta(-x_0) \Bigg). \end{aligned} \tag{4.77}$$

Additionally we use the definition of the theta function

$$\Theta(x^0) = -\frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \mathrm{d}\omega \frac{e^{-i\omega x^0}}{\omega + i\epsilon}. \tag{4.78}$$

to the time ordered product this to the form

$$\begin{aligned} T_{\mu\nu} = \frac{1}{4\pi m_B} \int \mathrm{d}^4 x \int \mathrm{d}\omega \Bigg(& \sum_{X_c} e^{-i(\omega+q^0-b^0_B+p^0_{X_c})x^0} e^{-i(\boldsymbol{q}+\boldsymbol{p}_{X_c})\cdot\boldsymbol{x}} \frac{\langle \bar{B} | J^\dagger_{h,\mu}(0) | X_c \rangle \langle X_c | J_{h,\nu}(0) | \bar{B} \rangle}{\omega + i\epsilon} \\ & + \sum_{X_{\bar{c}bb}} e^{i(\omega-q^0+p_{\bar{c}bb}-p^0_B)x^0} e^{-i(\boldsymbol{q}-\boldsymbol{p}_{\bar{c}bb})\cdot\boldsymbol{x}} \frac{\langle \bar{B} | J_{h,\nu}(0) | X_{\bar{c}bb} \rangle \langle X_{\bar{c}bb} | J^\dagger_{h,\mu}(0) | \bar{B} \rangle}{\omega + i\epsilon} \Bigg). \end{aligned} \tag{4.79}$$

To perform the integration we also need the definition

$$\delta(y) = 2\pi \int\limits_{-\infty}^{\infty} \mathrm{d}x\, e^{-ixy} \tag{4.80}$$

of the delta function. Using this on (4.79) we obtain

$$\begin{aligned} T_{\mu\nu} = \frac{(2\pi)^3}{2m_B} \int \mathrm{d}\omega \Bigg(& \sum_{X_c} \delta(\omega + q^0 - b^0_B + p^0_{X_c}) \delta(\boldsymbol{q} + \boldsymbol{p}_{X_c}) \frac{\langle \bar{B} | J^\dagger_{h,\mu}(0) | X_c \rangle \langle X_c | J_{h,\nu}(0) | \bar{B} \rangle}{\omega + i\epsilon} \\ & + \sum_{X_{\bar{c}bb}} \delta(\omega - q^0 + p_{\bar{c}bb} - p^0_B) \delta(\boldsymbol{q} - \boldsymbol{p}_{\bar{c}bb}) \frac{\langle \bar{B} | J_{h,\nu}(0) | X_{\bar{c}bb} \rangle \langle X_{\bar{c}bb} | J^\dagger_{h,\mu}(0) | \bar{B} \rangle}{\omega + i\epsilon} \Bigg). \end{aligned} \tag{4.81}$$

The delta functions now present a relation between between ω and the phase space. The integration gives

$$\begin{aligned} T_{\mu\nu} = & \sum_{X_c} \frac{\langle \bar{B} | J^\dagger_{h,\mu}(0) | X_c \rangle \langle X_c | J_{h,\nu}(0) | \bar{B} \rangle}{2m_B(m_B - E_{X_c} - q^0 + i\epsilon)} (2\pi)^3 \delta(\boldsymbol{q} + \boldsymbol{p}_{X_c}) \\ & - \sum_{X_{\bar{c}bb}} \frac{\langle \bar{B} | J_{h,\nu}(0) | X_{\bar{c}bb} \rangle \langle X_{\bar{c}bb} | J^\dagger_{h,\mu}(0) | \bar{B} \rangle}{2m_B(E_{X_{\bar{c}bb}} - m_B - q^0 - i\epsilon)} (2\pi)^3 \delta(\boldsymbol{q} - \boldsymbol{p}_{X_{\bar{c}bb}}), \end{aligned} \tag{4.82}$$

where we additionally used that we are in the B-meson rest frame ($p^0_B = m_B$) and the 0th components of the momenta of the hadronic states can be written by their energies ($p^0_{X_c} = E_{X_c}$

and $p^0_{X_{\bar{c}bb}} = E_{X_{\bar{c}bb}}$). As the final states X_c and $X_{\bar{c}bb}$ can contain different values of energies, the denominators of (4.82) do not create poles but cuts along the real axis of the complex q^0 plain. The cuts can be found in the regions

$$-\infty < q^0 \le m_B - E_{X_c^{\min}} = m_B - \sqrt{m^2_{X_c^{\min}} + |\boldsymbol{q}^2|} \tag{4.83}$$

for the case of X_c and

$$E_{X_{\bar{c}bb}^{\min}} - m_B = \sqrt{m^2_{X_{\bar{c}bb}^{\min}} + |\boldsymbol{q}^2|} - m_B \le q^0 < \infty \tag{4.84}$$

for the case of $X_{\bar{c}bb}$. For the kinematics of our decay $B \to X_c\, e\, \nu_e$ these cuts are widely separated for all values of q which justifies the use of Cauchy's integral formula (4.223). Using the optical theorem (4.220) we can rewrite (4.82) to

$$\begin{aligned} -\frac{1}{\pi}\mathrm{Im}\, T_{\mu\nu} =& \frac{1}{2m_B}\sum_{X_c}\langle \bar{B}|J^\dagger_{h,\mu}(0)|X_c\rangle\langle X_c|J_{h,\nu}(0)|\bar{B}\rangle\delta(p_B - q - p_{X_c}) \\ &- \frac{1}{2m_B}\sum_{X_{\bar{c}bb}}\langle \bar{B}|J_{h,\nu}(0)|X_{\bar{c}bb}\rangle\langle X_{\bar{c}bb}|J^\dagger_{h,\mu}(0)|\bar{B}\rangle\delta(p_B + q - p_{X_{\bar{c}bb}}). \end{aligned} \tag{4.85}$$

If we consider the kinematics $p_B = q + p_X \Rightarrow p_B + q > p_X$ of our decay we immediately see that the delta distribution of the second term always vanishes. We end up with the identity

$$-\frac{1}{\pi}\mathrm{Im}\, T_{\mu\nu} = \frac{1}{2m_B}\sum_{X_c}\langle \bar{B}|J^\dagger_{h,\mu}(0)|X_c\rangle\langle X_c|J_{h,\nu}(0)|\bar{B}\rangle\delta(p_B - q - p_{X_c}) = W_{\mu\nu}. \tag{4.86}$$

The time ordered product can now be decomposed into scalar structure function

$$T_{\mu\nu} = -g_{\mu\nu}T_1 + v_\mu v_\nu T_2 - i\epsilon_{\mu\nu\eta\lambda}v^\eta v^\lambda T_3 + q_\mu q_\nu T_4 + (v_\mu q_\nu + v_\nu q_\mu)T_5. \tag{4.87}$$

in the same way we already have done for the hadronic tensor $W_{\mu\nu}$. The scalar structure functions T_i are then just dependent on the total lepton energy $v \cdot q$ and the leptonic invariant mass q^2. Furthermore, we can use the linearity of the imaginary part formation to use (4.86) for the separate structure functions. Therefore we obtain the relation

$$-\frac{1}{\pi}\mathrm{Im}\, T_i = W_i, \tag{4.88}$$

which of course has to be applied on the left-hand cut only.

4.3.2. Operator product expansion

Using the results from above we can now rewrite the triple differential decay rate (4.68) to

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2\mathrm{d}E_e\mathrm{d}E_\nu} = -\frac{G_F^2|V_{cb}|^2}{4m_B\,\pi}L^{\mu\nu}\,\mathrm{Im}\, T_{\mu\nu}, \tag{4.89}$$

which relates it to the time ordered product. Since only the time ordered product depends on hadronic quantities and therefore contains all interactions with the background field, we separate it for our considerations regarding the operator product expansion. Thus we can take

$$T_{\mu\nu} = \int \mathrm{d}^4x\, e^{-iq\cdot x}\langle B(p)|T[\bar{b}(x)\Gamma^\dagger_\mu c(x)\bar{c}(0)\Gamma_\nu b(0)]|B(p)\rangle\,, \tag{4.90}$$

as a starting point, where

$$\begin{aligned}\Gamma_{h,\mu} =& c_L\ \gamma_\mu P_L + c_R\ \gamma_\mu P_R + g_L\ i\overleftrightarrow{D_\mu} P_L + g_R\ i\overleftrightarrow{D_\mu} P_R \\ &+ d_L\ i\overleftrightarrow{\partial}^\nu (\bar{c}\, i\sigma_{\mu\nu} P_L) + d_R\ i\overleftrightarrow{\partial}^\nu (i\sigma_{\mu\nu} P_R)\,,\end{aligned} \tag{4.91}$$

is the Dirac structure of the extended hadronic current which has been introduced in section 4.1. To be able to do the operator product expansion (3.83), we have to rewrite the time ordered product a bit. Since we like to describe a B-meson of momentum $p_B = m_B v$ containing a single heavy b-quark, it is reasonable to describe the momentum of the b-quark inside the meson by

$$p_b = m_b\, v + k, \tag{4.92}$$

where $m_b\, v$ is the momentum coming from the motion of the B-meson and k is the residual momentum which derives from the motion of the b-quark inside the meson due to the interactions with the mesons background field as we have introduced in section 3.3. Thus we know it makes sense to perform the field redefinition

$$b(x) = e^{-im_b v\cdot x}\, b_v(x), \tag{4.93}$$

to separate the momentum $m_b\, v$ from the residual momentum k which describes the interactions with the mesons background field we are interested in. Additionally we will use Wick's theorem to rewrite the time ordered product into normal ordering:

$$\begin{aligned}T_{\mu\nu} = \int \mathrm{d}^4x\, e^{-i(m_b v-q)x}\Big(&\langle\bar{B}|\,\mathbf{:}\bar{b}_v(x)\Gamma_\mu c(x)\bar{c}(0)\Gamma_\nu b_v(0)\mathbf{:}\,|\bar{B}\rangle + \langle\bar{B}|\,\mathbf{:}\bar{b}_v(x)\Gamma^\dagger_\mu c(x)\bar{c}(0)\Gamma_\nu b_v(0)\mathbf{:}\,|\bar{B}\rangle \\ &+ \langle\bar{B}|\,\mathbf{:}\bar{b}_v(x)\Gamma^\dagger_\mu c(x)\bar{c}(0)\Gamma_\nu b_v(0)\mathbf{:}\,|\bar{B}\rangle + \langle\bar{B}|\,\mathbf{:}\bar{b}_v(x)\Gamma^\dagger_\mu c(x)\bar{c}(0)\Gamma_\nu b_v(0)\mathbf{:}\,|\bar{B}\rangle\Big).\end{aligned} \tag{4.94}$$

The only combination which survives is the one containing the contraction of only the c-quark fields, since all the other combinations vanish because of the external states. Now we rewrite the contraction of the c-quark fields into the c-quark propagator

$$iS_c = \int \frac{\mathrm{d}^4p_c}{(2\pi)^4}\frac{ie^{-ip_c x}}{\not{p}_c - m_c + i\epsilon}, \tag{4.95}$$

and obtain

$$\begin{aligned}T_{\mu\nu} =& \int \mathrm{d}^4x\, e^{-i(m_b v-q-p_c)\cdot x}\langle\bar{B}|\bar{b}_v(x)\Gamma^\dagger_\mu\left(\int \frac{\mathrm{d}^4p_c}{(2\pi)^4}\frac{ie^{-ip_c x}}{\not{p}_c - m_c + i\epsilon}\right)\Gamma_\nu\, b_v(0)|\bar{B}\rangle \\ =& \langle\bar{B}|\int \mathrm{d}^4x \int \frac{\mathrm{d}^4p_c}{(2\pi)^4}\, ie^{-i(m_b v-q-p_c)\cdot x}\bar{b}_v(x)\Gamma^\dagger_\mu \frac{1}{\not{p}_c - m_c + i\epsilon}\Gamma_\nu\, b_v(0)|\bar{B}\rangle.\end{aligned} \tag{4.96}$$

From now on we will not mark the normal ordering any more, since the residual operators are already given in normal ordering. The next step is to rewrite the $b_v(x)$ fields into momentum space. This gives us

$$T_{\mu\nu} = \langle\bar{B}|\int \mathrm{d}^4x \int \frac{\mathrm{d}^4p_c}{(2\pi)^4}\, ie^{-i(m_b v-q-p_c)\cdot x}\int \frac{\mathrm{d}^4k}{(2\pi)^4}e^{ik\cdot x}\,\tilde{\bar{b}}_v(k)\Gamma^\dagger_\mu \frac{1}{\not{p}_c - m_c + i\epsilon}\Gamma_\nu\, \tilde{b}_v(0)|\bar{B}\rangle. \tag{4.97}$$

Only the exponential functions are still dependent on x and thus we can write

$$\begin{aligned} T_{\mu\nu} =& \langle \bar{B} | \int \frac{\mathrm{d}^4 p_c}{(2\pi)^4} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \, \tilde{\bar{b}}_v(k) \Gamma_\mu^\dagger \frac{i}{\not{p}_c - m_c + i\epsilon} \Gamma_\nu \, \tilde{b}_v(0) | \bar{B} \rangle \int \mathrm{d}^4 x e^{-i((m_b v - q + k) - p_c) \cdot x} \\ =& \langle \bar{B} | \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \int \mathrm{d}^4 p_c \, \tilde{\bar{b}}_v(k) \Gamma_\mu^\dagger \frac{i \delta^4\big((m_b v - q + k) - p_c\big)}{\not{p}_c - m_c + i\epsilon} \Gamma_\nu \, \tilde{b}_v(0) | \bar{B} \rangle \\ =& \langle \bar{B} | \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \, \tilde{\bar{b}}_v(k) \Gamma_\mu^\dagger \frac{i}{m_b \not{v} - \not{q} + \not{k} - m_c + i\epsilon} \Gamma_\nu \, \tilde{b}_v(0) | \bar{B} \rangle, \end{aligned} \tag{4.98}$$

where we used the definition (4.80) of the delta distribution in the second step and integrated over p_c in the last step. From this equation we can extract the c-quark propagator

$$i \, S_{\mathrm{BGF}} = \frac{1}{m_b \not{v} - \not{q} + \not{k} - m_c} = \frac{1}{\not{Q} + \not{k} - m_c} \tag{4.99}$$

describing the motion of a c-quark in the background field of the B meson. Here we introduced the quantity $\not{Q} = m_b \not{v} - \not{q}$ as a shortcut for further considerations. The residual momentum is because of $k \approx \Lambda_{\mathrm{QCD}} \ll m_b$ small, so it is reasonable to expand the propagator in k. A calculation to the order $1/m_b^n$ requires to expand this expression to n^{th} order in k according to

$$i S_{\mathrm{BGF}} = \frac{1}{\not{Q} - m_c} - \frac{1}{\not{Q} - m_c} \not{k} \frac{1}{\not{Q} - m_c} + \frac{1}{\not{Q} - m_c} \not{k} \frac{1}{\not{Q} - m_c} \not{k} \frac{1}{\not{Q} - m_c} + \ldots \tag{4.100}$$

Therefore we used the identity

$$\frac{1}{A + B} = \frac{1}{A} - \frac{1}{A} B \frac{1}{A + B} \tag{4.101}$$

recursively up to the desired order. Introducing the quantity $\Delta_0 = Q^2 - m_c^2 + i\epsilon$ we get

$$\begin{aligned} i S_{\mathrm{BGF}} =& \frac{1}{\Delta_0} (\not{Q} - m_c) + \frac{1}{\Delta_0^2} (\not{Q} - m_c) \not{k} (\not{Q} - m_c) \\ & - \frac{1}{\Delta_0^3} (\not{Q} - m_c) \not{k} (\not{Q} - m_c) \not{k} (\not{Q} - m_c) \\ & + \frac{1}{\Delta_0^4} (\not{Q} - m_c) \not{k} (\not{Q} - m_c) \not{k} (\not{Q} - m_c) \not{k} (\not{Q} - m_c) + \ldots . \end{aligned} \tag{4.102}$$

Inserting this into 4.98 we can transform this into the momentum space of the strong decay by evaluating the remaining integral over k. Using

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \, (k_\mu k_\nu \ldots) \hat{\bar{b}}(k) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{i x \cdot k} \, (k_\mu k_\nu \ldots) \hat{\bar{b}}(k) \Big|_{x=0} = (\partial_\mu \partial_\nu \ldots) \bar{b}(0) \tag{4.103}$$

we obtain

$$\begin{aligned} i S_{\mathrm{BGF}} =& \frac{1}{\Delta_0} (\not{Q} - m_c) + \frac{1}{\Delta_0^2} (\not{Q} - m_c) \not{\partial} (\not{Q} - m_c) \\ & - \frac{1}{\Delta_0^3} (\not{Q} - m_c) \not{\partial} (\not{Q} - m_c) \not{\partial} (\not{Q} - m_c) \\ & + \frac{1}{\Delta_0^4} (\not{Q} - m_c) \not{\partial} (\not{Q} - m_c) \not{\partial} (\not{Q} - m_c) \not{\partial} (\not{Q} - m_c) + \ldots . \end{aligned} \tag{4.104}$$

At this point we note that we have to replace all partial derivatives by covariant ones to ensure gauge covariance. Thus we end up with

$$
\begin{aligned}
T_{\mu\nu} = \langle \bar{B}|\bar{b}_v(0)\Gamma^\dagger_\mu \Bigg(& \frac{1}{\Delta_0}(\not{Q} - m_c) + \frac{1}{\Delta_0^2}(\not{Q} - m_c) i\not{D}(\not{Q} - m_c) \\
& - \frac{1}{\Delta_0^3}(\not{Q} - m_c) i\not{D}(\not{Q} - m_c) i\not{D}(\not{Q} - m_c) \\
& + \frac{1}{\Delta_0^4}(\not{Q} - m_c) i\not{D}(\not{Q} - m_c) i\not{D}(\not{Q} - m_c) i\not{D}(\not{Q} - m_c) + \dots \Bigg) \Gamma_\nu b_v(0)|\bar{B}\rangle,
\end{aligned}
\tag{4.105}
$$

which is obviously built up of local expressions. A crucial aspect of this kind of expansion is that it keeps track of the ordering of the covariant derivatives. This is astonishing at first sight, as we have dealt with momenta k in the first place which actually commute in momentum space, and transformed this quantity into a partial derivative in position space which is also a commutative quantity. But since all those quantities are contracted with noncommutative Dirac matrices we get the right ordering after the introduction of the A_μ fields of the covariant derivatives $iD_\mu = i\partial_\mu + g_s A_\mu$ by chance. This shows that a description of the interactions in a background field by a residual momentum which is not seen as an operator from the beginning, is a rather naive sight. However, in this case it works due to the ordering by the Dirac matrices and even prevents us from calculating one- or even more-gluon matrix elements (which is normally done in this case to retrieve the ordering).

4.3.3. Calculation of the forward matrix elements

The remaining task is to evaluate the forward matrix elements of operators of the form

$$\bar{b}_{v,\alpha}(iD_{\mu_1})\dots(iD_{\mu_n})b_{v,\beta}\,, \tag{4.106}$$

where $\bar{b}_v$ (b_v) carries the spinor index α (β). Note that we omit to write the dependence of x, since we have $x = 0$ for both fields. The field b_v is still the full QCD field, but redefined by a phase factor

$$b(x) = e^{-im_b\, v\cdot x} b_v(x) \tag{4.107}$$

to remove the large piece of the b-quark momentum according to (4.93). The argument of the fields will be suppressed in the following to improve the readability. To be able to work with this definition we first have to analyze the impact of this redefinition on the Dirac equation of motion. Therefore we start using the covariant derivative on both sides of (4.107). This gives us

$$
\begin{aligned}
iD_\mu b &= i\big(\partial_\mu + igA_\mu\big)b \\
&= i\big(\partial_\mu e^{-im_b\, v\cdot x}\big)b_v + ie^{-im_b\, v\cdot x}\big(\partial_\mu b_v\big) - gA_\mu b_v \\
&= m_b v_\mu b_v e^{-im_b\, v\cdot x} + e^{-im_b\, v\cdot x}\big(i\partial_\mu + igA_\mu\big)b_v \\
&= e^{-im_b\, v\cdot x}\big(m_b v_\mu + iD_\mu\big)b_v\,,
\end{aligned}
\tag{4.108}
$$

which can be used to rewrite the Dirac equation to

$$0 = (i\not{D} - m_b)b = e^{-im_b\, v\cdot x}\big(m_b(\not{v} - 1) + i\not{D}\big)b_v \Rightarrow \not{v} b_v = b_v - \frac{1}{m_b} i\not{D} b_v \tag{4.109}$$

Additionally applying the definition of the projectors $P_\pm = (1 \pm \not{v})/2$ on the large and small components of a Dirac spinor (which have already been discussed in section 3.3), this equation can also be written in the form

$$P_- b_v = \frac{1}{2m_b} i\not{D} b_v\,. \tag{4.110}$$

Using that the product of the projectors P_+ and P_- vanishes we additionally obtain

$$0 = P_+ P_- b_v = P_+ \frac{1}{2m_b} i\not{D} \Rightarrow P_+ i\not{D} b_v = 0. \tag{4.111}$$

from this. Furthermore, we can analyze

$$\begin{aligned} 0 &= (i\not{D} + m_b)(i\not{D} - m_b) b \\ &= (i\not{D} i\not{D} - m_b^2) b \\ &= e^{-im_b\, v\cdot x}\big((m_b \not{v} + i\not{D})(m_b \not{v} + i\not{D}) - m_b^2\big) b_v \\ &= e^{-im_b\, v\cdot x}\big(2m_b(ivD) + i\not{D} i\not{D}\big) b_v, \end{aligned} \tag{4.112}$$

which can be solved to

$$(ivD) b_v = -\frac{1}{2m_b}(i\not{D})^2 b_v. \tag{4.113}$$

Thus all in all the field satisfies the useful relations

$$\not{v} b_v = b_v - \frac{1}{m_b} i\not{D} b_v \tag{4.114}$$

$$P_+ b_v = -\frac{1}{2m_b} i\not{D} b_v + b_v \tag{4.115}$$

$$P_- b_v = \frac{1}{2m_b} i\not{D} b_v \tag{4.116}$$

$$(ivD) b_v = -\frac{1}{2m_b} i\not{D}\, i\not{D} b_v \tag{4.117}$$

As mentioned in section 3.4 the heavy quark expansion will yield only matrix elements of local operators. However, these matrix elements still contain a nontrivial mass dependence which will be discussed in the following. The evaluation of the matrix elements is performed in a new recursive way. This means that the starting point is always the matrix element of the highest dimension which contains the maximal number of derivatives. These matrix elements can be treated in the static limit $m_b \to \infty$ from section 3.3 which implies that we can neglect all contributions of $1/m_b$ relative to these matrix elements as they belong to higher orders in the operator product expansion. Thus we will at first consider the static limit of the forward matrix element of the highest dimensional operator which has the form [76]

$$\begin{aligned} \langle B(p)| b_{v,\alpha}(iD_{\mu_1})\dots(iD_{\mu_n}) b_{v,\beta} |B(p)\rangle &= \langle \mathrm{B}_v| h_{v,\alpha}(iD_{\mu_1})\dots(iD_{\mu_n}) h_{v,\beta} |\mathrm{B}_v\rangle + \mathcal{O}(1/m_b) \\ &= 1_{\beta\alpha} A_{\mu_1\mu_2\dots\mu_n} + s_\lambda B^\lambda_{\mu_1\mu_2\dots\mu_n} \end{aligned} \tag{4.118}$$

where $s_\lambda = P_+(-i\,\sigma_{\mu\nu})P_+$ is the generalization of the Pauli matrices to the case $v \neq (1,0,0,0)$ and $|\mathrm{B}_v\rangle$ is the static limit of the B meson state $|B(p)\rangle$. Like in section 3.3 the quantities P_+ denote the projectors $P_\pm = (1 \pm \not{v})/2$. The tensor structures A and B have to be related to a minimal set of fundamental matrix elements. To calculate the spin independent parameters the matrix element $\langle B| b_v(iD_{\mu_1})\dots.(iD_{\mu_n}) b_v |B\rangle$ has to be contracted in all possible ways with combinations of the metric tensor $g_{\mu,\nu}$ and the four velocity v_μ. For the spin dependent parameters we have to consider the all contractions of the matrix element $\langle B|b_v(iD_{\mu_1})\dots(iD_{\mu_n})(-i\sigma_{\rho\xi})b_v|B\rangle$ in an analogous way. Note that this leads to the same contractions as for the matrix elements of order $n+2$. The number of matrix elements can be reduced by the fact that we only have to consider the static matrix elements, since the other contributions are described by the nonperturbative parameters of higher orders because of (4.114-4.117). Thus all combinations $\langle B|b_v(iD_{\mu_1})\dots(ivD)b_v|B\rangle$ and $\langle B|b_v(ivD)....(iD_{\mu_n})b_v|B\rangle$ vanish, as 4.117 becomes $ivDb_v=0$ for the static case. In addition to that, we have to keep the antisymmetry of the $(-i\sigma_{\rho\xi})$ in mind. According to $P_+(-i\sigma^{\mu\nu})P_+ = i\,e^{\mu\nu\alpha\beta}\,v_\alpha\,s_\beta$ all contractions with the vector v_μ vanish. Of course also the full contractions of the $(-i\sigma_{\rho\xi})$ with the metric tensor disappear. This results in a further reduction of the number of contractions for the spin dependent case. All of our non perturbative operators have to be taken between forward matrix elements. From the OPE it is understood that the derivatives can act on both sides, since the momentum state is the same. So we can show that some of the nonperturbative matrix elements are actually related by the time transformation

$$\langle A_1|\hat{O}|A_2\rangle \stackrel{!}{=} T(\langle A_1|\hat{O}|A_2\rangle) = \langle A_1|\hat{O}|A_2\rangle^* = \langle A_2|\hat{O}^\dagger|A_1\rangle\,. \tag{4.119}$$

From this we get an additional reduction of the matrix elements, starting at dimension 7. The remaining fundamental matrix elements describe all the structures we retain after the evaluation of the traces. Reconsidering equation (3.57) we see that in the static the HQE contains the same quantities as the HQET, leading to the same fundamental matrix elements. However, the definitions of these matrix elements will differ in HQE and HQET for the lower dimensions, since we get further terms of (3.57) in HQET which do not appear in the HQE. As the fundamental matrix elements have to be extracted from experiment, it will make no difference how we choose to define the matrix elements, as they do not have to be calculated from first principles. Thus we find that it makes no sense to perform an expansion of the b_v fields in the sense of (3.57).

Once the tensors A and B for the matrix elements of the highest order in the $1/m_b$ expansion have been calculated, we proceed to the matrix elements of dimension $n-1$. Now we have to take into account all possible Dirac structures, as the lowest order $1/m_b$ corrections for this case are now of the same order as the higher order. So we get

$$\langle B(p)|b_{v,\alpha}(iD_{\mu_1})....(iD_{\mu_{n-1}})b_{v,\beta}|B(p)\rangle = \sum_i \hat{\Gamma}^{(i)}_{\beta\alpha} A^{(i)}_{\mu_1\mu_2\cdots\mu_{n-1}} \tag{4.120}$$

where $\hat{\Gamma}^{(i)}$ are the complete set of the sixteen Dirac matrices. The calculation of the spin independent and spin dependent matrix elements works nearly in the same manner as in the static case, beside the fact that the relations (4.93-4.117) now connect different orders of the $1/m_b$ expansion instead of eliminating the combinations $\langle B|b_v(iD_{\mu_1})\dots(ivD)b_v|B\rangle$ and $\langle B|b_v(ivD)....(iD_{\mu_n})b_v|B\rangle$. Thus we may now express the tensor coefficients $A^{(i)}$ in terms of the basic parameters of the order $1/m_b^{n-1}$ and the ones of the order $1/m_b^n$. This prescription defines a way to recursively compute the relevant matrix elements of the $1/m_b$ expansion up to order $1/m_b^n$ at tree level, starting from the operator of the highest dimension. Thus the leading matrix element of dimension 3 will then be expressed by this recursive method as a series in $1/m_b^n$ involving all the basic parameters up to this order.

The number of independent parameters can be calculated by the formula

$$f(n) = \sum_{n_g=1}^{[\frac{n}{2}]} (2n_g-1)!! \binom{n-2}{n-2n_g} \tag{4.121}$$

for the spin independent terms and

$$f_\sigma(n) = \sum_{n_g=1}^{[\frac{n}{2}]-1} (2n_g-1)!! \binom{n-2}{n-2n_g-2} \binom{2+2n_g}{2} \tag{4.122}$$

for the spin dependent terms. Here the brackets denote the binomial coefficient

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}. \tag{4.123}$$

For both cases n is again the number of covariant derivatives which have to be contracted with either v_μ or one index of $g_{\mu\nu}$. n_g denotes the number of $g_{\mu\nu}$ used for the contraction. Thus the number of v_μ is determined by $n - 2n_g$ since each $g_{\mu\nu}$ contains two indices. We sum over all possible numbers of n_g, where we start by 1 (as the case of only v_μ is excluded by the equations of motion) and sum up to $[n/2]$ denoting the integer part of $n/2$ which is the maximum number of $g_{\mu\nu}$ we can get. The first part of the sum represents the number of different orderings of the indices which can occur for n_g contractions with $g_{\mu\nu}$. As an example we can take the case $n_g = 2$, where we can have three orderings, namely $g_{\mu\nu}g_{\rho\sigma}$, $g_{\mu\sigma}g_{\nu\rho}$ and $g_{\mu\rho}g_{\nu\sigma}$. To calculate the number of possible contractions begin with a separation of the $g_{\mu\nu}$ into their indices. Denoting with every g a position in the matrix element which is contracted with one side of a $g_{\mu\nu}$ we write

$$ggg\ldots gg \tag{4.124}$$

containing $2n_g$ entries of g. Contracting one pair of g's we get for example

$$\overbrace{ggg\ldots g}\ldots g \tag{4.125}$$

For this first contraction we had $2n_g - 1$ possibilities to link two of the g's, since we always have to subtract one possibility, as we first have to choose one g to contract with another and then one g which is contracted to the chosen g. For the next contraction we obviously have $2n_g - 3$ possibilities of contraction left. All in all we get the double factorial of $(2n_g - 1)$. Note that this formula respects that the $g_{\mu\nu}$ are indistinguishable as well as their symmetric structure. The second part of (4.121) gives the possibilities to distribute the v_μ between the various contractions of $g_{\mu\nu}$. The v's are indistinguishable, while the gaps between the g's are distinguishable. The number of different distributions is the given by the binomial coefficient

$$\binom{k+j-1}{k}, \tag{4.126}$$

where k denotes the number of v's while j denotes the number of gaps between the g's. Using $j = n_g - 1$ for the number of gaps and $k = n - n_g$ for the number of v's we obtain the binomial given in (4.121). The situation is nearly the same for (4.122), since the $\sigma_{\mu\nu}$ behaves equal to the $g_{\mu\nu}$. The only difference is that it can be distinguished from the $g_{\mu\nu}$. Therefore we set $n_g \to n_g - 1$ and allocate the two indices of the $\sigma_{\mu\nu}$ between the g's as we did for the v's. This time we have to consider additionally the position outside the g's, thus we have to rise the number of places by two for the $\sigma_{\mu\nu}$'s indices (of course not for the v's). Furthermore, the number of v's is now described by $n - n_g - 2$, as we have to take into account the indices of the $\sigma_{\mu\nu}$ which also reduce the v's numbers. Note that the formulae (4.121-4.122) do not contain reduction of the parameters due to the time invariance (4.119). Up to now we have not found a way to easily implement this into the formula.

Derivatives	2	3	4	5	6	7	8
Spin dependent	1	1	4	7	24	60	216
Spin independent	1	1	5	11	48	150	624
Total	2	2	9	18	72	210	840

Table 4.1.: Number of parameters for different numbers of covariant derivatives.

Besides the number of independent parameters we can explicitly calculate how the parameters look by a PYTHON script. This script calculates the contractions by setting up all possible contractions and eliminating entries which are the same by e.g. time invariance by comparison. Since this script of course also retains the correct number of parameters we can extract that for high n their number is reduced by time invariance by about a factor of 2. Even though the script uses hashes for comparison and therefore should go less than quadratic in time with increasing n, it will start to get slow at about $n = 9$. The calculation time could possibly be decreased with other algorithms or using a faster programming language like c or fortran. However, since we like to perform our calculations up to $n = 4$, this script should be sufficient for our needs. Table 4.1 shows the number of parameters for the different numbers of covariant derivatives as obtained from the PYTHON script. Here we immediately see, that calculations for operators of higher dimensions than 7, or 4 covariant derivatives respectively, will make no sense, since the number of constants becomes much to high – at least if no way is found to calculate or resum the constants.

In the following we explicitly perform the recursive calculation up to order $1/m_b^4$. For each dimension of the expansion we get a set of nonperturbative parameters which can be calculated by the PYTHON script.

Dimension 7

For the matrix elements of order $1/m_b^4$ in the expansion we get matrix elements of dimension 7 which contain 4 covariant derivatives. According to (4.118) the general structure of the forward matrix element is given by

$$\langle \bar{B}_v | \bar{b}_{v,\xi} (iD_\rho)(iD_\sigma)(iD_\lambda)(iD_\delta) b_{v,\eta} | \bar{B}_v \rangle = A^{(1)}_{\rho\sigma\lambda\delta} P_+ + A^{(4)}_{\rho\sigma\lambda\delta\alpha\beta} P_+ (-i\sigma^{\alpha\beta}) P_+ . \tag{4.127}$$

Note that the states still carry spinor indices. Thus we can multiply any desired Dirac matrices into the matrix element. In the following we will observe the structure of the $A^{(i)}$. These quantities can only depend on v_ρ and $g_{\rho\sigma}$ as the right hand side of (4.127) does not contain any Dirac structure by definition. The most common structure for the spin independent term turns out to be

$$\begin{aligned} A^{(1)}_{\rho\sigma\lambda\delta} = & a_1 g^{\lambda\delta} g^{\rho\sigma} + b_1 v^\delta v^\lambda g^{\rho\sigma} + c_1 v^\delta g^{\rho\lambda} v^\sigma + d_1 v^\lambda g^{\rho\delta} v^\sigma + e_1 v^\delta v^\rho g^{\sigma\lambda} \\ & + f_1 v^\lambda v^\rho g^{\sigma\delta} + g_1 v^\rho g^{\lambda\delta} v^\sigma + h_1 v^\delta v^\lambda v^\rho v^\sigma + i_1 g^{\rho\lambda} g^{\sigma\delta} + j_1 g^{\rho\delta} g^{\sigma\lambda} , \end{aligned} \tag{4.128}$$

while we can write

$$
\begin{aligned}
A^{(4)}_{\rho\sigma\lambda\delta\alpha\beta} = {} & a_4 \left(g^{\delta\beta} g^{\lambda\alpha} g^{\rho\sigma} - g^{\delta\alpha} g^{\lambda\beta} g^{\rho\sigma}\right) + b_4 \left(g^{\delta\beta} g^{\rho\lambda} g^{\sigma\alpha} - g^{\delta\alpha} g^{\rho\lambda} g^{\sigma\beta}\right) \\
& + c_4 \left(g^{\lambda\beta} g^{\rho\delta} g^{\sigma\alpha} - g^{\lambda\alpha} g^{\rho\delta} g^{\sigma\beta}\right) + d_4 \left(g^{\delta\beta} g^{\rho\alpha} g^{\sigma\lambda} - g^{\delta\alpha} g^{\rho\beta} g^{\sigma\lambda}\right) \\
& + e_4 \left(g^{\lambda\beta} g^{\rho\alpha} g^{\sigma\delta} - g^{\lambda\alpha} g^{\rho\beta} g^{\sigma\delta}\right) + f_4 \left(g^{\lambda\delta} g^{\rho\alpha} g^{\sigma\beta} - g^{\lambda\delta} g^{\rho\beta} g^{\sigma\alpha}\right) \\
& + g_4 \left(v^{\delta} v^{\lambda} g^{\rho\alpha} g^{\sigma\beta} - v^{\delta} v^{\lambda} g^{\rho\beta} g^{\sigma\alpha}\right) + h_4 \left(v^{\delta} v^{\sigma} g^{\lambda\beta} g^{\rho\alpha} - v^{\delta} v^{\sigma} g^{\lambda\alpha} g^{\rho\beta}\right) \\
& + i_4 \left(v^{\lambda} v^{\sigma} g^{\delta\beta} g^{\rho\alpha} - v^{\lambda} v^{\sigma} g^{\delta\alpha} g^{\rho\beta}\right) + j_4 \left(v^{\delta} v^{\rho} g^{\lambda\beta} g^{\sigma\alpha} - v^{\delta} v^{\rho} g^{\lambda\alpha} g^{\sigma\beta}\right) \\
& + k_4 \left(v^{\lambda} v^{\rho} g^{\delta\beta} g^{\sigma\alpha} - v^{\lambda} v^{\rho} g^{\delta\alpha} g^{\sigma\beta}\right) + l_4 \left(v^{\rho} v^{\sigma} g^{\delta\beta} g^{\lambda\alpha} - v^{\rho} v^{\sigma} g^{\delta\alpha} g^{\lambda\beta}\right)
\end{aligned}
\tag{4.129}
$$

for the spin dependent term. Under consideration of (4.114-4.117) in the static limit and (4.119) we obtain

$$2m_B\, s_1 = \langle \bar{B}(p)|\bar{b}_v iD_\rho (ivD)^2 iD^\rho b_v|\bar{B}(p)\rangle \tag{4.130}$$
$$2m_B\, s_2 = \langle \bar{B}(p)|\bar{b}_v iD_\rho (iD)^2 iD^\rho b_v|\bar{B}(p)\rangle \tag{4.131}$$
$$2m_B\, s_3 = \langle \bar{B}(p)|\bar{b}_v ((iD)^2)^2 b_v|\bar{B}(p)\rangle \tag{4.132}$$
$$2m_B\, s_4 = \langle \bar{B}(p)|\bar{b}_v iD_\mu iD_\nu iD^\mu iD^\nu b_v|\bar{B}(p)\rangle \tag{4.133}$$
$$2m_B\, s_5 = \langle \bar{B}(p)|\bar{b}_v iD_\mu iD_\nu (iD)^2 (-i\sigma^{\mu\nu}) b_v|\bar{B}(p)\rangle \tag{4.134}$$
$$2m_B\, s_6 = \langle \bar{B}(p)|\bar{b}_v iD_\mu (iD)^2 iD_\nu (-i\sigma^{\mu\nu}) b_v|\bar{B}(p)\rangle \tag{4.135}$$
$$2m_B\, s_7 = \langle \bar{B}(p)|\bar{b}_v iD_\mu (ivD)^2 iD_\nu (-i\sigma^{\mu\nu}) b_v|\bar{B}(p)\rangle \tag{4.136}$$
$$2m_B\, s_8 = \langle \bar{B}(p)|\bar{b}_v iD_\rho iD_\mu iD_\nu iD^\rho (-i\sigma^{\mu\nu}) b_v|\bar{B}(p)\rangle \tag{4.137}$$
$$2m_B\, s_9 = \langle \bar{B}(p)|\bar{b}_v iD_\rho iD_\mu iD^\rho iD_\nu (-i\sigma^{\mu\nu}) b_v|\bar{B}(p)\rangle \tag{4.138}$$

as a list of independent parameters. To calculate the coefficients in (4.128) and (4.129) we use the orthogonality of the Dirac matrices. Therefore we multiply both sides of (4.127) with the corresponding coefficient and express the left hand side by the independent parameters (4.130-4.138). This gives us a set of equations which we can solve according to the constants in (4.128) and (4.129). Note that the structures which contain v_ρ or v_δ vanish because of the equations of motion (4.114-4.117) since we analyze the static limit of the forward matrix element. Using the results in (4.127), we obtain the trace formulae. The calculation has been performed within a Mathematica notebook. We do not display the result for the trace formula here, as it is far too lengthy. However, it is given in the appendix C.

Besides the structure of the parameters and their number, we are also interested in their physical meaning. Therefore we have to rewrite the parameters (4.130-4.138) in terms of the electric field $\boldsymbol{E}$, the magnetic field $\boldsymbol{B}$ and the momentum $\boldsymbol{p}$ of the external state for the spin independent terms. For the spin dependent terms we also have the spin vector $\boldsymbol{s} = (\tau_1, \tau_2, \tau_2)$ containing the Pauli matrices τ_i. Additionally we will encounter the covariant derivative $i\boldsymbol{\nabla}$ which denotes the derivative with respect to the QCD fields. Remember that the dimensions of the quantities $\dim[\boldsymbol{v}] = \dim[\boldsymbol{p}] = \dim[\boldsymbol{\nabla}] = 1$, $\dim[\boldsymbol{E}] = \dim[\boldsymbol{B}] = 2$ and $\dim[b_v] = 3/2$ as already discussed in section B.3. When we set up the list of physical quantities, we have to remember that we are dealing with noncommutative quantities. Thus combinations as for example $\boldsymbol{E} \times \boldsymbol{E}$ do not vanish. For dimension 7 we can therefore set up the 9 independent parameters

$$2m_B\, m_1^4 = (\boldsymbol{p}^2)^2 \qquad (4.139)$$
$$2m_B\, m_2^4 = \boldsymbol{E}^2 \qquad (4.140)$$
$$2m_B\, m_3^4 = \boldsymbol{B}^2 \qquad (4.141)$$
$$2m_B\, m_4^4 = \boldsymbol{p}\cdot(i\boldsymbol{\nabla}\times\boldsymbol{B}) \qquad (4.142)$$
$$2m_B\, m_5^4 = \boldsymbol{s}\cdot(\boldsymbol{E}\times\boldsymbol{E}) \qquad (4.143)$$
$$2m_B\, m_6^4 = \boldsymbol{s}\cdot(\boldsymbol{B}\times\boldsymbol{B}) \qquad (4.144)$$
$$2m_B\, m_7^4 = \boldsymbol{p}^2(\boldsymbol{s}\cdot\boldsymbol{B}) \qquad (4.145)$$
$$2m_B\, m_8^4 = (\boldsymbol{s}\times\boldsymbol{p})\cdot(\boldsymbol{p}\times\boldsymbol{B}) \qquad (4.146)$$
$$2m_B\, m_9^4 = (\boldsymbol{s}\times i\boldsymbol{\nabla})\cdot(i\boldsymbol{\nabla}\times\boldsymbol{B}) \qquad (4.147)$$

The next step consists of expressing the parameters (4.130-4.138) which have been used throughout the calculation in the MATHEMATICA notebook by the parameters (4.139-4.147). Therefore we will make use of a covariant form of the Maxwell equations. In classical electrodynamics the Maxwell equations take the form

$$\partial_\mu F^{\mu\nu} = j^\nu \quad \text{and} \quad \partial_\mu F^{\alpha\beta}\frac{1}{2}\epsilon^{\mu\nu\alpha\beta} = 0. \qquad (4.148)$$

Here we have defined the field strength tensor $F_{\mu\nu}$ in such a way that

$$E_i = F_{0i} \quad \text{and} \quad B_i = \frac{1}{2}\epsilon_{ijk}F_{jk} = \widetilde{F}_{0i} \quad (F_{ij} = \epsilon_{ijk}B_k)\,, \qquad (4.149)$$

where

$$\widetilde{F}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\mu\nu}. \qquad (4.150)$$

denotes the dual field strength tensor. Likewise we can define analog equations for the non-commutative case of QCD. We get

$$[D_\mu, G^{\mu\nu}] = j^\nu \quad \text{and} \quad [D_\mu, G^{\mu\nu}]\,\frac{1}{2}\epsilon^{\alpha\beta\mu\nu} = \tilde{j}^\nu, \qquad (4.151)$$

where we introduced the additional current $\tilde{j}^\nu$. The commutators have been used to ensure that the derivatives act only on the field strength tensors, and not on any outer states. From (A.16) directly follows the Bianchi identity which we can use to show that $\tilde{j}^\nu$ must vanish for the case of QCD:

$$\begin{aligned}
&[iD_\mu,[iD_\alpha,iD_\beta]] + [iD_\alpha,[iD_\beta,iD_\mu]] + [iD_\beta,[iD_\mu,iD_\alpha]] = 0\\
\Rightarrow\ &([iD_\mu,[iD_\alpha,iD_\beta]] + [iD_\alpha,[iD_\beta,iD_\mu]] + [iD_\beta,[iD_\mu,iD_\alpha]])\,\epsilon^{\mu\alpha\beta\nu} = 0\\
\Rightarrow\ &3\,[iD_\mu,[iD_\alpha,iD_\beta]]\,\epsilon^{\mu\alpha\beta\nu} = 0\\
\Rightarrow\ &-\frac{i}{2g_s}\,[iD_\mu,[iD_\alpha,iD_\beta]]\,\epsilon^{\mu\alpha\beta\nu} = [D_\mu, G_{\alpha,\beta}]\,\frac{1}{2}\epsilon^{\mu\alpha\beta\nu} = 0.
\end{aligned} \qquad (4.152)$$

Thus we obtain

$$[D_\mu, G^{\mu\nu}] = j^\nu \quad \text{and} \quad [D_\mu, G^{\mu\nu}]\,\frac{1}{2}\epsilon^{\alpha\beta\mu\nu} = 0 \qquad (4.153)$$

as QCD Maxwell equations. Therefore we have again a set of homogeneous and inhomogeneous equations - just like in classical electrodynamics. Furthermore, we shall make use of the conventions

$$iD_\mu = i\partial_\mu + g_s\, A_\mu \tag{4.154}$$

$$[iD_\mu, iD_\nu] = ig_s G_{\mu\nu} \tag{4.155}$$

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig_s\,[A_\mu, A_\nu] \tag{4.156}$$

$$\widetilde{G}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}G^{\mu\nu} \tag{4.157}$$

$$P_+(-i\sigma^{\mu\nu})P_+ = i\, e^{\mu\nu\alpha\beta}\, v_\alpha\, s_\beta \tag{4.158}$$

$$\Pi^{\mu\nu} = g^{\mu\nu} - v^\mu v^\nu\,. \tag{4.159}$$

The last identity defines a projector on the spacial components which we will use to rewrite the contractions with $g_{\mu\nu}$ in (4.130-4.138) into a form which is more appropriate for the representation (4.139-4.147). The relation (4.158) connects the vector of Pauli matrices $\boldsymbol{s}$ to the $\sigma_{\mu\nu}$ contained in the spin dependent terms for the case of a static matrix element. At this point we should remark that all parameters are defined in the static limit. Therefore this relation is also valid for the lower dimensional operators which are calculated in the following sections. Before we are able to rewrite the parameters (4.130-4.138) into the representation (4.139-4.147), we have to provide (4.139-4.147) in terms of iD, $G_{\mu\nu}$, $\sigma_{\mu\nu}$ and $\Pi_{\mu\nu}$. Using the conventions given above, we get

$$2m_B\, m_1^4 = \frac{1}{3}\langle\bar{B}|\bar{b}_v\, iD_\rho iD_\sigma iD_\lambda iD_\delta\, b_v|\bar{B}\rangle\, \left(\Pi^{\rho\sigma}\Pi^{\lambda\delta} + \Pi^{\rho\lambda}\Pi^{\sigma\delta} + \Pi^{\rho\delta}\Pi^{\sigma\lambda}\right) \tag{4.160}$$

$$2m_B\, m_2^4 = -\frac{1}{g_s^2}\langle\bar{B}|\bar{b}_v\, \left[iD_\rho, iD_\sigma\right]\left[iD_\lambda, iD_\delta\right]\, b_v|\bar{B}\rangle\, \Pi^{\rho\delta}v^\sigma v^\lambda \tag{4.161}$$

$$2m_B\, m_3^4 = -\frac{1}{2g_s^2}\langle\bar{B}|\bar{b}_v\, \left[iD_\rho, iD_\sigma\right]\left[iD_\lambda, iD_\delta\right]\, b_v|\bar{B}\rangle\, \Pi^{\rho\lambda}\Pi^{\sigma\delta} \tag{4.162}$$

$$2m_B\, m_4^4 = \frac{1}{ig_s}\langle\bar{B}|\bar{b}_v\, \Big[iD_\rho, \left[iD_\sigma, iD_\lambda\right]\Big] iD_\delta\, b_v|\bar{B}\rangle\, \Pi^{\rho\lambda}\Pi^{\sigma\delta} \tag{4.163}$$

$$2m_B\, m_5^4 = \frac{1}{ig_s^2}\langle\bar{B}|\bar{b}_v\, \left[iD_\rho, iD_\sigma\right]\left[iD_\lambda, iD_\delta\right]\big(-i\sigma_{\alpha\beta}\big)\, b_v|\bar{B}\rangle\ \Pi^{\alpha\rho}\Pi^{\beta\delta}v^\sigma v^\lambda \tag{4.164}$$

$$2m_B\, m_6^4 = -\frac{1}{ig_s^2}\langle\bar{B}|\bar{b}_v\, \left[iD_\rho, iD_\sigma\right]\left[iD_\lambda, iD_\delta\right]\big(-i\sigma_{\alpha\beta}\big)\, b_v|\bar{B}\rangle\, \left(\Pi^{\alpha\sigma}\Pi^{\beta\lambda}\Pi^{\rho\delta} - \Pi^{\alpha\rho}\Pi^{\beta\delta}v^\sigma v^\lambda\right) \tag{4.165}$$

$$2m_B\, m_7^4 = -\frac{1}{g_s}\langle\bar{B}|\bar{b}_v\, \left[iD_\rho, iD_\sigma\right] iD_\lambda\, iD_\delta\big(-i\sigma_{\alpha\beta}\big)\, b_v|\bar{B}\rangle\ \Pi^{\lambda\delta}\Pi^{\alpha\rho}\Pi^{\beta\sigma} \tag{4.166}$$

$$2m_B\, m_8^4 = -\frac{1}{g_s}\langle\bar{B}|\bar{b}_v\, \left[iD_\rho, iD_\sigma\right] iD_\lambda\, iD_\delta\big(-i\sigma_{\alpha\beta}\big)\, b_v|\bar{B}\rangle\ \Pi^{\sigma\lambda}\Pi^{\alpha\rho}\Pi^{\beta\delta} \tag{4.167}$$

$$2m_B\, m_9^4 = \frac{1}{g_s}\langle\bar{B}|\bar{b}_v\, \bigg[iD_\rho, \Big[iD_\sigma, \left[iD_\lambda, iD_\delta\right]\Big]\bigg]\big(-i\sigma_{\alpha\beta}\big)\, b_v|\bar{B}\rangle\ \Pi^{\rho\beta}\Pi^{\lambda\alpha}\Pi^{\sigma\delta}\,, \tag{4.168}$$

where the $[,]$ denote the commutators between the operators. By rewriting the projection operators $\Pi_{\mu\nu}$ back to $g_{\mu\nu} - v_\mu v_\nu$ and contracting we find the operators (4.139-4.147) given in terms of (4.130-4.138). Solving to (4.130-4.138) we obtain

$$s_1 = -\frac{g_s^2}{2} m_2^4 \tag{4.169}$$

$$s_2 = \frac{1}{3} m_1^4 - \frac{g_s^2}{2} m_2^4 - \frac{2\, g_s^2}{3} m_3^4 + \frac{i g_s}{3} m_4^4 \tag{4.170}$$

$$s_3 = m_1^4 \tag{4.171}$$

$$s_4 = \frac{1}{3} m_1^4 + \frac{g_s^2}{3} m_3^4 + \frac{i g_s}{3} m_4^4 \tag{4.172}$$

$$s_5 = -\frac{g_s}{2} m_7^4 \tag{4.173}$$

$$s_6 = i g_s^2\, m_5^4 - \frac{g_s}{2}\, m_7^4 - 2 g_s\, m_8^4 - \frac{g_s}{2} m_9^4 \tag{4.174}$$

$$s_7 = i g_s^2\, m_5^4 \tag{4.175}$$

$$s_8 = -i g_s^2\, m_5^4 - i g_s^2\, m_6^4 - \frac{g_s}{2} m_7^4 - \frac{g_s}{2} m_9^4 \tag{4.176}$$

$$s_9 = -\frac{g_s}{2} m_7^4 - g_s\, m_8^4 - \frac{g_s}{2} m_9^4 \tag{4.177}$$

However, all the calculations have been done in terms of $s_1, \ldots, s_9$ which simplifies the calculations considerably. The physical interpretation has been introduced only for convenience and makes more sense when we continue with the dimensions 6 and 5, where we compare the standard parameters μ_π^2, μ_G^2, ρ_D^3 and ρ_{LS}^3 from the literature with our newly introduced parameters μ_1, μ_2, ρ_1 and ρ_2 which follow our notation conventions. In other words: Besides the intuition which fields are contained in the parameters listed in (4.139 - 4.147) we do not get any benefit from rewriting the parameters - not even for experimental reasons in a fit, but we have shown that we should indeed have 9 parameters, since we have parameters as in (4.143 - 4.144) which can exist only due to the noncommutativity of the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields.

Dimension 6

For the order $1/m_b^3$ we have to introduce new matrix elements of dimension 6 containing 3 derivatives. The calculation for the trace formula of dimension 6 goes along the same lines as the dimension 7 one to a large extent. Again we start with a common formula for the matrix element which matches dimension 6 and has therefore 3 derivatives. The difference to the dimension 7 calculation is given by the fact that we cannot use the static limit any more. Thus we have to use (4.120) rather than (4.118) and obtain

$$\begin{aligned}\langle \bar{B}(p)|\bar{b}_{v,\xi}(iD_\rho)(iD_\sigma)(iD_\lambda)b_{v,\eta}|\bar{B}(p)\rangle =& A^{(1)}_{\rho\sigma\lambda} + A^{(2)}_{\rho\sigma\lambda\alpha}\gamma^\alpha + A^{(3)}_{\rho\sigma\lambda\alpha}\gamma^\alpha\gamma_5 \\ &+ A^{(4)}_{\rho\sigma\lambda\alpha\beta}(-i\sigma^{\alpha\beta}) + A^{(5)}_{\rho\sigma\lambda}\gamma_5.\end{aligned} \tag{4.178}$$

As for dimension 7 we can now set up the $A^{(i)}$ in their most common form. For the scalar structure we obtain

$$A^{(1)}_{\rho\sigma\lambda} = a_1 v^\lambda g^{\rho\sigma} + b_1 g^{\rho\lambda} v^\sigma + c_1 v^\rho g^{\sigma\lambda} + d_1 v^\lambda v^\rho v^\sigma, \tag{4.179}$$

while the vector structure looks like

$$\begin{aligned} A^{(2)}_{\rho\sigma\lambda\alpha} = & a_2 g^{\lambda\alpha} g^{\rho\sigma} + b_2 g^{\rho\lambda} g^{\sigma\alpha} + c_2 g^{\rho\alpha} g^{\sigma\lambda} + d_2 v^\alpha v^\lambda g^{\rho\sigma} \\ & + e_2 v^\alpha g^{\rho\lambda} v^\sigma + f_2 v^\lambda g^{\rho\alpha} v^\sigma + g_2 v^\alpha v^\rho g^{\sigma\lambda} \\ & + h_2 v^\lambda v^\rho g^{\sigma\alpha} + i_2 v^\rho g^{\lambda\alpha} v^\sigma + j_2 v^\alpha v^\lambda v^\rho v^\sigma . \end{aligned} \tag{4.180}$$

For the axial vector structure we encounter an additional subtlety. The term $i\epsilon_{\rho\sigma\lambda\alpha}\gamma^\alpha\gamma_5$ is not independent of the other possibilities since the relation

$$i\epsilon_{\rho\sigma\lambda\kappa}\gamma^\alpha\gamma_5 = i\epsilon^{\alpha\lambda\sigma\kappa} v^\kappa v^\rho - i\epsilon^{\alpha\lambda\rho\kappa} v^\kappa v^\sigma - i\epsilon^{\alpha\rho\sigma\kappa} v^\kappa v^\lambda + i\epsilon^{\lambda\rho\sigma\kappa} v^\kappa v^\alpha \tag{4.181}$$

shows its linear dependence. Thus we end up with the relation

$$A^{(3)}_{\rho\sigma\lambda\alpha}\gamma^\alpha\gamma_5 = a_3\, i\epsilon^{\alpha\lambda\sigma\kappa} v^\kappa v^\rho - b_3\, i\epsilon^{\alpha\lambda\rho\kappa} v^\kappa v^\sigma - c_3\, i\epsilon^{\alpha\rho\sigma\kappa} v^\kappa v^\lambda + d_3\, i\epsilon^{\lambda\rho\sigma\kappa} v^\kappa v^\alpha \tag{4.182}$$

as a description for the axial vector structure. However, the description of the tensor current is again straight forward. We obtain

$$\begin{aligned} A^{(4)}_{\rho\sigma\lambda\alpha\beta} = & a_4 \left(v^\lambda g^{\rho\alpha} g^{\sigma\beta} - v^\lambda g^{\rho\beta} g^{\sigma\alpha} \right) + b_4 \left(v^\sigma g^{\lambda\beta} g^{\rho\alpha} - v^\sigma g^{\lambda\alpha} g^{\rho\beta} \right) \\ & + c_4 \left(v^\rho g^{\lambda\beta} g^{\sigma\alpha} - v^\rho g^{\lambda\alpha} g^{\sigma\beta} \right) + d_4 \left(v^\beta g^{\lambda\alpha} g^{\rho\sigma} - v^\alpha g^{\lambda\beta} g^{\rho\sigma} \right) \\ & + e_4 \left(v^\beta g^{\rho\lambda} g^{\sigma\alpha} - v^\alpha g^{\rho\lambda} g^{\sigma\beta} \right) + f_4 \left(v^\beta g^{\rho\alpha} g^{\sigma\lambda} - v^\alpha g^{\rho\beta} g^{\sigma\lambda} \right) \\ & + g_4 \left(v^\beta v^\lambda v^\sigma g^{\rho\alpha} - v^\alpha v^\lambda v^\sigma g^{\rho\beta} \right) + h_4 \left(v^\beta v^\lambda v^\rho g^{\sigma\alpha} - v^\alpha v^\lambda v^\rho g^{\sigma\beta} \right) \\ & + i_4 \left(v^\beta v^\rho v^\sigma g^{\lambda\alpha} - v^\alpha v^\rho v^\sigma g^{\lambda\beta} \right) . \end{aligned} \tag{4.183}$$

The only structure that is left is the pseudoscalar structure. There is only one combination possible, namely

$$A^{(5)} = a_5\, i\epsilon_{\rho\sigma\lambda\kappa} v^\kappa . \tag{4.184}$$

Like for dimension 7 we use (4.114-4.117) as well as (4.119), but this time of course not in the static limit. We find that some of the matrix elements therefore can be expressed by $s_1, \ldots, s_9$ and end up with

$$2m_B\, \rho_1 = \langle \bar{B}(p) | \bar{b}_v (iD_\mu)(ivD)(iD^\mu) b_v | \bar{B}(p) \rangle \tag{4.185}$$

$$2m_B\, \rho_2 = \langle \bar{B}(p) | \bar{b}_v (iD_\mu)(ivD)(iD_\nu)(-i\sigma^{\mu\nu}) b_v | \bar{B}(p) \rangle \tag{4.186}$$

$$\tag{4.187}$$

defining our remaining parameters. The calculation of the indices goes along the same lines as the calculation performed for dimension 7. The only difference is that the terms which contain a contraction of the outer indices with v_ρ do not vanish any more, but can be rewritten to a matrix element of higher dimension by the equation of motion (4.114-4.117). Here we clearly see, why we have started our calculation with dimension 7, since these matrix elements can be derived with the trace formulae we have calculated before. Again we will not present the lengthy results here but refer to appendix C.

Like for dimension 7 we can now go on and reveal the physical meaning of (4.185) and (4.186). This time the operators are the already known operators ρ_D and ρ_{LS} (see e.g. [54]) which can be expressed in terms of (4.154-4.159) by

$$2m_B\,\rho_D^3 = \frac{1}{2g_s}\langle\bar{B}|\bar{b}_v\Big[iD_\rho,\big[iD_\sigma,iD_\lambda\big]\Big]\,b_v|\bar{B}\rangle\;\Pi^{\rho\lambda}v^\sigma \tag{4.188}$$

$$2m_B\,\rho_{LS}^3 = -\frac{1}{g_s}\langle\bar{B}|\bar{b}_v\big[iD_\rho,iD_\sigma\big]iD_\lambda\big(-i\sigma_{\alpha\beta}\big)\,b_v|\bar{B}\rangle\;\Pi^{\alpha\rho}\Pi^{\beta\lambda}v^\sigma\,. \tag{4.189}$$

The physical interpretation then reads

$$2m_B\rho_D^3 = -\frac{1}{2}\nabla\cdot\boldsymbol{E}+\mathcal{O}(1/m_b,g_s) \tag{4.190}$$

$$2m_B\rho_{LS}^3 = -\boldsymbol{s}\cdot(\boldsymbol{p}\times\boldsymbol{B})+\mathcal{O}(1/m_b,g_s)\,. \tag{4.191}$$

The higher order corrections arise from the fact that the matrix element is not static any more. Thus the normalization of the matrix element is now given by

$$\langle\bar{B}|\bar{b}_v\not{v}b_v|\bar{B}\rangle = 2m_B \tag{4.192}$$

$$\langle\bar{B}|\bar{b}_v b_v|\bar{B}\rangle = 1-\frac{1}{2m_b^2}(\mu_\pi^2-\mu_G^2)+\dots\,, \tag{4.193}$$

instead of $\langle B|\bar{b}_v b_v|B\rangle = 1$. Replacing the $\Pi_{\mu\nu}$ in (4.190) and (4.191) by the relation (4.159), contracting and solving to ρ_1 and ρ_2 provides the dependencies

$$\rho_1 = g_s\rho_D^3-\frac{2m_1^4-g_s m_7^4}{4m_b} \tag{4.194}$$

$$\rho_2 = -g_s\rho_{\rm LS}^3-\frac{-g_s^2m_2^4-2g_s^2m_3^4+6ig_s^2m_5^4+4ig_s^2m_6^4-g_sm_7^4-6g_sm_8^4}{4m_b} \tag{4.195}$$

between ρ_1, ρ_2, ρ_D and ρ_{LS}.

Dimension 5

The calculation of the matrix elements of order $1/m_b^2$ in the expansion is completely analog to the one of order $1/m_b^3$. Like for dimension 6, the parametrization for dimension 5 is done in terms of (4.120) which gives

$$\langle\bar{B}(p)|\bar{b}_{v,\xi}(iD_\rho)(iD_\sigma)b_{v,\eta}|\bar{B}(p)\rangle = A^{(1)}_{\rho\sigma}+A^{(2)}_{\rho\sigma\alpha}\gamma^\alpha+A^{(3)}_{\rho\sigma\alpha}\gamma^\alpha\gamma_5+A^{(4)}_{\rho\sigma\alpha\beta}(-i\sigma^{\alpha\beta})+A^{(5)}_{\rho\sigma}\gamma_5. \tag{4.196}$$

The derivation of the $A^{(i)}$ is absolutely analog to the calculation of the dimension 6 terms. Again we have to set up the generic structures of the terms at first. Using $g_{\rho\sigma}$ and v_ρ, there are only two possible terms to set up the scalar structure which turns out to be

$$A^{(1)}_{\rho\sigma} = a_1g_{\rho\sigma}+b_1v_\rho v_\sigma. \tag{4.197}$$

The vector structure has an additional index and therefore a more extensive structure. The according coefficient can be parametrized by

$$A^{(2)}_{\rho\sigma\alpha} = a_2g_{\rho\sigma}v_\alpha+b_2g_{\rho\alpha}+c_3g_{\sigma\alpha}v_\rho+d_2v_\rho v_\sigma v_\alpha, \tag{4.198}$$

while for the pseudotensor structure we obtain only the single term

$$A^{(3)}_{\rho\sigma\alpha} = a_3\,i\epsilon_{\rho\sigma\alpha\kappa}v^\kappa. \tag{4.199}$$

For the parametrization of $A^{(4)}_{\rho\sigma\alpha\beta}$ we directly respect the antisymmetry of the indices α and β. This gives us the result

$$A^{(4)}_{\rho\sigma\alpha\beta} = a_4 \left(g^{\rho\alpha} g^{\sigma\beta} - g^{\rho\beta} g^{\sigma\alpha}\right) + b_4 \left(v^\beta v^\sigma g^{\rho\alpha} - v^\alpha v^\sigma g^{\rho\beta}\right) + c_4 \left(v^\beta v^\rho g^{\sigma\alpha} - v^\alpha v^\rho g^{\sigma\beta}\right). \tag{4.200}$$

There is no possibility to form a pseudoscalar quantity with two indices. Therefore we have

$$A^{(5)}_{\rho\sigma} = 0. \tag{4.201}$$

Considering dimension 5 matrix elements with two covariant derivatives we obtain

$$2m_B\, \mu_1 = \langle \bar{B}(p)|\bar{b}_v (iD)^2 b_v|\bar{B}(p)\rangle \tag{4.202}$$

$$2m_B\, \mu_2 = \langle \bar{B}(p)|\bar{b}_v (iD_\mu)(iD_\nu)(-i\sigma^{\mu\nu}) b_v|\bar{B}(p)\rangle. \tag{4.203}$$

as new parameters. The calculation of the trace formula is now completely analog to the one of dimension 6 and the result is again given in appendix C.

Like in the previous sections we will shortly discuss the physical meaning of the parameters here. As in the case of ρ_1 and ρ_2 the physical interpretations are well known. These read according to [54]

$$2m_B\, \mu_\pi^2 = -\langle \bar{B}|\bar{b}_v\, iD_\rho iD_\sigma\, b_v|\bar{B}\rangle\ \Pi^{\rho\sigma} \tag{4.204}$$

$$2m_B\, \mu_G^2 = \frac{i}{2g_s}\langle \bar{B}|\bar{b}_v \left[iD_\rho, iD_\sigma\right]\left(-i\sigma_{\alpha\beta}\right) b_v|\bar{B}\rangle\ \Pi^{\alpha\rho}\Pi^{\beta\sigma}\,, \tag{4.205}$$

which is equivalent to

$$2m_B\mu_\pi^2 = p^2 + \mathcal{O}(1/m_b, g_s) \tag{4.206}$$

$$2m_B\mu_G^2 = -\frac{i}{2}\boldsymbol{s}\cdot\boldsymbol{B} + \mathcal{O}(1/m_b, g_s). \tag{4.207}$$

Again we have higher order corrections due to the non static matrix element. The same procedure as in the earlier sections gives us

$$\mu_1 = -\,\mu_\pi^2 - \frac{-2m_1^4 + g_s^2 m_2^4 + 2g_s^2 m_3^4 - 6ig_s^2 m_5^4 - 4ig_s^2 m_6^4 + 2g_s m_7^4 + 6g_s m_8^4}{8m_b^2} \tag{4.208}$$

$$\mu_2 = -\,ig_s\mu_G^2 + \frac{g_s}{2m_b}\rho_D^3 - \frac{-4m_1^4 - 3g_s^2 m_2^4 - 4g_s^2 m_3^4 + 2ig_s m_4^4 + 6ig_s^2 m_5^4 - 12g_s m_8^4 - 3g_s m_9^4}{24m_b^2} \tag{4.209}$$

Dimension 4

The next step is the calculation of the matrix elements of order $1/m_b$. Thus we have a dimension 4 matrix element with a single covariant derivative. The only contraction which is possible is given by the contraction with a single v_μ. Thus we have only a matrix element of the form

$$\langle \bar{B}(p)|\bar{b}_v (ivD) b_v|\bar{B}(p)\rangle \tag{4.210}$$

which can be related to parameters of dimension 5 by (4.117). Thus we do not get any new parameters below dimension 5. Anyway, the calculation of the dimension 4 trace formula proceeds along the same lines as for the higher dimensions. The most common form of the trace formula is given by

$$\langle \bar{B}(p)|\bar{b}_{v,\xi}(iD_\rho)b_{v,\eta}|\bar{B}(p)\rangle = A^{(1)}_\rho + A^{(2)}_{\rho\alpha}\gamma^\alpha + A^{(3)}_{\rho\alpha}\gamma^\alpha\gamma_5 + A^{(4)}_{\rho\alpha\beta}(-i\sigma^{\alpha\beta}) + A^{(5)}_\rho\gamma_5, \tag{4.211}$$

where we have again used equation (4.120). Going along the same lines as before we obtain the nonvanishing parametrizations

$$A^{(1)}_\rho = a_1\, v_\rho \tag{4.212}$$

$$A^{(2)}_{\rho\alpha} = a_2\, g_{\rho\alpha} + b_2\, v_\rho v_\alpha \tag{4.213}$$

$$A^{(4)}_{\rho\alpha\beta} = a_2(g_{\rho\alpha}v_\beta + g_{\rho\beta}v_\alpha). \tag{4.214}$$

The other coefficients turn out to be zero, since there are not enough indices to parametrize them. For the results please refer to appendix C again.

Dimension 3

The 0^{th} order in the $1/m_b$ expansion does not contain any covariant derivatives. Thus we cannot construct any new parameters. The parametrization of the trace formula can be written in the simple form

$$\langle \bar{B}(p)|\bar{b}_{v,\xi}\, b_{v,\eta}|\bar{B}(p)\rangle = A^{(1)} + A^{(2)}_\alpha\gamma^\alpha, \tag{4.215}$$

as there are not enough indices for the other structures. As parametrizations we obtain

$$A^{(1)} = a_1 \tag{4.216}$$

$$A^{(2)}_\alpha = a_2\, v_\alpha \tag{4.217}$$

This dimension corresponds to the parton model of the free quark. Thus we have to take the normalization

$$\langle \bar{B}(p)|\bar{b}_v \slashed{v} b_v|\bar{B}(p)\rangle = 2m_B \tag{4.218}$$

into account which gives us the result for the parts in the calculation which cannot be expressed by our parameters.

4.3.4. Calculation of the hadronic tensor

Finally, the time ordered product is obtained from the trace formula

$$\begin{aligned} T_{\rho\sigma} &= \langle \bar{B}(p)|\bar{b}_v\,\Gamma^\dagger_\rho\, iS_{\text{BGF}}\,\Gamma_\sigma\, b_v|\bar{B}(p)\rangle \\ &= \sum_i \operatorname{Tr}\left\{\Gamma^\dagger_\rho \frac{1}{\slashed{Q}-m_c}\Gamma_\sigma\,\hat{\Gamma}^{(i)}\right\} A^{(i,0)} + \sum_i \operatorname{Tr}\left\{\Gamma^\dagger_\rho \frac{1}{\slashed{Q}-m_c}\gamma^{\mu_1}\frac{1}{\slashed{Q}-m_c}\Gamma_\sigma\,\hat{\Gamma}^{(i)}\right\} A^{(i,1)}_{\mu_1} \\ &\quad + \sum_i \operatorname{Tr}\left\{\Gamma^\dagger_\rho \frac{1}{\slashed{Q}-m_c}\gamma^{\mu_1}\frac{1}{\slashed{Q}-m_c}\gamma^{\mu_2}\frac{1}{\slashed{Q}-m_c}\Gamma_\sigma\,\hat{\Gamma}^{(i)}\right\} A^{(i,2)}_{\mu_1\mu_2} + \cdots \end{aligned}$$

where the tree level expansion of the background-field propagator (4.100) automatically yields the correct ordering of the covariant derivatives induced by the Dirac structure. The ordering of the covariant derivatives in the quantities $A^{(i,\xi)}_{\mu_1,\ldots,\mu_\xi}$ obeys the ordering of the γ_{μ_ξ} to deliver consistent results. As discussed above, this prevents us from calculating one or even more gluon matrix elements which is normally done to retrieve the ordering.

The remaining task is to rewrite this result for the time-ordered product into a result for the hadronic tensor. As we have already discussed in section (4.3.1) those two quantities can be related by the optical theorem. As a result we could extract the formula

$$W_j = -\frac{1}{\pi} \operatorname{Im} T_j \tag{4.219}$$

at the end of this section. The only imaginary parts which can be found throughout the T_j are contained in the n^{th} power of the denominators $\Delta_0 = Q^2 - m_c^2 + i\epsilon$ which are generated by the expansion 4.105 of the time ordered product. The imaginary part of these objects is given by the identity

$$-\frac{1}{\pi} \operatorname{Im} \left(\frac{1}{\Delta_0} \right)^{n+1} = \frac{(-1)^n}{n!} \delta^{(n)} \big(Q^2 - m_c^2 \big), \tag{4.220}$$

where $\delta^{(n)}(Q^2 - m_c^2)$ denotes the n^{th} derivative to the delta distribution concerning the argument. For the derivation of this relation we will define $Q^2 = -x$ and $a = -m_c^2$ to get a more streamlined notation. Multiplying the left side of equation (4.220) it with a test function $f(x)$ and integrating leads to

$$\begin{aligned} -\frac{1}{\pi} \int\limits_{-\infty}^{+\infty} f(x) \operatorname{Im} \frac{1}{(a - x + i\epsilon)^{n+1}} \mathrm{d}x &= -\frac{1}{2\pi i} \int\limits_{-\infty}^{+\infty} f(x) \left(\frac{1}{(a - x + i\epsilon)^{n+1}} - \frac{1}{(a - x - i\epsilon)^{n+1}} \right) \mathrm{d}x \\ &= -\frac{1}{2\pi i} \left(\int\limits_{-\infty}^{+\infty} \frac{f(x)}{(a - x + i\epsilon)^{n+1}} \mathrm{d}x - \int\limits_{-\infty}^{+\infty} \frac{f(x)}{(a - x - i\epsilon)^{n+1}} \mathrm{d}x \right) . \end{aligned} \tag{4.221}$$

The difference which remains after the second step, is called “discontinuity across the cut”. In this case the cut is degenerated to a pole. This can now be rewritten to

$$\begin{aligned} -\frac{1}{\pi} \int\limits_{-\infty}^{+\infty} f(x) \operatorname{Im} \frac{1}{(a - x + i\epsilon)^{n+1}} \mathrm{d}x &= -\frac{1}{2\pi i} \int\limits_{-\infty}^{+\infty} \frac{f(x)}{(a - x + i\epsilon)^{n+1}} \mathrm{d}x - \frac{1}{2\pi i} \int\limits_{+\infty}^{-\infty} \frac{f(x)}{(a - x - i\epsilon)^{n+1}} \mathrm{d}x \\ &= -\frac{1}{2\pi i} \oint \frac{f(x)}{(a - x)^{n+1}} \mathrm{d}x, \end{aligned} \tag{4.222}$$

where interchanged the integral limits of the second integral in the first step and used in the second step that the integrals cancel each other besides the complex curve integral around the pole. Using Cauchy's integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \tag{4.223}$$

we obtain

$$
\begin{aligned}
-\frac{1}{\pi}\int_{-\infty}^{+\infty} f(x)\mathrm{Im}\frac{1}{(a-x+i\epsilon)^{n+1}}\mathrm{d}x &= -\frac{1}{2\pi i}\oint\frac{f(x)}{(a-x)^{n+1}}\mathrm{d}x = \frac{(-1)^{n+2}}{2\pi i}\oint\frac{f(x)}{(x-a)^{n+1}}\mathrm{d}x \\
&= \frac{(-1)^n}{n!}f^{(n)}(a) = \frac{(-1)^n}{n!}\int_{-\infty}^{+\infty} f(x)^{(n)}\delta(a-x)\mathrm{d}x \\
&= \frac{(-1)^n}{n!}\int_{-\infty}^{+\infty} f(x)\delta^{(n)}(a-x)\mathrm{d}x.
\end{aligned}
\tag{4.224}
$$

In the second step we have extracted the factor $(-1)^{n+1}$ out of the $(n+1)^{\mathrm{th}}$ power of the propagator to reproduce the right sign for Cauchy's integral formulae which has been used in the second step. In the third step we rewrite the resolution function $f^{(n)}(a)$ back to an integration of $f^{(n)}(x)$ over x alongside the delta distribution. In the last step we performed n partial integrations. Removing the integral and the test function we obtain the equation (4.220). All these calculations have been done within the Mathematica notebook. For the calculation of the traces we additionally used an interface for the computer algebra system Form written in Perl, since the calculation within FeynCalc would have taken too much time. The source code can be obtained from the author.

4.4. Calculation of the total decay rate and hadronic and leptonic energy moments

In this section we will calculate the total decay rate as well as the hadronic and leptonic energy moments. The calculation of the moments is done since they play a crucial role in the determination of the parameters m_c, m_b and V_{cb}. Additionally exhaustive data has been collected which can be used to determine the Michel parameters c_L, c_R, g_L, g_R, d_L and d_R that we have introduced to present a way to test the weak standard model current. We will start our considerations with the calculation of the triple differential decay rate. The results for the W_i of the last section alongside (4.72) and the introduction of the charged lepton energy $\hat{E}_l = E_l/m_b$, the leptonic invariant mass $\hat{q}^2 = q^2/m_b^2$ and the rescaled total lepton energy $v\cdot\hat{q} = v\cdot q/m_b$ as independent variables give us the triple differential decay rate in terms of the scalar functions $\widetilde{W}_i$

$$
\begin{aligned}
\frac{\mathrm{d}^3\Gamma}{\mathrm{d}\hat{q}^2\,\mathrm{d}(v\cdot\hat{q})\,\mathrm{d}\hat{E}_l} = &\sum_{n=0}^{4}\frac{G_F^2 m_b^2|V_{cb}|^2}{2\pi^3}\theta\left(\hat{q}^2\right)\theta\left(2\,v\cdot\hat{q}-2\hat{E}_l-\frac{\hat{q}^2}{2\hat{E}_l}\right)\delta^{(n)}\left(v\cdot\hat{q}-\frac{1-\hat{m}_c^2+\hat{q}^2}{2}\right) \\
&\times\left(\widetilde{W}_1^{(n)}\,\hat{q}^2+\widetilde{W}_2^{(n)}\left(v\cdot\hat{q}\,\hat{E}_l-\hat{E}_l^2-\frac{\hat{q}^2}{4}\right)+\widetilde{W}_3^{(n)}\,\hat{q}^2\left(\hat{E}_l-v\cdot\hat{q}\right)\right).
\end{aligned}
\tag{4.225}
$$

Note that we have explicitly installed the theta functions $\theta(q^2)$ and $\theta(4E_eE_{\bar{\nu}_e}-q^2)$ which have been rewritten to be functions of $\hat{q}^2$, $v\cdot\hat{s}$ and $\hat{E}_l$ respectively. Furthermore, from now on the quantities marked with a head are always the dimensionless variant of the quantity. The $\widetilde{W}_i$ denote the coefficients of W_i according to the derivatives of the delta distributions

$\delta^{(n)}((m_b v - q)^2 - m_c^2)$ induced by the optical theorem in the last section which are now displayed explicitly. Those delta functions have been rewritten by the useful equation

$$\frac{\mathrm{d}^n \delta(ax)}{\mathrm{d}(ax)^n} = \frac{1}{a^n |a|} \frac{\mathrm{d}^n \delta(x)}{\mathrm{d}x^n} \tag{4.226}$$

to match the new variables. In addition to that all coefficients arising by the change of the notation to the new variables have been absorbed into the $\hat{W}_i$. As we have already stated, we are interested in the calculation of the hadronic and leptonic energy moments in addition to the calculation of the total decay rate. The charged lepton energy moments are simply given by

$$L_n = \frac{1}{\Gamma^{(0,0)}} \int\limits_{E_{\mathrm{cut}}} \mathrm{d}\hat{E}_l\, \hat{E}_l^n \frac{\mathrm{d}\Gamma}{\mathrm{d}\hat{E}_l}\,, \tag{4.227}$$

where the $\Gamma^{(0,0)}$ in the normalization denotes the partonic total decay rate

$$\Gamma^{(0,0)} = \frac{G_{\mathrm{F}}^2 |V_{cb}|^2 m_b^5}{192\pi^3} \big(1 - 8\hat{m}_c^2 - 12\hat{m}_c^4 \ln(\hat{m}_c^2) + 8\hat{m}_c^6 - \hat{m}_c^8\big). \tag{4.228}$$

Additionally we have included a cut E_{cut} on the charged lepton energy $\hat{E}_l$ as such a cut has to be used in the experimental analysis. To define the hadronic energy moments we first have to introduce the hadronic energy and invariant mass

$$\begin{aligned} E_{\mathrm{had}} &= v \cdot (p_B - q) = m_B - v \cdot q \\ m_{\mathrm{had}}^2 &= (p_B - q)^2 = m_B^2 - 2 m_B\, v \cdot q + q^2, \end{aligned} \tag{4.229}$$

where m_B and $p_B = m_B\, v$ are the mass and the momentum of the B meson and q is the momentum of the leptonic system. Since our calculation are completely performed in partonic quantities and the conversion of the partonic to the leptonic quantities is well known it makes sense to expand the B meson mass as

$$m_B = m_b + \bar{\Lambda} + \frac{\mu_\pi^2 + \mu_g^2}{2 m_b} + \cdots\,. \tag{4.230}$$

Thus it is possible to relate the hadronic variables in (4.229) to the partonic ones

$$\begin{aligned} \hat{E}_{\mathrm{part}} &= \frac{E_{\mathrm{part}}}{m_b} = \frac{v \cdot (p_b - q)}{m_b} = 1 - v \cdot \hat{q} \\ \hat{m}_{\mathrm{part}}^2 &= \frac{m_{\mathrm{part}}}{m_b^2} = \frac{(p_b - q)^2}{m_b^2} = 1 - 2\, v \cdot \hat{q} + \hat{q}^2, \end{aligned} \tag{4.231}$$

where p_b is the b quark momentum which we have defined in the earlier sections. The moments in the partonic invariant mass and the partonic energy are then given by

$$\begin{aligned} H_{ij} &= \frac{1}{\Gamma^{(0,0)}} \int\limits_{E_{\mathrm{cut}}} \mathrm{d}\hat{E}_l \int \mathrm{d}\hat{m}_{\mathrm{part}}^2\, \mathrm{d}\hat{E}_{\mathrm{part}}\, \frac{\mathrm{d}^3\Gamma}{\mathrm{d}\hat{E}_{\mathrm{part}}\, \mathrm{d}\hat{m}_{\mathrm{part}}^2\, \mathrm{d}\hat{E}_l}\, (\hat{m}_{\mathrm{part}}^2 - \hat{m}_c^2)^i\, E_{\mathrm{part}}^j \\ &= \frac{1}{\Gamma^{(0,0)}} \int\limits_{E_{\mathrm{cut}}} \mathrm{d}\hat{E}_l \int \mathrm{d}(v \cdot \hat{q}) \mathrm{d}\hat{q}^2 \frac{\mathrm{d}^3\Gamma}{\mathrm{d}\hat{q}^2\, \mathrm{d}(v \cdot \hat{q})\, \mathrm{d}\hat{E}_l}\; 2\, (1 - 2\, v \cdot \hat{q} + \hat{q}^2 - \hat{m}_c^2)^i\, (1 - v \cdot \hat{q})^j \end{aligned} \tag{4.232}$$

where we have rewritten the variables of the integration in terms of the variables $v \cdot \hat{q}$ and $\hat{q}^2$ by using (4.231). The additional factor of 2 comes from the slater determinant. As the integration over the leptonic energy is the last one we will perform, the calculation of the leptonic and hadronic moments is completely analog to the one of the differential rate up to a factor of $2\,\hat{E}_l^n\,(1-2\,v\cdot\hat{q}+\hat{q}^2-\hat{m}_c^2)^i\,(1-v\cdot\hat{q})^j$ which obviously is one for n, j, $i=0$ and therefore retrieves the total rate for this case. Thus we only have to install an additional theta distribution $\theta(E_l - E_{\text{cut}})$ in the last integration. Therefore we define

$$\frac{\mathrm{d}^3\widetilde{\Gamma}}{\mathrm{d}\hat{q}^2\,\mathrm{d}(v\cdot\hat{q})\,\mathrm{d}\hat{E}_l} = \frac{\mathrm{d}^3\Gamma}{\mathrm{d}\hat{q}^2\,\mathrm{d}(v\cdot\hat{q})\,\mathrm{d}\hat{E}_l} 2\,\hat{E}_l^n\,(1-2\,v\cdot\hat{q}+\hat{q}^2-\hat{m}_c^2)^i\,(1-v\cdot\hat{q})^j \tag{4.233}$$

as a short hand notation for the rate including the quantities for partonic and leptonic moments.

4.4.1. Calculation to the double differential decay rate

For the calculation of the double differential decay rate we have decided to integrate over the rescaled total lepton energy $v{\cdot}\hat{q}$ at first. Since we do not know how to integrate over a differential delta distribution, we first have to rewrite the integral by partial integration. Therefore we decompose the triple differential rate into parts according to the number of derivatives acting on the delta function:

$$\frac{\mathrm{d}\widetilde{\Gamma}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l} = \int\limits_{-\infty}^{\infty} \sum_{n=0}^{4} \frac{\mathrm{d}\widetilde{\Gamma}^{(n)}}{\mathrm{d}\hat{q}^2\,\mathrm{d}(v\cdot\hat{q})\,\mathrm{d}\hat{E}_l} \delta^{(n)}\left(\hat{v}\cdot\hat{q} - \frac{1-\hat{m}_c^2+\hat{q}^2}{2}\right)\theta(\hat{q}^2)\theta\left(2\,v\cdot\hat{q}-2\hat{E}_l-\frac{\hat{q}^2}{2\hat{E}_l}\right)\mathrm{d}(v{\cdot}\hat{q}). \tag{4.234}$$

The superscript n at the triple differential decay rates just denotes the coefficient of the n^{th} derivative of the delta function (not to be confused with the n^{th} derivative of the triple differential decay rate itself). Note that the partial integrations also act on the theta functions which limit the phase space. This will lead to new delta distributions and their derivatives which have to be considered in further integrations. Without those terms we would obtain $1/\hat{m}_c^{2k}$ divergencies for the limit $\hat{m}_c \to 0$ in the total rate. Furthermore, we have to remember that the derivative of the delta function is done with respect to its argument. Thus we will rewrite it by

$$\frac{\mathrm{d}^n\delta\big(g(y)\big)}{\mathrm{d}g^n(y)} = \left(\frac{\mathrm{d}y}{\mathrm{d}g(y)}\frac{\mathrm{d}}{\mathrm{d}y}\right)^n \delta\big(g(y)\big), \tag{4.235}$$

with an arbitrary function $g(y)$. However, in this case we will not run into any difficulties since we can identify

$$y = v\cdot\hat{q} \quad \text{and} \quad g(y) = \hat{v}\cdot\hat{q} - \frac{1-\hat{m}_c^2+\hat{q}^2}{2}, \tag{4.236}$$

which implies

$$\frac{\mathrm{d}g(y)}{\mathrm{d}y} = 1 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}g(y)} = 1. \tag{4.237}$$

Thus we just have the relation

$$\delta^{(n)}\left(\hat{v}\cdot\hat{q} - \frac{1-\hat{m}_c^2+\hat{q}^2}{2}\right) = \frac{\mathrm{d}^n}{\mathrm{d}(v\cdot\hat{q})^n}\delta\left(\hat{v}\cdot\hat{q} - \frac{1-\hat{m}_c^2+\hat{q}^2}{2}\right) \tag{4.238}$$

and do not have to worry about additional terms coming from the conversion of the delta function. In the following lines we will perform the integrations separately for each number of derivatives n of the delta function, starting with $n = 0$. For the 0th derivative of the delta distribution (the delta distribution itself) we just have to perform the integration by the usual replacement rules. Therefore we get

$$\begin{aligned}\frac{\mathrm{d}\widetilde{\Gamma}^{(0)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l} &= \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\widetilde{\Gamma}^{(0)}}{\mathrm{d}\hat{q}^2\,\mathrm{d}(v\cdot\hat{q})\,\mathrm{d}\hat{E}_l}\delta^{(0)}\left(\hat{v}\cdot\hat{q} - \frac{1-\hat{m}_c^2+\hat{q}^2}{2}\right)\theta(\hat{q}^2)\theta\left(2\,v\cdot\hat{q} - 2\hat{E}_l - \frac{\hat{q}^2}{2\hat{E}_l}\right)\mathrm{d}(v\cdot\hat{q}) \\ &= \left.\frac{\mathrm{d}\widetilde{\Gamma}^{(0)}}{\mathrm{d}\hat{q}^2\,\mathrm{d}(v\cdot\hat{q})\,\mathrm{d}\hat{E}_l}\right|_{\hat{v}\cdot\hat{q}=(1-\hat{m}_c^2+\hat{q}^2)/2}\theta(\hat{q}^2)\theta\left(\frac{2\hat{E}_l-1}{2\hat{E}_l}\hat{q}^2 - 2\hat{E}_l + 1 - \hat{m}_c^2\right).\end{aligned} \tag{4.239}$$

To improve the readability we will use the shortcut

$$\hat{z} = \frac{2\hat{E}_l-1}{2\hat{E}_l}\hat{q}^2 - 2\hat{E}_l + 1 - \hat{m}_c^2 \tag{4.240}$$

for the argument of the second theta distribution after the replacement in the further calculations. Encountering the first derivative of the delta distribution we have to partially integrate once. This gives us

$$\begin{aligned}\frac{\mathrm{d}\widetilde{\Gamma}^{(1)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l} &= \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\widetilde{\Gamma}^{(1)}}{\mathrm{d}\hat{q}^2\,\mathrm{d}(v\cdot\hat{q})\,\mathrm{d}\hat{E}_l}\delta^{(1)}\left(\hat{v}\cdot\hat{q} - \frac{1-\hat{m}_c^2+\hat{q}^2}{2}\right)\theta(\hat{q}^2)\theta\left(2\,v\cdot\hat{q} - 2\hat{E}_l - \frac{\hat{q}^2}{2\hat{E}_l}\right)\mathrm{d}(v\cdot\hat{q}) \\ &= \left.\left(-\frac{\mathrm{d}}{\mathrm{d}\hat{s}}\frac{\mathrm{d}\widetilde{\Gamma}^{(1)}}{\mathrm{d}\hat{q}^2\mathrm{d}(v\cdot\hat{q})\mathrm{d}\hat{E}_l}\theta(\hat{z}) - \frac{\mathrm{d}\widetilde{\Gamma}^{(1)}}{\mathrm{d}\hat{q}^2\mathrm{d}(v\cdot\hat{q})\mathrm{d}\hat{E}_l}\delta(\hat{z})\right)\right|_{\hat{v}\cdot\hat{q}=(1-\hat{m}_c^2+\hat{q}^2)/2}\theta(\hat{q}^2).\end{aligned} \tag{4.241}$$

Analogously we have to perform two partial integrations for the terms containing the second derivative of the delta function. We get

$$\begin{aligned}\frac{\mathrm{d}\widetilde{\Gamma}^{(2)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l} &= \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\widetilde{\Gamma}^{(2)}}{\mathrm{d}\hat{q}^2\,\mathrm{d}(v\cdot\hat{q})\,\mathrm{d}\hat{E}_l}\delta^{(2)}\left(\hat{v}\cdot\hat{q} - \frac{1-\hat{m}_c^2+\hat{q}^2}{2}\right)\theta(\hat{q}^2)\theta\left(2\,v\cdot\hat{q} - 2\hat{E}_l - \frac{\hat{q}^2}{2\hat{E}_l}\right)\mathrm{d}(v\cdot\hat{q}) \\ &= \left(\frac{\mathrm{d}^2}{\mathrm{d}\hat{s}^2}\frac{\mathrm{d}\widetilde{\Gamma}^{(2)}}{\mathrm{d}\hat{q}^2\mathrm{d}(v\cdot\hat{q})\mathrm{d}\hat{E}_l}\theta(\hat{z}) + 2\frac{\mathrm{d}}{\mathrm{d}\hat{s}}\frac{\mathrm{d}\widetilde{\Gamma}^{(2)}}{\mathrm{d}\hat{q}^2\mathrm{d}(v\cdot\hat{q})\mathrm{d}\hat{E}_l}\delta(\hat{z})\right. \\ &\qquad\left.\left. + \frac{\mathrm{d}\widetilde{\Gamma}^{(3)}}{\mathrm{d}\hat{q}^2\mathrm{d}(v\cdot\hat{q})\mathrm{d}\hat{E}_l}\delta'(\hat{z})\right)\right|_{\hat{v}\cdot\hat{q}=(1-\hat{m}_c^2+\hat{q}^2)/2}\theta(\hat{q}^2).\end{aligned} \tag{4.242}$$

In the same sense we can calculate the term containing three derivatives of the delta function:

$$\begin{aligned}\frac{\mathrm{d}\widetilde{\Gamma}^{(3)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l} &= \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\widetilde{\Gamma}^{(3)}}{\mathrm{d}\hat{q}^2\,\mathrm{d}(v\cdot\hat{q})\,\mathrm{d}\hat{E}_l}\delta^{(3)}\left(\hat{v}\cdot\hat{q} - \frac{1-\hat{m}_c^2+\hat{q}^2}{2}\right)\theta(\hat{q}^2)\theta\left(2\,v\cdot\hat{q} - 2\hat{E}_l - \frac{\hat{q}^2}{2\hat{E}_l}\right)\mathrm{d}(v\cdot\hat{q}) \\ &= \left(-\frac{\mathrm{d}^3}{\mathrm{d}\hat{s}^3}\frac{\mathrm{d}\widetilde{\Gamma}^{(3)}}{\mathrm{d}\hat{q}^2\mathrm{d}(v\cdot\hat{q})\mathrm{d}\hat{E}_l}\theta(\hat{z}) - 3\frac{\mathrm{d}^2}{\mathrm{d}\hat{s}^2}\frac{\mathrm{d}\widetilde{\Gamma}^{(3)}}{\mathrm{d}\hat{q}^2\mathrm{d}(v\cdot\hat{q})\mathrm{d}\hat{E}_l}\delta(\hat{z})\right. \\ &\qquad\left.\left. - 3\frac{\mathrm{d}}{\mathrm{d}\hat{s}}\frac{\mathrm{d}\widetilde{\Gamma}^{(3)}}{\mathrm{d}\hat{q}^2\mathrm{d}(v\cdot\hat{q})\mathrm{d}\hat{E}_l}\delta'(\hat{z}) - \frac{\mathrm{d}\widetilde{\Gamma}^{(3)}}{\mathrm{d}\hat{q}^2\mathrm{d}(v\cdot\hat{q})\mathrm{d}\hat{E}_l}\delta''(\hat{z})\right)\right|_{\hat{v}\cdot\hat{q}=(1-\hat{m}_c^2+\hat{q}^2)/2}\theta(\hat{q}^2).\end{aligned} \tag{4.243}$$

For the fourth derivative of the delta function we get

$$
\begin{aligned}
\frac{\mathrm{d}\widetilde{\Gamma}^{(4)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l} &= \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\widetilde{\Gamma}^{(4)}}{\mathrm{d}\hat{q}^2\,\mathrm{d}(v\cdot\hat{q})\,\mathrm{d}\hat{E}_l}\delta^{(4)}\left(\hat{v}\cdot\hat{q}-\frac{1-\hat{m}_c^2+\hat{q}^2}{2}\right)\theta(\hat{q}^2)\theta\left(2\,v\cdot\hat{q}-2\hat{E}_l-\frac{\hat{q}^2}{2\hat{E}_l}\right)\mathrm{d}(v\cdot\hat{q}) \\
&=\Bigg(\frac{\mathrm{d}^4}{\mathrm{d}\hat{s}^4}\frac{\mathrm{d}\widetilde{\Gamma}^{(4)}}{\mathrm{d}\hat{q}^2\mathrm{d}(v\cdot\hat{q})\mathrm{d}\hat{E}_l}\theta(\hat{z})+4\frac{\mathrm{d}^3}{\mathrm{d}\hat{s}^3}\frac{\mathrm{d}\widetilde{\Gamma}^{(4)}}{\mathrm{d}\hat{q}^2\mathrm{d}(v\cdot\hat{q})\mathrm{d}\hat{E}_l}\delta(\hat{z})+6\frac{\mathrm{d}^2}{\mathrm{d}\hat{s}^2}\frac{\mathrm{d}\widetilde{\Gamma}^{(4)}}{\mathrm{d}\hat{q}^2\mathrm{d}(v\cdot\hat{q})\mathrm{d}\hat{E}_l}\delta'(\hat{z}) \\
&\qquad +4\frac{\mathrm{d}}{\mathrm{d}\hat{s}}\frac{\mathrm{d}\widetilde{\Gamma}^{(4)}}{\mathrm{d}\hat{q}^2\mathrm{d}(v\cdot\hat{q})\mathrm{d}\hat{E}_l}\delta''(\hat{z})+\frac{\mathrm{d}\widetilde{\Gamma}^{(4)}}{\mathrm{d}\hat{q}^2\mathrm{d}(v\cdot\hat{q})\mathrm{d}\hat{E}_l}\delta'''(\hat{z})\Bigg)\Bigg|_{\hat{v}\cdot\hat{q}=(1-\hat{m}_c^2+\hat{q}^2)/2}\theta(\hat{q}^2)
\end{aligned}
\tag{4.244}
$$

in an analog way. Resumming the single terms (4.239-4.244) we finally obtain the double differential decay rate

$$
\frac{\mathrm{d}\widetilde{\Gamma}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l}=\sum_{n=0}^{4}\frac{\mathrm{d}\widetilde{\Gamma}^{(n)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l} \tag{4.245}
$$

The result is again very lengthy and therefore will not be displayed here, but can be obtained by the author in form of a Mathematica notebook.

4.4.2. Calculation to the differential decay rate

To obtain the charged-lepton energy spectrum rate we have to integrate over the invariant mass $\hat{q}^2$ of the leptonic system. Again we will decompose the double differential rate, but this time according to the derivatives of $\theta(\hat{z})$. This gives us

$$
\frac{\mathrm{d}\widetilde{\Gamma}}{\mathrm{d}\hat{E}_l}=\int\limits_{-\infty}^{\infty}\sum_{i=0}^{4}\frac{\mathrm{d}\widetilde{\Gamma}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l}\theta(\hat{q}^2)\theta^{(i)}(\hat{z})\mathrm{d}\hat{q}^2, \tag{4.246}
$$

where $\theta^{(i)}(\hat{z})$ denotes the i^{th} derivative of the theta distribution according to the argument $\hat{z}$. The reader shall be warned not to confuse the decomposition given in (4.246) with the one given in (4.245). While (4.245) contains arbitrary combinations for the derivatives $\theta^{(i)}(\hat{z})$ of the theta distribution, the equation (4.246) is newly decomposed according to the numbers of derivatives acting on $\theta(\hat{z})$. Again the derivatives acting on the theta distribution have to be rewritten to derivatives by $\hat{q}^2$ to be able to perform the partial integrations. Therefore we will again use the formula (4.226). For the current case this leads to

$$
\begin{aligned}
\delta^{(n)}(\hat{z}) &= \delta^{(n)}\left(\frac{2\hat{E}_l-1}{2\hat{E}_l}\hat{q}^2-2\hat{E}_l+1-\hat{m}_c^2\right) \\
&= -\left(\frac{2\hat{E}_l}{2\hat{E}_l-1}\right)^{n+1}\delta^{(n)}\left(\hat{q}^2-\frac{2\hat{E}_l}{1-2\hat{E}_l}(1-2\hat{E}_l-\hat{m}_c^2)\right).
\end{aligned}
\tag{4.247}
$$

The derivatives of the delta distribution are still acting on its argument. With this in mind we recall the formula (4.235) and identify

$$
y=\hat{q}^2 \quad \text{and} \quad g(y)=\hat{q}^2-\frac{2\hat{E}_l}{1-2\hat{E}_l}(1-2\hat{E}_l-\hat{m}_c^2). \tag{4.248}
$$

which leads to the result

$$\frac{\mathrm{d}g(y)}{\mathrm{d}y} = 1 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}g(y)} = 1 \tag{4.249}$$

as seen in the last section. Thus we can rewrite the derivative regarding the argument of the delta function into a derivative by $\hat{q}^2$ without obtaining any further terms and end up with

$$\delta^{(n)}(\hat{z}) = -\left(\frac{2\hat{E}_l}{2\hat{E}_l - 1}\right)^{n+1} \frac{\mathrm{d}^n}{\mathrm{d}(\hat{q}^2)^n}\delta\left(\hat{q}^2 - \frac{2\hat{E}_l}{1-2\hat{E}_l}(1 - 2\hat{E}_l - \hat{m}_c^2)\right). \tag{4.250}$$

Now we are ready to integrate the various terms of the decomposition separately like we did in the last section. The term proportional to the 0th derivative of the theta distribution, and thus to the theta function itself, does not contain any delta functions. Thus we just have to perform the integration

$$\frac{\mathrm{d}\widetilde{\Gamma}^{(0)}}{\mathrm{d}\hat{E}_l} = \int\limits_0^{\frac{2\hat{E}_l}{1-2\hat{E}_l}(1-2\hat{E}_l-\hat{m}_c^2)} \frac{\mathrm{d}\widetilde{\Gamma}^{(0)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l}\mathrm{d}\hat{q}^2. \tag{4.251}$$

The first derivative of the theta distribution confronts us with a delta distribution again. However, as there are no derivatives acting on this delta distribution, we can just integrate the term using the normal replacement rules for delta distributions. Thus we obtain

$$\begin{aligned}\frac{\mathrm{d}\widetilde{\Gamma}^{(1)}}{\mathrm{d}\hat{E}_l} &= \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\widetilde{\Gamma}^{(1)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l}\theta(\hat{q}^2)\delta(\hat{z})\mathrm{d}\hat{q}^2 \\ &= \frac{\mathrm{d}\widetilde{\Gamma}^{(1)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l}\theta(\hat{q}^2)\bigg|_{\hat{q}^2=\frac{2\hat{E}_l}{1-2\hat{E}_l}(1-2\hat{E}_l-\hat{m}_c^2)}.\end{aligned} \tag{4.252}$$

For the case, where two derivatives act on the theta function, we find one delta distribution. Therefore we have to perform one partial integration and end up with

$$\begin{aligned}\frac{\mathrm{d}\widetilde{\Gamma}^{(2)}}{\mathrm{d}\hat{E}_l} &= \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\widetilde{\Gamma}^{(2)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l}\theta(\hat{q}^2)\delta(\hat{z})\mathrm{d}\hat{q}^2 \\ &= \left(-\frac{\mathrm{d}}{\mathrm{d}\hat{q}^2}\frac{\mathrm{d}\widetilde{\Gamma}^{(2)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l}\theta(\hat{q}^2) - \frac{\mathrm{d}\widetilde{\Gamma}^{(2)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l}\delta(\hat{q}^2)\right)\bigg|_{\hat{q}^2=\frac{2\hat{E}_l}{1-2\hat{E}_l}(1-2\hat{E}_l-\hat{m}_c^2)}.\end{aligned} \tag{4.253}$$

Performing two partial integrations we get

$$\begin{aligned}\frac{\mathrm{d}\widetilde{\Gamma}^{(3)}}{\mathrm{d}\hat{E}_l} &= \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\widetilde{\Gamma}^{(3)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l}\theta(\hat{q}^2)\delta(\hat{z})\mathrm{d}\hat{q}^2 \\ &= \left(\frac{\mathrm{d}^2}{\mathrm{d}(\hat{q}^2)^2}\frac{\mathrm{d}\widetilde{\Gamma}^{(3)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l}\theta(\hat{q}^2) + 2\frac{\mathrm{d}}{\mathrm{d}\hat{q}^2}\frac{\mathrm{d}\widetilde{\Gamma}^{(3)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l}\delta(\hat{q}^2) + \frac{\mathrm{d}\widetilde{\Gamma}^{(3)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l}\delta'(\hat{q}^2)\right)\bigg|_{\hat{q}^2=\frac{2\hat{E}_l}{1-2\hat{E}_l}(1-2\hat{E}_l-\hat{m}_c^2)}.\end{aligned} \tag{4.254}$$

as a result for the fourth term. Finally, we conclude our calculations with three partial integrations for the last term

$$\begin{aligned}\frac{\mathrm{d}\widetilde{\Gamma}^{(4)}}{\mathrm{d}\hat{E}_l} &= \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\widetilde{\Gamma}^{(4)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l}\theta(\hat{q}^2)\delta(\hat{z})\mathrm{d}\hat{q}^2 \\ &= \Bigg(- \frac{\mathrm{d}^3}{\mathrm{d}(\hat{q}^2)^3}\frac{\mathrm{d}\widetilde{\Gamma}^{(4)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l}\theta(\hat{q}^2) - 3\frac{\mathrm{d}^2}{\mathrm{d}(\hat{q}^2)^2}\frac{\mathrm{d}\widetilde{\Gamma}^{(4)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l}\delta(\hat{q}^2) \\ &\qquad - 3\frac{\mathrm{d}}{\mathrm{d}\hat{q}^2}\frac{\mathrm{d}\widetilde{\Gamma}^{(4)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l}\delta'(\hat{q}^2) - \frac{\mathrm{d}\widetilde{\Gamma}^{(4)}}{\mathrm{d}\hat{q}^2\mathrm{d}\hat{E}_l}\delta''(\hat{q}^2)\Bigg)\Bigg|_{\hat{q}^2=\frac{2\hat{E}_l}{1-2\hat{E}_l}(1-2\hat{E}_l-\hat{m}_c^2)}.\end{aligned} \tag{4.255}$$

The total differential rate is again the sum

$$\frac{\mathrm{d}\widetilde{\Gamma}}{\mathrm{d}\hat{E}_l} = \sum_{i=0}^{4}\frac{\mathrm{d}\widetilde{\Gamma}^{(i)}}{\mathrm{d}\hat{E}_l} \tag{4.256}$$

of all the terms calculated above. The rather lengthy result is displayed in appendix D.1.

4.4.3. Calculation of the total rate

Last but not least we will present the calculation of the total decay rate. Like in the sections before we can decompose the differential rate according to the number derivatives acting on the theta distributions contained in it. The result looks like

$$\frac{\mathrm{d}\widetilde{\Gamma}}{\mathrm{d}\hat{E}_l} = \sum_{n=0}^{4}\frac{\mathrm{d}\widetilde{\Gamma}^{(n)}}{\mathrm{d}\hat{E}_l}\theta^{(n)}\left(\frac{2\hat{E}_l}{1-2\hat{E}_l}(1-2\hat{E}_l-\hat{m}_c^2)\right), \tag{4.257}$$

where the superscript of the theta distribution again denotes the n^{th} derivative while the superscript of the differential rate denotes just the coefficient of the according theta distribution. The argument of the theta distribution automatically delivers the right phase space region, since we have

$$\frac{2\hat{E}_l}{1-2\hat{E}_l}(1-2\hat{E}_l-\hat{m}_c^2) \geq 0 \quad \Longleftrightarrow \quad 0 \leq \hat{E}_l \leq \frac{1-\hat{m}_c^2}{2}. \tag{4.258}$$

To be able to perform a partial integration of the delta distributions like in the last sections, we first have to rewrite them into derivations according to $\hat{E}_l$. This time this step will be a bit more tricky than in the last sections, since the argument of the delta distribution is not already solved to $\hat{E}_l$. Therefore we will consider the common case

$$\begin{aligned}\delta^{(n)}\big(g(y)\big) &= \frac{\mathrm{d}^n}{\mathrm{d}g^n(y)}\delta\big(g(y)\big) \\ &= \frac{\mathrm{d}^n}{\mathrm{d}g^n(y)}\sum_i\frac{1}{|g'(y_i)|}\delta(y-y_i) \\ &= \left(\frac{1}{g'(y)}\frac{\mathrm{d}}{\mathrm{d}y}\right)^n\sum_i\frac{1}{|g'(y_i)|}\delta(y-y_i)\end{aligned} \tag{4.259}$$

at first, where $g(y)$ is an arbitrary function with $g'(y) \neq 0$. For our case we obviously have

$$g(\hat{E}_l) = \frac{2\hat{E}_l}{1-2\hat{E}_l}(1-2\hat{E}_l-\hat{m}_c^2). \tag{4.260}$$

This function has two roots, namely $\hat{E}_l^{(1)} = 0$ and $\hat{E}_l^{(2)} = (1-\hat{m}_c^2)/2$. Thus we can show by

$$g'(\hat{E}_l) = \frac{8\hat{E}_l^2 - 8\hat{E}_l - 2\hat{m}_c^2 + 2}{(1-2\hat{E}_l)^2} \tag{4.261}$$

$$\Rightarrow \frac{1}{|g'(\hat{E}_l^{(1)})|} = \frac{2}{1-\hat{m}_c^2} \quad \text{and} \quad \frac{1}{|g'(\hat{E}_l^{(2)})|} = \frac{2\,\hat{m}_c^2}{1-\hat{m}_c^2} \tag{4.262}$$

that (4.259) is valid. Up to this point we still have not introduced the lepton-energy cut $\hat{E}_{\text{cut}}$ at low energies which we wanted to install because of experimental reasons. The only thing we have to do is to multiply everything by the theta distribution $\theta(\hat{E}_l - \hat{E}_{\text{cut}})$. As all terms containing two different delta functions vanish, we do not have to worry about the derivatives of this distribution because of the partial integrations and just get the lower boundary $\hat{E}_{\text{cut}}$ for our integrals. This results in the elimination of all parts containing the delta function $\delta\big(\hat{E}_l - \hat{E}_l^{(1)}\big)$. If we like to retain the evaluation for the uncut case, we only have to set $\hat{E}_{\text{cut}} = 0$. Using (4.259) to rewrite (4.257) and integrating partially over $\hat{E}_l$ like in the last sections we finally get the parts

$$\widetilde{\Gamma}^{(0)} = \int\limits_{\hat{E}_{\text{cut}}}^{(1-\hat{m}_c^2)/2} \frac{\mathrm{d}\widetilde{\Gamma}^{(0)}}{\mathrm{d}\hat{E}_l}\mathrm{d}\hat{E}_l \tag{4.263}$$

$$\widetilde{\Gamma}^{(1)} = \sum_{i=1}^{2} \int\limits_{\hat{E}_{\text{cut}}}^{\infty} \frac{2}{1-\hat{m}_c^2}\frac{\mathrm{d}\widetilde{\Gamma}^{(1)}}{\mathrm{d}\hat{E}_l}\delta\big(\hat{E}_l - \hat{E}_l^{(i)}\big)\mathrm{d}\hat{E}_l \tag{4.264}$$

$$\widetilde{\Gamma}^{(2)} = \sum_{i=1}^{2} \int\limits_{\hat{E}_{\text{cut}}}^{\infty} \left(\frac{\mathrm{d}}{\mathrm{d}\hat{E}_l}\frac{1}{g'(\hat{E}_l)}\frac{2}{1-\hat{m}_c^2}\frac{\mathrm{d}\widetilde{\Gamma}^{(2)}}{\mathrm{d}\hat{E}_l}\right)\delta\big(\hat{E}_l - \hat{E}_l^{(i)}\big)\mathrm{d}\hat{E}_l \tag{4.265}$$

$$\begin{aligned}\widetilde{\Gamma}^{(3)} = &\sum_{i=1}^{2} \int\limits_{\hat{E}_{\text{cut}}}^{\infty} \left(\frac{\mathrm{d}}{\mathrm{d}\hat{E}_l}\left(\frac{g''(\hat{E}_l)}{g'(\hat{E}_l)}\right)^3 \frac{2}{1-\hat{m}_c^2}\frac{\mathrm{d}\widetilde{\Gamma}^{(2)}}{\mathrm{d}\hat{E}_l}\right)\delta\big(\hat{E}_l - \hat{E}_l^{(i)}\big)\mathrm{d}\hat{E}_l \\ &+ \sum_{i=1}^{2} \int\limits_{\hat{E}_{\text{cut}}}^{\infty} \frac{2\,\hat{m}_c^2}{1-\hat{m}_c^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}\hat{E}_l^2}\frac{1}{\big(g'(\hat{E}_l)\big)^2}\frac{\mathrm{d}\widetilde{\Gamma}^{(2)}}{\mathrm{d}\hat{E}_l}\right)\delta\big(\hat{E}_l - \hat{E}_l^{(i)}\big)\mathrm{d}\hat{E}_l\,,\end{aligned} \tag{4.266}$$

which have been calculated a Mathematica notebook like the triple, double and single differential rates. The total rate is the sum

$$\widetilde{\Gamma} = \sum_{i=0}^{4} \widetilde{\Gamma}^{(i)} \tag{4.267}$$

of these terms. Again we do not display the results here, but refer to appendix D.2.

4.5. Radiative corrections

The calculation of the QCD radiative corrections to the inclusive $\bar{B} \to X_c\, e\, \bar{\nu}_e$ decay is already well known for the standard model current. It has been performed in [77, 70, 78] and the results for the semileptonic moments in the kinetic scheme have been given in [55]. However, since these

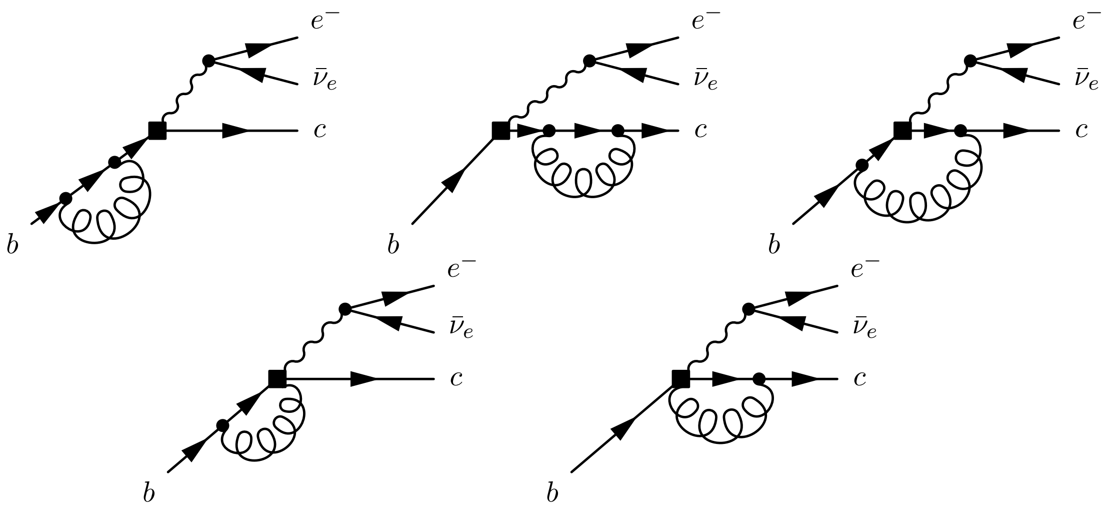


Figure 4.3.: Virtual corrections

results contain only the standard model currents, we have to recalculate the radiative corrections for the non standard currents up to order α_s. While doing this, we will also recalculate the radiative corrections for the standard model currents to be able to compare our results with [77, 70, 78] and [55]. Thus we have to evaluate the Feynman diagrams shown in fig. 4.3 and 4.4. The real and virtual corrections are both divergent in the infrared region. These divergencies can be regularized by the introduction of a gluon mass which drops out, when the contributions from the real and virtual corrections are summed up. Additionally, the wave-function renormalization of the b and c fields has to be taken into account.

The total amplitude consists of the sum of the standard model contribution and the one of the newly introduced operators of higher mass dimension. Since the new-physics contributions are of order $1/\Lambda^2$ we shall include only the interference terms of the standard model with these contributions, neglecting the squares of the new-physics terms are already of order $1/\Lambda^4$. Thus we compute

$$\mathrm{d}\Gamma = \frac{1}{2m_b}\Big(\langle c\, e\, \bar{\nu}_e|\mathcal{H}_{\text{eff}}^{\text{SM}}|b\rangle\langle c\, e\, \bar{\nu}_e|\mathcal{H}_{\text{eff}}|b\rangle^* + \langle c\, e\, \bar{\nu}_e|\mathcal{H}_{\text{eff}}|b\rangle\langle c\, e\, \bar{\nu}_e|\mathcal{H}_{\text{eff}}^{\text{SM}}|b\rangle^*\Big)\mathrm{d}\phi_{\text{PS}} \tag{4.268}$$

where $\mathrm{d}\phi_{\text{PS}}$ is the corresponding phase space (PS) element and

$$\mathcal{H}_{\text{eff}}^{\text{SM}} = \frac{4G_F V_{cb}}{\sqrt{2}}(\bar{c}\gamma_\mu P_L b)(\bar{e}\gamma^\mu P_L \nu_e) \tag{4.269}$$

$$\mathcal{H}_{\text{eff}} = \frac{4G_F V_{cb}}{\sqrt{2}} J_{h,\mu}(\bar{e}\gamma^\mu P_L \nu_e) \tag{4.270}$$

are the effective Hamiltonians of the standard model and the new physics contributions (as introduced in section 4.1) respectively, and

$$\begin{aligned} J_{h,\mu} = {} & c_L\ \bar{c}\gamma_\mu P_L b + c_R\ \bar{c}\gamma_\mu P_R b + g_L\ \bar{c}\, i\overleftrightarrow{D_\mu} P_L b + g_R\ \bar{c}\, i\overleftrightarrow{D_\mu} P_R b \\ & + d_L\ i\partial^\nu(\bar{c}\, i\sigma_{\mu\nu}P_L b) + d_R\ i\partial^\nu(\bar{c}\, i\sigma_{\mu\nu}P_R b) \end{aligned} \tag{4.271}$$

denotes again the extended hadronic current. From this current we can read off the relevant Feynman rules for the new-physics operators at tree level. At this point we shall note that the scalar current (i.e. the terms that are proportional to the constants g_L and g_R) introduces new boson-gluon-quark-quark vertices in order to maintain QCD gauge invariance. These are shown in the lower section of fig. 4.3 and on the right hand side of 4.4.

In the following we will perform a renormalization group analysis to rewrite the parameters c_L, c_R, g_L, g_R, d_L and d_R from the high scale Λ, where they were calculated, to a lower scale, where the actual experiments take place.

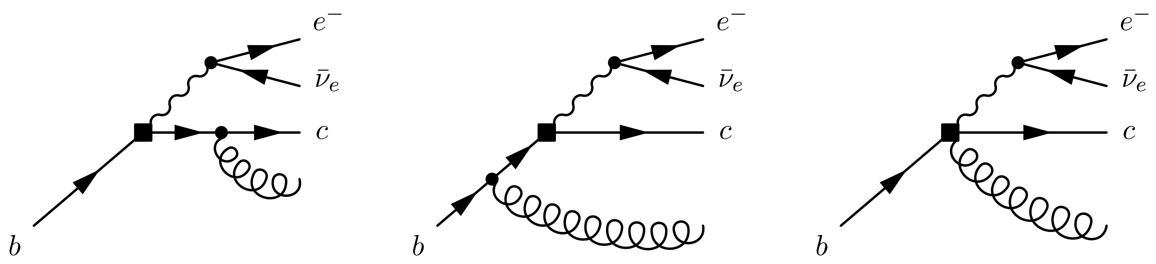


Figure 4.4.: Real corrections

4.5.1. Renormalization group analysis

Up to this point our calculations have been done only for a certain fixed scale Λ. To be able to perform QCD radiative corrections, we have to perform a renormalization group analysis as it has been discussed in the sections 3.2 and 3.2 to see how our parameters c_L, c_R, g_L, g_R, d_L and d_R change with the scale in our integrations. Due to current conservation (in the massless case) the left- and right-handed currents do not have an anomalous dimension and hence the parts of our effective Lagrange density proportional to c_L and c_R are not renormalized. However, the scalar and tensor contributions have anomalous dimensions. This means that we have to normalize then at some high scale Λ (e.g. the scale of the gauge bosons) and run them down to the scale of the bottom quark. Thus we have to calculate the anomalous dimension matrix, as it has been discussed in section (3.2). The whole calculation has been performed within a MATHEMATICA notebook which can be obtained by the author. Starting from equation (3.4) we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}\ln\mu} \langle c\, e\, \bar{\nu}_e | \mathcal{H}_{\text{eff}} | b \rangle. \tag{4.272}$$

Using the current 4.271 the matrix element reads

$$\begin{aligned} \langle c\, e\, \bar{\nu}_e | \mathcal{H}_{\text{eff}} | b \rangle = \frac{4 G_{\text{F}} V_{\text{cb}}}{\sqrt{2}} \Big(& \langle c\, e\, \bar{\nu}_e | \big[c_L (\bar{c}\, \gamma_\mu P_- b)(\bar{e}\, \gamma^\mu P_- v_e) + c_R (\bar{c}\, \gamma_\mu P_+ b)(\bar{e}\, \gamma^\mu P_- v_e) \big] | b \rangle \\ & + \boldsymbol{C} \cdot \langle c\, e\, \bar{\nu}_e | \boldsymbol{O} | b \rangle \Big) \end{aligned} \tag{4.273}$$

after the OPE. Here we have defined

$$\boldsymbol{C} = \begin{pmatrix} g_L \\ g_R \\ d_L \\ d_R \\ c_L^{m_b} \\ c_R^{m_b} \\ c_L^{m_c} \\ c_R^{m_c} \end{pmatrix} \qquad \boldsymbol{O} = \begin{pmatrix} (\bar{c}\, i \overleftrightarrow{D}_\mu P_- b)(\bar{e}\,\gamma^\mu P_- v_e) \\ (\bar{c}\, i \overleftrightarrow{D}_\mu P_+ b)(\bar{e}\,\gamma^\mu P_- v_e) \\ (i\partial^\nu(\bar{c}\, i\sigma_{\mu\nu} P_- b))(\bar{e}\,\gamma^\mu P_- v_e) \\ (i\partial^\nu(\bar{c}\, i\sigma_{\mu\nu} P_+ b))(\bar{e}\,\gamma^\mu P_- v_e) \\ (m_b\, \bar{c}\,\gamma_\mu P_- b)(\bar{e}\,\gamma^\mu P_- v_e) \\ (m_b\, \bar{c}\,\gamma_\mu P_+ b)(\bar{e}\,\gamma^\mu P_- v_e) \\ (m_c\, \bar{c}\,\gamma_\mu P_- b)(\bar{e}\,\gamma^\mu P_- v_e) \\ (m_c\, \bar{c}\,\gamma_\mu P_+ b)(\bar{e}\,\gamma^\mu P_- v_e) \end{pmatrix}, \tag{4.274}$$

where the operators $\boldsymbol{O}$ are of dimension seven. Notice that the left- and right-handed vector currents are not integrated into the vector $\boldsymbol{O}$ of the operators in (4.272) but are rather written separately, since they do not have an anomalous dimension. Despite this fact the reader may have noticed that we have introduced four additional operators marked by the coefficients $c_L^{m_b}, c_R^{m_b}, c_L^{m_c}, c_R^{m_c}$ which look very similar to the left- and right-handed vector currents but feature an additional m_c or m_b mass. These operators appear during the calculation of the off-diagonal elements of the anomalous dimension matrix for the scalar and tensor currents and are therefore purely mixing induced. This means that these operators are evanescent at the original scale Λ, while they mix into the scalar and tensor currents for the lower scale μ. This behavior cannot be fixed by redefining the constants completely to have the same mass dimension, as the operators occur with both masses m_c and m_b. This would indeed make the anomalous dimension matrix look smaller, but at the price of having mass terms included into it. Thus we decided to leave these terms in a separate position. For the calculation of the anomalous-dimension matrix of these dimension 7 operators we can now use the standard way introduced in section 3.2. Therefore we define the anomalous-dimension matrix γ by

$$\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}\ln\mu} = \gamma^T(\mu)\,\boldsymbol{C}\,, \tag{4.275}$$

to renormalize the Wilson coefficients rather than the operators. Remember therefore that the renormalization constants of the coefficients Z^c_{ij} form the inverse matrix of the renormalization constants of the operators $Z_{ij} = (Z^c_{ij})^{-1}$. The anomalous dimension γ is then computed from the divergencies of the renormalization constants using

$$\gamma = Z^{-1}\frac{\mathrm{d}}{\mathrm{d}\ln(\mu)}Z\,, \tag{4.276}$$

which ends up in the relation

$$\gamma^{(0)}_{ij} = -2[2a_1\delta_{ij} + (b_1)_{ij}] \tag{4.277}$$

in one-loop order, as shown in section 3.2. Here a_1 marks again the constant arising from the first divergent term in the $1/\epsilon$-expansion of the mass renormalization, while the constants b_1 are introduced by the divergencies at $1/\epsilon$ order in the expansion of the renormalization constants of the Green functions. Thus the entries of the anomalous dimension matrix can be read off directly from the one-loop calculation of the divergent terms resulting from the self-energy and vertex diagrams shown in fig. 4.5 which are sub-diagrams of 4.3 and 4.4, under consideration of the mass renormalization terms for the diagonal terms. The calculation of the anomalous dimension has been performed within the MATHEMATICAnotebook. In this notebook we have

Figure 4.5.: Feynman diagrams for self energies and vertex corrections

calculated the one-loop diagrams for the self energies and for the vertex corrections for each part of the current concerning the newly introduced scalar and tensor vertices as well as the standard model vector vertex. From this we obtain

$$\gamma^T(\mu) = \frac{2\alpha_s(\mu)}{3\pi} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{3} & 0 & 0 & \mathbf{3} & 0 & 0 & 0 \\ \mathbf{3} & 0 & 0 & 0 & 0 & \mathbf{3} & 0 & 0 \\ \mathbf{3} & 0 & 0 & 0 & 0 & 0 & \mathbf{3} & 0 \\ 0 & \mathbf{3} & 0 & 0 & 0 & 0 & 0 & \mathbf{3} \end{pmatrix} . \tag{4.278}$$

Note that the anomalous-dimension matrix is given in transposed form because of the definition (3.8). The renormalization group equation (3.8) for the wilson coefficients now reads

$$\left(\frac{\mathrm{d}}{\mathrm{d}\ln\mu} + \gamma^T(\mu) \right) \boldsymbol{C}(\Lambda/\mu) = 0 \tag{4.279}$$

where we can replace the derivative with respect to $\ln\mu$ with

$$\frac{\mathrm{d}}{\mathrm{d}\ln(\mu)} = -2\beta_0 \frac{\alpha_s^2}{4\pi} \frac{\mathrm{d}}{\mathrm{d}\alpha_s} \,, \tag{4.280}$$

which results form the one-loop order of (2.85). Thus we end up with

$$\left(-2\beta_0 \frac{\alpha_s^2}{4\pi} \frac{\mathrm{d}}{\mathrm{d}\alpha_s} + \gamma^T(\mu) \right) \boldsymbol{C}(\Lambda/\mu) = 0 \,, \tag{4.281}$$

whose solution represents the resummation of the first row of 3.18. Since the entries for scalar currents within the anomalous dimension matrix vanish (and therefore the coefficients for the scalar parts are constant regarding the scale), and the scalar currents are the only currents which are mixed into other ones, the solution to this differential equation turns out to be quite simple, as the resulting equations can be solved separately instead of having to solve a set of coupled differential equation. A natural starting point is therefore the calculation of the running of the coefficients to the scalar currents. For those currents (4.281) shows up as

$$-2\beta_0 \frac{\alpha_s^2}{4\pi} \frac{\mathrm{d}}{\mathrm{d}\alpha_s} g_L(\alpha_s(\mu)) = 0 \tag{4.282}$$

$$\Rightarrow \quad \frac{\mathrm{d}g_L}{\mathrm{d}\alpha_s} = 0 \,, \tag{4.283}$$

where we have indicated the μ dependence of the strong coupling in the first step. This can be solved easily by integration to give

$$g_L(\mu) = K \,, \tag{4.284}$$

where K is an arbitrary constant. This constant can be determined by using the fact that for the scale $\mu = \Lambda$ we have to retain the results which we have actually calculated at this scale. Thus we immediately see that $K = g_L(\Lambda)$. The calculation for the right-handed current proceeds completely analog to the steps as described above. Proceeding with the tensor currents we find that equation (4.281) becomes

$$-2\beta_0 \frac{\alpha_s^2}{4\pi} \frac{\mathrm{d}}{\mathrm{d}\alpha_s} d_L = \frac{2\alpha_s(\mu)}{3\pi}(d_L + g_L) \tag{4.285}$$

for the left-handed tensor current. This inhomogeneous differential equation of first order is solved in two steps. The first one consists of finding the general solution of the homogeneous part of (4.285) and the second part of constructing a partial solution for the inhomogeneous equation by varying the constants. The homogeneous equation

$$-\frac{3\beta_0}{4}\alpha_s \frac{\mathrm{d}d_L}{\mathrm{d}\alpha_s} = d_L \tag{4.286}$$

can be solved by separation of the variables. This results in

$$d_L = K\alpha_s^{-\frac{4}{3\beta_0}}\,, \tag{4.287}$$

where we have introduced an arbitrary constant K again. The next step consists of the variation of this constant. Inserting the upper equation with $K = K(\alpha_s)$ into (4.285) results in

$$\frac{\mathrm{d}K}{\mathrm{d}\alpha_s} = -\frac{3\beta_0}{4} g_L \alpha_s^{\frac{4}{3\beta_0}-1} \tag{4.288}$$

The solution is again obtained by integration. It reads

$$K = -g_L \alpha_s^{\frac{4}{3\beta_0}} + K_1\,. \tag{4.289}$$

The insertion of this equation into 4.287 provides us with

$$d_L(\mu) = K_1 \alpha_s^{-\frac{4}{3\beta_0}} - g_L(\mu)\,. \tag{4.290}$$

Again we have to adjust K_1 so that our initial condition $d_L(\mu) = d_L(\Lambda)$ is fulfilled for the scale $\mu = \Lambda$. Thus K_1 has to contain $d_L(\Lambda)$ as well as $\alpha_s(\Lambda)^{\frac{4}{3\beta_0}}$ to cancel $\alpha_s(\mu)^{-\frac{4}{3\beta_0}}$ at $\mu = \Lambda$ and $g_L(\mu)$ to cancel the last term of the upper equation at $\mu = \Lambda$. Taking into account that $g_L(\mu) = g_L(\Lambda)$ we obtain

$$d_L(\mu) = \Big(g_L(\Lambda) + d_L(\Lambda)\Big)\left(\frac{\alpha_s(\Lambda)}{\alpha_s(\mu)}\right)^{\frac{4}{3\beta_0}} - g_L(\Lambda) \tag{4.291}$$

as a result for the running of d_L. The calculation of the corresponding running of d_R is completely analog to the calculation above, and we obtain the same result with the subscript R instead of L. Since the calculations of the additional constants do not differ much from each other, we will present only one example. The renormalization group equation of $c_L^{m_b}$ reads

$$-2\beta_0 \frac{\alpha_s^2}{4\pi} \frac{\mathrm{d}}{\mathrm{d}\alpha_s} c_L^{m_b} = \frac{2\alpha_s(\mu)}{\pi}(c_L^{m_b} + g_R)\,. \tag{4.292}$$

Its solution equation is similar to that of (4.285). We obtain

$$c_L^{m_b}(\mu) = K_1 \alpha_s^{-\frac{4}{\beta_0}} - g_L(\Lambda)\,. \tag{4.293}$$

The only difference to the calculation performed above stems from the initial conditions

$$c_{L/R}^{m_b}(\Lambda) = 0 = c_{L/R}^{m_c}(\Lambda)\,, \tag{4.294}$$

which occur as the matching of the left- and right-handed currents is performed by fixing the coefficients c_L and c_R and all additional contributions are induced only by the renormalization group running. Thus the constant K_1 may contain only a factor to cancel $\alpha_s^{-\frac{4}{\beta_0}}$ at $\mu = \Lambda$ and $g_L(\Lambda)$. In the end we find

$$c_L^{m_b}(\mu) = g_R(\Lambda) \left(\left(\frac{\alpha_s(\Lambda)}{\alpha_s(\mu)} \right)^{\frac{4}{\beta_0}} - 1 \right) \tag{4.295}$$

Consolidating all our results we end up with

$$c_{L/R}(\mu) = c_{L/R}(\Lambda) \tag{4.296}$$

$$g_{L/R}(\mu) = g_{L/R}(\Lambda) \tag{4.297}$$

$$d_{L/R}(\mu) = \Big(g_{L/R}(\Lambda) + d_{L/R}(\Lambda) \Big) \left(\frac{\alpha_s(\Lambda)}{\alpha_s(\mu)} \right)^{\frac{4}{3\beta_0}} - g_{L/R}(\Lambda) \tag{4.298}$$

$$c_{L/R}^{m_b}(\mu) = g_{R/L}(\Lambda) \left(\left(\frac{\alpha_s(\Lambda)}{\alpha_s(\mu)} \right)^{\frac{4}{\beta_0}} - 1 \right) \tag{4.299}$$

$$c_{L/R}^{m_c}(\mu) = g_{L/R}(\Lambda) \left(\left(\frac{\alpha_s(\Lambda)}{\alpha_s(\mu)} \right)^{\frac{4}{\beta_0}} - 1 \right) \tag{4.300}$$

as scale dependent parameters. These equations may be re-expanded using the one-loop expression for the strong coupling constant 2.89 and obtain the logarithmic terms of the one-loop calculation. Nevertheless, the complete one-loop calculation we have performed additionally yields non-logarithmic terms. Except for the vector and axial vector currents these contributions depend on the renormalization scheme. In order to fix this dependence on the renormalization scheme we would have to include the renormalization group running at two loops. However, the two-loop calculation of the michel parameter analysis is beyond the scope of this work. Here we use the $\overline{MS}$ scheme and fix the scale to be $\mu = m_b$ which is the relevant scale of the decay processes we analyze. Nonetheless it also has advantages to keep the non-logarithmic terms. Since they lead to kinematic effects like e.g. a bremsstrahlung spectrum for the hadronic invariant mass, they give rise to a distortion of the spectra and therefore also to the moments. In particular the partonic mass moments $\langle (M_x^2 - m_c^2)^n \rangle$ with $n > 0$ will start at order α_s.

4.5.2. Mass Scheme

The calculation of the process is usually set up using the pole scheme for the masses of the particles. However, it bears problems since the pole mass is known as not being a well-defined mass, when it is used in calculations resulting in abnormally large radiative corrections. Because of that reason an appropriately defined short distance mass is better suited for the OPE calculation of an inclusive semileptonic rate [79]. In the present analysis we will therefore use the kinetic mass scheme, where the mass is defined by a nonrelativistic sum rule for the kinetic energy [58]. At one-loop level the kinetic mass is related to the pole mass by

$$m_q^{\text{kin}}(\mu_f) = m_q^{\text{pole}} \left[1 - \frac{4}{3}\frac{\alpha_s}{\pi}\left(\frac{\mu_f}{m_b} + \frac{\mu_f^2}{2m_b^2}\right)\right] , \tag{4.301}$$

where μ_f is a factorization scale for removing contributions below from the mass definition. As $1\,\text{GeV}$ is the typical energy release in the process we choose to set μ_f to this value in the following. This low renormalization scale is in fact the reason why the $\overline{MS}$ scheme is inappropriate for our purposes. However, the ratio $\hat{m}_c^2 = m_c^2/m_b^2$ is rather stable under the choice of schemes. Therefore the choice of the mass scheme affects the decay rates only through their m_b^5 dependence. Within this order equation (4.301) gives us the ratio

$$\left(\frac{m_q^{\text{pole}}}{m_q^{\text{kin}}(1\,\text{GeV})}\right) \approx 1 + 2.0899\frac{\alpha_s}{\pi} \tag{4.302}$$

of the masses expressed in the pole mass scheme and the kinetic mass scheme respectively. From the calculation in the standard model we find that the $\mathcal{O}(\alpha_s)$ corrections of the relation between the kinetic and the pole mass compensate the radiative corrections to the rates computed in the pole mass scheme to a large extent, leaving only small QCD radiative corrections. It turns out that this is also the case for our calculation including the anomalous couplings.

4.5.3. Combined corrections and reparametrization invariance

Up to this point we have calculated the radiative QCD corrections up to order α_s as well as the nonperturbative corrections up to order $1/m_b^4$. What is still missing is a description of the combined corrections to order α_s^i/m_b^j. In section 4.3.2 we have discussed how to perform an operator product expansion in a new systematic way. Within this calculation we have defined the residual momentum k_μ which describes the motion of the heavy b quark in the background field of a meson of velocity v. This has lead to the expansion

$$\begin{aligned} iS_{\text{BGF}} =& \frac{1}{\Delta_0}(\not{Q} - m_c) + \frac{1}{\Delta_0^2}(\not{Q} - m_c)\not{k}(\not{Q} - m_c) \\ &- \frac{1}{\Delta_0^3}(\not{Q} - m_c)\not{k}(\not{Q} - m_c)\not{k}(\not{Q} - m_c) \\ &+ \frac{1}{\Delta_0^4}(\not{Q} - m_c)\not{k}(\not{Q} - m_c)\not{k}(\not{Q} - m_c)\not{k}(\not{Q} - m_c) + \dots . \end{aligned} \tag{4.303}$$

where $\not{Q} = m_b\not{v} - \not{q}$ and $\Delta_0 = Q^2 - m_c^2 + i\epsilon$ which prevented us form calculating one- or even more-gluon matrix elements, since the ordering of the residual momentum was retained. However, as we have seen, this ordering is retained only due to the fact that every k_μ comes with a non-commutating Dirac matrix γ^μ. Everything works fine up to the point, where a gluon propagator is needed for loop corrections. This propagator is of the form

$$\frac{i\delta_{ab}}{q^2 + i\epsilon}\left(-g_{\mu\nu} + (1-\xi)\frac{q^\mu q^\nu}{q^2 + i\epsilon}\right) , \tag{4.304}$$

where q denotes the gluon momentum and ξ denotes the gauge parameter. The expansion of this propagator obviously leads to problems, since the momenta are not contracted to the noncommutative Dirac matrices. Therefore we do not get the right ordering of the residual momentum for free as for the quark propagator and thus gluon matrix elements would have to be included. All in all the description of the heavy quark's motion inside the hadron by a residual momentum is to naive to deliver the right results for calculations beyond tree level. However, the ordering could possibly be fixed by introducing a noncommutative residual momentum operator instead of the naive residual momentum to retain the right ordering. Another problem arises, as we do not know how the residual momentum behaves, when getting in contact with a gluon loop. Naively there are three possibilities how the residual momentum could be split up: The residual momentum stays completely in the quark line, the residual momentum flows completely through the gluon or the residual momentum is split up. The first two cases are definitely not right since they obviously would not reproduce the right gluon matrix elements. For the first case the gluons coupling to the gluon loop are completely absent and for the second case the higher order $1/m_b$ corrections which we have found for the tree-level diagram, are completely absent. Thus the question arises how to split the residual momentum up into two parts which would strengthen one contribution and weaken another. An idea for that would be to think of the residual momentum to be created by the background field of the meson for all contents of the diagram which underly the strong interaction. Thus all components of the diagram would contain their own residual momentum. Note that this would only work if the corrections due to the residual momentum were small compared to the momentum of the corresponding component. However, none of the above mentioned methods has been tested and it is not clear if they retain the right one gluon matrix elements.

Thus only the symmetric matrix elements which do not include gluon matrix elements can be calculated at present. A calculation of the one-loop α_s corrections for μ_π^2 has been performed in [72]. However, here we will present a much simpler and more elegant way of calculating these corrections using reparametrization invariance. The starting point is again the total decay rate

$$\Gamma = \frac{1}{2m_b}\left(\prod_{f=c,e,\nu}\int \frac{d^3\boldsymbol{p}_f}{(2\pi)^3\, 2E_f}\right)|\mathcal{M}(m_b \to \{p_f\})|^2 (2\pi)^4 \delta^4\left(p_b - \sum p_f\right) . \tag{4.305}$$

Additionally we define

$$\Gamma^{(0,0)} = \frac{G_F^2 |V_{cb}|^2 m_b^5}{192\pi^3} f(\hat{m}_c^2) \quad \text{with} \quad f(\hat{m}_c^2) = (1 - 8\hat{m}_c^2 - 12\hat{m}_c^4 \ln \hat{m}_c^2 + 8\hat{m}_c^6 - \hat{m}_c^8)\,, \tag{4.306}$$

which is the well known tree-level parton result [80]. The normalization factor $1/m_b$ in equation (4.305) results from $1/E_b$ in the rest system of the b-Quark, where we have $p_b = m_b v$ which means that we have not installed the residual momentum k describing the movement of the quark inside of the hadron yet. This residual momentum is acquired by the transformation

$$m_b v \to m_b v + k \tag{4.307}$$

to the rest system of the B meson. In this system we have $v^\alpha = (1, \mathbf{0})$, and thus the transformation reads

$$\frac{1}{2m_b} \to \frac{1}{2(m_b v_0 + k_0)} = \frac{1}{2m_b}\frac{1}{1 + \frac{v\cdot k}{m_b}}\,, \tag{4.308}$$

where the second fraction is the time-dilatation factor induced by the transformation. Expanding this time-dilatation factor up to the second order we obtain

$$\frac{1}{1+\frac{v\cdot k}{m_b}} = 1 - \frac{v\cdot k}{m_b} + \frac{(v\cdot k)^2}{m_b^2} + \dots\,. \tag{4.309}$$

The matrix elements of the residual momentum have already been calculated in section 4.3.3. Considering only the μ_π^2 dependent part which is symmetric and thus independent of the gluon matrix elements, they read

$$\langle k^\alpha\rangle = \frac{\mu_\pi^2}{2m_b}v^\alpha \qquad \text{and} \qquad \langle k^\alpha k^\beta\rangle = -\frac{\mu_\pi^2}{3}(g^{\alpha\beta} - v^\alpha v^\beta)\,. \tag{4.310}$$

Note that the trace formulae for the μ_G^2 dependent part indeed look the same. Thus one could think that the technique of reparametrization invariance also works for the μ_G^2 dependent part. However, the differences between the μ_π^2 and the μ_G^2 dependent parts arise at another point, namely the expansion of the c-propagator (4.303) in the background field. While the expansion gives the same structure $f(\hat{m}_c^2)$ for the partonic rate and the μ_π^2 dependent terms, the gluon matrix elements contained in the μ_G^2 dependent terms trigger results with a completely different structure. Therefore the linear ansatz from above does not work properly for μ_G^2 and the corresponding terms have to be removed from the trace formulae to avoid the calculation of meaningless μ_G^2 dependent terms in the rate. The quadratic term of (4.310) vanishes because of $v^2 = 1$ and we get

$$\frac{1}{1+\frac{v\cdot k}{m_b}} \to 1 - \frac{\mu_\pi^2}{2m_b^2} \tag{4.311}$$

This means that the decay rate just is shifted by the factor $1 - \mu_\pi^2/2m_b^2$ and we obtain the result

$$\Gamma \to \Gamma\left(1 - \frac{\mu_\pi^2}{2m_b^2}\right) \tag{4.312}$$

for any decay rate Γ - even for order α_s^n terms. However, as stated above this method has one major flaw. The averaging (4.310) does not retain the ordering of the covariant derivatives and therefore does not reproduce the one and more gluon-matrix elements correctly. Therefore we only obtain the corrections which are symmetric as they do not contain gluon-matrix elements. Nevertheless, since we do not have any better results from more sophisticated theoretical descriptions, we will shortly discuss the impact of these corrections on the moments of the lepton-energy spectrum. First we shall remember the definition

$$L_n = \frac{1}{\Gamma_0}\int\limits_{E_{\text{cut}}} \mathrm{d}E_\ell E_\ell^n \frac{\mathrm{d}\Gamma}{\mathrm{d}E_\ell} \tag{4.313}$$

of the lepton-energy moments normalized to factor

$$\Gamma_0 = \frac{G_F^2|V_{cb}|^2 m_b^5}{192\pi^3}. \tag{4.314}$$

The time dilatation of the $\mathrm{d}\Gamma$ retains the same factor as we have already calculated for Γ. The reparametrization of the lepton energy E_ℓ is calculated by

$$E_\ell = v\cdot p_\ell \to (m_b v + k)\cdot p_\ell = \left(E_\ell + \frac{k\cdot p_\ell}{m_b}\right) \to E_\ell\left(1 + \frac{\mu_\pi^2}{2m_b^2}\right)\,. \tag{4.315}$$

For $n = 1$ this factor cancels with the time-dilatation factor $1 - \mu_\pi^2/2m_b^2$ of $\mathrm{d}\Gamma$. Therefore the $1/m_b^2$ correction for L_1 vanishes.

4.6. Discussions and conclusions

In this chapter we have introduced new contributions to the quark-quark-boson vertex of the inclusive semileptonic decay $\bar{B} \to X_c\, e^-\, \bar{\nu}_e$ within an effective field approach. These new contributions consist of a right-handed vector current as well as right- and left-handed scalar and tensor currents. For these contributions we have calculated the radiative corrections up to order α_s and the nonperturbative corrections up to order $1/m_b^4$. From this we have calculated the triple, double and single differential rates as well as the total decay rate. The calculation for the nonperturbative corrections has been performed completely analytical and the results for the differential rate and the total decay rates are given in appendix D. However, the results for the triple and double differential rate have not been shown, since they are much too lengthy. In contrast to that, the results for the radiative corrections have been performed completely numerical, and are presented in the tables at the end of this section. These tables also include the results of the calculation of the moments for the nonperturbative corrections for the new contributions up to order $1/m_b^2$. Although these contributions have been calculated completely analytical up to order $1/m_b^3$, we will not present the results in this work due to the same reason for which we do not show the triple and double differential rates. Furthermore, we have shown how to calculate combined $1/m_b$ and α_s corrections - at least for the scope of the constant μ_1 and one-loop corrections - and discussed the difficulties of our new method of calculating the $1/m_b$ corrections when combined with radiative corrections.

In order to have a qualitative discussion of the results all results are presented in the tables 4.2, 4.3, 4.4, 4.5 and 4.6 for various moments with and without an energy cut for the charged lepton energy E_l at 1 GeV,as it has been introduced in section 4.4. The entries of the tables contain the coefficients corresponding to the expansion

$$L_n = c_L^2 L_n^{(c_L c_L)} + c_L c_R L_n^{(c_L c_R)} + c_L d_L L_n^{(c_L d_L)} + c_L d_R L_n^{(c_L d_R)} + c_L g_L L_n^{(c_L g_L)} + c_L g_R L_n^{(c_L g_R)} \tag{4.316}$$

$$H_{ij} = c_L^2 H_{ij}^{(c_L c_L)} + c_L c_R H_{ij}^{(c_L c_R)} + c_L d_L H_{ij}^{(c_L d_L)} + c_L d_R H_{ij}^{(c_L d_R)} + c_L g_L H_{ij}^{(c_L g_L)} + c_L g_R H_{ij}^{(c_L g_R)} \tag{4.317}$$

of the various moments. All of the coefficients above have an expansion in α_s and $1/m_b$, and therefore read

$$L_n^{(c_1 c_2)} = L_n^{(c_1 c_2;\, m_b^0, \alpha_s^0)} + \frac{\mu_\pi^2}{m_b^2} L_n^{(c_1 c_2;\, m_b^2, \alpha_s^0)} + \frac{\mu_g^2}{3\, m_b^2} L_n^{(c_1 c_2;\, m_b^2, \alpha_s^0)} + \cdots + \frac{\alpha_s}{\pi} L_n^{(c_1 c_2;\, m_b^0, \alpha_s^1)} + \cdots \tag{4.318}$$

$$H_{ij}^{(c_1 c_2)} = H_{ij}^{(c_1 c_2;\, m_b^0, \alpha_s^0)} + \frac{\mu_\pi^2}{m_b^2} H_{ij}^{(c_1 c_2;\, m_b^2, \alpha_s^0)} + \frac{\mu_g^2}{3\, m_b^2} H_{ij}^{(c_1 c_2;\, m_b^2, \alpha_s^0)} + \cdots + \frac{\alpha_s}{\pi} H_{ij}^{(c_1 c_2;\, m_b^0, \alpha_s^1)} + \cdots \tag{4.319}$$

where we have only shown the terms which we have included in the tables. Note that the one-loop calculation of the moments yield results of the (schematic) form

$$L_n = L_n^{\mathrm{Born}}(\mu) \left[1 + a_n \frac{\alpha_s}{\pi} \ln\left(\frac{\Lambda^2}{m_b^2} \right) + b_n^{\overline{MS}} \frac{\alpha_s}{\pi} \right] \tag{4.320}$$

for the leptonic moments and a similar form for the hadronic moments. The non-logarithmic contribution b_n depends on the chosen scheme. Therefore the first part

$$1 + a_n \frac{\alpha_s}{\pi} \ln \left(\frac{\Lambda^2}{m_b^2} \right) \tag{4.321}$$

of (4.320) is replaced by the renormalization-group-improved results given by the relations (4.296) - (4.300).

Within the tables 4.2 - 4.6 we have chosen the scales to be $\Lambda = M_W$ and $\mu = m_b$ to be sure to have a high scale that is safely apart from the low scale of the b decay which naturally manifests itself at the energy scale m_b.

Table 4.2 contains the results for the leptonic moments normalized to the total leptonic rate at tree-level (4.306) for the whole lepton energy spectrum and for a cut of 1 GeV on the lepton energy. We find that the radiative corrections to the scalar and tensor currents are sizable, i.e. the α_s/π coefficients are large. Since the sign is opposed to the tree-level rate, a substantial reduction of the tree-level result is expected.

The various hadronic moments which have been calculated with and without a cut of 1 GeV are contained in the tables 4.4 and 4.3 respectively. For the moments of the hadronic energy without the hadronic mass moments ($i = 0$, j arbitrary) we find similar results as for the leptonic moments. Again we obtain large coefficients heavily reducing the tree-level results.

The tables 4.5 and 4.6 show the summed tree level and α_s/π coefficients for the three different scales $\mu = 2.3\,\text{GeV}, 4.6\,\text{GeV}, 9.2\,\text{GeV}$, when calculated without E_l cuts. Since the full next-to-leading order expressions have not yet been calculated, the expressions for the scalar and tensor couplings are not available up to now. The resulting residual scale dependence can then be estimated by calculating the results for some points $\mu = m_b/2, m_b, 2m_b$ around our designated scale $\mu = m_b$ which corresponds to the scales given above for $m_b = 4.6\,\text{GeV}$ in the kinetic scheme. The scale dependence of c_L^2 and $c_L c_R$ turns out to be weak and originates from yet unknown NNLO effects. Because of the large α_s dependence the scale dependence of the tensor couplings is sizable, while it is huge for the scalar couplings, since the tree contribution is almost cancelled by the radiative correction. It seems not very likely that a NLO calculation will improve this situation. Thus it seems as if we will not have a good sensitivity to the tensor couplings and practically no sensitivity to the scalar couplings.

Comparing the radiative corrections to the nonperturbative ones we find that because of $\alpha_s/\pi \approx \mu_\pi/m_b^2$ the nonperturbative corrections are of similar importance as the perturbative ones. Within the leptonic moments both corrections are small compared to the tree-level contributions. However, for the hadronic moments with $i > 0$ the tree-level contributions vanish at the partonic level as the tree-level partonic rate is proportional to the mass shell delta function $\delta(\hat{s}_0 - \rho)$. Thus for these moments the leading order contributions are of order α_s or $1/m_b^2$ respectively. This also implies that their dependence on the scale is given by the dependence of α_s. Since the radiative corrections are small in this region, the nonperturbative corrections become dominant for this case. These nonperturbative corrections contain derivatives of the mass shell delta function $\delta(\hat{s}_0 - \rho)$, where at leading order in $1/m_b$ the maximum number of derivatives is two. Because of this the first and second i moments are of order $1/m_b^2$; higher moments with $i > 2$ will only have contributions of order $1/m_b^3$ or higher.

The precision of the standard model contributions is the main limiting factor of the sensitivity to a possible new physics contribution. Current analyses usually use up to second moments in both, the leptonic energy and invariant mass squared. Thus the highest masses included in the standard model analyses are approximately sensitive to terms of order $1/m_b^3$. The size of these terms alongside the size of the α_s^2 corrections may serve as a conservative estimate to the uncertainties of the standard model calculation which determines the sensitivity to a possible new-physics contribution. Anyway, future experiments may be sensitive to deliver data that can be fitted with even higher moments of i and thus higher order $1/m_b$ corrections to the moments could be needed. The calculation has been performed up to order $1/m_b^3$ and is (even if those results are not shown in the tables) available from the author. The extension to higher order corrections is straightforward and can be implemented into the corresponding Mathematica notebook if needed.

			n	c_L^2	$c_L c_R$	$c_L g_L$	$c_L g_R$	$c_L d_L$	$c_L d_R$
no E_l cuts	Tree-level	parton	0	1.0000	−0.6685	0.2212	0.5400	0.3315	−0.6597
			1	0.3072	−0.2092	0.0613	0.1372	0.0977	−0.2307
			2	0.1030	−0.0708	0.0188	0.0388	0.0314	−0.0845
			3	0.0365	−0.0252	0.0062	0.0118	0.0107	−0.0319
		μ_π^2/m_b^2 coeff.	0	−0.5000	0.3342	−0.0017	0.1703	−0.1652	0.3288
			1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
			2	0.0858	−0.0590	0.0365	0.1146	0.0261	−0.0702
			3	0.0730	−0.0503	0.0210	0.0575	0.0214	−0.0637
		μ_g^2/m_b^2 coeff.	0	−1.9449	4.9934	1.0232	1.5624	−2.1536	3.7106
			1	−0.9625	1.8578	0.3253	0.6011	−0.7986	1.5873
			2	−0.4495	0.7237	0.1124	0.2427	−0.3081	0.6840
			3	−0.2052	0.2902	0.0410	0.1008	−0.1220	0.2966
	α_s/π coeff.		0	0.3125	0.8009	−2.6592	−8.8212	−2.1497	4.3637
			1	0.0908	0.2284	−0.7171	−2.3141	−0.5594	1.4880
			2	0.0276	0.0739	−0.2174	−0.6843	−0.1660	0.5394
			3	0.0085	0.0260	−0.0711	−0.2189	−0.0538	0.2039
$E_l > 1$ GeV cut	Tree-level	parton	0	0.8148	−0.5617	0.1621	0.3586	0.2631	−0.6161
			1	0.2776	−0.1919	0.0520	0.1089	0.0867	−0.2232
			2	0.0979	−0.0678	0.0172	0.0340	0.0296	−0.0831
			3	0.0356	−0.0246	0.0059	0.0109	0.0104	−0.0317
		μ_π^2/m_b^2 coeff.	0	−0.4504	0.3225	0.0433	0.3440	−0.1479	0.3631
			1	0.0087	−0.0021	0.0564	0.2247	0.0031	0.0059
			2	0.0874	−0.0594	0.0377	0.1194	0.0267	−0.0691
			3	0.0733	−0.0504	0.0213	0.0583	0.0215	−0.0635
		μ_g^2/m_b^2 coeff.	0	−2.1029	4.6903	0.8592	1.4595	−2.0451	3.7102
			1	−0.9883	1.8078	0.2989	0.5845	−0.7805	1.5871
			2	−0.4540	0.7149	0.1078	0.2398	−0.3049	0.6840
			3	−0.2060	0.2886	0.0401	0.1003	−0.1214	0.2966
	α_s/π coeff.		0	0.2640	0.5740	−1.8506	−5.9374	−1.3992	3.9213
			1	0.0828	0.1930	−0.5920	−1.8692	−0.4440	1.4126
			2	0.0262	0.0679	−0.1964	−0.6098	−0.1467	0.5260
			3	0.0083	0.0249	−0.0674	−0.2058	−0.0504	0.2014

Table 4.2.: Tree level and α_s/π coefficients of the leptonic moments without E_l cuts and with a cut $E_l > 1\,\mathrm{GeV}$. Note that we have redefined $\boldsymbol{d_{L/R}} = m_B\, d_{L/R}$ and $\boldsymbol{g_{L/R}} = m_B\, g_{L/R}$ with $m_B = 5.279\,\mathrm{GeV}$ in order to tabulate dimensionless quantities.

		i	j	$\boldsymbol{c_L^2}$	$\boldsymbol{c_L c_R}$	$\boldsymbol{c_L g_L}$	$\boldsymbol{c_L g_R}$	$\boldsymbol{c_L d_L}$	$\boldsymbol{c_L d_R}$
Tree-level	parton	0	0	1.0000	-0.6685	0.2212	0.5400	0.3315	-0.6597
		0	1	0.4220	-0.2500	0.0961	0.2556	0.1217	-0.2559
		0	2	0.1832	-0.0964	0.0429	0.1219	0.0461	-0.1021
		0	3	0.0815	-0.0383	0.0196	0.0586	0.0180	-0.0418
		$i > 0$	j	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	μ_π^2/m_b^2 coeff.	0	0	-0.5000	0.3342	-0.0017	0.1703	-0.1652	0.3288
		0	1	-0.5000	0.3342	-0.100	-0.2229	-0.1652	0.3288
		0	2	-0.2902	0.1836	-0.0773	-0.2119	-0.0899	0.1840
		0	3	-0.1382	0.0837	-0.0448	-0.1348	-0.0406	0.0853
		1	0	-0.5780	0.4185	-0.2038	-0.5937	-0.2091	0.4025
		1	1	-0.1584	0.1172	-0.0695	-0.2158	-0.0585	0.1129
		1	2	-0.0283	0.0280	-0.0217	-0.0718	-0.0143	0.0258
		2	0	0.1609	-0.0728	0.0386	0.1159	0.0337	-0.0809
		2	1	0.0735	-0.0302	0.0180	0.0561	0.0138	-0.0343
		3	0	0.0000	0.0000	0.0000	0.0000	0.0000	-0.0000
	μ_g^2/m_b^2 coeff.	0	0	-1.9449	4.9934	1.0232	1.5624	-2.1536	3.7106
		0	1	-0.3850	1.2777	0.4097	0.4782	-0.5223	0.9700
		0	2	-0.0302	0.2833	0.1576	0.1391	-0.1109	0.2254
		0	3	0.0298	0.0342	0.0578	0.0350	-0.0146	0.0347
		1	0	0.3143	-0.6395	-0.1100	-0.2167	0.2027	-0.4360
		1	1	0.1195	-0.2561	-0.0529	-0.0925	0.0744	-0.1709
		1	2	0.0466	-0.1059	-0.0254	-0.0405	0.0282	-0.0689
		$i > 1$	j	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
α_s/π coeff.		0	0	0.3128	0.8007	-2.6592	-8.8212	-2.1497	4.3637
		0	1	0.1631	0.3441	-1.2391	-4.1901	-0.8839	1.8575
		0	2	0.0910	0.1477	-0.5850	-2.0067	-0.3694	0.8017
		0	3	0.0526	0.0632	-0.2793	-0.9681	-0.1568	0.3505
		1	0	0.0901	-0.0363	0.0028	0.0176	0.0032	-0.0095
		1	1	0.0470	-0.0178	0.0014	0.0093	0.0015	-0.0046
		1	2	0.0251	-0.0090	0.0007	0.0050	0.0007	-0.0023
		2	0	0.0091	-0.0033	0.0001	0.0015	0.0002	-0.0008
		2	1	0.0053	-0.0019	0.0000	0.0009	0.0001	-0.0004
		3	0	0.0018	-0.0006	0.0000	0.0003	0.0000	-0.0001

Table 4.3.: Tree level and α_s/π coefficients of the hadronic moments without E_l cuts. The partonic tree-level moments for $i > 1$ are all zero. Note that we have redefined $\boldsymbol{d_{L/R}} = m_B\, d_{L/R}$ and $\boldsymbol{g_{L/R}} = m_B\, g_{L/R}$ with $m_B = 5.279\,\mathrm{GeV}$ in order to tabulate dimensionless quantities.

		i	j	c_L^2	$c_L c_R$	$c_L g_L$	$c_L g_R$	$c_L d_L$	$c_L d_R$
Tree-level	parton	0	0	0.8148	−0.5617	0.1621	0.3586	0.2631	−0.6161
		0	1	0.3341	−0.2037	0.0682	0.1676	0.0922	−0.2365
		0	2	0.1411	−0.0761	0.0295	0.0789	0.0332	−0.0933
		0	3	0.0612	−0.0293	0.0131	0.0375	0.0123	−0.0378
		$i>0$	j	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	μ_π^2/m_b^2 coeff.	0	0	−0.4504	0.3225	0.0433	0.3440	−0.1479	0.3631
		0	1	−0.4505	0.2921	−0.0597	−0.0843	−0.1329	0.3332
		0	2	−0.2673	0.1561	−0.0532	−0.1300	−0.0695	0.1841
		0	3	−0.1337	0.0706	−0.0327	−0.0935	−0.0308	0.0859
		1	0	−0.5424	0.3590	−0.1687	−0.4845	−0.1685	0.3887
		1	1	−0.1639	0.1022	−0.0598	−0.1852	−0.0478	0.1115
		1	2	−0.0417	0.0262	−0.0204	−0.0678	−0.0126	0.0273
		2	0	0.1203	−0.0547	0.0258	0.0742	0.0223	−0.0729
		2	1	0.0538	−0.0221	0.0118	0.0355	0.0087	−0.0306
		3	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	μ_g^2/m_b^2 coeff.	0	0	−2.1029	4.6903	0.8592	1.4595	−2.0451	3.7102
		0	1	−0.4609	1.2205	0.3461	0.4476	−0.5005	0.9855
		0	2	−0.0660	0.2921	0.1348	0.1332	−0.1119	0.2391
		0	3	0.0131	0.0538	0.0507	0.0363	−0.0194	0.0439
		1	0	0.3074	−0.5095	−0.0803	−0.1804	0.1654	−0.4093
		1	1	0.1171	−0.1971	−0.0381	−0.0751	0.0583	−0.1590
		1	2	0.0458	−0.0789	−0.0180	−0.0321	0.0211	−0.0635
		$i>1$	j	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
α_s/π coeff.		0	0	0.2642	0.5739	−1.8506	−5.9373	−1.3992	3.9213
		0	1	0.1216	0.2462	−0.8449	−2.7806	−0.5529	1.6572
		0	2	0.0608	0.1057	−0.3919	−1.3149	−0.2221	0.7103
		0	3	0.0323	0.0455	−0.1842	−0.6272	−0.0907	0.3086
		1	0	0.0576	−0.0231	0.0018	0.0101	0.0018	−0.0079
		1	1	0.0288	−0.0108	0.0009	0.0052	0.0008	−0.0038
		1	2	0.0147	−0.0052	0.0004	0.0027	0.0004	−0.0018
		2	0	0.0046	−0.0016	0.0001	0.0007	0.0001	−0.0006
		2	1	0.0026	−0.0009	0.0000	0.0004	0.0000	−0.0003
		3	0	0.0007	−0.0002	0.0000	0.0001	0.0000	−0.0001

Table 4.4.: Tree level and α_s/π coefficients of the hadronic moments with a cut $E_l > 1\,\text{GeV}$. The partonic tree-level moments for $i > 1$ are all zero. Note that we have redefined $\boldsymbol{d_{L/R}} = m_B\, d_{L/R}$ and $\boldsymbol{g_{L/R}} = m_B\, g_{L/R}$ with $m_B = 5.279\,\text{GeV}$ in order to tabulate dimensionless quantities.

μ	n	c_L^2	$c_L c_R$	$c_L g_L$	$c_L g_R$	$c_L d_L$	$c_L d_R$
2.3 GeV	0	1.0253	−0.6037	0.0042	−0.1916	0.1533	−0.2983
	1	0.3145	−0.1907	0.0028	−0.0552	0.0512	−0.1074
	2	0.1052	−0.0648	0.0011	−0.0182	0.0176	−0.0397
	3	0.0372	−0.0231	0.0004	−0.0065	0.0062	−0.0150
4.6 GeV	0	1.0208	−0.6151	0.0441	−0.0474	0.1883	−0.3692
	1	0.3132	−0.1940	0.0135	−0.0169	0.0604	−0.1317
	2	0.1048	−0.0658	0.0043	−0.0068	0.0204	−0.0485
	3	0.0371	−0.0234	0.0015	−0.0028	0.0071	−0.0184
9.2 GeV	0	1.0177	−0.6231	0.0715	0.0752	0.2146	−0.4223
	1	0.3123	−0.1963	0.0208	0.0164	0.0674	−0.1499
	2	0.1046	−0.0666	0.0065	0.0034	0.0225	−0.0552
	3	0.0370	−0.0237	0.0022	0.0005	0.0078	−0.0209

Table 4.5.: Summed up tree level and α_s/π coefficients of the leptonic moments without E_l cuts for $\mu = 2.3, 4.6$ and $9.2\,\mathrm{GeV}$.

μ	i	j	c_L^2	$c_L c_R$	$c_L g_L$	$c_L g_R$	$c_L d_L$	$c_L d_R$
2.3 GeV	0	0	1.0253	−0.6037	0.0042	−0.1916	0.1533	−0.2983
	0	1	0.4352	−0.2222	−0.0051	−0.0914	0.0486	−0.1024
	0	2	0.1906	−0.0845	−0.0049	−0.0440	0.0156	−0.0360
	0	3	0.0857	−0.0331	−0.0033	−0.0214	0.0050	−0.0129
4.6 GeV	0	0	1.0208	−0.6151	0.0441	−0.0474	0.1883	−0.3692
	0	1	0.4329	−0.2271	0.0136	−0.0234	0.0628	−0.1322
	0	2	0.1892	−0.0866	0.0040	−0.0117	0.0215	−0.0487
	0	3	0.0850	−0.0340	0.0010	−0.0059	0.0075	−0.0185
9.2 GeV	0	0	1.0177	−0.6231	0.0715	0.0752	0.2146	−0.4223
	0	1	0.4312	−0.2305	0.0269	0.0335	0.0734	−0.1545
	0	2	0.1883	−0.0880	0.0104	0.0151	0.0258	−0.0582
	0	3	0.0845	−0.0347	0.0041	0.0069	0.0093	−0.0226

Table 4.6.: Summed up tree level and α_s/π coefficients of the non-zero-tree-level hadronic moments without E_l cuts for $\mu = 2.3, 4.6$ and $9.2\,\mathrm{GeV}$.

5. Lepton flavor violating τ-decays

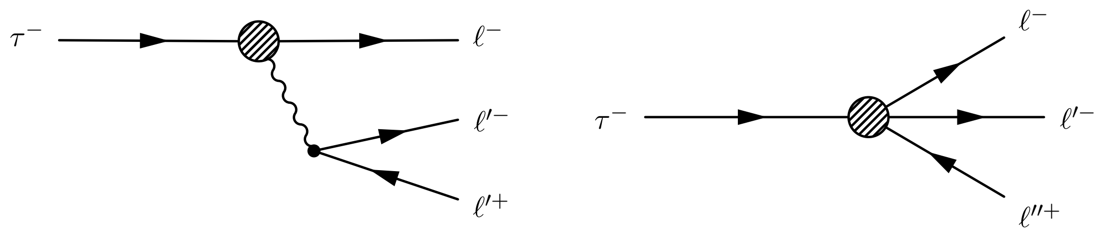


Figure 5.1.: Feynman diagrams for the radiative τ-decay (left) and the effective four fermion vertices from higher mass dimensions (right)

Within this chapter we will discuss lepton flavor violating τ-decays which provide a different approach to test the weak current of the standard model in a model independent way. On the one hand the minimal extension of the standard model including neutrino oscillations indeed predicts lepton flavor violation (LFV) for the charged leptons, though at a complete unobservable level. On the other hand many extensions of the standard model predict lepton flavor violation at much higher rates which may in some cases even be in conflict with existing experimental boundaries (see [8] or [81] for a recent summary of B-Factory results). With the upcoming experiments [82] this boundary will be pushed further, unless a discovery will be made. Considering especially the LHC it will be possible to detect LFV decays of a τ lepton, especially into channels with three leptons, where $\tau^- \to \mu^- \mu^- \mu^+$ will be one of the cleanest signatures [83, 84].

There are many models which predict LFV τ-decays of the form $\tau \to \ell\, \ell'\, \ell''$ with $\ell, \ell', \ell'' = e, \mu$ [85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100]. All these models will eventually match onto a set of local four-fermion operators resulting in $\tau \to \ell\, \ell'\, \ell''$ (see Fig. 5.1 right) or radiative operators, the latter mediating $\tau \to \ell\, \gamma^* \to \ell\, \ell'^+\, \ell'^-$ with subsequent decay of the (virtual) photon into a charged lepton pair(see Fig. 5.1 left). Note that this subsequent decay does not work for arbitrary decays of the form $\tau \to \ell\, \ell'\, \ell''$, since we assume charge conservation at every vertex. Thus decays of the form $\tau \to e^-\, e^-\, \mu^+$ are forbidden for the radiative decay, while they are allowed for the four fermion decay. In this dissertation we will provide a model independent way to compare these models with experimental data. Therefore we will use a bottom-up approach which allows us to consider all possible four-fermion and radiative operators with arbitrary coupling constants. These coupling constants can then be extracted by studying the decay distributions of the three leptons in the final state. In this work we will exclusively use the particle content of the standard model which has the advantage that we will be able to use our results for analyses, even if no additional particles will be found in future experiments. Similar analysis for the supersymmetric field content have been published by other groups (see e.g. [89, 87]).

5.1. The effective interaction for $\tau \to \ell\ell'\ell''$-decays

To be able to discuss lepton flavor violating τ-decays in a most generic way - without generating a specific model - we will create effective four fermion vertices in a bottom-up approach using an effective field theory picture. This means we have to generate a Lagrange density compatible with the observed $SU(2)_L \times U(1)_Y$ gauge symmetry which incorporates the terms of the standard model Lagrangian as these have shown to be valid up to energies which are accessible now. The task is now to add terms to the standard model Lagrange density in a structured way, while being as generic as possible. Therefore we expand our Lagrangian at a high scale Λ, where we expect some "new physics". We obtain

$$\mathcal{L} = \mathcal{L}_{4D} + \frac{1}{\Lambda}\mathcal{L}_{5D} + \frac{1}{\Lambda^2}\mathcal{L}_{6D} + \frac{1}{\Lambda^3}\mathcal{L}_{7D} + \frac{1}{\Lambda^4}\mathcal{L}_{8D} + \dots\,, \tag{5.1}$$

which represents a series of operators with increasing mass dimension. The dimension of these operators is then compensated by the dimension of the scale parameter Λ. Remember that the Lagrange density of the standard model is of dimension 4. This means that the term $\mathcal{L}_{4D}$ is completely represented by the standard model, since all additional terms of dimension 4 which one could imagine are excluded by the past experiments. Before discussing how the specific operators are set up, we first have to introduce a new notation which we will use within the chapter. This notation is useful for the description of minimal flavor violation which we will discuss in a later section. Therefore we group the left-handed leptons in an $SU(2)_L$ doublet, while the right-handed charged leptons (which are singlets under $SU(2)_L$) are put into an incomplete doublet, as a reminiscent of the right-handed $SU(2)_R$ related to the custodial symmetry. Writing also the Higgs boson in matrix form, we have

$$L = \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix}, \quad R = \begin{pmatrix} 0 \\ \ell_R \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} v + h_0 + i\chi_0 & \sqrt{2}\phi_+ \\ -\sqrt{2}\phi_- & v + h_0 - i\chi_0 \end{pmatrix}. \tag{5.2}$$

For simplicity, we have suppressed the family indices which will be specified once we consider a particular decay mode. The reader might note that the columns of the Higgs matrix are just given by the Higgs doublet Φ and its charge conjugate $\widetilde{\Phi}$ that have been introduced in section 2.1. Multiplying it with the right-handed doublet introduced above, we obtain just the same terms as we had in the notation with right-handed singlets and the Higgs doublet from section 2.1. From these fields we will now construct effective four-fermion operators (for decays of the form $\tau \to \ell\ell'\ell''$) and radiative operators with two fermions (for subsequent decays of the form $\tau \to \ell\gamma^* \to \ell\bar{\ell}'\ell'$) to describe the higher order effects contained in 5.1. For the description of the radiative operators we additionally need the field strengths $B_{\mu\nu}$ and $W^a_{\mu\nu}$ with the corresponding gauge couplings g and g' which belong to the $U(1)_Y$ and $SU(2)_L$ respectively. To obtain the mass dimensions of the fields which we will use, we point out that the action

$$S = \int \mathrm{d}^4x\, \mathcal{L}(\psi, \partial_\mu\psi) \tag{5.3}$$

is dimensionless in the units defined in section 1.3. Since four mass dimensions are integrated out the d^4x has the mass dimension -4. This has to be compensated by the mass dimension of the Lagrangian which therefore has the mass dimension 4. Regarding the Lagrangian (B.46) we find

$$\dim[\psi] = \frac{3}{2}, \quad \dim[D_\mu] = \dim[\partial_\mu] = 1, \quad \dim[H] = 1, \quad \dim[B_\mu] = 1 \quad \text{and} \quad \dim[F_{\mu\nu}] = 2 \tag{5.4}$$

as mass dimensions for its various contents. Since the charged lepton fields are of mass dimension 3/2 and the Higgs matrix is of dimension 1 it is obviously not possible to construct a four-fermion vertex or a radiative operator of dimension 5. Note that it would indeed be possible to construct operators of dimension 5 if we included right-handed neutrinos into our calculation, since we could make use of charge conjugated neutrino fields $\nu_R^c = C\,\gamma_0\nu_R^*$, like we did within the construction of the Majorana masses in section 2.2. Since we desire to retain charge conservation the only kind of four-fermion operators which we can produce in this way is of the form $\tau^- \to e^-\,\nu_\tau\,\bar{\nu}_\mu$, where one of the two neutrinos which are normally produced by the standard model interaction, is replaced in such a way that the lepton flavor is violated. However, there are at the moment no experiments which can resolve the neutrino flavor within high energy physics. Thus we will not consider those operators here. Hence we get the first new operators from the dimension 6 part of the Lagrange density:

dimension 6 leptonic:

$$O_1 = (\bar{L}\gamma_\mu L)(\bar{L}\gamma^\mu L) \tag{5.5}$$
$$O_2 = (\bar{L}\tau^a\gamma_\mu L)(\bar{L}\tau^a\gamma^\mu L) \tag{5.6}$$
$$O_3 = (\bar{R}\gamma_\mu R)(\bar{R}\gamma^\mu R) \tag{5.7}$$
$$O_4 = (\bar{R}\gamma_\mu R)(\bar{L}\gamma^\mu L) \tag{5.8}$$

dimension 6 radiative:

$$R_1 = g'(\bar{L}H\sigma_{\mu\nu}R)B^{\mu\nu} \tag{5.9}$$
$$R_2 = g(\bar{L}\tau^a H\sigma_{\mu\nu}R)W^{\mu\nu,a} \tag{5.10}$$

In this dimension we obtain only vector and pseudovector currents, since tensor and pseudotensor currents would require an additional Higgs matrix insertion. Thus we find that we have only three possible combinations of helicity structures, namely the cases where all fields are left-handed or right-handed and a mixed case, where we have two parts of which one is left- and one is right-handed. All other structures vanish due to the projectors. Beside the four-fermion operators we are able to build up radiative operators which can be set up by combinations of left- and right-handed spinors in combination with a tensor and a Higgs field. These are coupled to the gauge fields of the $U(1)$ and $SU(2)$ respectively. Going on with the higher dimension operators we note that we cannot create pure vector, scalar or tensor couplings, as the only component we can add to get the right mass dimension is one Higgs matrix. This would not give us the possibility to create fully contracted operators. For the dimension 8 part we obtain the scalar and pseudoscalar as well as the tensor current, since we now have the opportunity to insert another Higgs matrix. We cannot build up radiative operators as there is no possibility to set up a two fermion operator of dimension 6 which can be fully contracted with the gauge fields. Therefore we end up with the following list of dimension 8 operators:

dimension 8 leptonic:

$$P_1 = (\bar{L}HR)(\bar{L}HR) \tag{5.11}$$
$$P_2 = (\bar{L}\tau^a HR)(\bar{L}\tau^a HR) \tag{5.12}$$
$$Q_1 = (\bar{L}HR)(\bar{R}H^\dagger L) \tag{5.13}$$
$$Q_2 = (\bar{L}\tau^a HR)(\bar{R}H^\dagger\tau^a L) \tag{5.14}$$
$$T_1 = (\bar{L}H\sigma_{\mu\nu}R)(\bar{L}H\sigma^{\mu\nu}R) \tag{5.15}$$
$$T_2 = (\bar{L}\tau^a H\sigma_{\mu\nu}R)(\bar{L}\tau^a H\sigma^{\mu\nu}R) \tag{5.16}$$

dimension 8 radiative:

$$S_1 = g'(\bar{L}HH^\dagger H\sigma_{\mu\nu}R)B^{\mu\nu} \tag{5.17}$$
$$S_2 = g(\bar{L}\tau^a HH^\dagger H\sigma_{\mu\nu}R)W^{\mu\nu,a} \tag{5.18}$$

Note that we have only listed operators that have tree-level contributions to leptonic τ-decays. More operators which are bi-linear in the lepton fields operators and contribute at loop level can be found in [87, 89, 101, 102]. Furthermore, we neglected operators of dimension 8 involving additional covariant derivatives, since these become proportional to fermion masses when acting on fermions and thus are suppressed by the small lepton Yukawa couplings. The most general effective Hamiltonian at the electroweak scale is now obtained by summing over all operators, multiplied by arbitrary coefficients for every flavor combination. For a particular physics scenario these coefficients should be obtained by matching at the new physics scale Λ and evolving down to the scale M_W within the standard model as an effective theory manifesting in the dimension 4 part of the Hamiltonian.

In the following we are interested in LFV decays of a τ lepton into three charged leptons. Therefore we have to construct the effective interaction at the scale of the τ lepton by integrating out the weak gauge boson mass. For the following analysis we will focus on τ^- decays, as the decay distributions of the τ^+ decays are identical. We will therefore project on operators which are nonzero when sandwiched between the final and initial conditions $\langle \ell\ell'\ell''|$ and $|\tau^-\rangle$, where ℓ, ℓ' and ℓ'' are arbitrary combinations of the charged $e^\pm$ and $\mu^\pm$ leptons which obey the charge conservation. Furthermore, we will consider only operators of dimension 6, as the dimension 8 contributions are expected to be small, since they are additionally suppressed by two powers of the scale of new physics Λ. For four-fermion operators of dimension 6 we obtain the structures (5.5-5.8). Projecting on charged leptons only, we find that O_2 becomes equivalent to O_1. These two operators both match onto a purely left-handed operator

$$\mathcal{H}_{\text{eff}}^{(LL)(LL)} = g_V^{(LL)(LL)} \frac{(\bar{\ell}_L\gamma_\mu\tau_L)(\bar{\ell}'_L\gamma^\mu\ell''_L)}{\Lambda^2}\,, \tag{5.19}$$

where here and in the following the superscript of the couplings denotes the combinations of chiralities involved and the subscript denotes the Dirac structure. In analogy to that the operator O_3 corresponds to the purely right-handed interaction

$$\mathcal{H}_{\text{eff}}^{(RR)(RR)} = g_V^{(RR)(RR)} \frac{(\bar{\ell}_R\gamma_\mu\tau_R)(\bar{\ell}'_R\gamma^\mu\ell''_R)}{\Lambda^2}\,, \tag{5.20}$$

while we get a mixed term from the operator O_4

$$\mathcal{H}_{\text{eff}}^{(LL)(RR)} = g_V^{(LL)(RR)} \frac{(\bar{\ell}_L\gamma_\mu\tau_L)(\bar{\ell}'_R\gamma^\mu\ell''_R)}{\Lambda^2} + g_V^{(RR)(LL)} \frac{(\bar{\ell}_R\gamma_\mu\tau_R)(\bar{\ell}'_L\gamma^\mu\ell''_L)}{\Lambda^2}\,. \tag{5.21}$$

The dimension 6 radiative operators contain charged as well as neutral currents. Since we like to retain charge conservation and want to calculate decays into three charged leptons we are only interested in the neutral-current component coupling to a charged lepton pair. Thus only the third component of the $SU(2)_L$ gauge field is maintained and we obtain

$$R_1 \to g'(\bar{L}H\sigma_{\mu\nu}R)B^{\mu\nu} \tag{5.22}$$

$$R_2 \to g(\bar{L}\tau^3 H\sigma_{\mu\nu}R)W_3^{\mu\nu} \tag{5.23}$$

as operators to consider for the neutral weak decay. As we are not interested in interactions including the Higgs bosons here, we just can replace the Higgs matrix by its vacuum expectation value

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.24}$$

Since the upper components of the spinors are set to 0 (as we neither like to implement right-handed neutrinos nor want to have any neutrinos in our decay) the τ^3 in R_2 just gives a negative sign. Thus we get

$$R_1 \to \frac{v}{\sqrt{2}}\, g'(\bar{\ell}\sigma_{\mu\nu}\tau_R)B^{\mu\nu} \tag{5.25}$$

$$R_2 \to -\frac{v}{\sqrt{2}}\, g(\bar{\ell}\sigma_{\mu\nu}\tau_R)W_3^{\mu\nu} \tag{5.26}$$

using the notation from above. The next step is to rewrite $B^{\mu\nu}$ and $W_3^{\mu\nu}$ into their interaction basis. As already seen in section 2.1 we have

$$W_\mu^3 = \cos\theta_W Z_\mu + \sin\theta_W A_\mu \tag{5.27}$$

$$B_\mu = \cos\theta_W A_\mu - \sin\theta_W Z_\mu \tag{5.28}$$

for the fields and the relation $g\sin\theta_W = g'\cos\theta_W$ for the coupling constants. The gauge fields re-expressed in the interaction basis now read

$$B_{\mu\nu} = \cos\theta_W\,(\partial_\mu A_\nu - \partial_\nu A_\mu) - \sin\theta_W\,(\partial_\mu Z_\nu - \partial_\nu Z_\mu) \tag{5.29}$$

$$W_{\mu\nu}^3 = \sin\theta_W\,(\partial_\mu A_\nu - \partial_\nu A_\mu) + \cos\theta_W\,(\partial_\mu Z_\nu - \partial_\nu Z_\mu) + \dots\,, \tag{5.30}$$

where the dots mark terms with two field strengths which we do not need in the following as they would vanish because of the initial and final conditions due to the additional external Z_0 and γ fields. Additionally using the antisymmetry of $\sigma_{\mu\nu}$, we find

$$R_1 \to 2\,\frac{v}{\sqrt{2}}\, g'\sin\theta_W(\bar{\ell}\sigma^{\mu\nu}\tau_R)q_\mu A_\nu - 2\,\frac{v}{\sqrt{2}}\, g'\cos\theta_W(\bar{\ell}\sigma^{\mu\nu}\tau_R)q_\mu^Z Z_\nu \tag{5.31}$$

$$R_2 \to -2\,\frac{v}{\sqrt{2}}\, g\cos\theta_W(\bar{\ell}\sigma^{\mu\nu}\tau_R)q_\mu A_\nu - 2\,\frac{v}{\sqrt{2}}\, g\sin\theta_W(\bar{\ell}\sigma^{\mu\nu}\tau_R)q_\mu^Z Z_\nu\,, \tag{5.32}$$

where q_μ and q_μ^Z denote the momentum transfer to the leptons by the γ and Z_0 bosons respectively. This momentum transfer is proportional to the lepton masses, and thus this contribution scales as $1/(y\Lambda^2)$ where y is a Yukawa coupling of the leptons which would lead to an enhancement unless an additional Yukawa coupling appears in the numerator as e.g. in minimal flavor violation. A reordering of the operators (5.31) and (5.32) concerning the γ and Z_0 fields leads to

$$R_\gamma = \sqrt{2}\,v\left(g'\sin\theta_W - g\cos\theta_W\right)\left(\bar{\ell}\sigma^{\mu\nu}\tau_R\right)q_\mu A_\nu \tag{5.33}$$

$$R_Z = -\sqrt{2}\,v\left(g\sin\theta_W + g'\cos\theta_W\right)\left(\bar{\ell}\sigma^{\mu\nu}\tau_R\right)q_\mu^Z Z_\nu\,. \tag{5.34}$$

In the next step we will install the second part of the subsequent decay $\tau^- \to \ell^-\gamma^* \to \ell^-(\ell'^+\ell'^-)$. Therefore we require the parts

$$A^\mu\left(\bar{\ell}'\gamma_\mu\ell'\right) \qquad \text{and} \qquad Z^\mu\left(\bar{\ell}'\gamma_\mu P_L\ell'\right)\,, \tag{5.35}$$

which describe the weak and electromagnetic decay in the second vertex. The propagators are given by

$$P_{\mu\nu}^\gamma = A_\mu A_\nu = \frac{1}{q^2+i\epsilon}\left(-g_{\mu\nu} + (1-\xi_0)\frac{q^\mu q^\nu}{q^2+i\epsilon}\right) \tag{5.36}$$

for the photon and

$$P_{\mu\nu}^Z = Z_\mu Z_\nu = \frac{1}{(q^Z)^2 - M_Z^2 + i\epsilon}\left(-g_{\mu\nu} + (1-\xi_0)\frac{q^{Z,\mu}q^{Z,\nu}}{(q^Z)^2 - \xi_0 M_Z^2 + i\epsilon}\right) \tag{5.37}$$

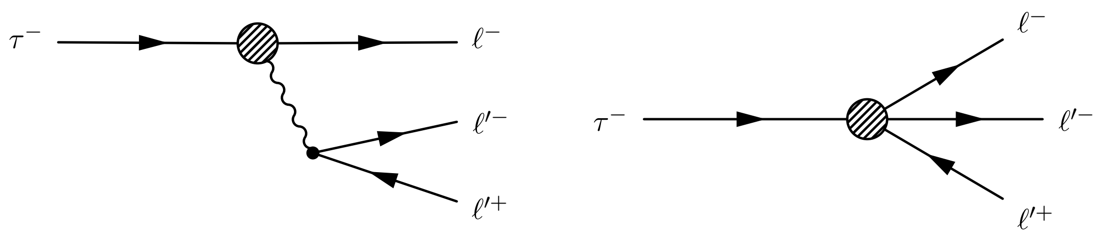


Figure 5.2.: Feynman diagrams for the radiative operators before (left) and after (right) the expansion of the W^--propagator

in arbitrary gauge. For our purposes we will choose the Feynman gauge $\xi_0 = 1$. For the Z_0 propagator we use $(q^Z)^2 < m_\tau^2 \ll M_Z^2$ we can expand the denominator in powers of $1/M_Z^2$ like we did in section 4.1. This gives us

$$\frac{1}{q^2 - M_Z^2} = \frac{1}{M_Z^2}\left(1 + \frac{q^2}{M_Z^2} + \dots\right), \tag{5.38}$$

where we can safely neglect the terms in the brackets as the expansion converges very rapidly because of $(m_\tau/M_Z)^2 \approx 0.0004$. Thus we obtain

$$P^\gamma_{\mu\nu} = -\frac{g^{\mu\nu}}{q^2 + i\epsilon} \qquad \text{and} \qquad P^Z_{\mu\nu} = -\frac{g^{\mu\nu}}{M_Z^2 + i\epsilon} \tag{5.39}$$

for the propagators. Combining everything we obtain the radiative operators

$$\mathcal{H}^{\text{rad}}_{\text{eff}} = \frac{e^2}{4\pi}\frac{v}{\Lambda^2}\frac{q^\nu}{q^2}\sum_{h,s} g^{(s,h)}_{\text{rad}} \left(\bar{\ell}_h(-i\sigma_{\mu\nu})\tau_s\right)\left(\bar{\ell}'\gamma^\mu\ell'\right) \tag{5.40}$$

for photon interactions and

$$\mathcal{H}^{\text{rad},\,Z}_{\text{eff}} = \frac{g_W^2}{4\pi}\frac{v}{\Lambda^2}\frac{q^\nu}{M_Z^2}\sum_{h,s} g^{(s,h)}_{\text{rad},\,Z} \left(\bar{\ell}_h(-i\sigma_{\mu\nu})\tau_s\right)\left(\bar{\ell}'\gamma^\mu P_L\ell'\right) \tag{5.41}$$

for the weak interactions, where the constants $g^{(L,R)}_{\text{rad}}$ and $g^{(R,L)}_{\text{rad}}$ denote the two possible chirality combinations. Taking into account the fact that $|q^\mu|$ is of the order of the τ mass, we find that the operator containing the neutral weak current is suppressed relative to the leading ones by the small Yukawa coupling of the τ lepton. Therefore only the photonic contribution has to be considered. As we can see in fig. 5.2, the radiative decay contains only two different types of particles, namely ℓ and ℓ'. This is a result of the standard vertices (5.35) which mediate a decay of the photon into a particle and its antiparticle (and therefore conserve the flavor). This leads to further constraints regarding the occurrence of radiative decays. First of all decays into three different particles cannot be described in this way. For our case this does not matter since for the standard model we only have three families, and thus only two types of particles, in which the τ can decay. Additionally we cannot obtain decays of the form $\tau \to \ell^-\ell^-\ell'^+$ from radiative decays. Since we do not wish to calculate such decays in the following, we do not need to expand our considerations further. However, if one desires to include more particles in the theory or to calculate decays of the form $\tau \to \ell^-\ell^-\ell'^+$ it is straightforward to extend the vertex for the τ decay with higher dimensional operators like the one for our b decay. Note that this would further increase the dimension of the operators. Thus the dimension would at least be 12, as the dimension 6 radiative operators would have to be connected to their counterparts in the vertex of the photon decay. Furthermore, note that for simplicity we have neglected possible form factor effects for decays into virtual photons from long distance lepton or quark loops. In the most general case, the $\tau \to \ell\gamma^*$ vertex could be parametrized as

$$\frac{e}{4\pi}\,\frac{v}{\Lambda^2}\sum_{h,s}\bar{\ell}_h\left\{g_{\rm rad}^{(s,h)}(q^2)\,(-i\sigma_{\mu\nu})\,q^\mu + m_\tau\, f_{\rm rad}^{(s,h)}(q^2)\,\left(\gamma_\nu - \frac{q_\nu}{q^2}\,\not{q}\right)\right\}\tau_s\,, \tag{5.42}$$

where $g_{\rm rad}^{(s,h)}(0) \equiv g_{\rm rad}^{(s,h)}$ and $f_{\rm rad}^{(s,h)}(0) = 0$, see, for instance [103].

Finally, we turn to the four-fermion operators with the chirality structure $(RL)(RL)$ or $(LR)(LR)$. At tree-level they only receive contributions from the dimension 8 operators $P_{1,2}$, $Q_{1,2}$ and $T_{1,2}$. Therefore, their matching coefficients are further suppressed by v^2/Λ^2. The one-loop matching coefficients at the electroweak scale may also receive contributions from the dimension 6 operators O_{1-4} and R_{1-2}, but in this case the required chirality flips induce an additional suppression by m_ℓ^2/v^2. Ignoring contributions suppressed by v^2/Λ^2 or m_ℓ^2/v^2 reduces the number of possible Dirac structures already to six in the case where the radiative operator can contribute and to four in cases like $\tau^- \to \mu^-\mu^-e^+$, where the radiative contribution is absent. The corresponding couplings

$$g_V^{(LL)(LL)},\quad g_V^{(RR)(RR)},\quad g_V^{(LL)(RR)},\quad g_V^{(RR)(LL)},\quad g_{\rm rad}^{(LR)},\quad g_{\rm rad}^{(RL)}, \tag{5.43}$$

are matrices in lepton flavor space. There are in total six different decay modes of the τ^- to consider:

$$\tau^- \to e^-e^-e^+ \tag{5.44}$$
$$\tau^- \to \mu^-\mu^-\mu^+ \tag{5.45}$$
$$\tau^- \to e^-e^-\mu^+ \tag{5.46}$$
$$\tau^- \to \mu^-\mu^-e^+ \tag{5.47}$$
$$\tau^- \to \mu^-e^-e^+ \tag{5.48}$$
$$\tau^- \to e^-\mu^-\mu^+ \tag{5.49}$$

Notice that (5.44 - 5.47) contain two identical particles (e^-e^- or $\mu^-\mu^-$) in the final state, whereas (5.48 + 5.49) do not. Moreover, only (5.44, 5.45, 5.48, 5.49) receive contributions from the radiative operator (5.40) by

$$\tau^- \to \ell^-\gamma^* \to \ell^-(\ell'^+\ell'^-)\,. \tag{5.50}$$

5.2. Constraints from minimal flavor violation

Here we will discuss the constraints on the parameters appearing in the next section by minimal lepton flavor violation (MLFV). Therefore we will start discussing what minimal flavor violation is. Our starting point shall be a theory without Yukawa couplings. The symmetry of the quark sector of the standard model would be $SU(3)_Q \times SU(3)_U \times SU(3)_D$ corresponding to the individual rotations of the Q_L^i, u_R^i and d_R^i fields (the left-handed quark doublet ant two right-handed quark singlets) for $i = 1,2,3$. In models with minimal flavor violation this symmetry is broken by the two Yukawa couplings λ_U and λ_D. These couplings act as spurion fields which induce the breaking of the symmetry by their transformation behavior. The transformation behavior is chosen such that the broken symmetry reflects the experimental observations. The two couplings transform in two different ways: in the spurion sense λ_U transforms as a

$(3, \bar{3}, 1)$, while λ_D transforms as $(1, 1, \bar{3})$, breaking the $SU(3)_U$ and $SU(3)_D$ respectively. Any higher order operator in a minimal flavor violating model describing long distance remnants of very short distance physics must be invariant under the full flavor symmetry group, when the couplings λ_U and λ_D are taken to transform as spurions. In this work we will define a similar structure of lepton flavor violation for the leptons. Therefore we will only consider the case of a minimal field content, where we use the particles contained in the standard model exclusively, in contrast to the extended field content (which is additionally discussed in e.g. [101] and [102]), where new particles (e.g. right-handed neutrinos) are added to the calculations. Thus we have three left-handed doublets L_L^i and three right-handed charged lepton singlets e_R^i which transform under the lepton flavor symmetry group

$$G_{\mathrm{LF}} = SU(3)_L \times SU(3)_{E_R}. \tag{5.51}$$

The leptonic sector is also invariant under two $U(1)$ symmetries which can be identified with the total lepton number and the weak hypercharge. Within a model with this particle content the neutrino mass matrix stems from left-handed Majorana mass terms. Therefore it transforms as $(6, 1)$ under G_{LF}. Due to the $SU(2)_L$ gauge symmetry, this mass term cannot be generated by renormalizable interactions. Furthermore, the absence of right-handed neutrino fields requires the breaking of the total lepton number. Additionally we assume that the breaking of the lepton number $U(1)_{\mathrm{LN}}$ takes place at a very high scale Λ_{LN} and is independent from the breaking of the lepton flavor symmetry G_{LF}. The Lagrange density for MLFV is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{MLFV}} =& -\lambda_e^{ij}\bar{e}_R^i(H^\dagger L_L^j) - \frac{1}{2\Lambda_{\mathrm{LN}}} g_\nu^{ij}(\bar{L}_L^{c,i}\tau_2 H)(H^T\tau_2 L_L^j) + h.c. \\ \to& -v\lambda_e^{ij}\bar{e}_R^i e_L^i - \frac{v^2}{2\Lambda_{\mathrm{LN}}} g_\nu^{ij}\bar{\nu}_L^{c,i}\nu_L^j + h.c.\,,\end{aligned} \tag{5.52}$$

where we have performed spontaneous symmetry breaking in the last step. It contains the two possible irreducible sources λ_ν^{ij} and g_ν^{ij} of lepton-flavor symmetry breaking which can be expressed by

$$\lambda = \frac{m_\ell}{v} = \frac{1}{v}\,\mathrm{diag}(m_e, m_\mu, m_\tau)\,, \tag{5.53}$$

$$g_\nu = \frac{\Lambda_{\mathrm{LN}}}{v^2}\, V_{\mathrm{PMNS}}^*\,\mathrm{diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3})\, V_{\mathrm{PMNS}}^\dagger \tag{5.54}$$

in a flavor diagonal basis, where V_{PMNS} denotes again the PMNS matrix introduced in section 2.2. The matrix $\lambda \simeq (\bar{3}, 3)$ is induced by the standard model Yukawa couplings from the charged leptons, while the matrix $g_\nu \simeq (\bar{6}, 1)$ stems from a lepton-number violating term of dimension 5 which has vanishing quantum numbers under the complete standard model gauge group. The $\bar{6}$ transformation behavior of g_ν stems from the transformation behavior $\bar{3}$ of the fields which induces $\bar{3} \times \bar{3} = 3_a + \bar{6}_s$ according to the symmetric and antisymmetric part of the resulting 3×3 matrix. As only the symmetric terms survive in the structure of (5.52), we end up with $\bar{6}$. Because the transformation properties of the lepton fields under G_{LF} are given by

$$L_L \to V_L\, L_L \qquad \text{and} \qquad e_R \to V_R\, e_R\,, \tag{5.55}$$

the Lagrangian (5.52) is invariant under this symmetry, if the spurion fields transform as

$$\lambda_e \to V_R \lambda_e V_L^\dagger \qquad \text{and} \qquad g_\nu \to V_L^* g_v V_L^\dagger\,. \tag{5.56}$$

If the scale $\Lambda_{\rm LN}$, associated with lepton-number violation, is sufficiently large, the resulting neutrino masses $m_{\rm Maj} \sim v^2/\Lambda_{\rm LN}$ are small, even if the spurion g_ν has generic entries of order unity.

We are interested in 4-lepton processes, induced by operators with some flavor structure

$$L^i\, L^j\, L_k^*\, L_l^*\,, \quad L^i\, R^j\, L_k^*\, R_l^*\,, \quad \text{etc.}$$

To render these operators formally invariant under the flavor group, they have to be multiplied by appropriate factors of λ_e and g_ν. In the following we will consider the minimal number of spurion insertions only. This is justified as long as the spurion fields are characterized by some small expansion parameter [101, 104, 105], e.g. if the neutrino mass *differences* Δm_ν^2 are smaller than their average $\Delta \bar{m}_\nu^2$. Note that, unlike in the case of the quark CKM matrix, the off-diagonal entries of the PMNS matrix are not always small.

Starting with the purely laft-handed case of $L^i\, L^j\, L_k^*\, L_l^*$, we need at least two spurion insertions. The possible flavor structures can be read off the reduction of the $SU(3)_L$ tensor product for g_ν and $g_\nu^\dagger$. Creating a standard weight diagram we find

$$\bar{6} \otimes 6 = 1 + 8 + 27\,. \tag{5.57}$$

A more sophisticated derivation as well as a table for this and other tensor products can be found in [106]. The flavor singlet term of 5.57 corresponds to the trace

$$\mathrm{tr}[g_\nu^\dagger g_\nu] = \frac{\Lambda_{\rm LN}^2}{v^4}\left(m_{\nu_1}^2 + m_{\nu_2}^2 + m_{\nu_3}^2\right) \equiv 3\,\frac{\Lambda_{\rm LN}^2\,\bar{m}_\nu^2}{v^4} \tag{5.58}$$

of the tensor product $g_\nu^\dagger g_\nu$, where we have introduced the shorthand notation $\bar{m}_\nu^2 = m_{\nu_1}^2 + m_{\nu_2}^2 + m_{\nu_3}^2$ in the second step. As the resulting quantity is a constant, it does not induce any flavor transitions. The flavor octet term is obtained as

$$\Delta = \Delta^\dagger \;\; = \;\; g_\nu^\dagger g_\nu - \frac{1}{3}\,\mathrm{tr}[g_\nu^\dagger g_\nu] = \frac{\Lambda_{\rm LN}^2}{v^4}\, U\, \Delta m_\nu^2\, U^\dagger\,, \tag{5.59}$$

where we defined $\Delta m_\nu^2 = \mathrm{diag}[m_{\nu_1}^2, m_{\nu_2}^2, m_{\nu_3}^2] - \bar{m}_\nu^2$. Note that our definition of Δ differs from the one in [101], as we have chosen to remove the flavor singlet contributions from the diagonal matrix elements. However, for flavor transitions the results should be the same, as the diagonal matrix elements do not contribute to them. Using this definition, we clearly separate the singlet and octet representations from each other and have everything expressed in terms of Δm^2 rather than encountering additional terms of m^2. Using the standard conventions

$$m_2^2 = m_1^2 + \Delta m_{21}^2 = m_1^2 + \Delta m_{\rm sol}^2 \tag{5.60}$$

$$m_3^2 = m_1^2 + \Delta m_{31}^2 = m_1^2 + \Delta m_{\rm atm}^2 \tag{5.61}$$

which we have already introduced in section 2.2, we find in particular that

$$\Delta_\tau^\mu = \mathcal{O}\left(\frac{\Lambda_{\rm LN}^2}{v^4}\,\Delta m_{\rm atm}^2\right) \tag{5.62}$$

whereas Δ_μ^e and Δ_τ^e are further suppressed by the neutrino mixing angle θ_{13}. It is to be stressed that Δ does neither depend on the absolute neutrino mass scale $\bar{m}_\nu^2$, nor on potential Majorana phases $\alpha_{1,2}$ in the PMNS matrix. The flavor structure of the corresponding invariant four-fermion operator reads $(L^* \Delta L)(L^* L)$. Therefore, the coefficients of specific flavor transitions are given by the quadratic neutrino mass differences and PMNS elements. The operator $(L^* \Delta L)(R^* R)$ is invariant under the flavor group by the same argument. It has been shown in [101] all possible flavor transitions induced by operators that are bilinear in the lepton fields (which means that they contain admixtures of left- and right-handed fields) can be expressed by combinations containing Δ. In particular, the flavor structure of the radiative operators in (5.9) and (5.10) reads $(L^* \Delta \lambda^\dagger R)$ and $(R^* \lambda \Delta L)$. Note that the combination of a single right-handed field with a left-handed one requires the insertion of the Yukawa spurion λ which leads to an additional suppression factor m_ℓ/v. Furthermore, note that purely right-handed lepton-flavor violating decays require at least four spurion insertions, $(R^* R)(R^* \lambda\, g^\dagger g\, \lambda^\dagger R)$, and are therefore strongly suppressed in MLFV.

Turning to the 27-plet combination of g_ν and $g_\nu^\dagger$, we introduce the corresponding representation in terms of the trace-less tensor

$$\begin{aligned} G_{ij}^{kl} &= (g_\nu)_{ij}\,(g_\nu^*)^{kl} - \frac{1}{12}\left(\delta_i^k \delta_j^l + \delta_i^l \delta_j^k\right) \mathrm{tr}(g^\dagger g) \\ &\quad - \frac{1}{5}\left(\delta_i^a \delta_b^l \delta_j^k + \delta_j^a \delta_b^l \delta_i^k + \delta_i^a \delta_b^k \delta_j^l + \delta_j^a \delta_b^k \delta_i^l\right) \Delta_a^b \end{aligned} \tag{5.63}$$

with $G_{ij}^{kl} = G_{ji}^{kl} = G_{ij}^{lk}$ and $\sum_i G_{ij}^{il} = 0$. Note that we have presented this tensor in the original $6 \otimes \bar{6}$ representation and not in the reduced $1 + 8 + 27$ one. The first term of 5.63 contains the singlet and octet representations which are removed by the second and third terms respectively to obtain definition consistent to 5.59, where we have removed the flavor singlet term. The dependence on the PMNS matrix is hidden in the $(g_\nu)_{ij}$ and $(g_\nu^*)^{kl}$ parts of the first term when using the representation chosen above. The flavor structure of the corresponding invariant four-lepton operator reads

$$G_{ij}^{kl}\, L^i L^j\, L_k^* L_l^*\,. \tag{5.64}$$

In contrast to Δ, the off-diagonal matrix elements of G depend on the absolute neutrino mass scale $\bar{m}_\nu^2$ and the Majorana phases. As a consequence we find for the general case in which the radiative operators do not dominate the $\tau \to 3\ell$ decay amplitudes that the purely leptonic decay modes are not directly correlated with the radiative ones $\tau \to \ell\gamma$. Relatively simple expressions for G_{ij}^{kl} can be obtained in the limit of vanishing Majorana phases, where we also apply the approximations $\sin^2\theta_{13} \sim \Delta m^2_{\mathrm{sol}}/\Delta m^2_{\mathrm{atm}} \ll 1$ and $\theta_{23} = 45°$. For the normal neutrino hierarchy $(m_{\nu_1} \sim m_{\nu_2} \ll m_{\nu_3})$, we obtain the leading coefficients as

$$G_{\tau e}^{e\mu} \simeq -2G_{\tau\mu}^{\mu\mu} \simeq -\frac{\Lambda^2_{\mathrm{LN}}}{v^4}\,\frac{\Delta m^2_{\mathrm{atm}}}{10}\,, \tag{5.65}$$

and sub-leading effects from

$$\begin{aligned} G_{\tau e}^{ee} &\simeq -\frac{\Lambda^2_{\mathrm{LN}}}{v^4}\,\frac{\sqrt{2}}{5}\,e^{i\delta}\,\sin\theta_{13}\,\Delta m^2_{\mathrm{atm}}\,, \qquad G_{\tau\mu}^{ee} \simeq \frac{\Lambda^2_{\mathrm{LN}}}{v^4}\,\frac{\sqrt{\Delta m^2_{\mathrm{atm}}}}{2}\left(m_{\nu_{1,2}} - \cos 2\theta_{12}\,\frac{\Delta m^2_{\mathrm{sol}}}{2m_{\nu_{1,2}}}\right), \\ G_{\tau e}^{\mu\mu} &\simeq \frac{\Lambda^2_{\mathrm{LN}}}{v^4}\,\frac{\sqrt{\Delta m^2_{\mathrm{atm}}}}{2\sqrt{2}}\left(e^{i\delta}\,\sin\theta_{13}\sqrt{\Delta m^2_{\mathrm{atm}}} - \sin 2\theta_{12}\,\frac{\Delta m^2_{\mathrm{sol}}}{4m_{\nu_{1,2}}}\right), \\ G_{\tau\mu}^{e\mu} &\simeq \frac{\Lambda^2_{\mathrm{LN}}}{v^4}\,\frac{\sqrt{\Delta m^2_{\mathrm{atm}}}}{2}\left(\frac{3\cos\delta - 7i\sin\delta}{5}\,\sin\theta_{13}\sqrt{\Delta m^2_{\mathrm{atm}}} + \sin 2\theta_{12}\,\frac{\Delta m^2_{\mathrm{sol}}}{4m_{\nu_{1,2}}}\right). \end{aligned} \tag{5.66}$$

where we have used the PDG parameterization [8] of the CKM matrix. For the inverted hierarchy $(m_{\nu_1} \sim m_{\nu_2} \gg m_{\nu_3})$, we find

$$G_{\tau\mu}^{ee} \simeq -5G_{\tau e}^{e\mu} \simeq 10G_{\tau\mu}^{\mu\mu} \simeq -\frac{\Lambda^2_{\mathrm{LN}}}{v^4}\,\frac{\Delta m^2_{\mathrm{atm}}}{2}\,, \tag{5.67}$$

and the corresponding sub-leading terms

$$G_{\tau e}^{ee} \simeq -\frac{\Lambda_{\rm LN}^2}{v^4}\,\frac{\Delta m_{\rm atm}^2}{\sqrt{2}}\,\sin\theta_{13}\,\frac{3\cos\delta - 7i\sin\delta}{5}\,,$$
$$G_{\tau e}^{\mu\mu} \simeq -\frac{\Lambda_{\rm LN}^2}{v^4}\,\frac{\Delta m_{\rm atm}^2}{2\sqrt{2}}\,e^{i\delta}\,\sin\theta_{13}\,, \qquad G_{\tau\mu}^{e\mu} \simeq \frac{\Lambda_{\rm LN}^2}{v^4}\,\frac{7\Delta m_{\rm atm}^2}{10\sqrt{2}}\,e^{i\delta}\,\sin\theta_{13}\,. \tag{5.68}$$

The results given above can be used to express the coefficients in (5.19, 5.21, 5.40) by the spurion fields, when we approach the effective vertices from a MLFV point of view. Explicitly we find

$$g_V^{(L_k L^i)(L_l L^j)} \quad\rightarrow\quad 2c_1\,\Delta_i^k\,\delta_j^l + c_2\,G_{ij}^{kl}\,, \tag{5.69}$$
$$g_V^{(L_k L^i)(R_l R^j)} \quad\rightarrow\quad c_3\,\Delta_i^k\,\delta_j^l\,, \tag{5.70}$$
$$g_{\rm rad}^{(L_k R^i)} \quad\rightarrow\quad c_4\,\Delta_i^k \tag{5.71}$$

for the dominating flavor coefficients, whereas the chiral structures corresponding to $g_V^{(RR)(RR)}$, $g_V^{(RR)(LL)}$ and $g_{\rm rad}^{(RL)}$ are suppressed by small lepton masses. The spurion combination G_{ij}^{kl} represents a new source of lepton flavor violation for the four-lepton operators compared to the radiative transitions $\tau \to \ell\gamma$ which are induced completely by the spurion Δ. Hence the coefficients to the radiative operators are completely driven by the difference of the squared neutrino masses, while the flavor coefficients of purely left-handed four-lepton operators also involve the absolute neutrino mass scale as well as the Majorana phases in minimal lepton flavor violation scenarios. In particular, the decay modes (5.46 + 5.47) only receive contributions from $G_{\tau\mu}^{ee}$ and $G_{\tau e}^{\mu\mu}$ because of the absence of the radiative operators.

5.3. Dalitz-plot analysis

In the following we will present and discuss Dalitz plots for the various possibilities of lepton flavor violating τ decays which have been discussed in the last two sections. Thereby we will focus on the two specific cases $\tau^- \to e^-\mu^-\mu^+$ which represents the decay into three different particles, and $\tau^- \to \mu^-\mu^-\mu^+$ which represents the decay into just two different particles. The other possibilities of decays can be sorted into one of the two classes and result in similar results with equal Dalitz plots. The only change comes because of the different masses which will provide different borders of the Dalitz plots. In the following subsection we will discuss the kinematics of the three body decays we like to analyze and the corresponding structure of the Dalitz plots.

5.3.1. Kinematics

All decays displayed in (5.44-5.50) obey the same decay kinematics, since even the radiative operator can be written as a four fermion interaction due to the choice of Feynman gauge. We shall start our considerations with the partial decay rate

$$\mathrm{d}\Gamma = \frac{1}{2M}|\mathcal{M}|^2(2\pi)^4\mathrm{d}\Phi_n(P;p_1,p_2,\ldots,p_n), \tag{5.72}$$

of a mass into n bodies, where $\mathcal{M}$ denotes the Lorentz invariant matrix element which is in our case given by $\langle \ell\ell'\ell''|\mathcal{H}_{\text{eff}}|\tau^-\rangle$, and $\mathrm{d}\Phi_n$ the n-body phase space which reads

$$\mathrm{d}\Phi_n(P;p_1,p_2,\ldots,p_n) = \delta\left(P - \sum_{i=1}^{n} p_i\right)\prod_{i=1}^{n}\frac{\mathrm{d}^3\boldsymbol{p}_i}{(2\pi)^3 2E_i}. \tag{5.73}$$

This simplifies to

$$\mathrm{d}\Gamma = \frac{1}{2M}\prod_{i=1}^{3}\frac{\mathrm{d}^3\boldsymbol{p}_i}{(2\pi)^3 2E_i}|\mathcal{M}|^2(2\pi)^4\delta^{(4)}(P-p_1-p_2-p_3) \tag{5.74}$$

for the case of three body decays mediated by our effective four fermion vertices which we have calculated in section 5.1. At first we will rewrite the differential of $\boldsymbol{p}_1$ to

$$\frac{\mathrm{d}^3\boldsymbol{p}_1}{2E_1} = \mathrm{d}^4 p_1\delta(p_1^2 - m_1^2) \tag{5.75}$$

in order to eliminate the four dimensional delta function in 5.74. This gives us

$$\mathrm{d}\Gamma = \frac{1}{2M}\frac{1}{(2\pi)^5}\frac{\mathrm{d}^3\boldsymbol{p}_2}{2E_2}\frac{\mathrm{d}^3\boldsymbol{p}_3}{2E_3}|\mathcal{M}|^2\delta\big((P-p_2-p_3)^2 - m_1^2\big)\,. \tag{5.76}$$

The next step consists of rewriting the remaining differentials to spherical coordinates. This gives us

$$\frac{\mathrm{d}^3\boldsymbol{p}_i}{2E_i} = \frac{1}{2E_i}|\boldsymbol{p}_i|^2\mathrm{d}|\boldsymbol{p}_i|\,\mathrm{d}\phi_i\,\mathrm{d}\!\cos\theta_i = |\boldsymbol{p}_i|\mathrm{d}E_i\,\mathrm{d}\phi_i\,\mathrm{d}\!\cos\theta_i\,, \tag{5.77}$$

where we have rewritten the $\mathrm{d}|\boldsymbol{p}_i|$ to $\mathrm{d}E_i$ according to

$$|\boldsymbol{p}_i| = \sqrt{E_i^2 - m_i^2} \Rightarrow \mathrm{d}|\boldsymbol{p}_i| = \frac{E_i}{\sqrt{E_i^2 - m_i^2}}\mathrm{d}E_i \tag{5.78}$$

in the second step. Using this on (5.76) we obtain

$$\mathrm{d}\Gamma = \frac{1}{8M}\frac{|\mathcal{M}|^2}{(2\pi)^5}|\boldsymbol{p}_2||\boldsymbol{p}_3|\mathrm{d}E_2\,\mathrm{d}E_3\,\mathrm{d}\phi_2\mathrm{d}\phi_3\,\mathrm{d}\!\cos\theta_2\mathrm{d}\!\cos\theta_{23}\,\delta\big((P-p_2-p_3)^2 - m_1^2\big)\,. \tag{5.79}$$

Note that we have chosen the azimuthal angle θ_{12} of p_2 as an angle between p_1 and p_2 rather than in dependence of the z-axis (like e.g. the angle θ_1). In a few lines we will see that this is a natural choice, as it provides an easy integration of the delta function. Therefore we have to reconsider the content of the remaining delta function:

$$\begin{aligned}(P-p_2-p_3)^2 - m_1^2 =& M^2 - 2E_2 - 2E_3 - (p_2+p_3)^2 - m_1^2\\ =& M^2 - 2E_2 - 2E_3 - m_1^2 - m_2^2 - m_3^2 - 2p_2\cdot p_3\\ =& M^2 - 2E_2 - 2E_3 - m_1^2 - m_2^2 - m_3^2 - 2E_2E_3 + 2|\boldsymbol{p}_2||\boldsymbol{p}_3|\cos\theta_{23}\,.\end{aligned} \tag{5.80}$$

If we additionally take into account the relation $\delta(ax) = 1/|a|\delta(x)$ we obtain

$$\delta\big((P-p_2-p_3)^2 - m_1^2\big) = \frac{1}{2|\boldsymbol{p}_2||\boldsymbol{p}_3|}\delta(\cos\theta_{23} - \cos\theta_{23}^0)\,, \tag{5.81}$$

where we have consolidated all parts not including $\cos\theta_{12}$ in the constant $\cos\theta^0_{12}$. Inserting this into (5.79) and performing the corresponding integration we find

$$\mathrm{d}\Gamma = \frac{1}{16M}\frac{1}{(2\pi)^5}|\mathcal{M}|^2 \mathrm{d}E_2\,\mathrm{d}E_3\,\mathrm{d}\phi_2 \mathrm{d}\phi_3\,\mathrm{d}\cos\theta_2\,. \tag{5.82}$$

Of course it is important to make sure that all contents of $\mathcal{M}$ have to be replaced correctly while integrating over the delta functions. The remaining angles are the three Euler angles which determine the orientation of the final system relative to the initial particle. These can be integrated out, if we average over the spin states of the decaying particle. We obtain

$$\mathrm{d}\Gamma = \frac{1}{8M}\frac{1}{(2\pi)^3}|\overline{\mathcal{M}}|^2 \mathrm{d}E_2\,\mathrm{d}E_3\,, \tag{5.83}$$

where $|\overline{\mathcal{M}}|$ denotes the spin averaged matrix element. Introducing a more common notation we define the new parameters $p_{ij} = p_i + p_j$ which lead to the invariant masses $m^2_{ij} = p^2_{ij}$ per two particles. These variables are not independent, since we get

$$m^2_{12} + m^2_{23} + m^2_{13} = M^2 + m^2_1 + m^2_2 + m^2_3, \tag{5.84}$$

if we sum up all possibilities. In the following we will choose to express E_1 and E_2 by m^2_{12} and m^2_{23} respectively. Using the total momentum conservation $P = p_1 + p_2 + p_2$ we can rewrite the invariant masses m^2_{12} and m^2_{23} to

$$m^2_{23} = (p_2 + p_3)^2 = (P - p_1)^2 = M^2 + m^2_1 - 2E_1 \tag{5.85}$$

$$m^2_{12} = (p_1 + p_2)^2 = (P - p_3)^2 = M^2 + m^2_3 - 2E_3\,, \tag{5.86}$$

where E_2 and E_3 are the energies of the second and third particle in the rest frame of M. Taking into account that $P^2 = M^2$, $P = (M, \mathbf{0})$, $p_i = (E_i, p_i)$ and $p^2_i = m_i$, we find

$$E_2 = \frac{P\cdot p_2}{M} = \frac{M^2 + m^2_2 - m^2_{23}}{2M} \tag{5.87}$$

$$E_3 = \frac{P\cdot p_3}{M} = \frac{M^2 + m^2_3 - m^2_{12}}{2M}\,. \tag{5.88}$$

and thus

$$\mathrm{d}E_2 = -\frac{1}{2M}\mathrm{d}m^2_{23} \qquad \text{and} \qquad \mathrm{d}E_3 = -\frac{1}{2M}\mathrm{d}m^2_{12} \tag{5.89}$$

Substituting this into (5.83) we find

$$\mathrm{d}\Gamma = \frac{1}{32M^3}\frac{1}{(2\pi)^3}|\overline{\mathcal{M}}|^2 \mathrm{d}m^2_{12}\,\mathrm{d}m^2_{23}\,, \tag{5.90}$$

which is the standard form for the Dalitz plot. For a given value of m^2_{12} the range of m^2_{23} is determined by its values, when $\boldsymbol{p}_2$ is parallel or antiparallel to $\boldsymbol{p}_3$:

$$(m^2_{23})_{\max} = (E^*_2 + E^*_3)^2 - \left(\sqrt{E^{*2}_2 - m^2_2} - \sqrt{E^{*2}_3 - m^2_3}\right) \tag{5.91}$$

and

$$(m^2_{23})_{\min} = (E^*_2 + E^*_3)^2 - \left(\sqrt{E^{*2}_2 - m^2_2} + \sqrt{E^{*2}_3 - m^2_3}\right). \tag{5.92}$$

Here $E^*_2 = (m^2_{12} - m^2_1 + m^2_2)/(2m_{12})$ and $E^*_3 = (M^2 - m^2_{12} - m^2_3)/(2m_{12})$ are the energies of the particles 2 and 3 in the m_{12} rest frame. The scatter plot in m^2_{12} and m^2_{23} is called a Dalitz plot. If $|\overline{\mathcal{M}}|^2$ is constant, the kinematically allowed region of the plot will be uniformly populated with even events. Therefore a nonuniformity in the plot gives immediate information about $|\mathcal{M}|^2$.

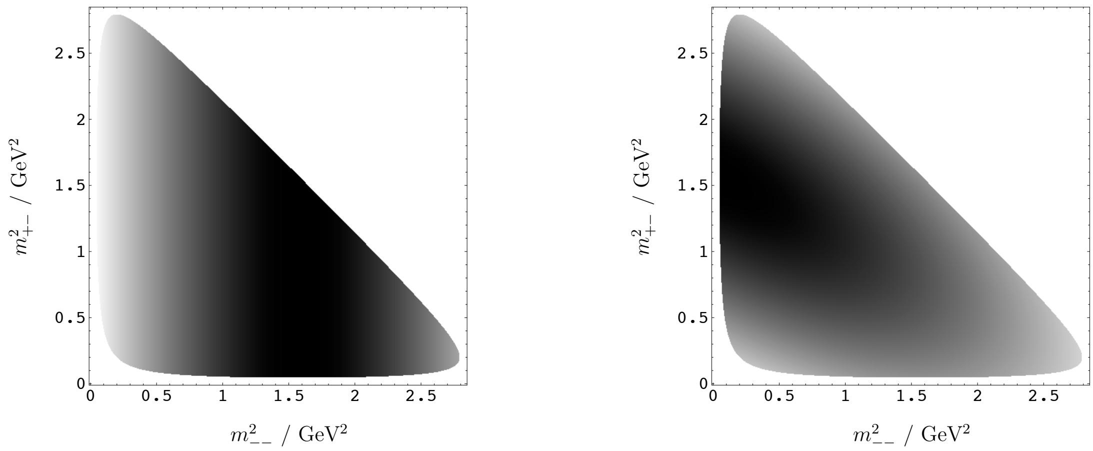


Figure 5.3.: Dalitz plot for $\mathrm{d}^2\Gamma_V^{(LL)(LL)}$ (left) and $\mathrm{d}^2\Gamma_V^{(LL)(RR)}$ (right) in $\tau^- \to \mu^-\mu^-\mu^+$.

5.3.2. The decay $\tau \to \mu^-\mu^-\mu^+$

At first we will give a detailed analysis of the decay $\tau^- \to \mu^-\mu^-\mu^+$ which is probably the most prominent channel to be looked for at the LHC. Therefore we shall examine the Dalitz distributions for the different chirality structures LLLL and LLRR of the dimension 6 effective Hamiltonian in terms of the variables

$$m_{--}^2 \equiv m_{12}^2 = (p_{\mu^-} + p'_{\mu^-})^2 \,, \qquad m_{+-}^2 \equiv m_{23}^2 = (p'_{\mu^-} + p_{\mu^+})^2 \,, \tag{5.93}$$

and $m_{13}^2 = m_\tau^2 + 3m_\mu^2 - m_{--}^2 - m_{+-}^2$. We will make use of (approximate) helicity conservation which implies that many of the interference terms between the operators with different chiralities are suppressed by powers of m_μ, and therefore can be ignored to first approximation.

In the following we will perform a subsequent calculation of all operators with the different helicity structures (5.19) and (5.21). Therefore we will start with the simplest case, namely where all four fermions are left-handed. The decay amplitude is then determined by $\mathcal{H}_{\text{eff}}^{(LL)(LL)}$ providing us with the decay distribution

$$\frac{\mathrm{d}^2\Gamma_V^{(LL)(LL)}}{\mathrm{d}m_{23}^2\,\mathrm{d}m_{12}^2} = \frac{|g_V^{(L_\mu L^\tau)(L_\mu L^\mu)}|^2}{\Lambda^4}\,\frac{(m_\tau^2 - m_\mu^2)^2 - (2m_{12}^2 - m_\tau^2 - 3m_\mu^2)^2}{256\,\pi^3\,m_\tau^3} \tag{5.94}$$

which is shown in Fig. 5.3 (left). The events are equally distributed along m_{+-}^2, while there is a rather flat maximum at $m_{--}^2 = m_{12}^2 \simeq m_\tau^2/2$. The case with all particles right-handed is completely analogous and yields the same distribution with

$$g_V^{(LL)(LL)} \to g_V^{(RR)(RR)}. \tag{5.95}$$

Remember that $g_V^{(RR)(RR)}$ is expected to be strongly suppressed within MLFV scenarios.

Next we will consider the operator $\mathcal{H}_{\text{eff}}^{(LL)(RR)}$ in (5.21). For a left-handed τ-lepton we obtain the Dalitz distribution

$$\begin{aligned}\frac{\mathrm{d}^2\Gamma_V^{(LL)(RR)}}{\mathrm{d}m_{23}^2\mathrm{d}m_{12}^2} = \frac{|g_V^{(L_\mu L^\tau)(R_\mu R^\mu)}|^2}{\Lambda^4}\Bigg[&\frac{(m_\tau^2 - m_\mu^2)^2 - 4m_\mu^2\,(m_\tau^2 + m_\mu^2 - m_{12}^2)}{512\,\pi^3\,m_\tau^3} \\ &- \frac{(2m_{13}^2 - m_\tau^2 - 3m_\mu^2)^2 + (2m_{23}^2 - m_\tau^2 - 3m_\mu^2)^2}{1024\,\pi^3\,m_\tau^3}\Bigg]\,,\end{aligned} \tag{5.96}$$

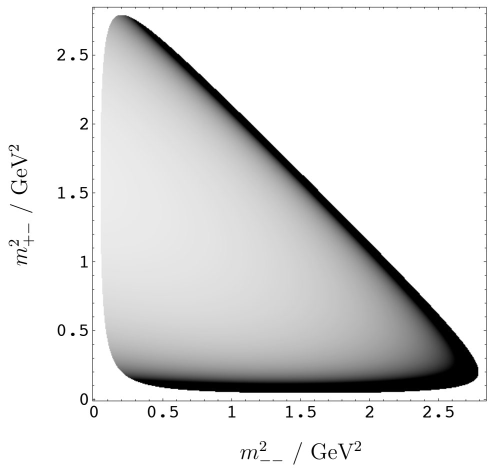


Figure 5.4.: Dalitz plot for $\mathrm{d}^2\Gamma^{(LR)}_{\mathrm{rad}}$ in $\tau^- \to \mu^-\mu^-\mu^+$.

shown in Fig. 5.3 (right). In this case the events are distributed around a flat maximum at $m^2_{+-} \simeq m^2_\tau/2$ and $m^2_{--} \simeq 0$. Again, the case of a right-handed τ yields the same distribution. As pointed out above, the interference terms between (5.19) and (5.21) are suppressed by m^2_μ/m^2_τ.

This exhausts the list of possible four-fermion operators. Now we will consider the contribution from the radiative operator (5.40). As a result we obtain the Dalitz distribution

$$\frac{\mathrm{d}^2\Gamma^{(LR)}_{\mathrm{rad}}}{\mathrm{d}m^2_{23}\mathrm{d}m^2_{12}} = \alpha^2_{\mathrm{em}}\,\frac{|g^{(L_\mu R^\tau)}_{\mathrm{rad}}|^2\,v^2}{\Lambda^4}\left[\frac{m^2_\mu\,(m^2_\tau - m^2_\mu)^2}{128\,\pi^3\,m^3_\tau}\left(\frac{1}{m^4_{13}} + \frac{1}{m^4_{23}}\right) + \frac{m^2_\mu(m^4_\tau - 3m^2_\tau m^2_\mu + 2m^4_\mu)}{128\,\pi^3\,m^2_{13}\,m^2_{23}\,m^3_\tau}\right.$$
$$\left. + \frac{(m^2_{13}+m^2_{23})(m^4_{12}+m^4_{13}+m^4_{23} - 6m^2_\mu(m^2_\mu + m^2_\tau))}{256\,\pi^3\,m^2_{13}\,m^2_{23}\,m^3_\tau} + \frac{2m^2_{12} - 3m^2_\mu}{128\,\pi^3\,m^3_\tau}\right], \tag{5.97}$$

for a right-handed τ lepton which is plotted in Fig. 5.4. This time the events are concentrated at low values of m^2_{23} or m^2_{13} resulting from the photon pole. Again, the decay distribution of the left-handed τ is calculated in complete analogy to that.

Finally, we have to determine the contribution arising from the interference terms between the radiative operators and four-fermion operators. To be more exact we should mention that we only have to analyze the cases where only the chirality of the τ lepton has to be flipped. The interference term between the four-fermion operator (5.19) and the radiative operator (5.40) reads

$$\frac{\mathrm{d}^2\Gamma^{(LL)(LL)}_{\mathrm{mix}}}{\mathrm{d}m^2_{23}\mathrm{d}m^2_{12}} = \alpha_{\mathrm{em}}\,\frac{2\,v\,\mathrm{Re}[g^{(L_\mu L^\tau)(L_\mu L^\mu)}_{\mathrm{V}}\;g^{*(L_\mu R^\tau)}_{\mathrm{rad}}]}{\Lambda^4}\left[\frac{m^2_{12} - 3m^2_\mu}{64\,\pi^3\,m^2_\tau} + \frac{m^2_\mu(m^2_\tau - m^2_\mu)(m^2_{13}+m^2_{23})}{128\,\pi^3\,m^2_\tau\,m^2_{13}\,m^2_{23}}\right], \tag{5.98}$$

while the interference term between (5.21) and (5.40) results in

$$\begin{aligned}\frac{\mathrm{d}^2\Gamma^{(LL)(RR)}_{\mathrm{mix}}}{\mathrm{d}m^2_{23}\mathrm{d}m^2_{12}} &= \alpha_{\mathrm{em}}\,\frac{2\,v\,\mathrm{Re}[g^{(L_\mu L^\tau)(R_\mu R^\mu)}_{\mathrm{V}}\;g^{*(L_\mu R^\tau)}_{\mathrm{rad}}]}{\Lambda^4}\\ &\quad\times\left[\frac{m^2_\tau - m^2_{12} - 3m^2_\mu}{256\,\pi^3\,m^2_\tau} + \frac{m^2_\mu(m^2_\tau - m^2_\mu)(m^2_{13}+m^2_{23})}{256\,\pi^3\,m^2_\tau\,m^2_{13}\,m^2_{23}}\right],\end{aligned} \tag{5.99}$$

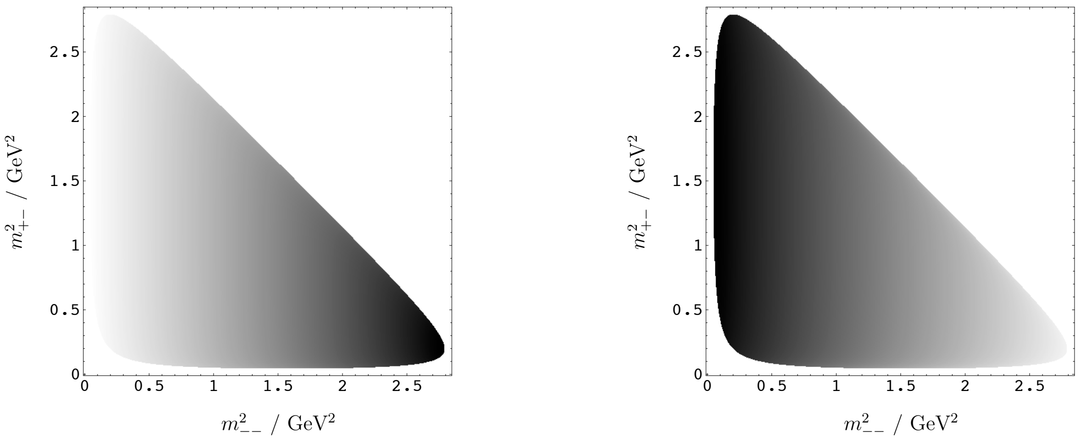


Figure 5.5.: Dalitz plot for $|\mathrm{d}^2\Gamma_{\mathrm{mix}}^{(LL)(LL)}|$ (left) and $|\mathrm{d}^2\Gamma_{\mathrm{mix}}^{(LL)(RR)}|$ (right) in $\tau^- \to \mu^-\mu^-\mu^+$.

which are shown in Fig. 5.5. In both cases the photon pole at $m_{13}^2 = 0$ or $m_{23}^2 = 0$ is suppressed by the small muon mass, while the remaining terms increase (decrease) monotonically with m_{12}^2. Combining the results (5.94, 5.96, 5.97, 5.98, 5.99) and integrating over the phase space, we obtain

$$\frac{\Gamma[\tau^- \to \mu^-\mu^-\mu^+]}{\Gamma[\tau^- \to \mu^-\bar{\nu}_\mu\nu_\tau]} = \frac{1}{G_F^2\Lambda^4}\left\{\frac{2\,|g_V^{(LL)(LL)}|^2 + 2\,|g_V^{(RR)(RR)}|^2 + |g_V^{(LL)(RR)}|^2 + |g_V^{(RR)(LL)}|^2}{8}\right.$$
$$+\frac{\alpha_{\mathrm{em}}^2\, v^2}{m_\tau^2}\left(\ln\frac{m_\tau^2}{m_\mu^2} - \frac{11}{4}\right)\left(|g_{\mathrm{rad}}^{(LR)}|^2 + |g_{\mathrm{rad}}^{(RL)}|^2\right) \tag{5.100}$$
$$\left.+\frac{\alpha_{\mathrm{em}}\, v}{2m_\tau}\mathrm{Re}\left[2\, g_{\mathrm{rad}}^{*(LR)}\, g_V^{(LL)(LL)} + g_{\mathrm{rad}}^{*(LR)}\, g_V^{(LL)(RR)} + (L \leftrightarrow R)\right]\right\}$$

for the total decay width (normalized to the standard model decay $\tau \to \mu\,\bar{\nu}_\mu\,\nu_\tau$ neclecting the muon mass). This result is consistent with the formula quoted, for instance, in [89].

5.3.3. The decay $\tau^- \to e^-\mu^-\mu^+$

The lepton violating τ decays into three different charged leptons will not deliver a signature as clean as $\tau \to \mu^-\mu^-\nu^+$, but however we will discuss the for completeness considering the example $\tau^- \to e^-\mu^-\mu^+$. Again, we will give the results in terms of the invariant masses

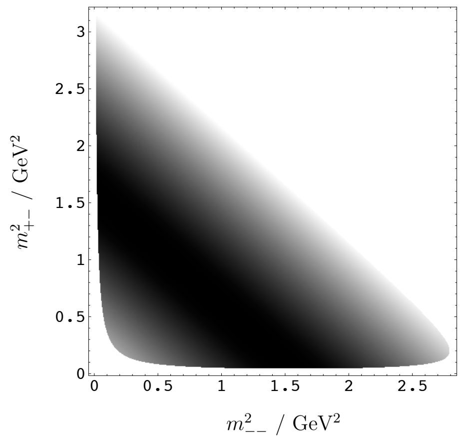
$$m_{--}^2 \equiv m_{12}^2 = (p_{e^-} + p'_{\mu^-})^2\,, \qquad m_{+-}^2 \equiv m_{23}^2 = (p'_{\mu^-} + p_{\mu^+})^2\,, \tag{5.101}$$

and $m_{13}^2 = m_\tau^2 + 2m_\mu^2 - m_{--}^2 - m_{+-}^2$, where we set the electron mass to zero. Note that the photon pole from $\tau^- \to e^-\gamma^* \to e^-\mu^-\mu^+$ is still regulated by the muon mass.

Starting again with the purely left-handed term in the effective Hamiltonian (5.19), we obtain the Dalitz distribution

$$\frac{\mathrm{d}^2\Gamma_V^{(LL)(LL)}}{\mathrm{d}m_{23}^2\,\mathrm{d}m_{12}^2} = \frac{|g_V^{(L_eL^\tau)(L_\mu L^\mu)}|^2}{\Lambda^4}\,\frac{m_\tau^4 - (2m_{12}^2 - m_\tau^2 - 2m_\mu^2)^2}{512\,\pi^3\, m_\tau^3}\,, \tag{5.102}$$

which coincides, apart from the region of the phase space boundaries, with (5.94) up to corrections of order m_μ^2/m_τ^2 and a statistical factor. Hence the corresponding Dalitz plot looks



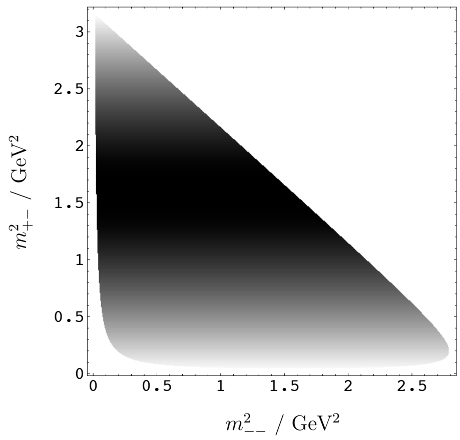


Figure 5.6.: Dalitz plots for the two contributions to $\mathrm{d}^2\Gamma_V^{(LL)(RR)}$ in $\tau^- \to e^-\mu^-\mu^+$. The term proportional to $|g_V^{(L_eL^\tau)(R_\mu R^\mu)}|^2$ is displayed on the left side while the term proportional to $|g_V^{(L_\mu L^\tau)(R_e R^\mu)}|^2$ is shown on the right side

almost identical to Fig. 5.3 (left). Like in the last section there is no need to calculate the purely right-handed term, since it gives the analog result.

From $\mathcal{H}_{\text{eff}}^{(LL)(RR)}$ in (5.21) we obtain for the case of a left-handed τ-lepton

$$\begin{aligned}\frac{\mathrm{d}^2\Gamma_V^{(LL)(RR)}}{\mathrm{d}m_{23}^2\mathrm{d}m_{12}^2} &= \frac{|g_V^{(L_eL^\tau)(R_\mu R^\mu)}|^2}{\Lambda^4}\,\frac{m_\tau^4-(2m_{13}^2-m_\tau^2-2m_\mu^2)^2}{512\,\pi^3\,m_\tau^3}\\ &\quad+\frac{|g_V^{(L_\mu L^\tau)(R_e R^\mu)}|^2}{\Lambda^4}\,\frac{(m_\tau^2-2m_\mu^2)^2-(2m_{23}^2-m_\tau^2-2m_\mu^2)^2}{512\,\pi^3\,m_\tau^3}\,.\end{aligned} \tag{5.103}$$

There is barely a need to mention that the structure for the right-handed τ is similar to that. The results are shown in Fig. 5.6. The events are distributed around $m_{--}^2 + m_{+-}^2 \simeq m_\tau^2/2$ or $m_{+-}^2 \simeq m_\tau^2/2$, respectively. For equal coupling constants in (5.103) we recover the $\tau \to 3\mu$ case in (5.96) (again up to mass corrections and a statistical factor).

For the radiative decay operators, we obtain

$$\frac{\mathrm{d}^2\Gamma_{\text{rad}}^{(LR)}}{\mathrm{d}m_{23}^2\mathrm{d}m_{12}^2} = \alpha_{\text{em}}^2\,\frac{|g_{\text{rad}}^{(L_eR^\tau)}|^2\,v^2}{\Lambda^4}\left[\frac{m_\mu^2\,(m_{23}^2-m_\tau^2)^2}{64\,\pi^3\,m_\tau^3\,m_{23}^4}+\frac{m_{12}^4+m_{13}^4-2m_\mu^4}{128\,\pi^3\,m_\tau^3\,m_{23}^2}+\frac{m_\tau^2-m_{23}^2}{128\,\pi^3\,m_\tau^3}\right], \tag{5.104}$$

and the corresponding Dalitz plot is shown in Fig: 5.7. In this case the photon pole extends the events at low values of $m_{+-}^2 = m_{23}^2$.

Last but not least we have to calculate the interference terms to complete our analysis. We obtain

$$\frac{\mathrm{d}^2\Gamma_{\text{mix}}^{(LL)(LL)}}{\mathrm{d}m_{23}^2\mathrm{d}m_{12}^2} = \alpha_{\text{em}}\,\frac{2\,v\,\mathrm{Re}[g_{\text{V}}^{(L_eL^\tau)(L_\mu L^\mu)}\,g_{\text{rad}}^{*(L_eR^\tau)}]}{\Lambda^4}\left[\frac{m_{12}^2-2m_\mu^2}{128\pi^3 m_\tau^2}+\frac{m_\mu^2}{128\pi^3 m_{23}^2}\right], \tag{5.105}$$

and

$$\frac{\mathrm{d}^2\Gamma_{\text{mix}}^{(LL)(RR)}}{\mathrm{d}m_{23}^2\mathrm{d}m_{12}^2} = \alpha_{\text{em}}\,\frac{2\,v\,\mathrm{Re}[g_{\text{V}}^{(L_eL^\tau)(R_\mu R^\mu)}\,g_{\text{rad}}^{*(L_eR^\tau)}]}{\Lambda^4}\left[\frac{m_{13}^2-2m_\mu^2}{128\pi^3 m_\tau^2}+\frac{m_\mu^2}{128\pi^3 m_{23}^2}\right]. \tag{5.106}$$

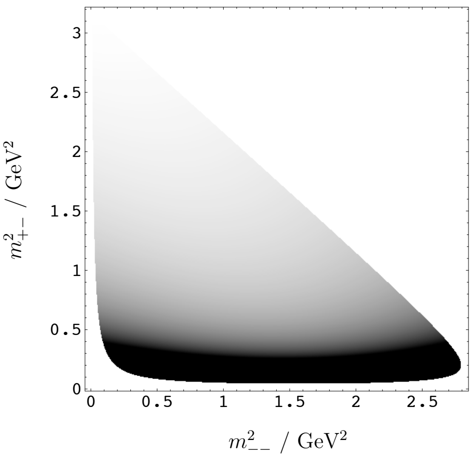


Figure 5.7.: Dalitz plot for $\mathrm{d}^2\Gamma^{(LR)}_{\mathrm{rad}}$ in $\tau^- \to e^-\mu^-\mu^+$.

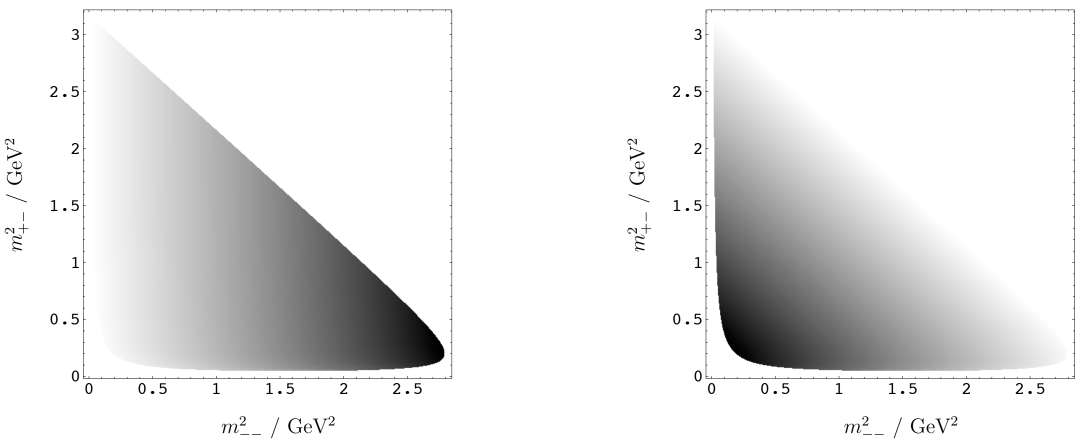


Figure 5.8.: Dalitz plot for $|\mathrm{d}^2\Gamma^{(LL)(LL)}_{\mathrm{mix}}|$ (left) and $|\mathrm{d}^2\Gamma^{(LL)(RR)}_{\mathrm{mix}}|$ (right) in $\tau^- \to e^-\mu^-\mu^+$.

The corresponding Dalitz plots are shown in Fig. 5.8.

5.4. Discussion and conclusions

Since many new physics models allow dramatic extensions to the tiny standard model effects in the sector of lepton flavor violating decays, there is the opportunity to falsify the standard model - or setting further constraints to the new physics models - by measuring such decays at future experiments. As we have chosen a model independent approach the results calculated above will be helpful in both cases. Especially the different structure of the Dalitz plots for the four fermion vertices which show a rather uniform behavior, and the radiative contributions which are concentrated at small values of $m^2_{+-} = (p_{\ell^+} + p_{\ell^-})^2$ will help to decide between new physics models with different sources of lepton flavor violation - even on the basis of rather few events - since these models give rather different predictions of the relative size of radiative and four fermion vertices.

As an example for models which show a completely different behavior in preferring either the radiative of four-fermion contributions we shall compare the case of supersymmetric extension to the standard model with the case little Higgs models. For supersymmetric models the photondipole operator which is induced by penguin diagrams usually dominates over the four-lepton operators. This leads to correlations like

$$\frac{\Gamma(\tau \to 3\mu)}{\Gamma(\tau \to \mu\gamma)} \simeq \frac{\alpha_{\rm em}}{3\pi}\left(\ln\frac{m_\tau^2}{m_\mu^2} - \frac{11}{4}\right) = \mathcal{O}(10^{-3}) \tag{5.107}$$

as presented in [89]. In this case we would expect Dalitz distributions as shown in fig. 5.4. It has been pointed out in [93] and [95] that Higgs mediated $\tau \to \mu$ transitions may lead to different results $\tan\beta$ and the off-diagonal slepton mass-matrix element $\delta_{3\ell}$ are large. For example the author of [93] and [95] finds

$$\frac{\Gamma(\tau \to \ell\mu\mu)}{\Gamma(\tau \to \mu\gamma)} \leq \frac{3 + 5\delta_{\ell\mu}}{36} \sim \mathcal{O}(0.1) \tag{5.108}$$

in the decoupling limit ($\cos(\beta - \alpha) = 0$, $m_{A^0} \gg M_Z$), where $\delta_{\ell\mu} = \tilde{m}_{\ell\mu}^2/\tilde{m}^2$. From an experimental point of view, this will lead to a more elaborate analysis, since generally the interplay of all contributions to the Dalitz distributions shown in fig. 5.3 - 5.5 have to be allowed. For the case to little Higgs models with T-parity (LHT) [98, 99, 100] we face a completely different situation. Here the Z_0 and box-diagram contributions dominate over the radiative operators [100] which is essentially induced by the constructive (destructive) interference term between the individual heavy gauge boson contributions. For a mass scale of the LHT mirror fermions of about 1 TeV one finds

$$\frac{\Gamma(\tau \to 3\mu)}{\Gamma(\tau \to \mu\gamma)} = \mathcal{O}(1)\,, \tag{5.109}$$

depending on the parameter values of the LHT. In this case we would expect rates for lepton flavor violating decays which are already close to the present bounds. Furthermore, the dominance of the four-fermion operator suggests rather flat Dalitz distributions for $\tau \to 3\mu$ as illustrated in fig. 5.3.

A. Mathematics

A.1. Distributions

Distributions are defined as generalized functions [75]. This means we have a linear and continuos map from a room of test functions $f(x)$ to the room of real numbers. The test functions have to be continuos and differentiable as often as required. We often make use of the Heaviside theta distribution $\theta(x)$ and the Dirac delta distribution $\delta(x)$. Both distributions are defined by an integral over a regular test function. Note that these distributions do not make any sense if they are not defined according to a specific integral. However, in this dissertation we often encounter differential rates that contain theta of delta distributions without including the corresponding integral. This does not lead to any problems, since the differential rates do not have any physical meaning, until they are integrated to show the actual decay rate.

The Dirac delta distribution is defined by the relation

$$\int\limits_a^b \mathrm{d}x\, \delta(x-x_0)\, f(x) = \begin{cases} f(x_0) & \text{for} \quad a \le x_0 \le b \\ 0 & \text{else} \end{cases}, \tag{A.1}$$

where a and b are arbitrary real numbers obeying $a < b$. This means it has to vanish for all values where $x \neq x_0$ and go to infinity at $x = x_0$ such that

$$\int\limits_{-\infty}^{\infty} \mathrm{d}x\, \delta(x-x_0) = 1\,, \tag{A.2}$$

which means nothing else, than that the area beneath the delta function is always normalized to 1. The derivative of the delta function is defined by the partial integration

$$\int\limits_{-\infty}^{+\infty} \mathrm{d}x\, \delta'(x-x_0)\, f(x) = -\int\limits_{-\infty}^{+\infty} \mathrm{d}x\, \delta(x-x_0)\, f'(x) = -f'(x_0)\,. \tag{A.3}$$

The n^{th} derivative of the delta function is then obviously given by performing n partial integrations

$$\int\limits_{-\infty}^{+\infty} \mathrm{d}x\, \delta^{(n)}(x-x_0)\, f(x) = (-1)^n f^{(n)}(x_0)\,, \tag{A.4}$$

The delta distribution with an argument $g(x)$ can be rewritten to a set of delta distributions with arguments $x - x_j$ by the Relation

$$\delta\big(g(x)\big) = \sum_j \frac{1}{|g'(x_j)|}\delta(x-x_j)\,, \tag{A.5}$$

where $g'(x)$ denotes the derivative of the delta functions argument and the summation is performed over all single roots x_j of $g(x)$. Another useful relation is

$$\frac{\mathrm{d}^n\delta(a\,x)}{\mathrm{d}(a\,x)^n} = \frac{1}{|a|a^n}\frac{\mathrm{d}^n\delta(x)}{\mathrm{d}x^n}\,, \tag{A.6}$$

which helps in many cases to simplify equations containing a delta function considerably. A special representation of the delta function is given by the Fourier transformation. Using the definition from above we find

$$\int \mathrm{d}^\xi x\, e^{ik\cdot x} = (2\pi)^\xi \delta^{[\xi]}(k). \tag{A.7}$$

Note that $\delta^{[n]}(k)$ now denotes the ξ-dimensional delta function which should not be confused with the n^{th} derivative defined above.

The Heaviside theta distribution is defined by

$$\theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \tag{A.8}$$

such that the delta distribution is its derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\theta(x) = \delta(x)\,. \tag{A.9}$$

It can be used to define the limits of an integration. For example we can write

$$\int_a^{+\infty} \mathrm{d}x\, f(x) = \int_{-\infty}^{+\infty} \mathrm{d}x\, \theta(x)\, f(x)\,. \tag{A.10}$$

This is important if the integral contains derivatives of a delta distribution, as in this case the theta distribution has to be derived alongside the function $f(x)$ while one performs the partial integrations to remove the derivatives from the delta distribution.

A.2. Lie groups and representations

In this chapter we will give a short review of Lie groups. It shall not be a complete introduction to group theory but rather a summary of definitions we need for the following sections. Some notes about group theory can be found in many books about quantum field theory or the standard model [9, 12]. However, a more complete introduction to group theory from a physicists point of view is given in [13].

The starting point of our considerations is obviously the definition of a group. A group is a (not empty) set $G = \{g\}$, for which we define a composition "$\circ$" with the following attributes:

- Completion: $g_1, g_2 \in G \rightarrow g_1 \circ g_2 \in G$
- Associative law: $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$
- An identity element e with $g \circ e = g = e \circ g$ exists.
- It exists exactly one inverse element g^{-1} for each g with $g \circ g^{-1} = e = g^{-1} \circ g$

The conjunctions of Lie groups additionally have a analytical structure given by

$$g(\alpha_i) \circ g(\beta_i) = g\big(f_i(\alpha_i, \beta_i)\big) \tag{A.11}$$

with analytical functions f_i in every variable. Up to now the definitions we have given are very general. In physics it is often useful to regard groups in a certain representation. A representation is a map $D\colon G \to GL(\mathcal{V})$ such that $D(g_i g_j) = D(g_i)D(g_j)$ for all $g_i, g_j \in G$, where $\mathcal{V}$ is an N dimensional linear vector space called representation space. Note that the dimension N of this vector space has generally nothing to do with the dimension of the group $\dim G$, since groups can be represented in different ways using vector spaces of different dimensions. The group's properties are now described by the general linear transformations

$$x_i \to M_{ij}(\alpha) x_j \tag{A.12}$$

in the representation space, where $i, j = 1, \ldots, N$. M is a $N \times N$ nonsingular ($\det M \neq 0$) and linear ($M(a\boldsymbol{x} + b\boldsymbol{y}) = a\, M\boldsymbol{x} + b\, M\boldsymbol{x}$) matrix. By means of the additional characteristics of those matrices one can classify the groups. The first class is the class of orthogonal groups marked with O. These groups have to fulfill $M^T M = M\, M^T = 1$. Analog to this U marks unitary groups with $M^\dagger M = M\, M^\dagger = 1$. The last class of groups which we will will introduce here is the class of special groups S which means that all group elements have to fulfill $\dim M = 1$. In this work we will mainly be encountered with groups of the structures SU and U.

Having categorized the properties of the groups representations we will now go on and talk about the objects which are transformed by a groups. Objects which have well defined transformation properties concerning a special group are called tensors. Depending on the representation of the group under which these objects transform one distinguishes:

Scalars are tensors of 0^{th} rank. They transform by $s \to s$ which means that $D(g) = 1$ for all group elements g contained in G and therefore $s' = D(g) = s$. This representation is called the trivial representation of the group.

Vectors are tensors of first rank. These objects transform like $v_i \to D_{ij}(g) v_j$ under the group G. $D_{ij}(g)$ is the defining representation of the group since all other representation can be constructed from its matrices. It is often called the fundamental representation of the group.

Tensors of higher rank n then transform by $t_{ij\ldots n} = D_{ii'}(g) D_{jj'}(g) \ldots D_{nn'}(g)\, t_{i'j'\ldots n'}$,which can be rewritten to $t_\alpha \to D_{\alpha\beta}(g) t_\beta$ by merging the indices $i \ldots n$ to a set of indices α. $D_{\alpha\beta}(g)$ is then a higher dimensional representation of G. Thus it is possible to consider the transformation behavior of a 3×3 matrix by its elements $\alpha = 1 \ldots 9$ instead by the matrix elements M_{ij}. Here $i, j = 1 \ldots 3$ denotes the location where the elements are placed in the matrix. For tensors of second rank the representation is called the adjoint representation.

For the classification of groups it is most common to use the dimension of the fundamental representation. The dimension of the group which corresponds to the number of independent parameters can then be concluded from the fundamental representations dimension and its additional properties. For example we have the $SU(N)$ which is of dimension $\dim SU(N) = N^2 - 1$ since its complex matrices are unitary and we have additionally the $\det M = 1$. All in all the number with which we classify the group should not be confused with the number of its free parameters. In the following we will observe how the transformations of Lie groups are described using generators. Therefore we consider a group $G = \{g(\alpha_a)\}$ with continuos parameters α_a numbered by $a = 1 \ldots n$, where $n = \dim G$. Remembering that we have an identity element $g(0) = 1$ we can taylor expand the other group elements in the defining representation around 1 for infinitesimal parameters α_a and obtain

$$g(\alpha) = 1 + i\alpha_a Q^a + \mathcal{O}(\alpha^2), \tag{A.13}$$

where the quantities

$$Q_a = \frac{1}{i}\frac{\partial g(\alpha)}{\partial \alpha^a} \tag{A.14}$$

are called the generators of the group. The factor i is introduced as a convention and makes the generators of unitary representations of g hermitian. In a further step one can use the analytical structure (A.11) for infinitesimal α and β. A taylor expansion in both parameters gives the commutation relation

$$[Q^a, Q^b] = i f^{abc} Q^c, \quad \text{where} \quad a, b, c \in \{1, \ldots, \dim G\}, \tag{A.15}$$

where the f^{abc} are the structure constants of the group. A Lie group is, in addition to this commutation relation, characterized by the Jacobi identity

$$\begin{aligned} & [[Q^a, Q^b], Q^c] + [[Q^b, Q^c], Q^a] + [[Q^c, Q^a], Q^b] = 0 \\ \Leftrightarrow \quad & f^{abe} f^{cde} + f^{ace} f^{dbe} + f^{ade} f^{bce} = 0\,. \end{aligned} \tag{A.16}$$

If $f^{abc} = 0$ is valid for all a, b and c the group is called abelian. After these definitions we can finally introduce finite transformations in an exponential representation. Using $g(\alpha) = 1 + i\alpha^a Q_a$ with infinitesimal α^a we get a finite transformation by composition of n infinitesimal transformations. Thus we get a new parameter $\theta^a = n\,\alpha^a$ which describes the finite transformations. Therefore we get

$$g(\theta) = \lim_{n\to\infty}\left(1 + i\frac{\theta^a}{n} Q_a\right)^n = e^{i\theta^a Q_a} \tag{A.17}$$

as an exponential representation of our transformation. To get a certain representation of the group $D(Q_a) = T_a$ has to be used in all equations above, where the T^a are $N \times N$ matrices describing the transformations of tensors in an N dimensional vector space.

The algebra of the symmetry group $SU(N)$ has a representation by $N \times N$ matrices which is called the fundamental representation of the $SU(N)$. In this work we need two examples which we will introduce in the following.

The Pauli matrices are the generators of the fundamental representation of the $SU(2)$. Thus we have $N^2 - 1 = 3$ two dimensional matrices. They read

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.18}$$

All three matrices are constructed such that their traces vanish:

$$\mathrm{Tr}(\sigma_a) = 0 \tag{A.19}$$

As commutators and anticommutators we find

$$[\sigma_a, \sigma_b]_- = 2\,i\epsilon_{abc}\sigma_c \qquad (a, b, c = 1, 2, 3) \tag{A.20}$$

$$[\sigma_a, \sigma_b]_+ = 2\,\delta_{ab} \cdot I \qquad (a, b = 1, 2, 3) \tag{A.21}$$

The Gell-Mann matrices are the generators of the fundamental representation of the $SU(3)$. For $N = 3$ we obtain $N^2 - 1 = 8$ of them. Thus we get the 8 matrices

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{aligned} \tag{A.22}$$

of dimension 3. Like the Pauli matrices the Gell-Mann matrices are traceless, while the trace of two matrices is nonzero only for two equal matrices:

$$\mathrm{Tr}(\lambda_a) = 0 \qquad (a = 1, \ldots, 8) \tag{A.23}$$

$$\mathrm{Tr}(\lambda_a \lambda_b) = 2\,\delta_{ab} \qquad (a, b = 1, \ldots, 8)\,. \tag{A.24}$$

Like for the Pauli matrices we find nontrivial commutation relations which take the form

$$[\lambda_a, \lambda_b]_- = 2\,i f_{abc}\,\lambda_c \qquad (a, b, c = 1, \ldots, 8) \tag{A.25}$$

$$[\lambda_a, \lambda_b]_+ = \frac{4}{3}\delta_{ab} I + 2\,d_{abc}\lambda_c \qquad (a, b, c = 1, \ldots, 8)\,. \tag{A.26}$$

Furthermore, we find the completeness relation

$$\lambda^a_{ij}\lambda^a_{kl} = -\frac{2}{3}\delta_{ij}\delta_{kl} + 2\delta_{il}\delta_{jk}\,. \tag{A.27}$$

Table A.1 presents the nonvanishing coefficients of f_{abc} and d_{abc}.

abc	f_{abc}	abc	d_{abc}	abc	d_{abc}
123	1	118	$1/\sqrt{3}$	355	$1/2$
147	$1/2$	146	$1/2$	366	$-1/2$
156	$-1/2$	157	$1/2$	377	$-1/2$
246	$1/2$	228	$1/\sqrt{3}$	448	$-1/(2\sqrt{3})$
257	$1/2$	247	$-1/2$	558	$-1/(2\sqrt{3})$
345	$1/2$	256	$1/2$	668	$-1/(2\sqrt{3})$
367	$-1/2$	338	$1/\sqrt{3}$	778	$-1/(2\sqrt{3})$
458	$\sqrt{3}/2$	334	$1/2$	888	$-1/\sqrt{3}$
678	$\sqrt{3}/2$				

Table A.1.: Nonvanishing f_{abc} and d_{abc} coefficients

B. Basics of relativistic quantum field theory

B.1. Free quantized fields

The quantum mechanical state of a free massive particle shall be given by $|m, p, j, j_3\rangle$, where m denotes the particles mass, p the momentum (which is connected to the energy E_p of the particle by (1.10)) and j the spin with the third component j_3. The normalization of these states is then given by

$$\langle m, p, j, j_3 | m, p', j', j_3' \rangle = 2E_p (2\pi)^3 \delta(\boldsymbol{p} - \boldsymbol{p}') \delta_{j_3 j_3'} \,. \tag{B.1}$$

These free one particle one particle states are connected to the vacuum $|0\rangle$ (the relativistically invariant ground state which is normalized to $\langle 0|0\rangle = 1$) by creation operators $a^\dagger_{j_3}(p)$ and annihilation operators $a_{j_3}(p)$. Acting on the vacuum these operators yield

$$a_{j_3}|0\rangle = 0 \qquad \text{and} \qquad a^\dagger_{j_3}|0\rangle = |m, p, j, j_3\rangle \tag{B.2}$$

While the annihilation operator acting on $|m, p, j, j_3\rangle$ gives

$$a_{j_3}|m, p, j, j_3\rangle = |0\rangle \,. \tag{B.3}$$

Multi-particle states can be created by repeated application of $a^\dagger_{j_3}$ on the vacuum. Those multi-particle states are symmetric under the exchange of identical bosons, while they are antisymmetric under the exchange of identical fermions. This results in the commutation relations

$$[a^\dagger_{j_3}(p), a^\dagger_{j_3'}(p')] = [a_{j_3}(p), a_{j_3'}(p')] = 0 \tag{B.4}$$

$$[a_{j_3}(p), a^\dagger_{j_3'}(p')] = 2E_p (2\pi)^3 \delta(\boldsymbol{p} - \boldsymbol{p}') \delta_{j_3 j_3'} \tag{B.5}$$

for the annihilation and creation operators for the bosons and the anticommutation relations

$$\{a^\dagger_{j_3}(p), a^\dagger_{j_3'}(p')\} = \{a_{j_3}(p), a_{j_3'}(p')\} = 0 \tag{B.6}$$

$$\{a_{j_3}(p), a^\dagger_{j_3'}(p')\} = 2E_p (2\pi)^3 \delta(\boldsymbol{p} - \boldsymbol{p}') \delta_{j_3 j_3'} \tag{B.7}$$

for the fermions respectively. The annihilation and creation operators of antiparticles shall be denoted by $b^\dagger_{j_3}(p)$ and $b_{j_3}(p)$ in the following. The propagation of free particles in space time is described by the Klein-Gordon equation for scalar and vector fields and the Dirac equation for spinor fields acting on free quantized fields.

Spin 0

Scalar fields can be described by the Klein-Gordon equation

$$(\partial^\nu \partial_\nu + m^2)\psi(x) = 0 \tag{B.8}$$

The fields fulfilling this equation read

$$\psi(x) = \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left[a_\lambda(p)\, e^{-ip\cdot x} + a^\dagger(p)\, e^{+ip\cdot x} \right] , \tag{B.9}$$

and obey, according to (B.4) and (B.5), the commutation relations

$$[\psi(x), \psi(x')] = [\psi(x)^\dagger, \psi(x')^\dagger] = 0 \tag{B.10}$$
$$[\psi(x), \psi^\dagger(x')] = i\Delta(x - x', m) , \tag{B.11}$$

where $\Delta(x, m)$ is the Pauli-Jordan-Schwinger function

$$i\Delta(x, m) = \int \frac{\mathrm{d}^4p}{(2\pi)^3}\, \delta(p^2 - m^2)\, \mathrm{sgn}(p^0)\, e^{-ip\cdot x} \tag{B.12}$$

with

$$\mathrm{sgn}(p_0) = \begin{cases} +1 & \text{for} \quad p_0 > 0 \\ -1 & \text{for} \quad p_0 < 0 \end{cases} \tag{B.13}$$

Spin 1

For the case of neutral free vector fields the Klein-Gordon Equation is changed to

$$(\Box\, g_{\mu\nu} + \partial_\mu \partial_\nu) A_\nu(x) + m^2 A_\mu(x) = 0 \tag{B.14}$$

with the free field connected to the annihilation and creation operators by the Fourier transformation

$$A_\mu(x) = \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left[a_\lambda(p)\, \epsilon_\mu(p, \lambda)\, e^{-ip\cdot x} + a_\lambda^\dagger(p)\, \epsilon_\mu^*(p, \lambda)\, e^{+ip\cdot x} \right] . \tag{B.15}$$

We note that this description contains the annihilation operator for particles and the creation operator for antiparticles. In this way antiparticles are described by exchanging $\psi(x)$ and the conjugate field $\psi(x)^\dagger$. The polarization vectors $\epsilon_\mu(p, \lambda)$ describe the polarization of the bosonic field. These vectors obey the relations

$$\epsilon(p, \lambda) \cdot \epsilon^*(p, \lambda') = -\delta_{\lambda\,\lambda'} \tag{B.16}$$
$$\sum_\lambda \epsilon_\mu^*(p, \lambda) \epsilon_\nu(k, \lambda) = -\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2} \right) , \tag{B.17}$$

where $\lambda, \lambda' = 1, 2, 3$. The commutator between two fields is therefore

$$[A_\mu(x), A_\nu(0)]_- = -\left(g_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{m^2} \right) \Delta(x, m) . \tag{B.18}$$

For charged vector fields one has to replace (B.15) by

$$A_\mu(x) = \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left[a_\lambda(p)\, \epsilon_\mu(p, \lambda)\, e^{-ip\cdot x} + b_\lambda^\dagger(p)\, \epsilon_\mu^*(p, \lambda)\, e^{+ip\cdot x} \right] , \tag{B.19}$$

where the operators $a_\lambda(p)$ and $b_\lambda(p)$ separately treat the particles and antiparticles. These operators separately fulfill the commutation relations (B.4-B.5). Additionally they commute:

$$[a_\lambda(p), b_\lambda(p)] = [a_\lambda(p), b_\lambda^\dagger(p)] = [a_\lambda^\dagger(p), b_\lambda(p)] = [a_\lambda^\dagger(p), b_\lambda^\dagger(p)] = 0 . \tag{B.20}$$

Spin 1/2

For spinor fields we have to consider the Dirac equation

$$\sum_{\beta=1}^{4}(i\gamma^{\mu}_{\alpha\beta}\partial_\mu - m\delta_{\alpha\beta})\psi_\beta(x) = 0 \tag{B.21}$$

including the Dirac matrices γ_μ which fulfill the anticommutation relation

$$\{\gamma_\mu, \gamma_\nu\} = 2\, g_{\mu\nu}. \tag{B.22}$$

Again we have used $a_\lambda(p)$ and $b_\lambda(p)$ to describe charged particles. In this case the field can be expressed by

$$\psi_\alpha(x) = \frac{1}{(2\pi)^3}\sum_{j_3}\int\frac{d^3p}{2E_p}\left[u_\alpha(p,j_3)a_{j_3}(p)e^{-ip\cdot x} + v_\alpha(p,j_3)b^\dagger(p)e^{+ip\cdot x}\right], \tag{B.23}$$

where the $u_\alpha(p,j_3)$ and $v_\alpha(p,j_3)$ are the four component Dirac spinors. These spinors are normalized by

$$\sum_\alpha \bar{u}_\alpha(p,j_3')u_\alpha(p,j_3) = 2m\,\delta_{j_3,j_3'} \tag{B.24}$$

$$\sum_\alpha \bar{v}_\alpha(p,j_3')v_\alpha(p,j_3) = -2m\,\delta_{j_3,j_3'} \tag{B.25}$$

for massive particles. For massless particles one usually replaces the mass m by the energy E of the particle. Furthermore, they obey

$$\sum_\alpha \bar{v}_\alpha(p,j_3')u_\alpha(p,j_3) = 0 = \sum_\alpha \bar{u}_\alpha(p,j_3')v_\alpha(p,j_3)\,. \tag{B.26}$$

Their polarization sums are given by

$$\sum_{j_3} u_\alpha(p,j_3)\bar{u}_\beta(p,j_3) = (\not{p}+m)_{\alpha\beta} \tag{B.27}$$

$$\sum_{j_3} v_\alpha(p,j_3)\bar{v}_\beta(p,j_3) = (\not{p}-m)_{\alpha\beta} \tag{B.28}$$

As a short-hand notation for products of four vectors with Dirac matrices we define the Feynman dagger $\not{k} = k_\mu\gamma^\nu$. Thus we obtain the anticommutation relations

$$\{\psi_\alpha(x), \psi_{\alpha'}\} = \{\bar{\psi}_\alpha(x), \bar{\psi}_{\alpha'}\} = 0 \tag{B.29}$$

$$\{\psi_\alpha(x), \bar{\psi}_{\alpha'}\} = (i\gamma^\mu_{\alpha\alpha'}\partial_\mu + m\delta_{\alpha\alpha'})i\Delta(x-x', m)\,, \tag{B.30}$$

which involve the conjugate Dirac field $\bar{\psi}_\alpha = \psi^\dagger_\beta\gamma^0_{\alpha\beta}$.

B.2. Gauge symmetry and interaction terms

The theories which we will describe in the next sections are invariant under local phase transformations. Let us consider a system of fermion fields ψ which shall be the solution of the Dirac equation of a massless particle $i\not{\partial}\,\psi = 0$. A phase transformation is now given by

$$\psi(x) \to e^{-i\alpha(x)}\psi(x) \tag{B.31}$$

where $\alpha(x)$ is a space time dependent phase. Such local mappings are called gauge transformations. Let us now consider the transformation of the Lagrange density for the fermions ψ:

$$i\bar{\psi}(x)\partial\!\!\!/\psi(x) \to i\bar{\psi}(x)\partial\!\!\!/\psi(x) + i\bar{\psi}(x)\gamma^\mu\psi(x) \cdot \partial_\mu\alpha(x). \tag{B.32}$$

The Lagrange density is therefore not invariant under local gauge transformations, since the term $\partial_\mu\alpha(x)$ does not necessarily vanish. In order to define a gauge invariant Lagrangian we have to introduce the gauge covariant derivative D_μ which transforms as

$$D_\mu\psi(x) \to e^{-i\alpha(x)}D_\mu\psi \tag{B.33}$$

under the local phase transformations. In the following lines we will show, how to set up these local phase transformations by using the formalism described in the last chapter. Therefore we take a (in general non-abelian) group G of dimension n with the group parameters α^a $(a = 1, \ldots, n)$. In an r dimensional representation the generators are given by $r \times r$ matrices T^a $(a = 1, \ldots, n)$. These generators have to obey the Lie algebra (A.15). Using the structure A.17 for finite transformations, the gauge transformation (B.31) can be written as

$$\psi' = U(\alpha)\psi \qquad \text{with} \qquad U(\alpha) = e^{-i\alpha_a T^a}. \tag{B.34}$$

A covariant derivative can now be constructed in terms of gauge fields B^a_μ $(a = 1, \ldots, n)$ as

$$D_\mu\psi = (\partial_\mu + igT_a B^a_\mu)\psi. \tag{B.35}$$

Recalling the transformation behavior

$$(D_\mu\psi)' = U(\alpha)(D_\mu\psi), \tag{B.36}$$

from (B.31) we find that the invariance of $(D_\mu\psi)$ demands the transformation behavior

$$B'_\mu = U(\alpha)B_\mu U^{-1}(\alpha) + \frac{i}{g}\partial_\mu U(\alpha) \cdot U^{-1}(\alpha) \tag{B.37}$$

for the gauge fields. Therefore the gauge field itself must have a kinetic contribution to the Lagrangian. This is written in terms of a field strength tensor $F_{\mu\nu}$ which is defined as the commutator

$$[D_\mu, D_\nu]\psi = igF_{\mu\nu}\psi, \tag{B.38}$$

of the covariant derivatives. This can be rewritten in terms of the gauge fields to

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + ig[B_\mu, B_\nu]. \tag{B.39}$$

The equations (B.37) and (B.39) provide the transformation property

$$F'_{\mu\nu} = U(\alpha)F_{\mu\nu}U(\alpha)^{-1} \tag{B.40}$$

of the field strength. Therefore the field strength tensor is in general not gauge invariant (one exception is the abelian case).

B.3. Lagrange densities

Relativistic equations of motion for the fields can be obtained from an invariant Lagrange density $\mathcal{L}(\psi, \partial_\mu \psi)$ which is sometimes called Lagrangian. Using Hamilton's principle

$$\delta \int \mathrm{d}^4 x \, \mathcal{L}(\psi, \partial_\mu \psi) = 0 \tag{B.41}$$

we obtain the Euler-Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = 0. \tag{B.42}$$

Combining these definitions with the results from the last section we finally find a generic gauge invariant Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(F^{\mu\nu} F_{\mu\nu}) + (D_\mu \phi)^* D^\mu \phi + i\bar{\psi} \not{D} \psi \,, \tag{B.43}$$

where ϕ and ψ are distinct multiplets of scalar and spin 1/2 fields. Note that this generic Lagrangian contains only the gauge fields. This means non-gauge interactions as well as mass terms have not been implemented. The trace in (B.43) acts in the space of the groups representation (not in the space of the Dirac matrices). For the specific calculations in our case it is convenient to rewrite (B.43) to a notation with individual fields rather than the matrix notation given above. Therefore we start observing the trace

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab} \quad \text{with} \quad a, b = 1, \ldots, n \,. \tag{B.44}$$

Rather than describing the transformation properties of the matrix $F_{\mu\nu}$ we will describe the transformation properties of the components $F^a_{\mu\nu}$ (a=1,. . . ,k). At this point we should remind that we talked about the transformation of tensors of n^{th} rank in the last section. While the ϕ and ψ fields transform as scalars and vectors, the field strength tensor $F_{\mu\nu}$ transforms as a tensor of second rank. This means the scalar quantities transform in the trivial representation of the group G, while the vector and tensor quantities transform in the fundamental and adjoint representation respectively. Thus in common we have $k \neq n$. Multiplying equation (B.39) with T^a from the right side we get

$$F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu - g c^{abc} B^a_\mu B^b_\nu \quad \text{with} \quad a, b, c = 1, \ldots, k \,, \tag{B.45}$$

where we have used the Lie algebra (A.15) to eliminate the commutator. The Lagrangian can then be rewritten to

$$\mathcal{L} = -\frac{1}{2} F^{\mu\nu}_a F^a_{\mu\nu} + (D^\mu_{lm} \phi_m)^\dagger (D_\mu)_{ln} \phi_n + i\bar{\psi}_i \not{D}_{ij} \psi_j \,. \tag{B.46}$$

In later notations we will often suppress the indices a, l, m, n, i and j to get a more streamlined notation. Of course one still has to sum over these indices then.

C. General matrix elements in terms of basic parameters

In the following we will present a list of the general matrix elements which we have calculated up to dimension 7 in section 4.3.3. We will present these matrix elements in terms of the dimensionless parameters $\hat{\mu}_1$, $\hat{\mu}_2$, $\hat{\rho}_1$, $\hat{\rho}_2$ and $\hat{s}_1, \ldots, \hat{s}_9$ which will improve the readability of the formulae in contrast to the dimensionful parameters which we have used in section 4.3.3. The dimensionless parameters are defined as $\hat{\mu}_x = \mu_x/m_b^2$, $\hat{\rho}_x = \rho_x/m_b^4$ and $\hat{s}_x = s_x/m_b^4$.

C.1. Dimension 3

$$\langle \bar{B}|\bar{b}_v b_v|\bar{B}\rangle = \frac{1}{2} m_B(\gamma \cdot v + 1) + \frac{m_B}{4}\left(\hat{\mu}_1 + \hat{\mu}_2\right) + \mathcal{O}(1/m_b^5) \tag{C.1}$$

C.2. Dimension 4

$$\begin{aligned}\langle \bar{B}|\bar{b}_v(iD_\rho)b_v|\bar{B}\rangle =& \frac{m_B\, m_b}{12}\left(v^\rho(-5\gamma \cdot v - 3) + 2\gamma^\rho\right)\left(\hat{\mu}_1 + \hat{\mu}_2\right) \\ &+ \frac{m_B\, m_b}{12}\left(\gamma^\rho - 4v^\rho \gamma \cdot v\right)\left(\hat{\rho}_1 + \hat{\rho}_2\right) + \mathcal{O}(1/m_b^5)\end{aligned} \tag{C.2}$$

C.3. Dimension 5

As a short hand notation we shall introduce

$$S = \hat{s}_2 + \hat{s}_3 - \hat{s}_4 + 2\hat{s}_5 + \hat{s}_6 - 2\hat{s}_8 + \hat{s}_9 \tag{C.3}$$

for this mass dimension.

$$\begin{aligned}\langle \bar{B}|\bar{b}_v(iD_\rho)(iD_\sigma)b_v|\bar{B}\rangle = \frac{m_B\, m_b^2}{48}\Big[&\left(\gamma^\sigma\gamma^\rho - \gamma^\rho\gamma^\sigma\right)\left(S + 2\left(\hat{\mu}_2 + \hat{\rho}_1 + \hat{\rho}_2\right)\right) - 2g^{\rho\sigma}\left(S - 4\hat{\mu}_1\right) \\ &+ \Big(\left(\gamma^\rho \not{v} - \not{v}\gamma^\rho\right)v^\sigma - \left(\gamma^\sigma \not{v} - \not{v}\gamma^\sigma\right)v^\rho\Big)\left(S + \hat{\mu}_2 + 2(\hat{\rho}_1 + \hat{\rho}_2)\right) \\ &+ 8v^\rho v^\sigma(S - \hat{\mu}_1) - 2(v^\sigma\gamma^\rho + v^\rho\gamma^\sigma)\Big(S - 2(\hat{s}_1 + \hat{s}_7) - 2(\hat{\rho}_1 + \hat{\rho}_2)\Big) \\ &+ 2\not{v} g^{\rho\sigma}\Big(2(\hat{s}_1 - \hat{s}_2 - \hat{s}_6 + \hat{s}_7) - S + 4\hat{\mu}_1\Big) \\ &- 4\not{v} v^\rho v^\sigma\Big(6\hat{s}_1 - \hat{s}_2 - \hat{s}_6 + 6\hat{s}_7 - 3S + 2\left(\hat{\mu}_1 + \hat{\rho}_1 + \hat{\rho}_2\right)\Big) \\ &+ 2i\gamma^\alpha\gamma^5\epsilon^{\alpha\rho\sigma v}\Big(\hat{s}_2 + \hat{s}_6 + 2\left(\hat{\mu}_2 + \hat{\rho}_1 + \hat{\rho}_2\right)\Big)\Big]\end{aligned} \tag{C.4}$$

C.4. Dimension 6

$$
\begin{aligned}
&\langle \bar{B}|\bar{b}_v(iD_\rho)(iD_\sigma)(iD_\lambda)b_v|\bar{B}\rangle \\
&\quad = \frac{m_B\, m_b^3}{240}\Big[10\left(-i\sigma^{\rho\sigma}v^\lambda - i\sigma^{\sigma\lambda}v^\rho\right)(-\hat{s}_1+\hat{s}_2-\hat{s}_4+\hat{s}_5+\hat{s}_6-\hat{s}_7-2\hat{s}_8+\hat{s}_9) \\
&\qquad + 10\left(-i\sigma^{\rho\beta}v^\lambda + i\sigma^{\lambda\beta}v^\rho\right)v_\beta\, v^\sigma(6\hat{s}_1-2\hat{s}_2+\hat{s}_3+\hat{s}_4-2\hat{s}_5+6\hat{s}_7+2\hat{s}_8-\hat{s}_9+2\hat{\rho}_2) \\
&\qquad + 10\left(-i\sigma^{\rho\beta}v_\beta\, g^{\rho\sigma} - i\sigma^{\rho\beta}v_\beta\, g^{\lambda\sigma}\right)(-\hat{s}_1+\hat{s}_2-\hat{s}_3-\hat{s}_5+\hat{s}_6-\hat{s}_7) \\
&\qquad + 20\, i\sigma^{\rho\lambda}v^\sigma(\hat{s}_1+\hat{s}_7+\hat{\rho}_2) - 20\, i\gamma^\alpha\gamma^5 v^\sigma\epsilon^{\alpha\lambda\rho v}(\hat{s}_1+\hat{s}_7+\hat{\rho}_2) \\
&\qquad + 10\, i\gamma^\alpha\gamma^5\left(v^\lambda\epsilon^{\alpha\rho\sigma\beta} - v^\rho\epsilon^{\alpha\lambda\sigma\beta}\right)v_\beta\,(\hat{s}_1-\hat{s}_2+\hat{s}_4-\hat{s}_5-\hat{s}_6+\hat{s}_7+2\hat{s}_8-\hat{s}_9) \\
&\qquad + i\not{v}\gamma^5\epsilon^{\lambda\rho\sigma v}(-20\hat{s}_5+20\hat{s}_8-20\hat{s}_9) + 40v^\lambda v^\rho v^\sigma(\hat{s}_3+\hat{s}_5-\hat{\rho}_1) \\
&\qquad + 8v^\lambda v^\rho v^\sigma\not{v}(-6\hat{s}_1+\hat{s}_2+6\hat{s}_3+\hat{s}_4+6\hat{s}_5+\hat{s}_6-6\hat{s}_7+\hat{s}_8-5\hat{\rho}_1) \\
&\qquad + \left(\gamma^\lambda v^\rho v^\sigma + \gamma^\rho v^\lambda v^\sigma\right)(26\hat{s}_1-6\hat{s}_2-6\hat{s}_3+4\hat{s}_4-6\hat{s}_5-6\hat{s}_6+26\hat{s}_7+4\hat{s}_8) \\
&\qquad + 4\gamma^\sigma\left(g^{\lambda\rho} - v^\lambda v^\rho\right)(\hat{s}_1-\hat{s}_2-\hat{s}_3+4\hat{s}_4-\hat{s}_5-\hat{s}_6+\hat{s}_7+4\hat{s}_8) \\
&\qquad + 4v^\sigma\not{v}g^{\lambda\rho}(-\hat{s}_1+\hat{s}_2+\hat{s}_3-4\hat{s}_4+\hat{s}_5+\hat{s}_6-\hat{s}_7-4\hat{s}_8+10\hat{\rho}_1) \\
&\qquad + v^\rho\not{v}g^{\lambda\sigma}(6\hat{s}_1-6\hat{s}_2-26\hat{s}_3+4\hat{s}_4-26\hat{s}_5-6\hat{s}_6+6\hat{s}_7+4\hat{s}_8) \\
&\qquad - 20v^\lambda g^{\rho\sigma}(\hat{s}_3+\hat{s}_5) - 20v^\rho g^{\lambda\sigma}(\hat{s}_3+\hat{s}_5-2\hat{\rho}_1) \\
&\qquad + \gamma^\lambda g^{\rho\sigma}(-6\hat{s}_1+6\hat{s}_2+6\hat{s}_3-4\hat{s}_4+6\hat{s}_5+6\hat{s}_6-6\hat{s}_7-4\hat{s}_8) \\
&\qquad + v^\lambda\not{v}g^{\rho\sigma}(6\hat{s}_1-26\hat{s}_3+4\hat{s}_4-26\hat{s}_5-6\hat{s}_6+6\hat{s}_7+4\hat{s}_8) \\
&\qquad + \gamma^\rho g^{\lambda\sigma}(-6\hat{s}_1+6\hat{s}_2+6\hat{s}_3-4\hat{s}_4+6\hat{s}_5+6\hat{s}_6-6\hat{s}_7-4\hat{s}_8)\Big]
\end{aligned}
\tag{C.5}
$$

C.5. Dimension 7

$$
\langle \bar{B}|\bar{b}_v(iD_\rho)(iD_\sigma)(iD_\lambda)(iD_\delta)b_v|\bar{B}\rangle \tag{C.6}
$$

$$
= \frac{m_B\, m_b^4}{240}\Bigg[\frac{1+\not{v}}{2}\Bigg(-16v^\delta v^\lambda v^\rho v^\sigma(6\hat{s}_1-\hat{s}_2-\hat{s}_3-\hat{s}_4) + 8v^\delta v^\rho g^{\lambda\sigma}(4\hat{s}_1-4\hat{s}_2+\hat{s}_3+\hat{s}_4) \tag{C.7}
$$

$$
+ 8(g^{\delta\lambda}g^{\rho\sigma} - v^\rho v^\sigma g^{\delta\lambda} - v^\delta v^\lambda g^{\rho\sigma})(\hat{s}_1-\hat{s}_2+4\hat{s}_3-\hat{s}_4) \tag{C.8}
$$

$$
+ g^{\delta\rho}g^{\lambda\sigma}(-4\hat{s}_1+4\hat{s}_2-\hat{s}_3-\hat{s}_4) + 8v^\lambda v^\sigma g^{\delta\rho}(14\hat{s}_1-4\hat{s}_2+\hat{s}_3+\hat{s}_4) \tag{C.9}
$$

$$
+ 8\big(g^{\delta\sigma}g^{\lambda\rho} - g^{\delta\sigma}v^\lambda v^\rho - v^\delta v^\sigma g^{\lambda\rho}\big)(\hat{s}_1-\hat{s}_2-\hat{s}_3+4\hat{s}_4)\Bigg) \tag{C.10}
$$

$$
+ 2\Big(2(\gamma^\delta v^\rho - \gamma^\rho v^\delta)\not{v}(v^\lambda v^\sigma + g^{\lambda\sigma}) + 2(-i\sigma^{\rho\delta})g^{\lambda\sigma} \tag{C.11}
$$

$$
+ (-i\sigma^{\rho\delta})\not{v}g^{\lambda\sigma} + \not{v}(-i\sigma^{\rho\delta})g^{\lambda\sigma}\Big)(2\hat{s}_5-3\hat{s}_6+3\hat{s}_7+2\hat{s}_8) \tag{C.12}
$$

$$
- 2\left(2(-i\sigma^{\rho\delta}) + (-i\sigma^{\rho\delta})\not{v} + \not{v}(-i\sigma^{\rho\delta})\right)v^\lambda v^\sigma(2\hat{s}_5-3\hat{s}_6+8\hat{s}_7+2\hat{s}_8) \tag{C.13}
$$

$$+ 4\left(\gamma^\sigma v^\lambda - \gamma^\lambda v^\sigma\right) \not{v}\, v^\delta v^\rho \,(\hat{s}_5 - \hat{s}_8 - \hat{s}_9) \tag{C.14}$$

$$+ 2\Bigg(\left(2(-i\sigma^{\rho\sigma}) + (-i\sigma^{\rho\sigma})\not{v} + \not{v}(-i\sigma^{\rho\sigma})\right)\left(v^\delta v^\lambda - g^{\delta\lambda}\right) \tag{C.15}$$

$$+ \left(2(-i\sigma^{\lambda\delta}) + (-i\sigma^{\lambda\delta})\not{v} + \not{v}(-i\sigma^{\lambda\delta})\right)(v^\rho v^\sigma - g^{\rho\sigma})\Bigg)(3\hat{s}_5 - \hat{s}_6 + \hat{s}_7 + \hat{s}_9) \tag{C.16}$$

$$+ 2\Bigg(\left(2(-i\sigma^{\delta\sigma}) + (-i\sigma^{\delta\sigma})\not{v} + \not{v}(-i\sigma^{\delta\sigma})\right)(v^\lambda v^\rho - g^{\lambda\rho}) \tag{C.17}$$

$$+ \left(2(-i\sigma^{\lambda\rho}) + (-i\sigma^{\lambda\rho})\not{v} + \not{v}(-i\sigma^{\lambda\rho})\right)(v^\delta v^\sigma - g^{\delta\sigma}) \tag{C.18}$$

$$+ 2\left(\gamma^\delta v^\sigma - \gamma^\sigma v^\delta\right)\not{v} g^{\lambda\rho}\Bigg)(\hat{s}_6 - \hat{s}_7 - 3\hat{s}_8 + \hat{s}_9) \tag{C.19}$$

$$+ 2\Bigg(\left(2(-i\sigma^{\sigma\lambda}) + (-i\sigma^{\sigma\lambda})\not{v} + \not{v}(-i\sigma^{\sigma\lambda})\right)(v^\delta v^\rho - g^{\delta\rho}) \tag{C.20}$$

$$+ 2(\gamma^\sigma v^\lambda - \gamma^\lambda v^\sigma)\not{v} g^{\delta\rho}\Bigg)(2\hat{s}_5 - 2\hat{s}_8 + 3\hat{s}_9)\Bigg] \tag{C.21}$$

D. Nonperturbative corrections to the standard model and nonstandard currents

Here we will present the results from the nonperturbative corrections which have been calculated in chapter 4 under usage of the general matrix elements shown in appendix C. To improve the readability of the outputs we will introduce the shortcuts $\rho = \hat{m}_c^2 = m_c^2/m_b^2$ and $y = 2\hat{E}_l = 2E_l/m_b$ which are also consistent with the notation in our publication [18]. Furthermore, we used the dimensionless parameters $\hat{\mu}_1$, $\hat{\mu}_2$, $\hat{\rho}_1$, $\hat{\rho}_2$ and $\hat{s}_1, \ldots, \hat{s}_9$, where $\hat{\mu}_x = \mu_x/m_b^2$, $\hat{\rho}_x = \rho_x/m_b^4$ and $\hat{s}_x = s_x/m_b^4$, like we have done in the last section. We will not present the full results of the hadronic or leptonic energy moments introduced in section 4.4. Numerical values for these moments can be found in the tables presented in section 4.6. As all currents except the left-handed vector current are expected to be small because of the results from past experiments we will only present the mixed terms of the newly introduced currents with this current. The names of the subsections denote the current which is mixed with the standard model term. "Right-handed vector current" means for example the mixed term of the right-handed and the left-handed vector term.

D.1. The electron energy spectrum

For this section we have also introduced the shortcuts $\theta(x)$ and $\delta^{(n)}(x)$ for the distributions, where $x = y(1-y-\rho)/(1-y)$ and the superscript (n) denotes again the n^{th} derivative of the delta distribution. Since the full formula is much to long to be presented in one piece, the differential rate has been split up into parts regarding the currents and order in the $1/m_b$ expansion. The complete electron spectrum can be obtained by

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}y} = \frac{G_F^2 |V_{cb}|^2 m_b^2}{192\pi^2} \sum_{i,j} \frac{1}{m_b^i} \left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,i)}_{c_L\, j}, \tag{D.1}$$

where $i = 1, \ldots, 4$ and $j = c_L, c_R, g_L, g_R, d_L, d_R$. This also gives the reader the opportunity to have a separated view on the different current combinations, if only special combinations are of interest.

D.1.1. Left-handed vector current

$$\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,0)}_{c_L c_L} = -\, 2y^2 (y+\rho-1)^2 \left(2y^2 - y(\rho+5) + 3(\rho+1)\right) \frac{\theta(x)}{(y-1)^3} \tag{D.2}$$

$$
\begin{aligned}
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,2)}_{c_L c_L} = & \Bigg(\Bigg(10y^3\left(4\rho^3-3\rho^2-1\right)+10y^6\left(y^2-5y+10\right) \\
& \quad +4y^5\left(\rho^3-3\rho^2-25\right)+2y^4\left(-10\rho^3+21\rho^2+25\right)\Bigg)\frac{\theta(x)}{3(y-1)^5} \\
& + \left(y^2-2y-\rho+1\right)^2(y+\rho-1)\frac{12\,\rho\,y^3\delta(x)}{(y-1)^6}\Bigg)\hat{\mu}_1 \\
& + \Bigg(\Bigg((y+\rho-1)\big(\left(-5y^2+8y-3\right)\rho+5(y-3)\rho^2+(5y+6)(y-1)^2\big) \\
& \quad +3\rho\big(\left(y^2+y-2\right)\rho+(y-3)(y-1)^2+5\rho^2\big)\Bigg)\frac{2y^2\theta(x)}{3(y-1)^4}\Bigg)\hat{\mu}_2
\end{aligned}
\tag{D.3}
$$

$$
\begin{aligned}
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,3)}_{c_L c_L} = & \Bigg(\Bigg(-y^5+y^4(5-\rho)-2y^3(5-2\rho)+2y^2(5-3\rho)-y(1-\rho)(\rho+5) \\
& \quad +(1-\rho)^2(\rho+1)\Bigg)\frac{12\,\rho\,y^3\delta(x)}{(y-1)^6} \\
& + \Bigg(4y^6-14y^5+2y^4(6\rho+5)+y^3(\rho^3+3\rho^2-54\rho+20)-6(1-\rho)^3 \\
& \quad -5y^2(\rho^3+3\rho^2-18\rho+8)+2y(1-\rho)(-5\rho^2-20\rho+13)\Bigg)\frac{2y^2\theta(x)}{3(y-1)^5}\Bigg)\hat{\rho}_2 \\
& + \Bigg(\Bigg(-y^7(\rho+1)+y^6(2\rho+7)-y^5(5\rho^2-\rho+21)+y^4(13\rho^2+35) \\
& \quad -y^3(\rho^3+9\rho^2+15\rho+35)+y^2(7-3\rho)(\rho+1)(2\rho+3) \\
& \quad -y(1-\rho)(-\rho^3+22\rho^2+24\rho+7)+(1-\rho)^2(11\rho^2+6\rho+1)\Bigg)\frac{4y^3\delta(x)}{(y-1)^7} \\
& \quad +\Bigg(4y^7-30y^6+8y^5(\rho+27)-y^4(-5\rho^3+5\rho^2+18\rho+230) \\
& \quad\quad +6y^3(-5\rho^3+5\rho^2+4\rho+50)-y^2(-75\rho^3+75\rho^2+4\rho+234) \\
& \quad\quad +4y(-21\rho^3+23\rho^2-3\rho+25)-6(1-\rho)(9\rho^2+2\rho+3)\Bigg)\frac{2y^2\theta(x)}{3(y-1)^6} \\
& \quad +\Bigg(y^7-y^6(7-\rho)+7y^5(3-\rho)-y^4(\rho^2-20\rho+35) \\
& \quad\quad +y^3(3\rho^2-30\rho+35)-y^2(1-\rho)(3-\rho)(\rho+7) \\
& \quad\quad +y(1-\rho)^2(3\rho+7)-(1-\rho)^3(\rho+1)\Bigg)\frac{2\rho y^4\delta'(x)}{3(y-1)^8}\Bigg)\hat{\rho}_1
\end{aligned}
\tag{D.4}
$$

$$
\Gamma^{(0,4)}_{c_L c_L} = \Bigg(\Bigg((2-\rho)y^5-(10-3\rho)(1-\rho)y^4+2\left(2\rho^2-25\rho+10\right)y^3-2\left(3\rho^3-41\rho+10\right)y^2
$$

$$+ \left(15\rho^3 + 12\rho^2 - 61\rho + 10\right) y - (1-\rho)\left(-9\rho^3 - 2\rho^2 - 15\rho + 2\right)\Bigg)\frac{2\,y^3\delta(x)}{(y-1)^6}$$

$$+ \Bigg(-2y^5 + 2(12\rho+7)y^4 - 3\left(-3\rho^3 - \rho^2 + 38\rho + 12\right)y^3 + \left(-45\rho^3 - 15\rho^2 + 210\rho + 44\right)y^2$$

$$- 2\left(-45\rho^3 - 15\rho^2 + 81\rho + 13\right)y + 6\left(-15\rho^3 + \rho^2 + 7\rho + 1\right)\Bigg)\frac{y^2\theta(x)}{3(y-1)^5}$$

$$+ \Bigg(2y^5 - 2(5-\rho)y^4 + 4(5-3\rho)y^3 - 4(1-\rho)(5-\rho)y^2$$

$$+ 10(1-\rho)^2 y - 2(1-\rho)^3\Bigg)\frac{\rho\,y^4\delta'(x)}{3(y-1)^7}\Bigg)\hat{s}_9$$

$$+ \Bigg(\Bigg(8(3\rho+1)y^7 - 4(39\rho+14)y^6 + 4\left(-7\rho^2 + 124\rho + 42\right)y^5 - 4\left(3\rho^2 + 267\rho + 70\right)y^4$$

$$+ 8\left(-14\rho^3 + 25\rho^2 + 199\rho + 35\right)y^3 - 4\left(-105\rho^3 + 14\rho^2 + 373\rho + 42\right)y^2$$

$$+ 4(1-\rho)\left(21\rho^3 + 131\rho^2 + 206\rho + 14\right)y - 4(1-\rho)^2\left(39\rho^2 + 45\rho + 2\right)\Bigg)\frac{y^3\delta(x)}{5(y-1)^7}$$

$$+ \Bigg(-80y^8 + 8(7\rho+60)y^7 - 2\left(-21\rho^3 + 11\rho^2 + 146\rho + 590\right)y^6$$

$$+ 4\left(-63\rho^3 + 33\rho^2 + 144\rho + 380\right)y^5 - 10\left(-63\rho^3 + 33\rho^2 + 44\rho + 108\right)y^4$$

$$+ 40(1-\rho)\left(21\rho^2 + 11\rho + 10\right)y^3 - 60(1-\rho)\left(3\rho^2 + 1\right)y^2\Bigg)\frac{\theta(x)}{15(y-1)^6}$$

$$+ \Bigg(-8\rho y^{11} + 8(9-\rho)\rho y^{10} - 4(66-17\rho)\rho y^9 + 4\rho\left(\rho^2 - 63\rho + 130\right)y^8$$

$$- 8\rho\left(4\rho^2 - 61\rho + 75\right)y^7 + 8(1-\rho)\rho\left(-2\rho^2 - 13\rho + 51\right)y^6$$

$$- 4(1-\rho)^2\rho(7\rho+38)y^5 + 12(1-\rho)^3\rho(\rho+2)y^4\Bigg)\frac{\delta'(x)}{15(y-1)^8}\Bigg)\hat{s}_5$$

$$+ \Bigg(\Bigg(-4(3-\rho)y^{10} + 8(13-12\rho)y^9 - 4\left(17\rho^2 - 134\rho + 93\right)y^8 + 12\left(29\rho^2 - 109\rho + 60\right)y^7$$

$$- 4\left(3\rho^3 + 125\rho^2 - 413\rho + 205\right)y^6 + 4\left(15\rho^3 + 11\rho^2 - 278\rho + 138\right)y^5$$

$$- 4\left(26\rho^4 + 45\rho^3 - 90\rho^2 - 92\rho + 51\right)y^4 + 4(1-\rho)\left(-16\rho^3 - 49\rho^2 - 3\rho + 8\right)y^3\Bigg)\frac{\delta(x)}{5(y-1)^7}$$

$$+ \Bigg(-8(5-12\rho)y^7 + 4\left(13\rho^3 - 3\rho^2 - 148\rho + 55\right)y^6 - 24\left(13\rho^3 - 3\rho^2 - 64\rho + 20\right)y^5$$

$$+ 20\left(39\rho^3 - 9\rho^2 - 102\rho + 26\right)y^4 - 40\left(26\rho^3 - 3\rho^2 - 34\rho + 7\right)y^3$$

$$+ 60\left(8\rho^3 - 6\rho + 1\right)y^2\Bigg)\frac{\theta(x)}{15(y-1)^6}$$

$$+ \Bigg(2\rho y^{11} - 2(9-\rho)\rho y^{10} + 22(3-\rho)\rho y^9 - 2(5-3\rho)(13-\rho)\rho y^8$$

$$+ 2\rho\left(19\rho^2 - 86\rho + 75\right)y^7 - 2(1-\rho)\rho\left(3\rho^2 - 38\rho + 51\right)y^6$$

$$+ 2(19-9\rho)(1-\rho)^2\rho y^5 - 2(1-\rho)^3(3-\rho)\rho y^4\Bigg)\frac{\delta'(x)}{15(y-1)^8}\Bigg)\hat{s}_6$$

$$
\begin{aligned}
&+\Bigg(\bigg(-4(3-\rho)y^{10}+4(6\rho+11)y^9-4\left(-13\rho^2+91\rho+3\right)y^8-12\left(21\rho^2-106\rho+15\right)y^7\\
&\quad+4\left(27\rho^3+85\rho^2-517\rho+95\right)y^6+4\left(49\rho^4+15\rho^3-30\rho^2-193\rho+39\right)y^4\\
&\quad-4\left(60\rho^3+19\rho^2-442\rho+87\right)y^5-4(1-\rho)\left(-59\rho^3-41\rho^2-27\rho+7\right)y^3\bigg)\frac{\delta(x)}{5(y-1)^7}\\
&\quad+\bigg(60y^8-8(18\rho+35)y^7+2\left(-49\rho^3+9\rho^2+454\rho+230\right)y^6\\
&\qquad-12\left(-49\rho^3+9\rho^2+202\rho+20\right)y^5-10\left(147\rho^3-27\rho^2-324\rho+14\right)y^4\\
&\qquad+40(\rho+1)\left(49\rho^2-58\rho+5\right)y^3-60\left(17\rho^3-3\rho^2-9\rho+1\right)y^2\bigg)\frac{\theta(x)}{15(y-1)^6}\\
&\quad+\bigg(2\rho y^{11}-2(9-\rho)\rho y^{10}+22(3-\rho)\rho y^9-2(5-3\rho)(13-\rho)\rho y^8\\
&\qquad+2\rho\left(19\rho^2-86\rho+75\right)y^7-2(1-\rho)\rho\left(3\rho^2-38\rho+51\right)y^6\\
&\qquad+2(19-9\rho)(1-\rho)^2\rho y^5-2(1-\rho)^3(3-\rho)\rho y^4\bigg)\frac{\delta'(x)}{15(y-1)^8}\Bigg)\hat{s}_8\\
&+\Bigg(\bigg(24(3-\rho)y^{10}-4(136-59\rho)y^9+4\left(7\rho^2-224\rho+438\right)y^8\\
&\quad-4\left(27\rho^2-437\rho+780\right)y^7+8\left(4\rho^3+5\rho^2-239\rho+415\right)y^6\\
&\quad-4\left(15\rho^3-74\rho^2-293\rho+528\right)y^5+4\left(21\rho^4+70\rho^3-105\rho^2-92\rho+186\right)y^4\\
&\quad-4(1-\rho)\left(39\rho^3-24\rho^2+17\rho+28\right)y^3\bigg)\frac{\delta(x)}{5(y-1)^7}\\
&\quad+\bigg(20y^8-16(\rho+10)y^7+2\left(-21\rho^3+\rho^2+66\rho+250\right)y^6\\
&\qquad-4\left(-63\rho^3+3\rho^2+114\rho+200\right)y^5+10\left(-63\rho^3+3\rho^2+76\rho+70\right)y^4\\
&\qquad-40\left(-21\rho^3-4\rho^2+15\rho+8\right)y^3+60\left(-3\rho^3-3\rho^2+3\rho+1\right)y^2\bigg)\frac{\theta(x)}{15(y-1)^6}\\
&\quad+\bigg(-12\rho y^{11}+4(22-3\rho)\rho y^{10}-12(23-6\rho)\rho y^9+4\rho\left(-\rho^2-47\rho+120\right)y^8\\
&\qquad-4\rho\left(-3\rho^2-68\rho+125\right)y^7+4\rho\left(\rho^3-2\rho^2-57\rho+78\right)y^6\\
&\qquad-4(1-\rho)\rho\left(2\rho^2+\rho+27\right)y^5+4(1-\rho)^2\rho\left(3\rho^2+3\rho+4\right)y^4\bigg)\frac{\delta'(x)}{15(y-1)^8}\Bigg)\hat{s}_7\\
&+\Bigg(\bigg(-2y^{12}+4(3-\rho)y^{11}-6(4-7\rho)y^{10}-6(47-3\rho)\rho y^9+2\left(-98\rho^2+537\rho+42\right)y^8\\
&\quad-2\left(-24\rho^3-299\rho^2+1165\rho+84\right)y^7+2\left(-81\rho^4-347\rho^3+202\rho^2+451\rho+15\right)y^4\\
&\quad-2\left(-43\rho^4-229\rho^3+13\rho^2+1111\rho+48\right)y^5+2\left(-79\rho^3-312\rho^2+1487\rho+84\right)y^6\\
&\quad-2(1-\rho)\left(-7\rho^3+166\rho^2+79\rho+2\right)y^3\bigg)\frac{\delta(x)}{5(y-1)^8}\\
&\quad+\bigg(-20(\rho+12)y^8-\left(43\rho^3-9\rho^2-120\rho-660\right)y^7+\left(301\rho^3-63\rho^2-280\rho-1100\right)y^6
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(-903\rho^3+189\rho^2+380\rho+1200\right)y^5 - 5\left(-301\rho^3+27\rho^2+72\rho+168\right)y^4 \\
&+ 20\left(-64\rho^3-6\rho^2+11\rho+17\right)y^3 - 60\left(-6\rho^3-2\rho^2+\rho+1\right)y^2 + 40y^9\Big)\frac{\theta(x)}{15(y-1)^7} \\
&+ \Big((\rho+1)y^{13} - 3(\rho+3)y^{12} + 4\left(\rho^2+2\rho+9\right)y^{11} - 4\left(3\rho^2+14\rho+21\right)y^{10} \\
&+ \left(-10\rho^3-35\rho^2+105\rho+63\right)y^9 - 2\left(9\rho^3-195\rho^2+203\rho+63\right)y^8 \\
&+ 4(1-\rho)\left(-\rho^3-57\rho^2+133\rho+21\right)y^7 - 4(1-\rho)^2\left(-13\rho^2+90\rho+9\right)y^6 \\
&+ (1-\rho)^3\left(-\rho^2+128\rho+9\right)y^5 - (1-\rho)^4(19\rho+1)y^4\Big)\frac{\delta'(x)}{30(y-1)^9} \\
&+ \Big(-y^{14} + (9-\rho)y^{13} - 12(3-\rho)y^{12} + 4\left(\rho^2-14\rho+21\right)y^{11} - 2\left(15\rho^2-70\rho+63\right)y^{10} \\
&+ 6(1-\rho)\left(\rho^2-14\rho+21\right)y^9 - 28(1-\rho)^2(3-\rho)y^8 + 4(1-\rho)^3(9-\rho)y^7 \\
&- 9(1-\rho)^4y^6 + (1-\rho)^5y^5\Big)\frac{\rho\,\delta''(x)}{30(y-1)^{10}}\Bigg)\hat{s}_4 \\
&+ \Bigg(\Big(-2y^{12} + 16(\rho+2)y^{11} - 2(49\rho+72)y^{10} + 2\left(-\rho^2+109\rho+140\right)y^9 \\
&- 2\left(38\rho^2+53\rho+98\right)y^8 - 2\left(26\rho^3-279\rho^2+215\rho+84\right)y^7 \\
&+ 2\left(151\rho^3-752\rho^2+477\rho+224\right)y^6 - 2\left(7\rho^4+401\rho^3-977\rho^2+441\rho+188\right)y^5 \\
&+ 6(1-\rho)\left(27\rho^3-114\rho^2+92\rho+25\right)y^4 - 2(1-\rho)^2\left(-43\rho^2+61\rho+12\right)y^3\Big)\frac{\delta(x)}{5(y-1)^8} \\
&+ \Big(20(23-\rho)y^8 - \left(-7\rho^3+\rho^2-140\rho+1520\right)y^7 + 7\left(-7\rho^3+\rho^2-60\rho+400\right)y^6 \\
&- \left(-147\rho^3+21\rho^2-620\rho+3100\right)y^5 + 5\left(-49\rho^3-37\rho^2-88\rho+412\right)y^4 \\
&- 20\left(-\rho^3-19\rho^2-6\rho+38\right)y^3 + 60(1-\rho)\left(-\rho^2+2\rho+2\right)y^2 - 60y^9\Big)\frac{\theta(x)}{15(y-1)^7} \\
&+ \Big((\rho+1)y^{13} - (23\rho+9)y^{12} + 4\left(-4\rho^2+32\rho+9\right)y^{11} - 12\left(-9\rho^2+28\rho+7\right)y^{10} \\
&+ 2\left(5\rho^3-145\rho^2+245\rho+63\right)y^9 - 2\left(9\rho^3-195\rho^2+203\rho+63\right)y^8 \\
&+ 4(1-\rho)(3\rho+1)\left(21-2\rho^2\right)y^7 - 4(1-\rho)^2\left(17\rho^2+20\rho+9\right)y^6 \\
&+ (1-\rho)^3\left(19\rho^2+8\rho+9\right)y^5 - (1-\rho)^5y^4\Big)\frac{\delta'(x)}{30(y-1)^9} \\
&+ \Big(-y^{14} + (9-\rho)y^{13} - 12(3-\rho)y^{12} + 4\left(\rho^2-14\rho+21\right)y^{11} \\
&- 2\left(15\rho^2-70\rho+63\right)y^{10} + 6(1-\rho)\left(\rho^2-14\rho+21\right)y^9 - 28(1-\rho)^2(3-\rho)y^8 \\
&+ 4(1-\rho)^3(9-\rho)y^7 - 9(1-\rho)^4y^6 + (1-\rho)^5y^5\Big)\frac{\rho\,\delta''(x)}{30(y-1)^{10}}\Bigg)\hat{s}_3 \\
&+ \Bigg(\Big(-4\left(-116\rho^2+339\rho+189\right)y^8 + 4\left(-3\rho^3-323\rho^2+705\rho+308\right)y^7 - 2y^{12} \\
&- 4\left(2\rho^3-504\rho^2+869\rho+308\right)y^6 + 4\left(-31\rho^4+52\rho^3-464\rho^2+637\rho+186\right)y^5
\end{aligned}
$$

$$
\begin{aligned}
&-2\left(-129\rho^4+202\rho^3-472\rho^2+514\rho+125\right)y^4+4(3-\rho)y^{11}-4(7\rho+16)y^{10}\\
&+4(1-\rho)\left(56\rho^3+2\rho^2+53\rho+9\right)y^3+4\left(-18\rho^2+87\rho+70\right)y^9\Bigg)\frac{\delta(x)}{5(y-1)^8}\\
+\Bigg(&-10y^9+10(6\rho+5)y^8-2\left(-31\rho^3+18\rho^2+185\rho+65\right)y^7\\
&+2\left(-217\rho^3+126\rho^2+455\rho+125\right)y^6-2\left(-651\rho^3+378\rho^2+570\rho+175\right)y^5\\
&+10\left(-217\rho^3+114\rho^2+76\rho+31\right)y^4-10\left(-187\rho^3+87\rho^2+25\rho+15\right)y^3\\
&+30\left(-23\rho^3+9\rho^2+\rho+1\right)y^2\Bigg)\frac{\theta(x)}{15(y-1)^7}\\
+\Bigg(&(\rho+1)y^{13}-3(\rho+3)y^{12}+4\left(\rho^2-3\rho+9\right)y^{11}-4\left(8\rho^2-21\rho+21\right)y^{10}\\
&+2\left(-5\rho^3+45\rho^2-105\rho+63\right)y^9-2\left(-11\rho^3+55\rho^2-147\rho+63\right)y^8\\
&+4\left(\rho^4+11\rho^3+10\rho^2-63\rho+21\right)y^7-4(1-\rho)\left(8\rho^3-33\rho^2-24\rho+9\right)y^6\\
&+(1-\rho)^2\left(\rho^3-89\rho^2-21\rho+9\right)y^5-(1-\rho)^3\left(-19\rho^2-2\rho+1\right)y^4\Bigg)\frac{\delta'(x)}{30(y-1)^9}\\
+\Bigg(&-y^{14}+(9-\rho)y^{13}-12(3-\rho)y^{12}+4\left(\rho^2-14\rho+21\right)y^{11}+(1-\rho)^5y^5\\
&-2\left(15\rho^2-70\rho+63\right)y^{10}+6(1-\rho)\left(\rho^2-14\rho+21\right)y^9\\
&-14(1-\rho)^2(3-\rho)y^8+2(1-\rho)^3(9-\rho)y^7-3(1-\rho)^4y^6\Bigg)\frac{\rho\,\delta''(x)}{30(y-1)^{10}}\Bigg)\hat{s}_2\\
+\Bigg(\Bigg(&-12\left(52\rho^2-93\rho-238\right)y^8+4\left(38\rho^3+253\rho^2-505\rho-1148\right)y^7+4(76-13\rho)y^{10}\\
&+4\left(-113\rho^3-154\rho^2+489\rho+1148\right)y^6-4\left(-51\rho^4-43\rho^3-9\rho^2+267\rho+696\right)y^5\\
&+4\left(-117\rho^4+191\rho^3+14\rho^2+77\rho+235\right)y^4-4\left(-33\rho^2+57\rho+280\right)y^9\\
&-4(1-\rho)\left(201\rho^3+42\rho^2+43\rho+34\right)y^3+12y^{12}-24(3-\rho)y^{11}\Bigg)\frac{\delta(x)}{5(y-1)^8}\\
+\Bigg(&80y^9-80(7-\rho)y^8+2\left(-51\rho^3+43\rho^2-270\rho+860\right)y^7\\
&-2\left(-357\rho^3+301\rho^2-770\rho+1500\right)y^6+2\left(-1071\rho^3+903\rho^2-1210\rho+1600\right)y^5\\
&-10\left(-357\rho^3+281\rho^2-222\rho+208\right)y^4+40\left(-78\rho^3+59\rho^2-28\rho+19\right)y^3\\
&-120\left(-12\rho^3+7\rho^2-2\rho+1\right)y^2\Bigg)\frac{\theta(x)}{15(y-1)^7}\\
+\Bigg(&2\left(7\rho^3+90\rho^2+84\rho+189\right)y^8-2\left(5\rho^3+100\rho^2+189\right)y^9+2\left(53\rho^2-21\rho+126\right)y^{10}\\
&-2\left(\rho^4-9\rho^3+35\rho^2+147\rho+126\right)y^7+2\left(-13\rho^4-19\rho^3+\rho^2+117\rho+54\right)y^6\\
&-(1-\rho)\left(-3\rho^4+100\rho^3+116\rho^2+120\rho+27\right)y^5-3(\rho+1)y^{13}+9(\rho+3)y^{12}\\
&+3(1-\rho)^2\left(19\rho^3+13\rho^2+7\rho+1\right)y^4-2\left(11\rho^2-3\rho+54\right)y^{11}\Bigg)\frac{\delta'(x)}{15(y-1)^9}\\
+\Bigg(&3y^{14}-3(9-\rho)y^{13}+2(54-13\rho)y^{12}-2\left(\rho^2-49\rho+126\right)y^{11}
\end{aligned}
$$

$$
\begin{aligned}
&+2\left(5\rho^2-105\rho+189\right)y^{10}-2\left(\rho^3+10\rho^2-140\rho+189\right)y^9\\
&+2\left(3\rho^3+10\rho^2-119\rho+126\right)y^8-2(1-\rho)\left(-\rho^3-4\rho^2-9\rho+54\right)y^7\\
&+(1-\rho)^2\left(7\rho^2+16\rho+27\right)y^6-(1-\rho)^3\left(3\rho^2+4\rho+3\right)y^5\Big)\frac{\delta''(x)}{15(y-1)^{10}}\Big)\hat{s}_1
\end{aligned}
\tag{D.5}
$$

D.1.2. Right-handed vector current

$$
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,0)}_{c_Lc_R}=-\left(y+\rho-1\right)^2\frac{12\,y^2\theta(x)}{(y-1)^2}\sqrt{\rho}
\tag{D.6}
$$

$$
\begin{aligned}
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,2)}_{c_Lc_R}=&\left(\left((y-1)^2-\rho\right)^2(y+\rho-1)\frac{24y^3\delta(x)}{(y-1)^5}-\left(3y^2+2y(\rho-4)-5\rho+5\right)\frac{4y^3\rho\,\theta(x)}{(y-1)^4}\right)\sqrt{\rho}\,\hat{\mu}_1\\
&+\left(4y^3+y^2(\rho-14)-y\left(3\rho^2+\rho-16\right)+6\left(\rho^2-1\right)\right)\frac{4y^2\theta(x)}{(y-1)^3}\sqrt{\rho}\,\hat{\mu}_2
\end{aligned}
\tag{D.7}
$$

$$
\begin{aligned}
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,3)}_{c_Lc_R}=&\Bigg(\Bigg(-y^6-3y^5\rho+5y^4(\rho+3)-2y^3\left(\rho^2-\rho+20\right)+y^2(15-7\rho)(\rho+3)\\
&-y(1-\rho)\left(-3\rho^2+23\rho+24\right)+(1-\rho)^2(11\rho+5)\Bigg)\frac{8\,y^3\delta(x)}{(y-1)^6}\\
&+\Bigg(24y^5-12y^4(11-2\rho)+y^3\left(-3\rho^2-98\rho+282\right)-30\left(1-\rho^2\right)\\
&-y^2\left(-15\rho^2-130\rho+294\right)+2y\left(-15\rho^2-28\rho+75\right)\Bigg)\frac{4y^2\theta(x)}{3(y-1)^5}\\
&+\Bigg(y^7-y^6(7-\rho)+7y^5(3-\rho)-y^4\left(\rho^2-20\rho+35\right)+y^3\left(3\rho^2-30\rho+35\right)\\
&-y^2(1-\rho)(3-\rho)(\rho+7)+y(1-\rho)^2(3\rho+7)-(1-\rho)^3(\rho+1)\Bigg)\frac{y^4\delta'(x)}{3(y-1)^7}\Bigg)\sqrt{\rho}\,\hat{\rho}_1\\
&+\Bigg(3y^2-2y(3-\rho)+3(1-\rho)\Bigg)\frac{8\,y^2\theta(x)}{(y-1)^2}\sqrt{\rho}\,\hat{\rho}_2
\end{aligned}
\tag{D.8}
$$

$$
\begin{aligned}
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,4)}_{c_Lc_R}=&\Bigg(\Bigg(-24y^8+8(22-3\rho)y^7-8(59-14\rho)y^6+8\left(-3\rho^2-28\rho+75\right)y^5\\
&-16\left(-2\rho^2-15\rho+23\right)y^4+8(1-\rho)\left(3\rho^2-2\rho+11\right)y^3\Bigg)\frac{\delta(x)}{(y-1)^5}\\
&+\Bigg(60y^6-8(31-9\rho)y^5+4\left(-3\rho^2-61\rho+96\right)y^4-16\left(-3\rho^2-16\rho+16\right)y^3\\
&+12\left(-6\rho^2-5\rho+5\right)y^2\Bigg)\frac{\theta(x)}{3(y-1)^4}
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(4y^9-4(5-\rho)y^8+8(5-3\rho)y^7-8(1-\rho)(5-\rho)y^6\\
&\quad+20(1-\rho)^2y^5-4(1-\rho)^3y^4\Big)\frac{\delta'(x)}{3(y-1)^6}\Bigg)\sqrt{\rho}\,\hat{s}_9\\
&+\Bigg(\Big(96y^8-464y^7+48(18-\rho)y^6-64(\rho+12)y^5\\
&\quad+16\left(-11\rho^2+21\rho+20\right)y^4-16(1-\rho)(17\rho+3)y^3\Big)\frac{\delta(x)}{5(y-1)^5}\\
&\quad+\Big(2(74-27\rho)y^3-3(51-40\rho)y^2+20(10-13\rho)y-60(1-2\rho)-33y^4\Big)\frac{8\,y^2\theta(x)}{15(y-1)^4}\\
&\quad+\Big(8y^9-8(5-\rho)y^8+16(5-3\rho)y^7-16(1-\rho)(5-\rho)y^6\\
&\quad+40(1-\rho)^2y^5-8(1-\rho)^3y^4\Big)\frac{\delta'(x)}{15(y-1)^6}\Bigg)\sqrt{\rho}\,\hat{s}_7\\
&+\Bigg(\Big(224y^8-16(91-15\rho)y^7+16(211-62\rho)y^6-16\left(-15\rho^2-74\rho+227\right)y^5\\
&\quad+16\left(-29\rho^2-26\rho+115\right)y^4-16(1-\rho)\left(15\rho^2+23\rho+22\right)y^3\Big)\frac{\delta(x)}{5(y-1)^5}\\
&\quad+\Big(-616y^6+48(57-16\rho)y^5-40\left(-3\rho^2-67\rho+118\right)y^4\\
&\quad+160\left(-3\rho^2-19\rho+23\right)y^3-360\left(-2\rho^2-3\rho+3\right)y^2\Big)\frac{\theta(x)}{15(y-1)^4}\\
&\quad+\Big(-8y^9+8(5-\rho)y^8-16(5-3\rho)y^7+16(1-\rho)(5-\rho)y^6\\
&\quad-8(1-\rho)^2y^5+8(1-\rho)^3y^4\Big)\frac{\delta'(x)}{15(y-1)^6}\Bigg)\sqrt{\rho}\,\hat{s}_8\\
&+\Bigg(\Big(-136y^8+8(88-15\rho)y^7-8(163-56\rho)y^6+8\left(-15\rho^2-32\rho+131\right)y^5\\
&\quad-32\left(-8\rho^2+13\rho+10\right)y^4+8(1-\rho)\left(15\rho^2+44\rho+1\right)y^3\Big)\frac{\delta(x)}{5(y-1)^5}\\
&\quad+\Big(344y^6-144(11-3\rho)y^5+20\left(-3\rho^2-76\rho+142\right)y^4\\
&\quad-80\left(-3\rho^2-22\rho+29\right)y^3+360\left(-\rho^2-2\rho+2\right)y^2\Big)\frac{\theta(x)}{15(y-1)^4}\\
&\quad+\Big(-8y^9+8(5-\rho)y^8-16(5-3\rho)y^7+16(1-\rho)(5-\rho)y^6\\
&\quad-40(1-\rho)^2y^5+8(1-\rho)^3y^4\Big)\frac{\delta'(x)}{15(y-1)^6}\Bigg)\sqrt{\rho}\,\hat{s}_6\\
&+\Bigg(\Big(-216y^9+40(38-3\rho)y^8-8(551-81\rho)y^7+16\left(-10\rho^2-109\rho+422\right)y^6
\end{aligned}
$$

$$
\begin{aligned}
&- 8\left(-47\rho^2 - 350\rho + 721\right) y^5 + 8(1-\rho)\left(15\rho^2 + 31\rho + 326\right) y^4 \\
&- 8(1-\rho)^2(25\rho + 61)y^3\Big)\frac{\delta(x)}{5(y-1)^6} \\
&+ \Big(464y^7 - 16(148 - 27\rho)y^6 + 4\left(-15\rho^2 - 458\rho + 1186\right) y^5 \\
&- 20\left(-15\rho^2 - 134\rho + 230\right) y^4 + 40(1-\rho)(15\rho + 53)y^3 \\
&- 120(1-\rho)(\rho + 3)y^2\Big)\frac{\theta(x)}{15(y-1)^5} \\
&+ \Big(- 20y^{11} + 4(43 - 5\rho)y^{10} - 4(153 - 43\rho)y^9 + 4\left(5\rho^2 - 156\rho + 295\right) y^8 \\
&- 4\left(31\rho^2 - 294\rho + 335\right) y^7 + 4(1-\rho)\left(-5\rho^2 - 76\rho + 225\right) y^6 \\
&- 4(1-\rho)^2(7\rho + 83)y^5 + 4(1-\rho)^3(5\rho + 13)y^4\Big)\frac{\delta'(x)}{15(y-1)^7}\Big)\sqrt{\rho}\,\hat{s}_5 \\
+ \Big(\Big(&112y^{10} - 8(113 - 12\rho)y^9 + 16(202 - 43\rho)y^8 - 8\left(-13\rho^2 - 217\rho + 815\right) y^7 \\
&+ 24\left(-23\rho^2 - 74\rho + 330\right) y^6 - 8\left(-12\rho^3 - 183\rho^2 - 44\rho + 719\right) y^5 \\
&+ 32\left(-4\rho^3 - 55\rho^2 + 17\rho + 72\right) y^4 - 8(1-\rho)\left(-11\rho^2 + 82\rho + 49\right) y^3\Big)\frac{\delta(x)}{5(y-1)^7} \\
&+ \Big(- 300y^8 + 360(5-\rho)y^7 - 4\left(-12\rho^2 - 509\rho + 1130\right) y^6 \\
&+ 24\left(-12\rho^2 - 194\rho + 255\right) y^5 - 60\left(-12\rho^2 - 92\rho + 79\right) y^4 \\
&+ 80\left(-9\rho^2 - 43\rho + 25\right) y^3 - 180\left(-\rho^2 - 5\rho + 2\right) y^2\Big)\frac{\theta(x)}{15(y-1)^6} \\
&+ \Big(y^{12} + 4(\rho + 3)y^{11} - 112y^{10} + 4\left(-\rho^2 - 34\rho + 91\right) y^9 \\
&- 18\left(3\rho^2 - 30\rho + 35\right) y^8 + 4(1-\rho)\left(\rho^2 - 74\rho + 161\right) y^7 \\
&- 8(49 - 9\rho)(1-\rho)^2y^6 + 4(1-\rho)^3(33-\rho)y^5 - 19(1-\rho)^4y^4\Big)\frac{\delta'(x)}{15(y-1)^8} \\
&+ \Big(- y^{14} + (9-\rho)y^{13} - 12(3-\rho)y^{12} + 4\left(\rho^2 - 14\rho + 21\right) y^{11} - 2\left(15\rho^2 - 70\rho + 63\right) y^{10} \\
&+ 6(1-\rho)\left(\rho^2 - 14\rho + 21\right) y^9 - 28(1-\rho)^2(3-\rho)y^8 \\
&+ 4(1-\rho)^3(9-\rho)y^7 - 9(1-\rho)^4y^6 + (1-\rho)^5y^5\Big)\frac{\delta''(x)}{15(y-1)^9}\Big)\sqrt{\rho}\,\hat{s}_4 \\
+ \Big(\Big(& - 88y^{10} + 8(62 - 3\rho)y^9 - 8(131 - 14\rho)y^8 + 8\left(-7\rho^2 + 2\rho + 110\right) y^7 \\
&+ 24\left(12\rho^2 - 34\rho + 5\right) y^6 - 8(1-\rho)\left(-3\rho^2 - 105\rho + 94\right) y^5 \\
&+ 8(63 - 26\rho)(1-\rho)^2y^4 - 112(1-\rho)^3y^3\Big)\frac{\delta(x)}{5(y-1)^7} \\
&+ \Big(180y^8 - 120(9-\rho)y^7 + 4\left(-3\rho^2 - 161\rho + 680\right) y^6 - 24\left(-3\rho^2 - 56\rho + 155\right) y^5
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 60\left(-3\rho^2 - 26\rho + 49\right) y^4 - 80(16 - 13\rho)y^3 + 60(1-\rho)(4-\rho)y^2\Big) \frac{\theta(x)}{15(y-1)^6} \\
&+ \Big(- 19y^{12} + 4(33 - 4\rho)y^{11} - 8(49 - 15\rho)y^{10} + 4\left(4\rho^2 - 89\rho + 161\right) y^9 \\
&\quad - 18\left(3\rho^2 - 30\rho + 35\right) y^8 + 4(1-\rho)\left(-4\rho^2 - 19\rho + 91\right) y^7 \\
&\quad - 16(1-\rho)^2(3\rho + 7)y^6 + 4(1-\rho)^3(4\rho + 3)y^5 + (1-\rho)^4 y^4\Big) \frac{\delta'(x)}{15(y-1)^8} \\
&+ \Big(- y^{14} + (9-\rho)y^{13} - 12(3-\rho)y^{12} + 4\left(\rho^2 - 14\rho + 21\right) y^{11} - 2\left(15\rho^2 - 70\rho + 63\right) y^{10} \\
&\quad + 6(1-\rho)\left(\rho^2 - 14\rho + 21\right) y^9 - 28(1-\rho)^2(3-\rho)y^8 + 4(1-\rho)^3(9-\rho)y^7 \\
&\quad - 9(1-\rho)^4 y^6 + (1-\rho)^5 y^5\Big) \frac{\delta''(x)}{15(y-1)^9}\Big) \sqrt{\rho}\, \hat{s}_3 \\
+ \Big(\Big(& - 128y^{10} + 8(127 - 18\rho)y^9 - 64(57 - 13\rho)y^8 + 8\left(-17\rho^2 - 273\rho + 935\right) y^7 \\
&\quad - 8\left(-61\rho^2 - 418\rho + 1160\right) y^6 + 8\left(-18\rho^3 - 57\rho^2 - 386\rho + 861\right) y^5 \\
&\quad - 16\left(-22\rho^3 + 5\rho^2 - 99\rho + 176\right) y^4 + 8(1-\rho)\left(41\rho^2 + 18\rho + 61\right) y^3\Big) \frac{\delta(x)}{5(y-1)^7} \\
&+ \Big(360y^8 - 120(19 - 4\rho)y^7 + 4\left(-18\rho^2 - 656\rho + 1485\right) y^6 \\
&\quad - 48\left(-9\rho^2 - 118\rho + 170\right) y^5 + 120\left(-9\rho^2 - 50\rho + 52\right) y^4 \\
&\quad - 40\left(-30\rho^2 - 77\rho + 63\right) y^3 + 60\left(-9\rho^2 - 10\rho + 7\right) y^2\Big) \frac{\theta(x)}{15(y-1)^6} \\
&+ \Big(y^{12} - 4(2-\rho)y^{11} + 4(7 - 5\rho)y^{10} - 4\left(\rho^2 - 6\rho + 14\right) y^9 + 2\left(-7\rho^2 + 20\rho + 35\right) y^8 \\
&\quad - 4\left(\rho^3 - 30\rho^2 + 35\rho + 14\right) y^7 + 4(1-\rho)\left(-13\rho^2 + 46\rho + 7\right) y^6 \\
&\quad - 4(1-\rho)^2\left(-\rho^2 + 24\rho + 2\right) y^5 + (1-\rho)^3(19\rho + 1)y^4\Big) \frac{\delta'(x)}{15(y-1)^8} \\
&+ \Big(- y^{14} + (9-\rho)y^{13} - 12(3-\rho)y^{12} + 4\left(\rho^2 - 14\rho + 21\right) y^{11} - 2\left(15\rho^2 - 70\rho + 63\right) y^{10} \\
&\quad + 6(1-\rho)\left(\rho^2 - 14\rho + 21\right) y^9 - 28(1-\rho)^2(3-\rho)y^8 + 4(1-\rho)^3(9-\rho)y^7 \\
&\quad - 9(1-\rho)^4 y^6 + (1-\rho)^5 y^5\Big) \frac{\delta''(x)}{15(y-1)^9}\Big) \sqrt{\rho}\, \hat{s}_2 \\
+ \Big(\Big(& 48y^{10} - 144(4-\rho)y^9 + 64(37 - 13\rho)y^8 - 16\left(-11\rho^2 - 99\rho + 305\right) y^7 \\
&\quad + 16\left(-48\rho^2 - 79\rho + 355\right) y^6 - 16\left(-9\rho^3 - 21\rho^2 - 28\rho + 238\right) y^5 \\
&\quad + 16\left(-42\rho^3 + 85\rho^2 - 9\rho + 86\right) y^4 - 16(1-\rho)\left(78\rho^2 + 9\rho + 13\right) y^3\Big) \frac{\delta(x)}{5(y-1)^7} \\
&+ \Big(- 360y^8 + 240(10 - 3\rho)y^7 - 8\left(-9\rho^2 - 493\rho + 835\right) y^6 \\
&\quad + 48\left(-9\rho^2 - 178\rho + 205\right) y^5 - 120\left(-9\rho^2 - 73\rho + 67\right) y^4
\end{aligned}
$$

$$
\begin{aligned}
&+80\left(-18\rho^2-49\rho+43\right)y^3-120(5-9\rho)(\rho+1)y^2\Big)\frac{\theta(x)}{15(y-1)^6}\\
&+\Big(-6y^{12}+8(1-3\rho)y^{11}+4(25\rho+28)y^{10}-8\left(2\rho^2+18\rho+63\right)y^9\\
&+4\left(\rho^2+15\rho+245\right)y^8-8\left(2\rho^3-10\rho^2-5\rho+133\right)y^7\\
&+4\left(-13\rho^3-26\rho^2-9\rho+168\right)y^6-8(1-\rho)\left(-3\rho^3+25\rho^2+29\rho+29\right)y^5\\
&+2(1-\rho)^2\left(57\rho^2+36\rho+17\right)y^4\Big)\frac{\delta'(x)}{15(y-1)^8}\\
&+\Big(6y^{14}-6(9-\rho)y^{13}+4(54-13\rho)y^{12}-4\left(\rho^2-49\rho+126\right)y^{11}\\
&+4\left(5\rho^2-105\rho+189\right)y^{10}-4\left(\rho^3+10\rho^2-140\rho+189\right)y^9\\
&+4\left(3\rho^3+10\rho^2-119\rho+126\right)y^8-4(1-\rho)\left(-\rho^3-4\rho^2-9\rho+54\right)y^7\\
&+2(1-\rho)^2\left(7\rho^2+16\rho+27\right)y^6-2(1-\rho)^3\left(3\rho^2+4\rho+3\right)y^5\Big)\frac{\delta''(x)}{15(y-1)^9}\Big)\sqrt{\rho}\,\hat{s}_1
\end{aligned}
\tag{D.9}
$$

D.1.3. Left-handed scalar current

$$
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,0)}_{c_Lg_L}=-(y+\rho-1)^2\frac{12\,y^2\theta(x)}{y-1}\sqrt{\rho}\,m_b \tag{D.10}
$$

$$
\begin{aligned}
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,2)}_{c_Lg_L}=&\Big(3y^3(1-2\rho)-y^2\left(4\rho^2-18\rho+9\right)+9y(1-\rho)^2-3(1-\rho)^2\Big)\frac{2\,y^2\theta(x)}{(y-1)^3}\sqrt{\rho}\,m_b\,\hat{\mu}_1\\
&+\Big(y^2(3-2\rho)-2y(1-\rho)(3-\rho)+3(1-\rho)^2\Big)\frac{10\,y^2\theta(x)}{(y-1)^2}\sqrt{\rho}\,m_b\,\hat{\mu}_2
\end{aligned}
\tag{D.11}
$$

$$
\begin{aligned}
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,3)}_{c_Lg_L}=&\bigg(\Big(y^6-6y^5+y^4(15-\rho)-4y^3(5-\rho)+y^2\left(-\rho^2-6\rho+15\right)\\
&-2y(1-\rho)(\rho+3)+(1-\rho)^2(\rho+1)\Big)\frac{8\,y^3\,\delta(x)}{(y-1)^5}\\
&+\Big(12y^4(2-\rho)-y^3\left(15\rho^2-86\rho+96\right)+2y^2\left(27\rho^2-95\rho+72\right)\\
&-2y\left(33\rho^2-85\rho+48\right)+6(4-5\rho)(1-\rho)\Big)\frac{4\,y^2\theta(x)}{3(y-1)^4}\bigg)\sqrt{\rho}\,m_b\,\hat{\rho}_1\\
&+\Big(2y^2(2-\rho)-y\left(\rho^2-8\rho+8\right)+2(1-\rho)(2-\rho)\Big)\frac{4\,y^3\theta(x)}{(y-1)^3}\sqrt{\rho}\,m_b\,\hat{\rho}_2
\end{aligned}
\tag{D.12}
$$

$$
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,4)}_{c_Lg_L}=\bigg(\Big(-48y^9+192y^8-48(5-\rho)y^7+48(1-\rho)(5-\rho)y^5
$$

$$
\begin{aligned}
&\quad - 192(1-\rho)^2 y^4 + 48(1-\rho)^3 y^3\Big)\frac{\delta(x)}{5(y-1)^5} \\
&+ \Big(8(10-9\rho)y^6 - 12\left(7\rho^2-36\rho+30\right)y^5 + 20\left(15\rho^2-46\rho+30\right)y^4 \\
&\quad - 40(11-9\rho)(1-\rho)y^3 + 120(1-\rho)^2 y^2\Big)\frac{\theta(x)}{5(y-1)^4}\Big)\,\hat{s}_5 \\
+ \Big(\Big(&-8y^7 + 32y^6 - 16(3-\rho)y^5 + 32(1-\rho)y^4 - 8(1-\rho)^2 y^3\Big)\frac{\delta(x)}{(y-1)^4} \\
&+ \Big(4(10-9\rho)y^5 - 2\left(27\rho^2-102\rho+70\right)y^4 + 8\left(18\rho^2-43\rho+20\right)y^3 \\
&\quad - 12(5-9\rho)(1-\rho)y^2\Big)\frac{\theta(x)}{3(y-1)^3}\Big)\sqrt{\rho}\,m_b\,\hat{s}_9 \\
+ \Big(\Big(&48y^9 - 112y^8 + 112\rho y^7 + 115(1-\rho)y^6 - 16(1-\rho)(7\rho+5)y^5 - 48(1-\rho)^2 y^4 \\
&\quad + 16(1-\rho)^2(3\rho+2)y^3\Big)\frac{\delta(x)}{5(y-1)^5} \\
&+ \Big(-8(20-27\rho)y^6 + 4\left(63\rho^2-304\rho+170\right)y^5 - 20\left(45\rho^2-122\rho+50\right)y^4 \\
&\quad + 120\left(9\rho^2-16\rho+5\right)y^3 - 120(1-3\rho)(1-\rho)y^2\Big)\frac{\theta(x)}{15(y-1)^4}\Big)\sqrt{\rho}\,m_b\,\hat{s}_7 \\
+ \Big(\Big(&-8y^9 + 32y^8 - 8(5-\rho)y^7 + 8(1-\rho)(5-\rho)y^5 - 32(1-\rho)^2 y^4 \\
&\quad + 8(1-\rho)^3 y^3\Big)\frac{\delta(x)}{5(y-1)^5} \\
&+ \Big(8(40-27\rho)y^6 - 4\left(78\rho^2-379\rho+345\right)y^5 + 60\left(19\rho^2-57\rho+37\right)y^4 \\
&\quad - 20\left(72\rho^2-157\rho+79\right)y^3 + 60(7-10\rho)(1-\rho)y^2\Big)\frac{\theta(x)}{15(y-1)^4}\Big)\sqrt{\rho}\,m_b\,\hat{s}_6 \\
+ \Big(\Big(&-8y^9 + 32y^8 - 8(5-\rho)y^7 + 8(1-\rho)(5-\rho)y^5 - 32(1-\rho)^2 y^4 \\
&\quad + 8(1-\rho)^3 y^3\Big)\frac{\delta(x)}{5(y-1)^5} \\
&+ \Big(-16(25-24\rho)y^6 + 4\left(147\rho^2-656\rho+510\right)y^5 - 120\left(18\rho^2-51\rho+31\right)y^4 \\
&\quad + 40\left(69\rho^2-148\rho+73\right)y^3 - 120(7-10\rho)(1-\rho)y^2\Big)\frac{\theta(x)}{15(y-1)^4}\Big)m_b\,\hat{s}_8 \\
+ \Big(\Big(&-32y^{10} + 176y^9 - 32(12-\rho)y^8 + 80(5-\rho)y^7 - 32\left(5-\rho^2\right)y^6
\end{aligned}
$$

$$
\begin{aligned}
&- 16(3-7\rho)(1-\rho)y^5 + 32(1-\rho)^2(2-\rho)y^4 - 16(1-\rho)^3y^3\Big)\frac{\delta(x)}{5(y-1)^6} \\
&+ \Big(20(2-3\rho)y^7 - 2\left(21\rho^2-150\rho+80\right)y^6 + 20\left(9\rho^2-29\rho+12\right)y^5 \\
&- 10\left(27\rho^2-50\rho+16\right)y^4 + 40(1-3\rho)(1-\rho)y^3\Big)\frac{\theta(x)}{15(y-1)^5} \\
&+ \Big(-y^{12} + 8y^{11} - 4(7-\rho)y^{10} + 8(7-3\rho)y^9 \\
&- 2\left(3\rho^2-30\rho+35\right)y^8 + 8(7-3\rho)(1-\rho)y^7 - 4(1-\rho)^2(7-\rho)y^6 \\
&+ 8(1-\rho)^3y^5 - (1-\rho)^4y^4\Big)\frac{\delta'(x)}{15(y-1)^7}\Big)\sqrt{\rho}\,m_b\,\hat{s}_3 \\
&+ \Big(\Big(8y^{10} - 64y^9 + 8(27-\rho)y^8 - 40(10-\rho)y^7 + 8\left(-\rho^2-10\rho+55\right)y^6 \\
&- 16\left(-3\rho^2-5\rho+18\right)y^5 + 8(1-\rho)\left(-\rho^2+8\rho+13\right)y^4 \\
&- 8(1-\rho)^2(3\rho+2)y^3\Big)\frac{\delta(x)}{5(y-1)^6} \\
&+ \Big(80(4-3\rho)y^7 - 4\left(93\rho^2-475\rho+410\right)y^6 \\
&+ 60\left(29\rho^2-90\rho+56\right)y^5 - 80\left(39\rho^2-90\rho+43\right)y^4 \\
&+ 40\left(63\rho^2-115\rho+44\right)y^3 - 60(6-13\rho)(1-\rho)y^2\Big)\frac{\theta(x)}{15(y-1)^5} \\
&+ \Big(-y^{12} + 8y^{11} - 4(7-\rho)y^{10} + 8(7-3\rho)y^9 \\
&- 2\left(3\rho^2-30\rho+35\right)y^8 + 8(7-3\rho)(1-\rho)y^7 \\
&- 4(1-\rho)^2(7-\rho)y^6 + 8(1-\rho)^3y^5 - (1-\rho)^4y^4\Big)\frac{\delta'(x)}{15(y-1)^7}\Big)\sqrt{\rho}\,m_b\,\hat{s}_2 \\
&+ \Big(\Big(8y^{10} - 24y^9 - 8(\rho+3)y^8 + 40(5-\rho)y^7 - 8\left(\rho^2-30\rho+45\right)y^6 \\
&+ 8(39-11\rho)(1-\rho)y^5 - 8(1-\rho)^2(17-\rho)y^4 + 24(1-\rho)^3y^3\Big)\frac{\delta(x)}{5(y-1)^6} \\
&+ \Big(-60(2-3\rho)y^7 + 2\left(129\rho^2-610\rho+300\right)y^6 \\
&- 20\left(60\rho^2-161\rho+60\right)y^5 + 10\left(213\rho^2-410\rho+120\right)y^4 \\
&- 120\left(14\rho^2-21\rho+5\right)y^3 + 120(1-4\rho)(1-\rho)y^2\Big)\frac{\theta(x)}{15(y-1)^5} \\
&+ \Big(-y^{12} + 8y^{11} - 4(7-\rho)y^{10} + 8(7-3\rho)y^9 \\
&- 2\left(3\rho^2-30\rho+35\right)y^8 + 8(7-3\rho)(1-\rho)y^7 \\
&- 4(1-\rho)^2(7-\rho)y^6 + 8(1-\rho)^3y^5 - (1-\rho)^4y^4\Big)\frac{\delta'(x)}{15(y-1)^7}\Big)\sqrt{\rho}\,m_b\,\hat{s}_4
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg(\bigg(-48y^{10}+304y^9-48(17-\rho)y^8+240(5-\rho)y^7-16\left(-3\rho^2-30\rho+65\right)y^6\\
&\quad+16\left(-13\rho^2-30\rho+33\right)y^5-16\left(3\rho^3-17\rho^2-15\rho+9\right)y^4\\
&\quad+16(1-\rho)\left(-9\rho^2-2\rho+1\right)y^3\bigg)\frac{\delta(x)}{5(y-1)^6}\\
&+\bigg(-40(8-9\rho)y^7+4\left(153\rho^2-700\rho+440\right)y^6\\
&\quad-40\left(72\rho^2-203\rho+96\right)y^5+20\left(261\rho^2-560\rho+208\right)y^4\\
&\quad-80(7-18\rho)(4-3\rho)y^3+480(1-3\rho)(1-\rho)y^2\bigg)\frac{\theta(x)}{15(y-1)^5}\\
&+\bigg(6y^{12}-48y^{11}+4(42-\rho)y^{10}-24(14-\rho)y^9+4\left(-\rho^2-15\rho+105\right)y^8\\
&\quad-16\left(-\rho^2-5\rho+21\right)y^7+4\left(-\rho^3-6\rho^2-15\rho+42\right)y^6\\
&\quad-8(1-\rho)\left(\rho^2+3\rho+6\right)y^5+2(1-\rho)^2\left(3\rho^2+4\rho+3\right)y^4\bigg)\frac{\delta'(x)}{15(y-1)^7}\Bigg)\sqrt{\rho}\,m_b\,\hat{s}_1
\end{aligned}
\tag{D.13}
$$

D.1.4. Right-handed scalar current

$$
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,0)}_{c_Lg_R}=-\,(y+\rho-1)^2\frac{12\,y^2\theta(x)}{y-1}m_b \tag{D.14}
$$

$$
\begin{aligned}
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,2)}_{c_Lg_R}=&\left(3y^3(1-2\rho)-y^2\left(4\rho^2-18\rho+9\right)+9y(\rho-1)^2-3(\rho-1)^2\right)\frac{2\,y^2\theta(x)}{(y-1)^3}\,m_b\,\hat{\mu}_1\\
&-\left(y^2(2\rho-3)+2y\left(\rho^2-4\rho+3\right)-3(\rho-1)^2\right)\frac{2\,y^2\theta(x)}{(y-1)^2}\,m_b\,\hat{\mu}_2
\end{aligned}
\tag{D.15}
$$

$$
\begin{aligned}
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,3)}_{c_Lg_R}=&\Bigg(\bigg(y^6-6y^5+y^4(15-\rho)-4y^3(5-\rho)+y^2\left(-\rho^2-6\rho+15\right)\\
&\qquad-2y(1-\rho)(\rho+3)+(1-\rho)^2(\rho+1)\bigg)\frac{8\,y^3\delta(x)}{(y-1)^5}\\
&+\bigg(-12y^4(1-\rho)+y^3\left(9\rho^2-58\rho+48\right)-2y^2\left(15\rho^2-49\rho+36\right)\\
&\qquad+2y\left(15\rho^2-35\rho+24\right)-6(1-\rho)(2-\rho)\bigg)\frac{4\,y^2\theta(x)}{3(y-1)^4}\Bigg)\hat{\rho}_1\\
&+\left(2y^2-y(4-\rho)+2(1-\rho)\right)\frac{4\,\rho\,y^3\theta(x)}{(y-1)^3}\,\hat{\rho}_2
\end{aligned}
\tag{D.16}
$$

$$
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,4)}_{c_Lg_R}=\Bigg(\bigg(-8y^7+32y^6-16(3-\rho)y^5+32(1-\rho)y^4-8(1-\rho)^2y^3\bigg)\frac{\delta(x)}{(y-1)^4}
$$

$$+\Bigg(-4(2-9\rho)y^5+2\left(15\rho^2-54\rho+14\right)y^4-8\left(9\rho^2-14\rho+4\right)y^3$$

$$+12(1-3\rho)(1-\rho)y^2\Bigg)\frac{\theta(x)}{3(y-1)^3}\Bigg)m_b\,\hat{s}_9$$

$$+\Bigg(\Bigg(-32y^9+128y^8-32(5-\rho)y^7+32(1-\rho)(5-\rho)y^5-128(1-\rho)^2y^4$$

$$+32(1-\rho)^3y^3\Bigg)\frac{\delta(x)}{5(y-1)^5}$$

$$+\Bigg(-8(20-27\rho)y^6+4\left(33\rho^2-224\rho+150\right)y^5-60\left(7\rho^2-22\rho+14\right)y^4$$

$$+40(13-9\rho)(1-\rho)y^3-120(1-\rho)^2y^2\Bigg)\frac{\theta(x)}{15(y-1)^4}\Bigg)m_b\,\hat{s}_5$$

$$+\Bigg(\Bigg(-48y^9+272y^8-16(40-3\rho)y^7+160(5-\rho)y^6-16\left(-3\rho^2-12\rho+35\right)y^5$$

$$+16(1-\rho)(7\rho+13)y^4-16(1-\rho)^2(3\rho+2)y^3\Bigg)\frac{\delta(x)}{5(y-1)^5}$$

$$+\Bigg(8(10-27\rho)y^6-12\left(21\rho^2-88\rho+30\right)y^5+20\left(45\rho^2-98\rho+30\right)y^4$$

$$-40\left(27\rho^2-40\rho+11\right)y^3+120(1-3\rho)(1-\rho)y^2\Bigg)\frac{\theta(x)}{15(y-1)^4}\Bigg)m_b\,\hat{s}_7$$

$$+\Bigg(\Bigg(8y^9-32y^8+8(5-\rho)y^7-8(1-\rho)(5-\rho)y^5+32(1-\rho)^2y^4-8(1-\rho)^3y^3\Bigg)\frac{\delta(x)}{5(y-1)^5}$$

$$+\Bigg(-8(10-27\rho)y^6+4\left(48\rho^2-229\rho+75\right)y^5-60\left(11\rho^2-25\rho+7\right)y^4$$

$$+20\left(36\rho^2-55\rho+13\right)y^3-60(1-4\rho)(1-\rho)y^2\Bigg)\frac{\theta(x)}{15(y-1)^4}\Bigg)m_b\,\hat{s}_6$$

$$+\Bigg(\Bigg(8y^9-32y^8+8(5-\rho)y^7-8(1-\rho)(5-\rho)y^5+32(1-\rho)^2y^4-8(1-\rho)^3y^3\Bigg)\frac{\delta(x)}{5(y-1)^5}$$

$$+\Bigg(32(5-12\rho)y^6-4\left(87\rho^2-416\rho+150\right)y^5+120\left(10\rho^2-23\rho+7\right)y^4$$

$$-40\left(33\rho^2-52\rho+13\right)y^3+120(1-4\rho)(1-\rho)y^2\Bigg)\frac{\theta(x)}{15(y-1)^4}\Bigg)m_b\,\hat{s}_8$$

$$+\Bigg(\Bigg(8y^{10}-24y^9-8(\rho+3)y^8+40(5-\rho)y^7-8\left(\rho^2-30\rho+45\right)y^6$$

$$+8(39-11\rho)(1-\rho)y^5-8(1-\rho)^2(17-\rho)y^4+24(1-\rho)^3y^3\Bigg)\frac{\delta(x)}{5(y-1)^6}$$

$$+\Bigg(-180\rho y^7+2\left(-81\rho^2+410\rho+60\right)y^6-20\left(-36\rho^2+73\rho+24\right)y^5$$

$$
\begin{aligned}
&\qquad + 10\left(-117\rho^2+130\rho+72\right)y^4 - 120\left(-7\rho^2+5\rho+4\right)y^3 \\
&\qquad + 120(1-\rho)(2\rho+1)y^2\Big)\frac{\theta(x)}{15(y-1)^5} \\
&\quad + \Big(-y^{12}+8y^{11}-4(7-\rho)y^{10}+8(7-3\rho)y^9-2\left(3\rho^2-30\rho+35\right)y^8 \\
&\qquad + 8(7-3\rho)(1-\rho)y^7-4(1-\rho)^2(7-\rho)y^6+8(1-\rho)^3y^5 \\
&\qquad - (1-\rho)^4y^4\Big)\frac{\delta'(x)}{15(y-1)^7}\Big)m_b\,\hat{s}_4 \\
&+ \Big(\Big(-32y^{10}+176y^9-32(12-\rho)y^8+80(5-\rho)y^7-32\left(5-\rho^2\right)y^6 \\
&\qquad - 16(3-7\rho)(1-\rho)y^5+32(1-\rho)^2(2-\rho)y^4-16(1-\rho)^3y^3\Big)\frac{\delta(x)}{5(y-1)^6} \\
&\quad + \Big(-20(4-3\rho)y^7+2\left(9\rho^2-150\rho+160\right)y^6-20\left(3\rho^2-25\rho+24\right)y^5 \\
&\qquad + 10\left(3\rho^2-34\rho+32\right)y^4-80(1-\rho)y^3\Big)\frac{\theta(x)}{15(y-1)^5} \\
&\quad + \Big(-y^{12}+8y^{11}-4(7-\rho)y^{10}+8(7-3\rho)y^9-2\left(3\rho^2-30\rho+35\right)y^8 \\
&\qquad + 8(7-3\rho)(1-\rho)y^7-4(1-\rho)^2(7-\rho)y^6+8(1-\rho)^3y^5 \\
&\qquad - (1-\rho)^4y^4\Big)\frac{\delta'(x)}{15(y-1)^7}\Big)m_b\,\hat{s}_3 \\
&+ \Big(\Big(8y^{10}-64y^9+8(27-\rho)y^8-40(10-\rho)y^7+8\left(-\rho^2-10\rho+55\right)y^6 \\
&\qquad - 16\left(-3\rho^2-5\rho+18\right)y^5+8(1-\rho)\left(-\rho^2+8\rho+13\right)y^4 \\
&\qquad - 8(1-\rho)^2(3\rho+2)y^3\Big)\frac{\delta(x)}{5(y-1)^6} \\
&\quad + \Big(-40(1-6\rho)y^7+4\left(57\rho^2-305\rho+40\right)y^6-60\left(17\rho^2-42\rho+4\right)y^5 \\
&\qquad + 80\left(21\rho^2-33\rho+2\right)y^4-40\left(30\rho^2-35\rho+1\right)y^3-300(1-\rho)\rho y^2\Big)\frac{\theta(x)}{15(y-1)^5} \\
&\quad + \Big(-y^{12}+8y^{11}-4(7-\rho)y^{10}+8(7-3\rho)y^9-2\left(3\rho^2-30\rho+35\right)y^8 \\
&\qquad + 8(7-3\rho)(1-\rho)y^7-4(1-\rho)^2(7-\rho)y^6+8(1-\rho)^3y^5 \\
&\qquad - (1-\rho)^4y^4\Big)\frac{\delta'(x)}{15(y-1)^7}\Big)m_b\,\hat{s}_2 \\
&+ \Big(\Big(-48y^{10}+304y^9-48(17-\rho)y^8+240(5-\rho)y^7-16\left(-3\rho^2-30\rho+65\right)y^6 \\
&\qquad + 16\left(-13\rho^2-30\rho+33\right)y^5-16\left(3\rho^3-17\rho^2-15\rho+9\right)y^4
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 16(1-\rho)\left(-9\rho^2-2\rho+1\right)y^3\Big)\frac{\delta(x)}{5(y-1)^6}\\
&+\Big(40(4-9\rho)y^7-4\left(117\rho^2-560\rho+220\right)y^6+40\left(54\rho^2-139\rho+48\right)y^5\\
&\quad -20\left(189\rho^2-340\rho+104\right)y^4+80\left(36\rho^2-51\rho+14\right)y^3\\
&\quad -240(1-3\rho)(1-\rho)y^2\Big)\frac{\theta(x)}{15(y-1)^5}\\
&+\Big(6y^{12}-48y^{11}+4(42-\rho)y^{10}-24(14-\rho)y^9+4\left(-\rho^2-15\rho+105\right)y^8\\
&\quad -16\left(-\rho^2-5\rho+21\right)y^7+4\left(-\rho^3-6\rho^2-15\rho+42\right)y^6\\
&\quad -8(1-\rho)\left(\rho^2+3\rho+6\right)y^5+2(1-\rho)^2\left(3\rho^2+4\rho+3\right)y^4\Big)\frac{\delta'(x)}{15(y-1)^7}\Bigg)m_b\,\hat{s}_1
\end{aligned}
\tag{D.17}
$$

D.1.5. Left-handed tensor current

$$
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,0)}_{c_L d_L} = -(y-3)y^2(y+\rho-1)^3\frac{4\,\theta(x)}{(y-1)^3}\sqrt{\rho}\,m_b \tag{D.18}
$$

$$
\begin{aligned}
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,2)}_{c_L d_L} =&(y+\rho-1)^2\left(5y^3-10y^2(\rho+1)+y(40\rho+17)-12(5\rho+1)\right)\frac{2\,y^2\theta(x)}{3(y-1)^4}\sqrt{\rho}\,m_b\,\hat{\mu}_2\\
&+\Bigg(\Big((y+\rho-1)\left(y\left(20\rho^2-63\rho-20\right)-40\rho^2+35\rho+5\right)\\
&\quad +y^2(y+\rho-1)\left(5y^2-5y(\rho+4)-4\rho^2+33\rho+30\right)\Big)\frac{2\,y^3\theta(x)}{3(y-1)^5}\\
&\quad -\left(\left(y^2-3y+2\right)\rho+(y-1)^3-\rho^2\right)^2\frac{24\,y^3\delta(x)}{(y-1)^6}\Bigg)\sqrt{\rho}\,m_b\,\hat{\mu}_1
\end{aligned}
\tag{D.19}
$$

$$
\begin{aligned}
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,3)}_{c_L d_L} =&\Bigg(\Big(y^6-2(3-\rho)y^5+\left(\rho^2-10\rho+15\right)y^4-4\left(\rho^2-5\rho+5\right)y^3\\
&\quad +5(1-\rho)(3-\rho)y^2-2(1-\rho)^2(\rho+3)y+(1-\rho)^3(\rho+1)\Big)\frac{24\,y^3\delta(x)}{(y-1)^6}\\
&+\Big(2y^6-16y^5+\left(-3\rho^2-6\rho+50\right)y^4-\left(\rho^3-12\rho^2-30\rho+80\right)y^3+6(1-\rho)^3\\
&\quad +5(1-\rho)\left(-\rho^2+2\rho+14\right)y^2-2(1-\rho)^2(5\rho+16)\Big)\frac{4\,y^2\theta(x)}{3(y-1)^5}\Bigg)\sqrt{\rho}\,m_b\,\hat{\rho}_2\\
&+\Bigg(\Big(y^8-8(6-\rho)y^7+20y^6-\left(-5\rho^2+3\rho+50\right)y^5+6\left(-2\rho^2-2\rho+15\right)y^4\\
&\quad -\left(-\rho^3-18\rho^2-43\rho+106\right)y^3+2(1-\rho)\left(-5\rho^2+14\rho+38\right)y^2
\end{aligned}
$$

$$
\begin{aligned}
&- (1-\rho)^2 \left(-\rho^2+37\rho+30\right) y + (1-\rho)^3(11\rho+5)\Big) \frac{8\,y^3\delta(x)}{(y-1)^7} \\
&+ \Big(2y^7 - 12y^6 + 2\left(-3\rho^2+5\rho+21\right) y^5 - \left(5\rho^3-47\rho^2+48\rho+94\right) y^4 \\
&\quad + 6\left(5\rho^3-26\rho^2+20\rho+21\right) y^3 - \left(75\rho^3-273\rho^2+178\rho+96\right) y^2 \\
&\quad + 2(1-\rho)\left(-42\rho^2+88\rho+19\right) y - 6(1-\rho)^2(9\rho+1)\Big) \frac{4\,y^2\theta(x)}{3(y-1)^6} \\
&+ \Big(- y^8 + 8(4-\rho)y^7 - \left(\rho^2-15\rho+28\right) y^6 + 8\left(\rho^2-6\rho+7\right) y^5 \\
&\quad - \left(-\rho^3+24\rho^2-85\rho+70\right) y^4 + 2(1-\rho)\left(\rho^2-17\rho+28\right) y^3 \\
&\quad - (1-\rho)^2\left(-\rho^2-\rho+28\right) y^2 + 4(1-\rho)^3(\rho+2)y \\
&\quad - (1-\rho)^4(\rho+1)\Big) \frac{4\,y^4\delta'(x)}{3(y-1)^8}\Bigg) \sqrt{\rho}\, m_b\, \hat{\rho}_1
\end{aligned} \tag{D.20}
$$

$$
\begin{aligned}
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,4)}_{c_L d_L} =& \Bigg(\Big(4y^9 - 16(4-\rho)y^8 + 4\left(3\rho^2-22\rho+67\right) y^7 - 8\left(-\rho^2-29\rho+64\right) y^6 \\
&\quad + 4\left(6\rho^3-19\rho^2-90\rho+127\right) y^5 - 8(1-\rho)\left(-3\rho^2-5\rho+32\right) y^4 \\
&\quad + 4(1-\rho)^2\left(9\rho^2+2\rho+13\right) y^3\Big) \frac{\delta(x)}{(y-1)^6} \\
&+ \Big(- 52y^7 + 2\left(-9\rho^2-10\rho+44\right) y^6 - 2\left(9\rho^3-48\rho^2-142\rho+274\right) y^5 \\
&\quad + 2\left(45\rho^3-105\rho^2-260\rho+286\right) y^4 - 4\left(45\rho^3-60\rho^2-95\rho+74\right) y^3 \\
&\quad + 12(1-\rho)\left(-15\rho^2-2\rho+5\right) y^2\Big) \frac{\theta(x)}{3(y-1)^5} \\
&+ \Big(- 4y^{10} + 8(3-\rho)y^9 - 4\left(\rho^2-12\rho+15\right) y^8 + 16(5-2\rho)(1-\rho)y^7 \\
&\quad - 4(15-2\rho)(1-\rho)^2y^6 + 24(1-\rho)^3y^5 - 4(1-\rho)^4y^4\Big) \frac{\delta'(x)}{3(y-1)^7}\Bigg) \sqrt{\rho}\, m_b\, \hat{s}_9 \\
&+ \Bigg(\Big(- 8y^{11} + 8(22-\rho)y^{10} - 16(62-19\rho)y^9 + 8\left(17\rho^2-201\rho+319\right) y^8 \\
&\quad - 24\left(26\rho^2-130\rho+145\right) y^7 + 8\left(3\rho^3+72\rho^2-267\rho+316\right) y^6 \\
&\quad - 16\left(-4\rho^3-56\rho^2+39\rho+51\right) y^5 - 8(1-\rho)\left(26\rho^3+33\rho^2-180\rho+1\right) y^4 \\
&\quad + 16(1-\rho)^2\left(-8\rho^2-25\rho+3\right) y^3\Big) \frac{\delta(x)}{5(y-1)^7} \\
&+ \Big(- 84y^8 + 4\left(-27\rho^2-15\rho+133\right) y^7 - 8\left(13\rho^3-96\rho^2-35\rho+204\right) y^6 \\
&\quad + 12\left(52\rho^3-195\rho^2-70\rho+242\right) y^5 - 20\left(78\rho^3-192\rho^2-78\rho+149\right) y^4 \\
&\quad + 20\left(104\rho^3-150\rho^2-71\rho+81\right) y^3 - 8(1-\rho)\left(-120\rho^2-\rho+3\right) y^2\Big) \frac{\theta(x)}{15(y-1)^6}
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg(-4y^{12}+8(5-\rho)y^{11}-4\left(\rho^2-21\rho+42\right)y^{10}+8\left(7\rho^2-44\rho+49\right)y^9\\
&\quad -4\left(-3\rho^3+66\rho^2-195\rho+140\right)y^8+8(1-\rho)\left(11\rho^2-62\rho+63\right)y^7\\
&\quad -4(5-3\rho)(1-\rho)^2(14-\rho)y^6+8(11-5\rho)(1-\rho)^3y^5\\
&\quad -4(1-\rho)^4(3-\rho)y^4\Bigg)\frac{\delta'(x)}{15(y-1)^8}\Bigg)\sqrt{\rho}\,m_b\,\hat{s}_6\\
&+\Bigg(\Bigg(-8y^{11}-8(\rho+8)y^{10}+16(58-11\rho)y^9-8\left(13\rho^2-189\rho+446\right)y^8\\
&\quad +48\left(7\rho^2-85\rho+140\right)y^7-8\left(27\rho^3+18\rho^2-633\rho+884\right)y^6\\
&\quad +16\left(4\rho^3-19\rho^2-189\rho+264\right)y^5-8(1-\rho)\left(-49\rho^3+48\rho^2+75\rho+166\right)y^4\\
&\quad +8(1-\rho)^2\left(59\rho^2+40\rho+21\right)y^3\Bigg)\frac{\delta(x)}{5(y-1)^7}\\
&\quad +\Bigg(336y^8-8\left(-24\rho^2-45\rho+241\right)y^7+12\left(49\rho^3-333\rho^2-440\rho+1242\right)y^6\\
&\quad -8\left(49\rho^3-165\rho^2-160\rho+304\right)y^5+20\left(147\rho^3-327\rho^2-222\rho+316\right)y^4\\
&\quad -40\left(98\rho^3-144\rho^2-65\rho+75\right)y^3+120(1-\rho)\left(5-17\rho^2\right)y^2\Bigg)\frac{\theta(x)}{15(y-1)^6}\\
&\quad +\Bigg(-4y^{12}+8(5-\rho)y^{11}-4\left(\rho^2-21\rho+42\right)y^{10}+8\left(7\rho^2-44\rho+49\right)y^9\\
&\quad -4\left(-3\rho^3+66\rho^2-195\rho+140\right)y^8+8(1-\rho)\left(11\rho^2-62\rho+63\right)y^7\\
&\quad -4(5-3\rho)(1-\rho)^2(14-\rho)y^6+8(11-5\rho)(1-\rho)^3y^5\\
&\quad -4(1-\rho)^4(3-\rho)y^4\Bigg)\frac{\delta'(x)}{15(y-1)^8}\Bigg)\sqrt{\rho}\,m_b\,\hat{s}_8\\
&+\Bigg(\Bigg(-48y^{11}+24(19-2\rho)y^{10}-16(132-29\rho)y^9+8\left(7\rho^2-226\rho+729\right)y^8\\
&\quad -8\left(-22\rho^2-535\rho+1260\right)y^7+8\left(28\rho^3-123\rho^2-862\rho+1371\right)y^6\\
&\quad -8(1-\rho)\left(-87\rho^2+20\rho+912\right)y^5+8(1-\rho)^2\left(21\rho^2+164\rho+339\right)y^4\\
&\quad -24(1-\rho)^3(13\rho+18)y^3\Bigg)\frac{\delta(x)}{5(y-1)^7}\\
&\quad +\Bigg(-244y^8+4\left(-27\rho^2-50\rho+338\right)y^7-4\left(21\rho^3-182\rho^2-240\rho+758\right)y^6\\
&\quad +4\left(126\rho^3-525\rho^2-390\rho+856\right)y^5-20\left(63\rho^3-168\rho^2-34\rho+97\right)y^4\\
&\quad +40(1-\rho)\left(-42\rho^2+23\rho+11\right)y^3-360(1-\rho)^2\rho y^2\Bigg)\frac{\theta(x)}{15(y-1)^6}\\
&\quad +\Bigg(16y^{12}-32(5-\rho)y^{11}+8\left(2\rho^2-37\rho+84\right)y^{10}-16\left(9\rho^2-73\rho+98\right)y^9\\
&\quad +8\left(-\rho^3+72\rho^2-315\rho+280\right)y^8-32(1-\rho)\left(\rho^2-37\rho+63\right)y^7\\
&\quad +8(1-\rho)^2(4-\rho)(4\rho+35)y^6-16(1-\rho)^3(5\rho+22)y^5
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 24(1-\rho)^4(\rho+2)y^4\Big)\frac{\delta'(x)}{15(y-1)^8}\Bigg)\sqrt{\rho}\,m_b\,\hat{s}_5\\
&+\Bigg(\Bigg(48y^{11}-8(47-6\rho)y^{10}+384(3-\rho)y^9-8\left(7\rho^2-146\rho+209\right)y^8\\
&\quad+8\left(28\rho^2-185\rho+110\right)y^7+8\left(-8\rho^3-17\rho^2+22\rho+79\right)y^6\\
&\quad-8\left(3\rho^3+107\rho^2-178\rho+148\right)y^5+8(1-\rho)\left(21\rho^3+103\rho^2-85\rho+81\right)y^4\\
&\quad-8(1-\rho)^2\left(39\rho^2-15\rho+16\right)y^3\Big)\frac{\delta(x)}{5(y-1)^7}\\
&\quad+\Big(4y^8-4\left(-27\rho^2-10\rho+38\right)y^7+4\left(21\rho^3-172\rho^2-80\rho+198\right)y^6\\
&\quad-4\left(126\rho^3-465\rho^2-270\rho+436\right)y^5+20\left(63\rho^3-138\rho^2-98\rho+97\right)y^4\\
&\quad-40\left(42\rho^3-37\rho^2-44\rho+27\right)y^3+120(1-\rho)\left(-3\rho^2-3\rho+2\right)y^2\Big)\frac{\theta(x)}{15(y-1)^6}\\
&\quad+\Big(24y^{12}-8(25-6\rho)y^{11}+8\left(3\rho^2-43\rho+91\right)y^{10}-8\left(17\rho^2-134\rho+189\right)y^9\\
&\quad+8\left(\rho^3+43\rho^2-235\rho+245\right)y^8-8\left(4\rho^3+63\rho^2-250\rho+203\right)y^7\\
&\quad+8(1-\rho)\left(\rho^3-56\rho+105\right)y^6-8(1-\rho)^2\left(5\rho^2+4\rho+31\right)y^5\\
&\quad+8(1-\rho)^3\left(3\rho^2+3\rho+4\right)y^4\Big)\frac{\delta'(x)}{15(y-1)^8}\Bigg)\sqrt{\rho}\,m_b\,\hat{s}_7\\
&+\Bigg(\Bigg(-36y^{12}+4(89-8\rho)y^{11}-8(174-37\rho)y^{10}+4\left(\rho^2-236\rho+680\right)y^9\\
&\quad-8\left(-22\rho^2-103\rho+311\right)y^8-8\left(-13\rho^3+177\rho^2-254\rho+3\right)y^7\\
&\quad+8(1-\rho)\left(73\rho^2-459\rho+298\right)y^6-4(1-\rho)^2\left(-7\rho^2-460\rho+592\right)y^5\\
&\quad+12(85-27\rho)(1-\rho)^3y^4-172(1-\rho)^4y^3\Big)\frac{\delta(x)}{5(y-1)^8}\\
&\quad+\Big(10\left(-3\rho^2+2\rho+4\right)y^8-2\left(7\rho^3-106\rho^2+48\rho+140\right)y^7+120(1-\rho)^3y^2\\
&\quad+14\left(7\rho^3-46\rho^2+8\rho+60\right)y^6-2\left(147\rho^3-546\rho^2-212\rho+680\right)y^5\\
&\quad+10(1-\rho)\left(-49\rho^2-2\rho+124\right)y^4-40(1-\rho)^2(\rho+15)y^3\Big)\frac{\theta(x)}{15(y-1)^7}\\
&\quad+\Big(-y^{14}+(28-\rho)y^{13}-(187-43\rho)y^{12}+8\left(2\rho^2-37\rho+76\right)y^{11}\\
&\quad-2\left(65\rho^2-462\rho+581\right)y^{10}+2\left(-5\rho^3+198\rho^2-805\rho+700\right)y^9\\
&\quad-2(1-\rho)\left(3\rho^2-294\rho+539\right)y^8+8(1-\rho)^2\left(-3\rho^2+2\rho+64\right)y^7\\
&\quad-(1-\rho)^3(83\rho+133)y^6+(1-\rho)^4(19\rho+12)y^5+(1-\rho)^5y^4\Big)\frac{\delta'(x)}{15(y-1)^9}\\
&\quad+\Big(y^{15}-2(5-\rho)y^{14}+\left(\rho^2-22\rho+45\right)y^{13}-8(3-2\rho)(5-\rho)y^{12}\\
&\quad+2\left(-2\rho^3+45\rho^2-140\rho+105\right)y^{11}-4(1-\rho)\left(9\rho^2-56\rho+63\right)y^{10}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2(1-\rho)^2\left(3\rho^2-56\rho+105\right)y^9 - 8(15-4\rho)(1-\rho)^3y^8 + (45-4\rho)(1-\rho)^4y^7 \\
&- 10(1-\rho)^5y^6 + (1-\rho)^6y^5\Big)\frac{\delta''(x)}{15(y-1)^{10}}\Bigg)\sqrt{\rho}\,m_b\,\hat{s}_3 \\
+ \Bigg(\Bigg(&8\left(3\rho^3+318\rho^2-1666\rho+1892\right)y^7 - 8\left(108\rho^2-788\rho+1181\right)y^8 \\
&- 16\left(-16\rho^3+294\rho^2-1059\rho+951\right)y^6 + 16\left(9\rho^2-104\rho+225\right)y^9 \\
&+ 8\left(31\rho^4-152\rho^3+675\rho^2-1608\rho+1174\right)y^5 + 8(\rho+7)y^{11} \\
&- 4(1-\rho)\left(-129\rho^3+323\rho^2-529\rho+815\right)y^4 - 16(47-11\rho)y^{10} \\
&+ 8(1-\rho)^2\left(56\rho^2+3\rho+61\right)y^3 + 4y^{12}\Bigg)\frac{\delta(x)}{5(y-1)^8} \\
+ \Bigg(&- 60y^9 + 20(7-6\rho)(\rho+1)y^8 + 4\left(-31\rho^3+258\rho^2-124\rho+155\right)y^7 \\
&- 4\left(-217\rho^3+966\rho^2-623\rho+775\right)y^6 + 4\left(-651\rho^3+2028\rho^2-1364\rho+1375\right)y^5 \\
&- 20\left(-217\rho^3+489\rho^2-305\rho+247\right)y^4 + 20\left(-187\rho^3+318\rho^2-172\rho+113\right)y^3 \\
&- 60(1-\rho)\left(23\rho^2-6\rho+7\right)y^2\Bigg)\frac{\theta(x)}{15(y-1)^7} \\
+ \Bigg(&- 2\left(-15\rho^2+8\rho+21\right)y^{10} - 2(5-\rho)\rho(5\rho+3)y^9 + 4(4-\rho)(\rho+3)y^{11} \\
&+ 2\left(-7\rho^3-57\rho^2+83\rho+21\right)y^8 - 4(1-\rho)\left(-\rho^3-30\rho^2+79\rho+12\right)y^7 \\
&+ (1-\rho)^2\left(-37\rho^2+270\rho+27\right)y^6 - (1-\rho)^3\left(-\rho^2+113\rho+8\right)y^5 \\
&+ (1-\rho)^4(19\rho+1)y^4 - y^{14} + (8-\rho)y^{13} - 3(9-\rho)y^{12}\Bigg)\frac{\delta'(x)}{15(y-1)^9} \\
+ \Bigg(&y^{15} - 2(5-\rho)y^{14} + \left(\rho^2-22\rho+45\right)y^{13} - 8(3-2\rho)(5-\rho)y^{12} \\
&+ 2\left(-2\rho^3+45\rho^2-140\rho+105\right)y^{11} - 4(1-\rho)\left(9\rho^2-56\rho+63\right)y^{10} \\
&+ 2(1-\rho)^2\left(3\rho^2-56\rho+105\right)y^9 - 8(15-4\rho)(1-\rho)^3y^8 \\
&+ 5(45-4\rho)(1-\rho)^4y^7 - 2(1-\rho)^5y^6 + (1-\rho)^6y^5\Bigg)\frac{\delta''(x)}{15(y-1)^{10}}\Bigg)\sqrt{\rho}\,m_b\,\hat{s}_2 \\
+ \Bigg(\Bigg(&- 4\left(9\rho^2-274\rho+670\right)y^9 - 4\left(43\rho^4+114\rho^3-645\rho^2-874\rho+1602\right)y^5 \\
&+ 24\left(14\rho^2-159\rho+278\right)y^8 - 48\left(2\rho^3+17\rho^2-144\rho+218\right)y^7 \\
&+ 8\left(17\rho^3-18\rho^2-852\rho+1303\right)y^6 + 4y^{12} - 4(21-2\rho)y^{11} \\
&+ 12(1-\rho)\left(-27\rho^3-121\rho^2+123\rho+185\right)y^4 + 72(9-2\rho)y^{10} \\
&- 4(1-\rho)^2\left(-7\rho^2+164\rho+83\right)y^3\Bigg)\frac{\delta(x)}{5(y-1)^8} \\
+ \Bigg(&60y^9 - 30\left(-3\rho^2-2\rho+2\right)y^8 - 2\left(-43\rho^3+354\rho^2-12\rho+500\right)y^7 \\
&+ 2\left(-301\rho^3+1218\rho^2-504\rho+1880\right)y^6 - 6\left(-301\rho^3+798\rho^2-384\rho+990\right)y^5 \\
&+ 10\left(-301\rho^3+513\rho^2-198\rho+490\right)y^4 - 40\left(-64\rho^3+63\rho^2-15\rho+52\right)y^3
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 360(1-\rho)\left(2\rho^2+\rho+1\right)y^2\Big)\frac{\theta(x)}{15(y-1)^7}\\
&+\Big(4(4-\rho)(\rho+13)y^{11} - 2\left(-5\rho^2-152\rho+301\right)y^{10} - y^{14} + (8-\rho)y^{13}\\
&\quad + 2\left(5\rho^3+78\rho^2-555\rho+560\right)y^9 - 2(1-\rho)\left(13\rho^2-404\rho+679\right)y^8\\
&\quad + 4(1-\rho)^2\left(-\rho^2-81\rho+268\right)y^7 - (533-57\rho)(1-\rho)^3y^6 - (47-3\rho)y^{12}\\
&\quad + (1-\rho)^4(152-\rho)y^5 - 19(1-\rho)^5y^4\Big)\frac{\delta'(x)}{15(y-1)^9}\\
&+\Big(y^{15} - 2(5-\rho)y^{14} + \left(\rho^2-22\rho+45\right)y^{13} - 8(3-2\rho)(5-\rho)y^{12}\\
&\quad + 2\left(-2\rho^3+45\rho^2-140\rho+105\right)y^{11} - 4(1-\rho)\left(9\rho^2-56\rho+63\right)y^{10}\\
&\quad + 2(1-\rho)^2\left(3\rho^2-56\rho+105\right)y^9 - 8(15-4\rho)(1-\rho)^3y^8\\
&\quad + 5(45-4\rho)(1-\rho)^4y^7 - 2(1-\rho)^5y^6 + (1-\rho)^6y^5\Big)\frac{\delta''(x)}{15(y-1)^{10}}\Big)\sqrt{\rho}\,m_b\,\hat{s}_4\\
+\Big(\Big(&-24y^{12} + 8(23-6\rho)y^{11} - 8(41-18\rho)y^{10} - 8\left(33\rho^2-58\rho+95\right)y^9\\
&\quad + 8\left(133\rho^2-323\rho+521\right)y^8 - 8(1-\rho)^2\left(201\rho^2+33\rho+26\right)y^3\\
&\quad - 8\left(38\rho^3+153\rho^2-581\rho+977\right)y^7 + 8\left(53\rho^3+98\rho^2-548\rho+997\right)y^6\\
&\quad - 8\left(51\rho^4-197\rho^3+265\rho^2-308\rho+589\right)y^5\\
&\quad + 8(1-\rho)\left(-117\rho^3+464\rho^2+83\rho+190\right)y^4\Big)\frac{\delta(x)}{5(y-1)^8}\\
&+\Big(-4\left(-1071\rho^3+3423\rho^2-2244\rho+655\right)y^5 - 4\left(-51\rho^3+403\rho^2-424\rho+360\right)y^7\\
&\quad + 4\left(-357\rho^3+1561\rho^2-1288\rho+620\right)y^6 - 60y^9 + 20\left(9\rho^2-12\rho+23\right)y^8\\
&\quad + 20\left(-357\rho^3+884\rho^2-476\rho+87\right)y^4 - 40(1-3\rho)\left(52\rho^2-91\rho+17\right)y^3\\
&\quad + 120(1-8\rho)(1-3\rho)(1-\rho)y^2\Big)\frac{\theta(x)}{15(y-1)^7}\\
&+\Big(6y^{14} - 6(8-\rho)y^{13} + 2(\rho+101)y^{12} - 2(1-\rho)^2\left(-3\rho^3+172\rho^2+165\rho+136\right)y^5\\
&\quad + 4\left(-50\rho^2+54\rho+343\right)y^{10} - 4\left(-5\rho^3-101\rho^2-5\rho+560\right)y^9\\
&\quad + 4\left(-4\rho^3-124\rho^2-149\rho+637\right)y^8 - 4\left(-\rho^4-4\rho^3-101\rho^2-222\rho+488\right)y^7\\
&\quad + 2(1-\rho)\left(-41\rho^3+71\rho^2+171\rho+479\right)y^6 - 4\left(-11\rho^2+26\rho+152\right)y^{11}\\
&\quad + 2(1-\rho)^3\left(57\rho^2+36\rho+17\right)y^4\Big)\frac{\delta'(x)}{15(y-1)^9}\\
&+\Big(-6y^{15} + 12(5-\rho)y^{14} - 2\left(3\rho^2-56\rho+135\right)y^{13} + 8\left(7\rho^2-58\rho+90\right)y^{12}\\
&\quad - 4\left(-\rho^3+55\rho^2-280\rho+315\right)y^{11} + 8\left(-2\rho^3+60\rho^2-217\rho+189\right)y^{10}\\
&\quad - 4\left(-\rho^4-6\rho^3+160\rho^2-448\rho+315\right)y^9 + 8(1-\rho)\left(\rho^3+3\rho^2-64\rho+90\right)y^8\\
&\quad - 2(1-\rho)^2\left(-2\rho^3-\rho^2-2\rho+135\right)y^7 + 20(1-\rho)^3\left(\rho^2+2\rho+3\right)y^6
\end{aligned}
$$

$$- 2(1-\rho)^4 \left(3\rho^2+4\rho+3\right) y^5 \Bigg) \frac{\delta''(x)}{15(y-1)^{10}} \Bigg) \sqrt{\rho}\, m_b\, \hat{s}_1 \tag{D.21}$$

D.1.6. Right-handed tensor current

$$\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,0)}_{c_L d_R} = - (y-2\rho-1)(y+\rho-1)^2 \frac{4\, y^3 \theta(x)}{(y-1)^3} m_b \tag{D.22}$$

$$\begin{aligned}
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,2)}_{c_L d_R} =& \Bigg(\Bigg(5y^5 - 25y^4 + y^3\left(9\rho^2+50\right) + y^2\left(8\rho^3-21\rho^2-50\right) \\
& + y\left(-22\rho^3-3\rho^2+25\right) - 5(\rho-1)^2(2\rho+1) \Bigg) \frac{2\, y^3\theta(x)}{3(y-1)^5} \\
& - \left(y^2-2y-\rho+1\right)^2 (y+\rho-1) \frac{24\, \rho\, y^4 \delta(x)}{(y-1)^6} \Bigg) m_b\, \hat{\mu}_1 \\
& - \Bigg(\Bigg(y^4 + 4y^3(\rho-1) - y^2\left(\rho^2+16\rho-6\right) - 4y\left(\rho^3-\rho^2-5\rho+1\right) \\
& + 10\rho^3 - 3\rho^2 - 8\rho + 1 \Bigg) \frac{2\, y^3\theta(x)}{(y-1)^4} \Bigg) m_b\, \hat{\mu}_2
\end{aligned} \tag{D.23}$$

$$\begin{aligned}
\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,3)}_{c_L d_R} =& \Bigg(\Bigg(y^7 - y^6(7-2\rho) + y^5\left(3\rho^2-4\rho+21\right) - y^4\left(3\rho^2+10\rho+35\right) \\
& + y^3\left(2\rho^3-11\rho^2+40\rho+35\right) - y^2\left(-6\rho^3-21\rho^2+50\rho+21\right) \\
& + y(1-\rho)\left(-3\rho^3+23\rho^2+35\rho+7\right) - (1-\rho)^2\left(13\rho^2+8\rho+1\right) \Bigg) \frac{8\, y^4\delta(x)}{(y-1)^7} \\
& + \Bigg(- 24y^9 + 48y^8(2-\rho) - 8y^7\left(3\rho^2-29\rho+15\right) - 8y^6\rho\left(-\rho^2-11\rho+52\right) \\
& + 24y^5\left(-2\rho^3-\rho^2+14\rho+5\right) - 8y^4\left(-15\rho^3+21\rho^2+14\rho+12\right) \\
& + 8y^3(1-\rho)\left(20\rho^2+4\rho+3\right) \Bigg) \frac{\theta(x)}{3(y-1)^6} \\
& + \Bigg(- y^7 + y^6(7-\rho) - 7y^5(3-\rho) + y^4\left(\rho^2-20\rho+35\right) - y^3\left(3\rho^2-30\rho+35\right) \\
& + y^2(1-\rho)(3-\rho)(\rho+7) - y(1-\rho)^2(3\rho+7) + (1-\rho)^3(\rho+1) \Bigg) \frac{4\, \rho\, y^5 \delta'(x)}{3(y-1)^8} \Bigg) m_b\, \hat{\rho}_1 \\
& + \Bigg(- 2y^3 + 3y^2(2-\rho) - y(1-\rho)(6-\rho) + 2(1-\rho)^2 \Bigg) \frac{4\, y^3\theta(x)}{(y-1)^3}\, m_b\, \hat{\rho}_2
\end{aligned} \tag{D.24}$$

$$\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\right)^{(0,4)}_{c_L d_R} = \Bigg(\Bigg(8y^{10} - 8(11-16\rho)y^9 + 40\left(3\rho^2-18\rho+8\right) y^8 - 16\left(27\rho^2-91\rho+35\right) y^7$$

$$
\begin{aligned}
&\quad + 8\left(15\rho^3 + 28\rho^2 - 166\rho + 65\right) y^6 - 8\left(33\rho^3 - 58\rho^2 - 66\rho + 31\right) y^5 \\
&\quad + 8(1-\rho)\left(-15\rho^3 - 49\rho^2 - 2\rho + 6\right) y^4\Bigg) \frac{\delta(x)}{5(y-1)^6} \\
&+ \Bigg(- 80y^8 + 4(105 - 43\rho)y^7 - 4\left(27\rho^2 - 218\rho + 230\right) y^6 \\
&\quad + 4\left(10\rho^3 + 111\rho^2 - 460\rho + 260\right) y^5 - 200\left(\rho^3 + 3\rho^2 - 9\rho + 3\right) y^4 \\
&\quad + 20\left(20\rho^3 + 18\rho^2 - 33\rho + 7\right) y^3\Bigg) \frac{\theta(x)}{15(y-1)^5} \\
&+ \Bigg(8\rho y^{10} - 8(5-\rho)\rho y^9 + 16(5-3\rho)\rho y^8 - 16(1-\rho)(5-\rho)\rho y^7 \\
&\quad + 40(1-\rho)^2\rho y^6 - 8(1-\rho)^3\rho y^5\Bigg) \frac{\delta'(x)}{15(y-1)^7}\Bigg) m_b\, \hat{s}_6 \\
+ \Bigg(\Bigg(&8y^{10} + 8(4-29\rho)y^9 - 40\left(6\rho^2 - 39\rho + 7\right) y^8 + 16\left(63\rho^2 - 239\rho + 40\right) y^7 \\
&\quad - 8\left(30\rho^3 + 167\rho^2 - 554\rho + 85\right) y^6 + 8\left(57\rho^3 + 88\rho^2 - 309\rho + 44\right) y^5 \\
&\quad - 8(1-\rho)\left(-30\rho^3 - 41\rho^2 - 58\rho + 9\right) y^4\Bigg) \frac{\delta(x)}{5(y-1)^6} \\
&+ \Bigg(160y^8 - 28(30 - 11\rho)y^7 + 4\left(48\rho^2 - 367\rho + 460\right) y^6 \\
&\quad - 4\left(20\rho^3 + 189\rho^2 - 725\rho + 520\right) y^5 + 20\left(20\rho^3 + 45\rho^2 - 129\rho + 60\right) y^4 \\
&\quad - 40\left(20\rho^3 + 6\rho^2 - 21\rho + 7\right) y^3\Bigg) \frac{\theta(x)}{15(y-1)^5} \\
&+ \Bigg(8\rho y^{10} - 8(5-\rho)\rho y^9 + 16(5-3\rho)\rho y^8 - 16(1-\rho)(5-\rho)\rho y^7 \\
&\quad + 40(1-\rho)^2\rho y^6 - 8(1-\rho)^3\rho y^5\Bigg) \frac{\delta'(x)}{15(y-1)^7}\Bigg) m_b\, \hat{s}_8 \\
+ \Bigg(\Bigg(& - 48y^{10} + 16(23 - 3\rho)y^9 - 280(4-\rho)y^8 + 8\left(-\rho^2 - 72\rho + 220\right) y^7 \\
&\quad - 16\left(-16\rho^2 - 33\rho + 95\right) y^6 + 8(1-\rho)\left(-13\rho^2 + 60\rho + 86\right) y^5 \\
&\quad - 8(1-\rho)^2(29\rho + 16)y^4\Bigg) \frac{\delta(x)}{5(y-1)^6} \\
&+ \Bigg(4(33\rho + 10)y^7 - 4\left(-27\rho^2 + 173\rho + 40\right) y^6 + 4\left(-121\rho^2 + 355\rho + 60\right) y^5 \\
&\quad - 20\left(-40\rho^2 + 67\rho + 8\right) y^4 + 40(1-\rho)(13\rho + 1)y^3\Bigg) \frac{\theta(x)}{15(y-1)^5} \\
&+ \Bigg(- 8\rho y^{10} + 8(5-\rho)\rho y^9 - 16(5-3\rho)\rho y^8 + 16(1-\rho)(5-\rho)\rho y^7 \\
&\quad - 40(1-\rho)^2\rho y^6 + 8(1-\rho)^3\rho y^5\Bigg) \frac{\delta'(x)}{15(y-1)^7}\Bigg) m_b\, \hat{s}_7
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg(\bigg(-8(1-3\rho)y^9+8\left(3\rho^2-23\rho+5\right)y^8-16\left(7\rho^2-33\rho+5\right)y^7\\
&\quad+8\left(3\rho^3+31\rho^2-90\rho+10\right)y^6-8\left(4\rho^3+38\rho^2-59\rho+5\right)y^5\\
&\quad+8(1-\rho)\left(-3\rho^3+4\rho^2-14\rho+1\right)y^4\bigg)\frac{\delta(x)}{(y-1)^6}\\
&\quad+\bigg(-16y^8+2(38-15\rho)y^7-2\left(9\rho^2-61\rho+72\right)y^6+2\left(4\rho^3+32\rho^2-89\rho+68\right)y^5\\
&\quad-2\left(20\rho^3+25\rho^2-43\rho+32\right)y^4+4\left(20\rho^3-11\rho^2+3\right)y^3\bigg)\frac{\theta(x)}{3(y-1)^5}\\
&\quad+\bigg(-4\rho y^{10}+4(5-\rho)\rho y^9-8(5-3\rho)\rho y^8+8(1-\rho)(5-\rho)\rho y^7\\
&\quad-20(1-\rho)^2\rho y^6+4(1-\rho)^3\rho y^5\bigg)\frac{\delta'(x)}{3(y-1)^7}\Bigg)m_b\,\hat{s}_9\\
&+\Bigg(\bigg(-32y^{11}+56(3\rho+4)y^{10}-8\left(-15\rho^2+151\rho+84\right)y^9\\
&\quad+16\left(-37\rho^2+231\rho+70\right)y^8-8\left(-20\rho^3-197\rho^2+758\rho+140\right)y^7\\
&\quad+8\left(-58\rho^3-345\rho^2+697\rho+84\right)y^6-8(1-\rho)\left(15\rho^3+47\rho^2+367\rho+28\right)y^5\\
&\quad+8(1-\rho)^2\left(35\rho^2+76\rho+4\right)y^4\bigg)\frac{\delta(x)}{5(y-1)^7}\\
&\quad+\bigg(-160y^9+8(110-29\rho)y^8-12\left(9\rho^2-92\rho+170\right)y^7\\
&\quad+4\left(10\rho^3+113\rho^2-488\rho+640\right)y^6-4\left(60\rho^3+111\rho^2-340\rho+460\right)y^5\\
&\quad+60\left(10\rho^3-3\rho^2-2\rho+12\right)y^4-40(1-\rho)(7\rho+3)y^3\bigg)\frac{\theta(x)}{15(y-1)^6}\\
&\quad+\bigg(20\rho y^{12}-4(43-5\rho)\rho y^{11}+4(153-43\rho)\rho y^{10}-4\rho\left(5\rho^2-156\rho+295\right)y^9\\
&\quad+4\rho\left(31\rho^2-294\rho+335\right)y^8-4(1-\rho)\rho\left(-5\rho^2-76\rho+225\right)y^7\\
&\quad+4(1-\rho)^2\rho(7\rho+83)y^6-4(1-\rho)^3\rho(5\rho+13)y^5\bigg)\frac{\delta'(x)}{15(y-1)^8}\Bigg)m_b\,\hat{s}_5\\
&+\Bigg(\bigg(-8\left(-14\rho^4-273\rho^3+172\rho^2+231\rho+4\right)y^5-8\left(-86\rho^2+373\rho+28\right)y^9\\
&\quad+8\left(-13\rho^3-203\rho^2+730\rho+35\right)y^8-8\left(-70\rho^3-150\rho^2+855\rho+28\right)y^7\\
&\quad+16\left(-6\rho^4-104\rho^3+44\rho^2+299\rho+7\right)y^6+4y^{12}-8(13\rho+4)y^{11}\\
&\quad+4(1-\rho)\left(-41\rho^3+203\rho^2+77\rho+1\right)y^4+16\left(-6\rho^2+53\rho+7\right)y^{10}\bigg)\frac{\delta(x)}{5(y-1)^8}\\
&\quad+\bigg(80y^{10}-10(56-15\rho)y^9+30\left(3\rho^2-33\rho+52\right)y^8-4\left(8\rho^3+149\rho^2-685\rho+550\right)y^7\\
&\quad+4\left(56\rho^3+413\rho^2-1045\rho+400\right)y^6-6\left(112\rho^3+436\rho^2-625\rho+80\right)y^5\\
&\quad-10\left(-72\rho^3-227\rho^2+187\rho+4\right)y^4+40\left(-3\rho^3-20\rho^2+10\rho+1\right)y^3\bigg)\frac{\theta(x)}{15(y-1)^7}
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(-y^{14}+(9-2\rho)y^{13}-4\left(\rho^2+\rho+9\right)y^{12}+84(\rho+1)y^{11}\\
&\quad-2\left(-2\rho^3-71\rho^2+154\rho+63\right)y^{10}+2\left(30\rho^3-285\rho^2+280\rho+63\right)y^9\\
&\quad-4(1-\rho)\left(\rho^3-82\rho^2+168\rho+21\right)y^8+4(1-\rho)^2\left(-20\rho^2+109\rho+9\right)y^7\\
&\quad-(1-\rho)^3\left(-4\rho^2+151\rho+9\right)y^6+(1-\rho)^4(22\rho+1)y^5\Big)\frac{\delta'(x)}{15(y-1)^9}\\
&+\Big(\rho y^{15}-(9-\rho)\rho y^{14}+12(3-\rho)\rho y^{13}-4\rho\left(\rho^2-14\rho+21\right)y^{12}\\
&\quad+2\rho\left(15\rho^2-70\rho+63\right)y^{11}-6(1-\rho)\rho\left(\rho^2-14\rho+21\right)y^{10}\\
&\quad+28(1-\rho)^2(3-\rho)\rho y^9-4(1-\rho)^3(9-\rho)\rho y^8+9(1-\rho)^4\rho y^7\\
&\quad-(1-\rho)^5\rho y^6\Big)\frac{\delta''(x)}{15(y-1)^{10}}\Bigg)m_b\,\hat{s}_4\\
+&\Bigg(\Big(-36y^{12}+8(7\rho+26)y^{11}-8\left(-3\rho^2+39\rho+56\right)y^{10}+8\left(-9\rho^2+77\rho+42\right)y^9\\
&\quad+8\left(7\rho^3-33\rho^2-45\rho+35\right)y^8-8\left(45\rho^3-180\rho^2+55\rho+98\right)y^7\\
&\quad+8\left(3\rho^4+137\rho^3-297\rho^2+103\rho+84\right)y^6-8(1-\rho)\left(34\rho^3-118\rho^2+95\rho+34\right)y^5\\
&\quad+4(1-\rho)^2\left(-29\rho^2+48\rho+11\right)y^4\Big)\frac{\delta(x)}{5(y-1)^8}\\
&+\Big(-80y^{10}+10(44-9\rho)y^9-10\left(3\rho^2-53\rho+92\right)y^8+4\left(2\rho^3+46\rho^2-325\rho+200\right)y^7\\
&\quad+28\rho\left(-2\rho^2-16\rho+65\right)y^6-2\left(-84\rho^3-462\rho^2+805\rho+260\right)y^5\\
&\quad+10\left(12\rho^3-103\rho^2+85\rho+36\right)y^4-40(1-\rho)\left(-3\rho^2+7\rho+2\right)y^3\Big)\frac{\theta(x)}{15(y-1)^7}\\
&+\Big(-y^{14}+9(2\rho+1)y^{13}-4\left(-4\rho^2+31\rho+9\right)y^{12}+4\left(-30\rho^2+91\rho+21\right)y^{11}\\
&\quad-2\left(8\rho^3-181\rho^2+294\rho+63\right)y^{10}+2\left(30\rho^3-285\rho^2+280\rho+63\right)y^9\\
&\quad-4(1-\rho)\left(-4\rho^3-27\rho^2+98\rho+21\right)y^8+4(1-\rho)^2\left(10\rho^2+39\rho+9\right)y^7\\
&\quad-(1-\rho)^3\left(16\rho^2+31\rho+9\right)y^6+(1-\rho)^4(2\rho+1)y^5\Big)\frac{\delta'(x)}{15(y-1)^9}\\
&+\Big(\rho y^{15}-(9-\rho)\rho y^{14}+12(3-\rho)\rho y^{13}-4\rho\left(\rho^2-14\rho+21\right)y^{12}\\
&\quad+2\rho\left(15\rho^2-70\rho+63\right)y^{11}-6(1-\rho)\rho\left(\rho^2-14\rho+21\right)y^{10}\\
&\quad+28(1-\rho)^2(3-\rho)\rho y^9-4(1-\rho)^3(9-\rho)\rho y^8+9(1-\rho)^4\rho y^7\\
&\quad-(1-\rho)^5\rho y^6\Big)\frac{\delta''(x)}{15(y-1)^{10}}\Bigg)m_b\,\hat{s}_3\\
+&\Bigg(\Big(8\left(-46\rho^4+43\rho^3-192\rho^2+249\rho+66\right)y^5y^{10}+8\left(-104\rho^2+407\rho+182\right)y^9\\
&\quad-8\left(-17\rho^3-267\rho^2+770\rho+315\right)y^8+8\left(-60\rho^3-400\rho^2+895\rho+322\right)y^7\\
&\quad-16\left(-9\rho^4-21\rho^3-184\rho^2+316\rho+98\right)y^6-16\left(-9\rho^2+62\rho+28\right)
\end{aligned}
$$

$$
\begin{aligned}
&- 4(1-\rho)\left(101\rho^3+17\rho^2+103\rho+19\right)y^4+4y^{12}+8(17\rho+6)y^{11}\Big)\frac{\delta(x)}{5(y-1)^8}\\
&+\Big(-80y^{10}+20(28-9\rho)y^9-40(5-3\rho)(8-\rho)y^8+4\left(12\rho^3+176\rho^2-765\rho+600\right)y^7\\
&-16\left(21\rho^3+98\rho^2-265\rho+125\right)y^6+4\left(252\rho^3+396\rho^2-815\rho+220\right)y^5\\
&-20\left(64\rho^3+31\rho^2-66\rho+8\right)y^4-20\rho\left(-34\rho^2-\rho+11\right)y^3\Big)\frac{\theta(x)}{15(y-1)^7}\\
&+\Big(-y^{14}+(9-2\rho)y^{13}-4\left(\rho^2-4\rho+9\right)y^{12}+4\left(5\rho^2-14\rho+21\right)y^{11}\\
&-2\left(-2\rho^3+9\rho^2-56\rho+63\right)y^{10}+2\left(10\rho^3-35\rho^2-70\rho+63\right)y^9\\
&-4\left(-\rho^4+38\rho^3-50\rho^2-28\rho+21\right)y^8+4(1-\rho)\left(15\rho^3-59\rho^2-5\rho+9\right)y^7\\
&-(1-\rho)^2\left(4\rho^3-115\rho^2+2\rho+9\right)y^6+(1-\rho)^3\left(-22\rho^2+\rho+1\right)y^5\Big)\frac{\delta'(x)}{15(y-1)^9}\\
&+\Big(\rho y^{15}-(9-\rho)\rho y^{14}+12(3-\rho)\rho y^{13}-4\rho\left(\rho^2-14\rho+21\right)y^{12}\\
&+2\rho\left(15\rho^2-70\rho+63\right)y^{11}-6(1-\rho)\rho\left(\rho^2-14\rho+21\right)y^{10}\\
&+28(1-\rho)^2(3-\rho)\rho y^9-4(1-\rho)^3(9-\rho)\rho y^8+9(1-\rho)^4\rho y^7\\
&-(1-\rho)^5\rho y^6\Big)\frac{\delta''(x)}{15(y-1)^{10}}\Big)m_b\,\hat{s}_2\\
&+\Big(\Big(-24y^{12}-32(3\rho+4)y^{11}-8\left(-96\rho^4+213\rho^3+33\rho^2-6\rho+256\right)y^5\\
&+8\left(-22\rho^3-132\rho^2+195\rho+1190\right)y^8-8\left(-95\rho^3-40\rho^2+120\rho+1232\right)y^7\\
&+8\left(-18\rho^4-37\rho^3+52\rho^2+27\rho+756\right)y^6+8\left(-18\rho^2+69\rho+196\right)y^{10}\\
&+8(1-\rho)\left(213\rho^3+36\rho^2+34\rho+37\right)y^4-8\left(-89\rho^2+162\rho+672\right)y^9\Big)\frac{\delta(x)}{5(y-1)^8}\\
&+\Big(180\rho y^9-20\left(-9\rho^2+57\rho+4\right)y^8+4\left(-12\rho^3-261\rho^2+740\rho+100\right)y^7\\
&-4\left(-84\rho^3-567\rho^2+980\rho+200\right)y^6+4\left(-252\rho^3-441\rho^2+675\rho+200\right)y^5\\
&-20\left(-84\rho^3+18\rho^2+43\rho+20\right)y^4+80\left(-21\rho^3+9\rho^2+\rho+1\right)y^3\Big)\frac{\theta(x)}{15(y-1)^7}\\
&+\Big(-4\left(20\rho^2+21\rho+126\right)y^{11}+4\left(4\rho^3+7\rho^2+147\rho+189\right)y^{10}\\
&-4\left(-55\rho^2+315\rho+189\right)y^9+4\left(4\rho^4-27\rho^3-100\rho^2+357\rho+126\right)y^8\\
&-4\left(-10\rho^4-41\rho^3-74\rho^2+231\rho+54\right)y^7-6(9-2\rho)y^{13}\\
&+2(1-\rho)\left(-12\rho^4+97\rho^3+139\rho^2+189\rho+27\right)y^6+6y^{14}\\
&-6(1-\rho)^2\left(22\rho^3+17\rho^2+10\rho+1\right)y^5+12\left(2\rho^2-3\rho+18\right)y^{12}\Big)\frac{\delta'(x)}{15(y-1)^9}\\
&+\Big(-4\left(5\rho^2-105\rho+189\right)y^{11}+4\left(\rho^3+10\rho^2-140\rho+189\right)y^{10}-6y^{15}\\
&-4\left(3\rho^3+10\rho^2-119\rho+126\right)y^9+4(1-\rho)\left(-\rho^3-4\rho^2-9\rho+54\right)y^8
\end{aligned}
$$

$$
\begin{aligned}
&- 2(1-\rho)^2\left(7\rho^2+16\rho+27\right)y^7+6(9-\rho)y^{14}-4(54-13\rho)y^{13}\\
&+2(1-\rho)^3\left(3\rho^2+4\rho+3\right)y^6+4\left(\rho^2-49\rho+126\right)y^{12}\Big)\frac{\rho\,\delta''(x)}{15(y-1)^{10}}\Bigg)m_b\,\hat{s}_1
\end{aligned}
\tag{D.25}
$$

D.2. The total decay rate

Like in the last section we will split up the total rate according to the Dirac structures of the currents and the order in the $1/m_b$ expansion. The complete total rate can be regained by the sum

$$
\Gamma = \frac{G_F^2|V_{cb}|^2 m_b^2}{192\pi^2}\sum_{i,j}\frac{1}{m_b^i}\Gamma^{(0,i)}_{c_L\,j}\,, \tag{D.26}
$$

where $i = 1, ..., 4$ and $j = c_L, c_R, g_L, g_R, d_L, d_R$

D.2.1. Left-handed vector current

$$
\Gamma^{(0,0)}_{c_Lc_L} = -\rho^4+8\rho^3-12\rho^2\log(\rho)-8\rho+1 \tag{D.27}
$$

$$
\begin{aligned}
\Gamma^{(0,2)}_{c_Lc_L} =& \frac{1}{2}\left(-\rho^4+8\rho^3-12\rho^2\log(\rho)-8\rho+1\right)\hat{\mu}_1\\
&+\frac{1}{2}\left(-5\rho^4+24\rho^3-24\rho^2-12\rho^2\log(\rho)+8\rho-3\right)\hat{\mu}_2
\end{aligned}
\tag{D.28}
$$

$$
\Gamma^{(0,3)}_{c_Lc_L} = -\frac{2}{3}\left(5\rho^4-16\rho^3+12\rho^2+16\rho-12\log(\rho)-17\right)\hat{\rho}_1 \tag{D.29}
$$

$$
\begin{aligned}
\Gamma^{(0,4)}_{c_Lc_L} =& \frac{8}{9}\left(9\rho^4-20\rho^3+9\rho^2+6\log(\rho)+2\right)\hat{s}_1\\
&+\frac{1}{9}\left(-39\rho^4+104\rho^3-108\rho^2+120\rho-60\log(\rho)-77\right)\hat{s}_3\\
&+\frac{8}{9}\left(3\rho^4-7\rho^3+9\rho^2-21\rho+12\log(\rho)+16\right)\hat{s}_4\\
&+\frac{2}{9}\left(-9\rho^4+32\rho^3-36\rho^2+12\log(\rho)+13\right)\hat{s}_5\\
&-\frac{2}{3}(5\rho+1)(\rho-1)^3\hat{s}_6+\frac{4}{3}(5\rho+1)(\rho-1)^3\hat{s}_8\\
&+\frac{2}{9}\left(9\rho^4-20\rho^3+36\rho-12\log(\rho)-25\right)\hat{s}_7\\
&+\frac{1}{9}\left(-27\rho^4+68\rho^3-36\rho^2-36\rho+12\log(\rho)+31\right)\hat{s}_9
\end{aligned}
\tag{D.30}
$$

D.2.2. Right-handed vector current

$$\Gamma^{(0,0)}_{c_L c_R} = 4\sqrt{\rho}\left(\rho^3 + 9\rho^2 - 9\rho - 6(\rho+1)\rho\log(\rho) - 1\right) \tag{D.31}$$

$$\begin{aligned}\Gamma^{(0,2)}_{c_L c_R} =& 2\sqrt{\rho}\left(\rho^3 + 9\rho^2 - 9\rho - 6(\rho+1)\rho\log(\rho) - 1\right)\hat{\mu}_1 \\ &+ \frac{2}{3}\sqrt{\rho}\left(13\rho^3 - 27\rho^2 - 6\left(3\rho^2 - 3\rho + 2\right)\log(\rho) + 27\rho - 13\right)\hat{\mu}_2\end{aligned} \tag{D.32}$$

$$\Gamma^{(0,3)}_{c_L c_R} = \frac{16}{3}\sqrt{\rho}\left(2\rho^3 - 9\rho^2 + 18\rho - 6\log(\rho) - 11\right)\hat{\rho}_1 \tag{D.33}$$

$$\begin{aligned}\Gamma^{(0,4)}_{c_L c_R} =& -24\sqrt{\rho}(\rho-1)^3\hat{s}_1 - \frac{8}{\sqrt{\rho}}(\rho+1)(\rho-1)^3\hat{s}_4 \\ &+ \frac{8}{3\sqrt{\rho}}\left(5\rho^4 - 15\rho^3 + 18\rho^2 - 5\rho - 6\rho\log(\rho) - 3\right)\hat{s}_3 \\ &+ \frac{8}{9\sqrt{\rho}}\left(7\rho^4 - 30\rho^3 + 54\rho^2 - 34\rho - 12\rho\log(\rho) + 3\right)\hat{s}_5 \\ &+ \frac{4}{9\sqrt{\rho}}\left(23\rho^4 - 72\rho^3 + 72\rho^2 - 32\rho + 12\rho\log(\rho) + 9\right)\hat{s}_6 \\ &- \frac{16}{3}\sqrt{\rho}(\rho-1)^3\hat{s}_7 + \frac{4}{3\sqrt{\rho}}(7\rho-1)(\rho-1)^3\hat{s}_9 \\ &- \frac{8}{9\sqrt{\rho}}\left(23\rho^4 - 72\rho^3 + 72\rho^2 - 32\rho + 12\rho\log(\rho) + 9\right)\hat{s}_8\end{aligned} \tag{D.34}$$

D.2.3. Left-handed scalar current

$$\Gamma^{(0,0)}_{c_L g_L} = -\sqrt{\rho}\left(\rho^4 - 8\rho^3 + 12\rho^2\log(\rho) + 8\rho - 1\right)m_b \tag{D.35}$$

$$\Gamma^{(0,2)}_{c_L g_L} = -\frac{10}{3}\sqrt{\rho}\left(\rho^4 - 6\rho^3 + 18\rho^2 - 10\rho - 12\rho\log(\rho) - 3\right)m_b\,\hat{\mu}_2 \tag{D.36}$$

$$\begin{aligned}\Gamma^{(0,3)}_{c_L g_L} =& \frac{2}{3}\sqrt{\rho}\left(-9\rho^4 + 46\rho^3 - 114\rho^2 + 66\rho + 60\rho\log(\rho) + 11\right)m_b\,\hat{\rho}_1 \\ &+ \frac{2}{3}\sqrt{\rho}\left(\rho^4 - 2\rho^3 - 18\rho^2 - 46\rho + 12(5\rho+2)\log(\rho) + 65\right)m_b\,\hat{\rho}_2\end{aligned} \tag{D.37}$$

$$\begin{aligned}\Gamma^{(0,4)}_{c_L g_L} =& \frac{8}{9}\sqrt{\rho}\left(18\rho^4 - 73\rho^3 + 117\rho^2 - 99\rho + 12\log(\rho) + 37\right) m_b\, \hat{s}_1 \\ &+ \frac{1}{9}\sqrt{\rho}\left(-69\rho^4 + 314\rho^3 - 630\rho^2 + 438\rho + 12(15\rho - 2)\log(\rho) - 53\right) m_b\, \hat{s}_2 \\ &+ \frac{2}{9}\sqrt{\rho}\left(3\rho^4 - 16\rho^3 + 36\rho^2 - 48\rho + 12\log(\rho) + 25\right) m_b\, \hat{s}_3 \\ &+ 4\sqrt{\rho}(\rho-1)^4 m_b\, \hat{s}_4 - \frac{2}{3}\sqrt{\rho}\left(3\rho^4 - 16\rho^3 + 36\rho^2 - 48\rho + 12\log(\rho) + 25\right) m_b\, \hat{s}_5 \\ &- \frac{2}{9}\sqrt{\rho}\left(24\rho^4 - 121\rho^3 + 279\rho^2 - 231\rho + (30 - 90\rho)\log(\rho) + 49\right) m_b\, \hat{s}_6 \\ &+ \frac{2}{9}\sqrt{\rho}\left(9\rho^4 - 56\rho^3 + 144\rho^2 - 216\rho + 60\log(\rho) + 119\right) m_b\, \hat{s}_7 \\ &+ \frac{2}{9}\sqrt{\rho}\left(51\rho^4 - 248\rho^3 + 504\rho^2 - 600\rho + 132\log(\rho) + 293\right) m_b\, \hat{s}_8 \\ &- \frac{5}{9}\sqrt{\rho}\left(9\rho^4 - 40\rho^3 + 72\rho^2 - 72\rho + 12\log(\rho) + 31\right) m_b\, \hat{s}_9 \end{aligned} \tag{D.38}$$

D.2.4. Right-handed scalar current

$$\Gamma^{(0,0)}_{c_L g_r} = -\left(\rho^4 - 8\rho^3 + 12\rho^2\log(\rho) + 8\rho - 1\right) m_b \tag{D.39}$$

$$\Gamma^{(0,2)}_{c_L g_r} = \frac{2}{3}\left(-\rho^4 + 6\rho^3 - 18\rho^2 + 10\rho + 12\rho\log(\rho) + 3\right) m_b\, \hat{\mu}_2 \tag{D.40}$$

$$\begin{aligned}\Gamma^{(0,3)}_{c_L g_r} =& -\frac{2}{3} m_b\left(\rho^4 + 2\rho^3 - 30\rho^2 + 14\rho + 36\rho\log(\rho) + 13\right) m_b\, \hat{\rho}_1 \\ &+ \frac{2}{3}\left(-\rho^4 + 6\rho^3 - 18\rho^2 + 10\rho + 12\rho\log(\rho) + 3\right) m_b\, \hat{\rho}_2\end{aligned} \tag{D.41}$$

$$\begin{aligned}\Gamma^{(0,4)}_{c_L g_r} =& -\frac{4}{9}\left(9\rho^4 - 46\rho^3 + 90\rho^2 - 90\rho + 12\log(\rho) + 37\right) m_b\, \hat{s}_1 \\ &+ \frac{1}{9}\left(15\rho^4 - 46\rho^3 + 18\rho^2 + 30\rho + 12(3\rho - 2)\log(\rho) - 17\right) m_b\, \hat{s}_2 \\ &- \frac{8}{9}\left(\rho^3 - 9\rho^2 - 9\rho + 6(3\rho + 1)\log(\rho) + 17\right) m_b\, \hat{s}_3 \\ &+ \frac{2}{3}\left(-3\rho^4 + 8\rho^3 - 24\rho + 12\log(\rho) + 19\right) m_b\, \hat{s}_4 \\ &+ \frac{2}{9}\left(3\rho^4 - 20\rho^3 + 72\rho^2 - 12\rho - 12(6\rho + 1)\log(\rho) - 43\right) m_b\, \hat{s}_5 \\ &+ \frac{2}{9}\left(6\rho^4 - 19\rho^3 + 9\rho^2 + 3\rho + 6(3\rho - 1)\log(\rho) + 1\right) m_b\, \hat{s}_6 \\ &- \frac{2}{9}\left(9\rho^4 - 40\rho^3 + 72\rho^2 - 72\rho + 12\log(\rho) + 31\right) m_b\, \hat{s}_7 \\ &- \frac{2}{9}\left(15\rho^4 - 56\rho^3 + 72\rho^2 - 24\rho - 12\log(\rho) - 7\right) m_b\, \hat{s}_8 \\ &+ \frac{1}{9}\left(9\rho^4 - 40\rho^3 + 72\rho^2 - 72\rho + 12\log(\rho) + 31\right) m_b\, \hat{s}_9\end{aligned} \tag{D.42}$$

D.2.5. Left-handed tensor current

$$\Gamma^{(0,0)}_{c_L d_L} = \sqrt{\rho}\left(\rho^4 - 12\rho^3 - 36\rho^2 + 44\rho + 12(3\rho+2)\rho\log(\rho) + 3\right) m_b \tag{D.43}$$

$$\begin{aligned}\Gamma^{(0,2)}_{c_L d_L} =& \frac{1}{2}\sqrt{\rho}\left(\rho^4 - 12\rho^3 - 36\rho^2 + 44\rho + 12(3\rho+2)\rho\log(\rho) + 3\right) m_b\,\hat{\mu}_1 \\ &+ \frac{1}{6}\sqrt{\rho}\left(15\rho^4 - 100\rho^3 + 36\rho^2 + 12\left(9\rho^2 + 6\rho + 4\right)\log(\rho) - 60\rho + 109\right) m_b\,\hat{\mu}_2\end{aligned} \tag{D.44}$$

$$\Gamma^{(0,3)}_{c_L d_L} = \frac{2}{3}\sqrt{\rho}\left(5\rho^4 - 22\rho^3 + 18\rho^2 - 122\rho + 12(7\rho+4)\log(\rho) + 121\right) m_b\,\hat{\rho}_1 \tag{D.45}$$

$$\begin{aligned}\Gamma^{(0,4)}_{c_L d_L} =& -\frac{8}{3}\sqrt{\rho}\left(3\rho^4 - 10\rho^3 + 9\rho^2 + 6\rho - 6\log(\rho) - 8\right) m_b\,\hat{s}_1 \\ &+ \frac{1}{3\sqrt{\rho}}\left(13\rho^5 - 50\rho^4 + 102\rho^3 - 22\rho^2 - 67\rho + 12(2-7\rho)\rho\log(\rho) + 24\right) m_b\,\hat{s}_2 \\ &- \frac{4}{3\sqrt{\rho}}\left(2\rho^5 - 7\rho^4 + 21\rho^3 + 25\rho^2 - 47\rho - 6(7\rho+1)\rho\log(\rho) + 6\right) m_b\,\hat{s}_4 \\ &+ \frac{2}{9\sqrt{\rho}}\left(9\rho^5 - 44\rho^4 + 84\rho^3 - 72\rho^2 + 35\rho - 12\rho\log(\rho) - 12\right) m_b\,\hat{s}_5 \\ &+ \frac{2}{3\sqrt{\rho}}\left(5\rho^5 - 19\rho^4 + 33\rho^3 + 49\rho^2 - 62\rho - 42(\rho+1)\rho\log(\rho) - 6\right) m_b\,\hat{s}_6 \\ &+ \frac{2}{9}\sqrt{\rho}\left(-9\rho^4 + 32\rho^3 - 36\rho^2 + 12\log(\rho) + 13\right) m_b\,\hat{s}_7 \\ &- \frac{4}{3\sqrt{\rho}}\left(5\rho^5 - 19\rho^4 + 33\rho^3 + 49\rho^2 - 62\rho - 42(\rho+1)\rho\log(\rho) - 6\right) m_b\,\hat{s}_8 \\ &+ \frac{1}{9\sqrt{\rho}}\left(27\rho^5 - 92\rho^4 + 84\rho^3 + 72\rho^2 - 79\rho - 84\rho\log(\rho) - 12\right) m_b\,\hat{s}_9\end{aligned} \tag{D.46}$$

D.2.6. Right-handed tensor current

$$\Gamma^{(0,0)}_{c_L d_R} = \left(-3\rho^4 - 44\rho^3 + 36\rho^2 + 12(2\rho+3)\rho^2\log(\rho) + 12\rho - 1\right) m_b \tag{D.47}$$

$$\begin{aligned}\Gamma^{(0,2)}_{c_L d_R} =& -\frac{1}{2}\left(3\rho^4 + 44\rho^3 - 36\rho^2 - 12(2\rho+3)\rho^2\log(\rho) - 12\rho + 1\right) m_b\,\hat{\mu}_1 \\ &- \frac{1}{6}\left(29\rho^4 - 12\rho^3 + 36\rho^2 - 12\left(6\rho^2 - 3\rho + 8\right)\rho\log(\rho) - 20\rho - 33\right) m_b\,\hat{\mu}_2\end{aligned} \tag{D.48}$$

$$\Gamma^{(0,3)}_{c_L d_R} = -\frac{2}{3}\left(7\rho^4 - 38\rho^3 + 90\rho^2 - 106\rho - 12(\rho-2)\log(\rho) + 47\right) m_b\,\hat{\rho}_1 \tag{D.49}$$

$$
\begin{aligned}
\Gamma^{(0,4)}_{c_L d_R} =& \frac{8}{3}\left(3\rho^4 - 10\rho^3 + 9\rho^2 + 6\rho - 6\log(\rho) - 8\right) m_b\, \hat{s}_1 \\
&+ \frac{1}{3}\left(-15\rho^4 + 50\rho^3 - 54\rho^2 - 66\rho + 12(3\rho+4)\log(\rho) + 85\right) m_b\, \hat{s}_2 \\
&+ \frac{4}{3}(\rho+3)\left(2\rho^3 - 9\rho^2 + 18\rho - 6\log(\rho) - 11\right) m_b\, \hat{s}_4 \\
&- \frac{2}{9}\left(11\rho^4 - 56\rho^3 + 108\rho^2 - 200\rho + 12(4\rho+5)\log(\rho) + 137\right) m_b\, \hat{s}_5 \\
&- \frac{2}{9}\left(17\rho^4 - 61\rho^3 + 45\rho^2 - 107\rho + 6(17\rho+5)\log(\rho) + 106\right) m_b\, \hat{s}_6 \\
&+ \frac{2}{9}\left(3\rho^4 - 16\rho^3 + 36\rho^2 - 48\rho + 12\log(\rho) + 25\right) m_b\, \hat{s}_7 \\
&+ \frac{4}{9}\left(17\rho^4 - 61\rho^3 + 45\rho^2 - 107\rho + 6(17\rho+5)\log(\rho) + 106\right) m_b\, \hat{s}_8 \\
&- \frac{1}{9}\left(33\rho^4 - 124\rho^3 + 144\rho^2 - 132\rho + 12(6\rho+1)\log(\rho) + 79\right) m_b\, \hat{s}_9
\end{aligned}
\tag{D.50}
$$

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